Chapter 2
Forecast Error

2.1 Introduction

The standard deviation of the 1-month-ahead forecast error is used to determine how much safety stock is needed to satisfy the level of service to customers. An exact measure of the standard deviation is not easy to generate, however, estimates are used in its place. The way to estimate the standard deviation varies depending on the forecasting method in use (moving average, regression, discounting, smoothing). This chapter describes how to estimate the standard deviation for each of the forecast models presented in Chap. 1.

Another important statistic for each forecast model is the coefficient of variation (cov). The closer the cov is to zero, the better. When cov is near 0.30, the distribution of the forecast errors is similar to normal distribution. When the cov is closer to 1.00, the distribution of the forecast errors is not like a normal distribution. In Chap. 4, the concept of the truncated normal is presented. The cov is used to estimate the parameter to use for the truncated normal. When the distribution of the forecast errors is not normally distributed, the truncated normal distribution should be applied to determine how much safety stock is needed. In determining the safety stock, the standard deviation and cov of the lead-time demand are used in the computations. The methods to generate these statistics are included in the chapter.

When the forecast and its standard deviation is generated for a total collection of SKUs, estimates are needed for each SKU. A way to measure the standard deviation for each SKU is presented. As an example, this need occurs for a style shoe where the forecast is based on all sizes combined, and estimates for inventory control are needed for each size of the style. Estimates of the standard deviation for each forecast method is provided.

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2.2 Standard Deviation for the 1-Month-Ahead Forecast Error

The forecast is the main guide on how much stock to have available for each item over the future planning months. But since the forecast is an estimate and is not definitive, an extra layer of stock is needed to fill the customer demands that may occur above the forecast. This added stock is called the safety stock. A common way to measure the accuracy of the forecast is by way of the standard deviation, σ, of the 1-month forecast error. The standard deviation is used in subsequent computations that determine how much safety stock is needed to satisfy a desired level of service set by the management.

This chapter shows how to estimate the standard deviation for a 1-month forecast error, on the various forecasting methods described in Chap. 1. In addition, the chapter describes another measure, called the coefficient of variation, denoted as cov, that is a relative way to measure the accuracy of the forecasts. The cov is the ratio of the standard deviation over the most current level. The closer cov is to zero, the more accurate the forecast. Also when the cov is 0.50 or lower, the forecast errors are distributed similar to a normal distribution, and when cov is 1.00 or higher, the forecast errors are more like an exponential distribution.

2.3 Standard Deviation for the Horizontal Moving Average Model

The data available for the horizontal moving average model is the following:

$$x_t \ \text{for} \ t = 1 \ \text{to} \ N$$

where x_t is the demand in history month t, N is the number of history months, and $t = N$ is the most current month. Also known from the forecast computation is the most current measure of the level, denoted as a`. The level is used to measure the residual error for each history month as below:

$$e_t = \left(x_t - a^{`}\right) \quad \text{for} \ t = 1 \ \text{to} \ N$$

So now, an estimate of the 1-month forecast error is computed as follows:

$$\sigma^{`} = \sqrt{\sum\nolimits_{t=1}^{N} e_t^2/(N-1)}$$

The coefficient of variation becomes:

$$\text{cov} = \sigma^{`}/a^{`}$$

2.4 Standard Deviation for the Trend Regression Model

The data available for the trend regression model is also the following:

$$x_t \ \text{for} \ t = 1 \ \text{to} \ N$$

where x_t identifies the history demand in month t, N is the number of history months, and $t = N$ is the most current month. Upon completing the forecast computations, the estimated coefficients are the following:

$$\begin{aligned} &a_o = \text{intercept at } t = 0 \\ &b = \text{slope} \\ &\grave{a} = \text{level at } t = N \end{aligned}$$

The fit measures the forecast going backwards over the history months. The fit for history month t is obtained as follows:

$$f_t = (a_o + bt) \quad \text{for } t = 1 \text{ to } N$$

The residual error for each history month is measured as below:

$$e_t = (x_t - f_t) \quad \text{for } t = 1 \text{ to } N$$

and the estimate of the 1-month forecast error is obtained as follows:

$$\grave{\sigma} = \sqrt{\sum\nolimits_{t=1}^{N} e_t^2/(N-2)}$$

In this situation, the estimate of the most current level is the fit at $t = N$, whereby,

$$\grave{a} = f_N = (a_o + bN)$$

and so, the measure of the coefficient of variations is as below:

$$\text{cov} = \grave{\sigma}/\grave{a}$$

2.5 Standard Deviation for the Horizontal Discount Model

The data available for the horizontal discount model is the following:

$$x_t \ \text{for} \ t = 1 \ \text{to} \ N$$

and

$$w_t \ \text{for} \ t = 1 \ \text{to} \ N$$

where x_t is the demand in history month t, w_t is the discount weight for month t, N is the number of history months, and t = N is the most current month.

Also known from the forecast computations is the most current measure of the level, denoted as a`. The level is used to measure the residual errors for each history month as below:

$$e_t = \left(x_t - a^{`}\right) \quad \text{for} \ t = 1 \ \text{to} \ N$$

So now, an estimate of the standard deviation of the 1-month forecast error is as follows:

$$\sigma^{`} = \sqrt{\sum_{t=1}^{N} w_t e_t^2 / \sum_{t=1}^{N} w_t}$$

The coefficient of variation becomes:

$$\text{cov} = \sigma^{`}/a^{`}$$

2.6 Standard Deviation for the Trend Discount Model

The data available for the trend discount model is the following:

$$x_t \ \text{for} \ t = 1 \ \text{to} \ N$$

and

$$w_t \ \text{for} \ t = 1 \ \text{to} \ N$$

where x_t is the demand in history month t, w_t is the discount weight for month t, N is the number of history months, and t = N is the most current month.

Also known from the forecast computations are the following coefficient estimates: a_o = intercept at t = 0, b = slope, and a` = the current measure of the level. With these coefficients, the fit of the past history months are obtained as follows:

$$f_t = (a_o + bt) \quad \text{for} \ t = 1 \ \text{to} \ N$$

The fit is used to measure the residual errors for each history month as below:

$$e_t = (x_t - f_t) \quad \text{for} \ t = 1 \ \text{to} \ N$$

With the N set of residual errors, the estimate of the standard deviation of the 1-month forecast error is estimated as follows:

$$\sigma` = \sqrt{\sum_{t=1}^{N} w_t e_t^2 / \sum_{t=1}^{N} w_t}$$

The corresponding coefficient of variation becomes:

$$\mathrm{cov} = \sigma`/a`$$

2.7 Standard Deviation for the Horizontal Smoothing Model

When horizontal smoothing is in use to forecast the monthly demands at month $t = N$, the data available are the following:

α = smoothing parameter
σ_{N-1} = estimate of the standard deviation from the prior month, $t = N-1$
a_{N-1} = estimate of the level from month $t = N-1$
x_N = demand at month $t = N$

The current 1-month-ahead forecast error is obtained as below:

$$e = (x_N - a_{N-1})$$

So now in keeping with the spirit of smoothing, the current forecast error, e, is blended with the prior estimate of the standard deviation, σ_{N-1}, to yield the current estimate of the standard deviation as shown below:

$$\sigma_N = \sqrt{\alpha e^2 + (1-\alpha)(\sigma_{N-1})^2}$$

The estimate of the level at $t = N$ is denoted as a_N, and thereby the coefficient of variation is obtained as follows:

$$\mathrm{cov} = \sigma_N / a_N$$

2.8 Standard Deviation for the Trend Smoothing Model

When the trend smoothing model is in use to forecast the monthly demands for month $t = N$, the data available are the following:

σ_{N-1} = estimate of the standard deviation from the prior month, $t = N-1$
a_{N-1} = estimate of the level from month $t = N-1$

b_{N-1} = estimate of the slope from month t = N−1
x_N = demand at month t = N

The forecast from month t = N−1 for month t = N is denoted as $x`_N$ and is obtained by the following:

$$x`_N = (a_{N-1} + b_{N-1})$$

Thereby, the current 1-month-ahead forecast error is derived as below:

$$e = \left(x_N - x`_N\right)$$

So now, the current forecast error, e, is blended with the prior estimate of the standard deviation, σ_{N-1}, to yield the current estimate of the standard deviation as shown below:

$$\sigma_N = \sqrt{\alpha e^2 + (1-\alpha)(\sigma_{N-1})^2}$$

At t = N, the level and slope are revised and are denoted as:

a_N = estimate of the level at month t = N
b_N = estimate of the slope at month t = N

The corresponding coefficient of variation is obtained as follows:

$$cov = \sigma_N / a_N$$

2.9 Standard Deviation for the Seasonal Multiplicative Model

When the seasonal-multiplicative model is in use to forecast the monthly demands for month t = N, the data available are the following:

σ_{N-1} = estimate of the standard deviation from the prior month, t = N−1
a_{N-1} = estimate of the level from month t = N−1
b_{N-1} = estimate of the slope from month t = N−1
$r_N, \ldots, r_{N+11}$ = estimate of 12 months of seasonal ratios as of t = N−1
x_N = demand at month t = N

The forecast from month t = N−1 for month t = N is denoted as $x`_N$ and is obtained by the following:

$$x`_N = (a_{N-1} + b_{N-1}) r_N$$

Thereby, the current 1-month-ahead forecast error is derived as below:

$$e = \left(x_N - x'_N\right)$$

The current forecast error, e, is blended with the prior estimate of the standard deviation, σ_{N-1}, to yield the current estimate of the standard deviation as shown below:

$$\sigma_N = \sqrt{\alpha e^2 + (1-\alpha)\,(\sigma_{N-1})^2}$$

At t = N, the level, slope and seasonal ratios are revised and denoted as:

a_N = estimate of the level at month t = N
b_N = estimate of the slope at month t = N
$r_{N+1}, \ldots, r_{N+12}$ = estimate of 12 months seasonal ratios as of t = N

The corresponding coefficient of variation is obtained as follows:

$$\text{cov} = \sigma_N / a_N$$

2.10 Standard Deviation for the Seasonal Additive Model

When the seasonal-additive model is in use to forecast the monthly demands for month t = N, the data available are the following:

σ_{N-1} = estimate of the standard deviation from the prior month, t = N−1
a_{N-1} = estimate of the level from month t = N−1
b_{N-1} = estimate of the slope from month t = N−1
$d_N, \ldots, d_{N+11}$ = estimate of 12 months of seasonal increments as of t = N−1
x_N = demand at month t = N

The forecast from month t = N−1 for month t = N is denoted as x'_N and is obtained by the following:

$$x'_N = (a_{N-1} + b_{N-1}) + d_N$$

Thereby, the current 1-month-ahead forecast error is derived as below:

$$e = \left(x_N - x'_N\right)$$

The current forecast error, e, is blended with the prior estimate of the standard deviation, σ_{N-1}, to yield the current estimate of the standard deviation as shown below:

$$\sigma_N = \sqrt{\alpha e^2 + (1-\alpha)(\sigma_{N-1})^2}$$

At $t = N$, the level, slope and seasonal additives are revised as noted below:

a_N = estimate of the level at month $t = N$
b_N = estimate of the slope at month $t = N$
$d_{N+1}, \ldots, d_{N+12}$ = estimate of 12 months of seasonal increments as of $t = N$

The corresponding coefficient of variation is obtained as follows:

$$\text{cov} = \sigma_N / a_N$$

2.11 Standard Deviation for the Cumulative Forecast

Recall, the cumulative forecast for time duration T is denoted as F_T, where $T > 0$. Whether T is an integer or a fraction, the estimate of the standard deviation of the cumulative forecast, denoted as σ_T, is the following:

$$\sigma_T = \sqrt{T}\sigma$$

The above relation assumes that the monthly future forecast errors are independent and each future month has the same standard deviation, σ.

Example 2.1 Suppose a situation where the monthly standard deviation of the forecast error is $\sigma = 10$. Below gives an example of the corresponding standard deviations for cumulative forecasts of length, $T = 0.25$, 1.7 and 4.0.

$$\sigma_{0.25} = \sqrt{0.25} \times 10 = 5.00$$
$$\sigma_{1.70} = \sqrt{1.70} \times 10 = 13.00$$
$$\sigma_{4.00} = \sqrt{4.00} \times 10 = 20.00$$

2.12 Standard Deviation for Weekly Forecasts

When the monthly forecasts are converted to weekly forecasts, the data available for the monthly forecasts are the following:

F = monthly forecast
σ = standard deviation for a monthly forecast error

The related notation for weekly forecasts is below:

f_w = weekly forecast
σ_w = standard deviation for the weekly forecast

In general, $f_w = F/(52/12) = F/4.33 = 0.23F$

Thereby, the weekly standard deviation becomes:

$$\sigma_w = \sqrt{0.23}\sigma = 0.48\sigma$$

2.13 Standard Deviation for an SKU

In many inventory situations, the units are stocked in a multitude of variations, like, models, sizes and locations. Baseball gloves are stocked in various models, shoes are in a variety of sizes, and cellular phones come in different colors and features. Each of the variations is called a stock-keeping-unit (SKU). A forecast is typically generated for the total but not directly for every SKU. For inventory control applications, however, a forecast and standard deviation is needed with each SKU.

Below shows how to generate the standard deviation by SKU as needed. The typical data known is the following:

$$\begin{aligned}
&F = \text{one-month forecast for the total of all SKUs}\\
&\sigma = \text{standard deviation for the one-month total}\\
&p_i = \text{portion of demand for SKU i}\\
&n = \text{number of SKUs}\\
&\Sigma p_i = 1
\end{aligned}$$

For a particular month, the following notation is used in the derivation:

$$\begin{aligned}
&Y = \text{total one-month forecast}\\
&x_i = \text{demand for SKU i}\\
&\Sigma x_i = Y
\end{aligned}$$

Let $x =$ demand for SKU i. Also, for notational ease, let $p = p_i$ represent the portion of demand for SKU i.

The steps to find the forecast and standard deviation for an SKU is listed below:

$$\begin{aligned}
&E[x|Y] = pY\\
&E[x] = pE[Y] = pF\\
&V[x|Y] = E[x^2|Y] - E[x|Y]^2 = Yp(1-p)\\
&E[x^2|Y] = V[x|Y] + E[x|Y]^2 = Yp(1-p) + p^2Y^2\\
&E[x^2] = E[Y]p(1-p) + p^2E[Y^2]\\
&\qquad = Fp(1-p) + p^2[F^2 + \sigma^2]\\
&V[x] = E[x^2] - E[x]^2\\
&\qquad = Fp(1-p) + p^2[F^2 + \sigma^2] - p^2F^2\\
&\qquad = Fp(1-p) + p^2\sigma^2
\end{aligned}$$

Finally, the 1-month forecast for SKU i, f_i, and the corresponding standard deviation, σ_i, for SKU i are below:

$$f_i = p_i F$$
$$\sigma_i = \left[F p_i (1 - p_i) + p_i^2 \sigma^2\right]^{0.5}$$

Example 2.2 Suppose a shoe store where the forecast for a style shoe is $F = 10$ per month, and the corresponding standard deviation is $\sigma = 3$. A particular size (SKU) has an average of ten percent ($p = 0.10$) of the sales. The forecast, f_i, and standard deviation, σ_i, for the size shoe is computed as below:

$$f_i = 0.10 \times 10 = 1.0$$
$$V(x) = 10 \times 0.10(1 - 0.10) + 0.10^2 3^2 = 0.99$$
$$\sigma_i = \sqrt{0.99} = 0.99$$

Note, the coefficient of variation for the style is:

$$\text{cov} = \sigma/F = 0.30$$

and the cov for the size (SKU i) is:

$$\text{cov}_i = \sigma_i/f_i = 0.99$$

2.14 Summary

Estimates of the 1-month-ahead standard deviation are developed for each of the forecast models of Chap. 1. The estimates vary depending on the method of forecast, as per moving average, regression, discounting and smoothing. Special estimates are also presented for the seasonal multiplicative model and for the seasonal additive model. The coefficient of variation (cov) is likewise an important statistic generated for each model. The way to estimate these statistics also varies by forecast model in use. The standard deviation and the cov are also generated when the forecasts are for time durations not of 1 month, but instead are for less than 1 month, or for more than a month. The chapter also describes how to estimate the standard deviation for an individual SKU, for a situation when the forecasting system measures the standard deviation for the aggregate collection of all the SKUs, and also for weekly forecasts.

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