A Note on Fuzzy-Valued Inference

Jorma K. Mattila

Abstract Fuzzy-valued inference is discussed. For that purpose, a theory of fuzzy-valued associative Kleene algebra is introduced. As an example, it is shown that fuzzy-valued Kleene algebras give a mathematical model for some fuzzy screening systems.

Keywords Fuzzy-Valued Inference · Fuzzy-Valued Kleene-Algebra · Screening Systems

1 Introduction

In fuzzy decision making, there are a lot of cases where scales of linguistic scores are used. For example, in fuzzy control linguistic expressions are generally used as control values in control processes. In fuzzy logic truth values are usually linguistic terms, like, for example, ‘true’, ‘almost true’, ‘not true and not false’, ‘almost false’, ‘false’. In fuzzy screening systems linguistic values are used, too, like for example ‘outstanding’, ‘very high’, ‘high’, ‘medium’, ‘low’, ‘very low’, ‘none’.

To be a scale, the scale values must have a reasonable order. For example, the truth values mentioned above are already listed in the reasonable order. Based on intuition, we ordered them using the relationship between a truth value and how near to the truth it is. The biggest difference is between truth and falsity. Similarly, the above mentioned scores for a screening system are ordered from the highest value to the lowest one. Hence, we may say that these kinds of scales are totally ordered, because the scales are finite and the ordering in any scale can be defined to be unique. However, these kinds of orderings are actually not mathematical, because they are based on intuition. Also, the use of these linguistic scores in calculations is based on intuition, even though the calculation rules can be given based on the order of the linguistic scores.

J.K. Mattila (✉)
Department of Mathematics and Physics, Lappennranta University of Technology,
Skinnarilankatu 34, FI-53850 Lappeenranta, Finland
e-mail: jorma.mattila@lut.fi

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If we want to have a formal theory for a system of this kind, we need a mathematical counterpart to the system. Very often in many-valued logics, the truth values are given as numbers. In fuzzy systems the scores can be given as fuzzy sets. These kind of things serve the link between intuitive and formal systems. There are a lot of research about algebras for numerical truth values of many-valued logics. This research creates mathematical models for many-valued logics involved in fuzzy systems. Also, we may investigate some suitable sets of *fuzzy numbers* or fuzzy intervals in order to find some algebraic models for systems with fuzzy scores. So, we have the following basic questions:

> Does there exist some mathematical models for inference systems using scales with linguistic scores?
>
> What are the logical and mathematical bases of this kind of systems?

We give an answer to these questions in the following sections.

2 A Mathematical Background

In this presentation we introduce an algebraic approach to cases where score values are fuzzy numbers or fuzzy intervals.

We recall some earlier results for manipulating fuzzy numbers. The main things are ways of representing, ordering, and defining meets and joins of a given set of fuzzy numbers.

The representation theorem for considering fuzzy sets by means of α-cuts has been given, for example, by V. Novák [9], p. 44. A. Kaufmann and M.M. Gupta [2] (cf. pp. 19–35) consider interval arithmetics applied to triangular and trapezoidal fuzzy numbers (or fuzzy intervals) presented by means of α-cuts. They also introduced some criteria for ordering of fuzzy numbers.

Besides Kaufmaan and Gupta, also R. Fullér [1] (cf. pp. 35–36) has considered ordering of fuzzy numbers by defining *fuzzy max* and *fuzzy min* operations by means of α-cuts, and V. Novák [9] (cf. pp. 98–100) by defining *join* '$\sqcup$' and *meet* '$\sqcap$' by means of Zadeh's extension principle, as follows.

Let $\mathcal{A}$, $\mathcal{B}$ be fuzzy numbers and $x, y \in \mathbb{R}$. *Join* $\mathcal{A} \sqcup \mathcal{B}$ is the fuzzy number

$$(\mathcal{A} \sqcup \mathcal{B})(z) = \bigvee_{z=x\vee y} (\mathcal{A}(x) \wedge \mathcal{B}(y)). \tag{1}$$

Meet $\mathcal{A} \sqcap \mathcal{B}$ is the fuzzy number

$$(\mathcal{A} \sqcap \mathcal{B})(z) = \bigvee_{z=x\wedge y} (\mathcal{A}(x) \wedge \mathcal{B}(y)). \tag{2}$$

These operations appear to be the same as Fullér's *fuzzy max* and *fuzzy min*.

Novák also present the following theorem.

Theorem 1 (Novák) *The fuzzy numbers form a distributive lattice with respect to the operations '$\sqcap$' and '$\sqcup$'. It means that*

$$\begin{aligned} \mathcal{A} \sqcup \mathcal{A} &= \mathcal{A} & \mathcal{A} \sqcap \mathcal{A} &= \mathcal{A} \\ \mathcal{A} \sqcup \mathcal{B} &= \mathcal{B} \sqcup \mathcal{A} & \mathcal{A} \sqcap \mathcal{B} &= \mathcal{B} \sqcap \mathcal{A} \\ (\mathcal{A} \sqcup \mathcal{B}) \sqcup \mathcal{C} &= \mathcal{A} \sqcup (\mathcal{B} \sqcup \mathcal{C}) & (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{C} &= \mathcal{A} \sqcap (\mathcal{B} \sqcap \mathcal{C}) \\ \mathcal{A} \sqcup (\mathcal{B} \sqcap \mathcal{C}) &= (\mathcal{A} \sqcup \mathcal{B}) \sqcap (\mathcal{A} \sqcup \mathcal{C}) & \mathcal{A} \sqcap (\mathcal{B} \sqcup \mathcal{C}) &= (\mathcal{A} \sqcap \mathcal{B}) \sqcup (\mathcal{A} \sqcap \mathcal{C}) \end{aligned}$$

Then he defines the ordering relation $\sqsubseteq$ for fuzzy numbers $\mathcal{A}$, $\mathcal{B}$ in the familiar way:

$$\mathcal{A} \sqsubseteq \mathcal{B} \quad \text{iff} \quad \mathcal{A} \sqcap \mathcal{B} = \mathcal{A} \quad (\mathcal{A} \sqcup \mathcal{B} = \mathcal{B} \quad \text{respectively}). \tag{3}$$

In general, fuzzy numbers do not form a linearly ordered set, except in some special cases. We will exploit some of these special cases in the following considerations.

Consider a finite set of fuzzy sets

$$T_n = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n\} \tag{4}$$

of the interval $[0, p]$, $p \in \mathbb{R}$, $p > 0$. The fuzzy sets $\mathcal{A}_i$ $(i = 1, \dots, n)$ of T_n satisfy the following properties:

(1°) $\mathcal{A}_i$ is either a fuzzy number or a fuzzy interval for all $1 \le i \le n$;
(2°) T_n is ordered, such that $\mathcal{A}_i \sqsubseteq \mathcal{A}_j$ for all $1 \le i \le j \le n$;
(3°) for every $\mathcal{A}_i \in T_n$ there exists a unique fuzzy set $\neg\mathcal{A}_i \in T_n$, such that the following condition holds:

$$\neg\mathcal{A}_i = \mathcal{A}_{n-i+1}, \text{ if } 1 \le i \le n. \tag{5}$$

where for all $i = 1, \dots, n$

$$\mathcal{A}_i(x) = \neg\mathcal{A}_i(p - x), \text{ if } x \in [0, p] \tag{6}$$

The operation symbol '$\neg$' is a *complementarity operation*, and we use the name *negation* for it.

The set T_n is linearly ordered, by the property (2°).

The definition of negation, i.e., the formulas (5) and (6) gives some presuppositions for the fuzzy sets in T_n. From the Eq. (6), it follows that the supports of $\mathcal{A}_i$ and $\neg\mathcal{A}_i$ satisfy the equivalency

$$\text{supp}\mathcal{A}_i = [a, b] \quad \Longleftrightarrow \quad \text{supp}\neg\mathcal{A}_i = [p - b, p - a] \tag{7}$$

There exist two special cases, namely $\alpha = 0$ and $\alpha = 1$. These conditions give the closures of the support and core of a given fuzzy set $\mathcal{A}$. Now, because

$[\mathcal{A}_i]^0 = \mathrm{cl}(\mathrm{supp}\,\mathcal{A}_i)$ and $[\mathcal{A}_i]^1 = \mathrm{cl}(\mathrm{core}\,\mathcal{A}_i)$ then the closures of the supports and the cores of $\mathcal{A}_i$ and $\neg\mathcal{A}_i$ satisfy the equivalencies

$$\mathrm{cl}(\mathrm{supp}\,\mathcal{A}_i) = [a_0, b_0] \iff \mathrm{cl}(\mathrm{supp}\,\neg\mathcal{A}_i) = [p - b_0, p - a_0] \tag{8}$$
$$\mathrm{cl}(\mathrm{core}\,\mathcal{A}_i) = [a_1, b_1] \iff \mathrm{cl}(\mathrm{core}\,\neg\mathcal{A}_i) = [p - b_1, p - a_1] \tag{9}$$

The lengths of the intervals $[\mathcal{A}_i]^\alpha$ and $[\neg\mathcal{A}_i]^\alpha$ are the same for any α, because if $[\mathcal{A}_i]^\alpha = [a_\alpha, b_\alpha]$ then its length is $b_\alpha - a_\alpha$, and the length of $[\neg\mathcal{A}_i]^\alpha$ is $p - a_\alpha - (p - b_\alpha) = b_\alpha - a_\alpha$, too.

By means of these considerations above, it is easy to see that the fuzzy sets $\mathcal{A}_i$ and $\neg\mathcal{A}_i$ are symmetric with respect to the vertical line $x = \frac{p}{2}$. The value $\frac{p}{2}$ is the centre of the interval $[0, p]$.

Our next purpose is to show that the set T_n (see (4)) forms a *Kleene algebra* of fuzzy numbers belonging to the set T_n. First, we have to show that the set T_n forms a DeMorgan algebra. About DeMorgan algebras, see Rasiowa [10]. (Rasiowa uses the name *quasi-Boolean algebra* for DeMorgan algebra.) To do this, we prove the following Lemmas.

Lemma 1 *The system $\mathcal{T}_n = \langle T_n, \sqcup, \sqcap\rangle$ is a distributive and complete lattice.*

Proof $\mathcal{T}_n$ is a distributive lattice by means of Theorem 1. It is also complete because for any two elements $\mathcal{A}_i, \mathcal{A}_j \in T_n$ $(1 \le i, j \le n)$ the expressions $\mathcal{A}_i \sqcup \mathcal{A}_j$ and $\mathcal{A}_i \sqcap \mathcal{A}_j$ are defined and T_n is closed under the operations $\sqcup$ and $\sqcap$, i.e., $\mathcal{A}_i \sqcup \mathcal{A}_j$ equals to either $\mathcal{A}_i$ or $\mathcal{A}_j$, and $\mathcal{A}_i \sqcap \mathcal{A}_j$ equals to either $\mathcal{A}_j$ or $\mathcal{A}_i$ respectively. □

Lemma 2 *The law of double negation*

$$\neg\neg\mathcal{A}_i = \mathcal{A}_i \tag{10}$$

for any $\mathcal{A}_i \in T_n$ holds in the lattice $\mathcal{T}_n$.

Proof The result follows from the formula (5) by an easy calculation. □

Lemma 3 *De Morgan Laws hold on T_n.*

Proof Let $\mathcal{A}_i, \mathcal{A}_j \in T_n$ be any two elements, such that $\mathcal{A}_i \sqsubseteq \mathcal{A}_j$. Hence, $i \le j$, by the property (2°). Further, $\neg\mathcal{A}_i = \mathcal{A}_{n-i+1}$ and $\neg\mathcal{A}_j = \mathcal{A}_{n-j+1}$, by (5). Comparing the subindices $n - i + 1$ and $n - j + 1$ we see that $n - j + 1 \le n - i + 1$ because $i \le j$, by assumption. Hence, $\mathcal{A}_{n-j+1} \sqsubseteq \mathcal{A}_{n-i+1}$, i.e., $\neg\mathcal{A}_j \sqsubseteq \neg\mathcal{A}_i$. So, the implication $\mathcal{A}_i \sqsubseteq \mathcal{A}_j \Longrightarrow \neg\mathcal{A}_j \sqsubseteq \neg\mathcal{A}_i$ holds. It is easy to see that this implication holds to the other direction, too. Hence, the equivalency

$$\mathcal{A}_i \sqsubseteq \mathcal{A}_j \iff \neg\mathcal{A}_j \sqsubseteq \neg\mathcal{A}_i \tag{11}$$

holds for any $\mathcal{A}_i, \mathcal{A}_j \in T_n$. Further, $\neg\mathcal{A}_j \sqcup \neg\mathcal{A}_i = \neg\mathcal{A}_i$, by (3), and hence,

$$\neg(\neg\mathcal{A}_j \sqcup \neg\mathcal{A}_i) = \neg\neg\mathcal{A}_i = \mathcal{A}_i = \mathcal{A}_i \sqcap \mathcal{A}_j$$

by assumption and by (10). Hence, one of De Morgan Laws,

$$\mathcal{A}_i \sqcap \mathcal{A}_j = \neg(\neg\mathcal{A}_j \sqcup \neg\mathcal{A}_i) \tag{12}$$

holds. The other De Morgan law, $\mathcal{A}_i \sqcup \mathcal{A}_j = \neg(\neg\mathcal{A}_j \sqcap \neg\mathcal{A}_i)$, follows from (12) by replacing $\mathcal{A}_i$ and $\mathcal{A}_j$ with $\neg\mathcal{A}_i$ and $\neg\mathcal{A}_j$, respectively, and applying the law of double negation. □

From the Lemmas 1, 2 and 3 it follows that the system $\mathcal{T}_n = \langle T_n, \sqcup, \sqcap, \neg, \mathcal{A}_n \rangle$ is *De Morgan algebra*, because T_n is a non-empty set, $\mathcal{T}_n = \langle T_n, \sqcup, \sqcap \rangle$ is a distributive lattice with top element $\mathcal{A}_n$, $\neg$ is a unary operation on T_n, and $\mathcal{T}_n$ satisfies the law of double negation and De Morgan laws.

The top and bottom elements exist in $\mathcal{T}_n$ because $\mathcal{T}_n$ is finite totally ordered set. Now we also know that the complementarity $\neg$ is *quasi-complementation*. (See closer considerations, for example, in Rasiowa [10], p. 44–45.) The top element $\mathcal{A}_n$ is the neutral element of the operation $\sqcap$. In $\mathcal{T}_n$, there exists a bottom element, too, namely $\mathcal{A}_1$, which is the neutral element of the operation $\sqcup$. Especially, by the definition of negation, the conditions $\neg\mathcal{A}_1 = \mathcal{A}_n$ and $\neg\mathcal{A}_n = \mathcal{A}_1$ hold in $\mathcal{T}_n$. It is a general case that any De Morgan algebra has top element and bottom element being the negations of each other.

If a De Morgan algebra satisfies so-called *Kleene condition*, it is a *Kleene algebra*. So, the last thing before getting a fuzzy-valued Kleene algebra is to check whether the Kleene condition

$$\mathcal{A}_i \sqcap \neg\mathcal{A}_i \sqsubseteq \mathcal{A}_j \sqcup \neg\mathcal{A}_j\,, \text{ if } 1 \le i, j \le n \tag{K}$$

holds in our De Morgan algebra $\mathcal{T}_n = \langle T_n, \sqcup, \sqcap, \neg, \mathcal{A}_n \rangle$. Here the condition (K) is constructed for lattices where the elements are fuzzy numbers or fuzzy intervals.

Theorem 2 *The algebra* $\mathcal{T}_n = \langle T_n, \sqcup, \sqcap, \neg, \mathcal{A}_n \rangle$ *is a Kleene algebra.*

Proof The algebra $\mathcal{T}_n$ is De Morgan algebra, as we have noticed above. So, we have to show that the algebra $\mathcal{T}_n$ satisfies the Kleene condition (K).

Let $\mathcal{A}_i, \mathcal{A}_j \in T_n$ be arbitrarily chosen, hence $\neg\mathcal{A}_i, \neg\mathcal{A}_j \in T_n$, too, because T_n is closed under negation. Suppose $\mathcal{A}_i \sqsubseteq \mathcal{A}_j$ whenever $i \le j$, for all $\mathcal{A}_i, \mathcal{A}_j \in T_n$.

If the number of fuzzy sets in T_n is $n = 2k + 1$ (i.e., n is odd) then the middle element of T_n is $\mathcal{A}_{k+1}$, and hence $\neg\mathcal{A}_{k+1} = \mathcal{A}_{k+1}$, by the definition of negation.

If $n = 2k$ (i.e., n is even) then $\neg\mathcal{A}_k = \mathcal{A}_{k+1}$ and $\neg\mathcal{A}_{k+1} = \mathcal{A}_k$, by the definition of negation.

We denote

$$\mathcal{A}_{[\frac{n}{2}]} = \begin{cases} \mathcal{A}_{k+1} & \text{if} \quad n = 2k + 1 \\ \mathcal{A}_k & \text{if} \quad n = 2k \end{cases}$$

Hence, for any $\mathcal{A}_i$, if $\mathcal{A}_i \sqsubseteq \mathcal{A}_{[\frac{n}{2}]}$ then $\mathcal{A}_{[\frac{n}{2}]} \sqsubseteq \neg\mathcal{A}_i$, and vice versa. Hence, for any i, $\mathcal{A}_i \sqcap \neg\mathcal{A}_i \sqsubseteq \mathcal{A}_{[\frac{n}{2}]}$ and for any j, $\mathcal{A}_{[\frac{n}{2}]} \sqsubseteq \mathcal{A}_j \sqcap \neg\mathcal{A}_j$. This completes the proof. □

Especially, $\mathcal{T}_n = \langle T_k, \sqcup, \sqcap, \neg, \mathcal{A}_n \rangle$ is an associative Kleene algebra, by Theorem 1.

3 Construction of Applicable Kleene Algebras

Examples about easily manipulable fuzzy sets in applications based on Kleene algebras of fuzzy sets are triangular fuzzy numbers, Gaussian fuzzy numbers, other bell-shaped fuzzy numbers and fuzzy intervals.

As an example, consider a trapezoidal fuzzy interval

$$\mathcal{A}(x) = \begin{cases} 0 & \text{if } x < a_1 \\ \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x < a_2 \\ 1 & \text{if } a_2 \leq x < a_3 \\ \frac{a_4-x}{a_4-a_3} & \text{if } a_3 \leq x \leq a_4 \\ 0 & \text{if } x > a_4 \end{cases} \tag{13}$$

on a closed real interval $[0, p]$ ($0 < p$, $p \in \mathbb{R}$) and $x, a_1, a_2, a_3, a_4 \in [0, p]$ and $0 \leq a_1 < a_2 < a_3 < a_4$. If $a_2 = a_3$ then $\mathcal{A}$ is a triangular fuzzy number. The support and the core of $\mathcal{A}(x)$ are the intervals $[a_1, a_4]$ and $[a_2, a_3]$, respectively. The increasing part on the left and the decreasing part on the right side of $\mathcal{A}$ have the membership functions

$$\mathcal{A}_L(x) = \begin{cases} 0 & \text{if } x < a_1, a_2 < x \\ \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x < a_2 \end{cases}, \quad \mathcal{A}_R(x) = \begin{cases} \frac{a_4-x}{a_4-a_3} & \text{if } a_3 \leq x \leq a_4 \\ 0 & \text{if } x < a_3, a_4 < x \end{cases} \tag{14}$$

If the supports $\mathrm{supp}(\mathcal{A}_L) = [a_1, a_2]$ and $\mathrm{supp}(\mathcal{A}_R) = [a_3, a_4]$ have the same size then the membership function of the fuzzy interval $\mathcal{A}$ is symmetric with respect to the vertical line $x = \frac{a_3+a_2}{2}$.

Applying, for example, some considerations, due to Novák [9] and Kaufmann and Gupta [2], $\mathcal{A}$ can be considered as an ordered 4-tuple $\mathcal{A} = (a_1, a_2, a_3, a_4)$ where $\mathcal{A}(x)$ is increasing if $x \in [a_1, a_2]$, $\mathcal{A}(x) = 1$ if $x \in [a_2, a_3]$, and $\mathcal{A}(x)$ is decreasing if $x \in [a_3, a_4]$. Further, the negation for $\mathcal{A}$ on the interval $[0, p]$ can be given as the ordered 4-tuple

$$\neg\mathcal{A} = (p - a_4, p - a_3, p - a_2, p - a_1) \tag{15}$$

which is a fuzzy interval on $[0, p]$ of x-axis. This fuzzy interval can be given in the form, similar to $\mathcal{A}$ in (13), as

$$\neg\mathcal{A}(x) = \begin{cases} 0 & \text{if } x < p - a_1 \\ \frac{x-p+a_3}{a_3-a_2} & \text{if } p - a_3 \leq x < p - a_2 \\ 1 & \text{if } p - a_3 \leq x < p - a_2 \\ \frac{p-a_1-x}{a_2-a_1} & \text{if } p - a_2 \leq x \leq p - a_1 \\ 0 & \text{if } x > p - a_1 \end{cases} \tag{16}$$

Consider a set of fuzzy trapezoidal intervals $T_n = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n\}$ of the interval $[0, p]$ where the *divisional points* of $[0, p]$ on x-axis, determined by the fuzzy intervals $\mathcal{A}_i$ $(i = 0, \dots, n)$ are $a_0, a_1, \dots, a_k$, where $k = 2n - 1$. The fuzzy intervals

$$\mathcal{A}_1(x) = \begin{cases} 1 & \text{if } 0 \leq x < a_1 \\ \frac{a_2-x}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2 \\ 0 & \text{if } x > a_2 \end{cases}, \quad \mathcal{A}_n(x) = \begin{cases} 0 & \text{if } x < a_{k-2} \\ \frac{x-a_{k-2}}{a_{k-1}-a_{k-2}} & \text{if } a_{k-2} \leq x < a_{k-1} \\ 1 & \text{if } a_{k-1} \leq x < a_k \end{cases} \tag{17}$$

are the first and the last fuzzy interval, respectively, which can be given by means of the 4-tuples

$$\mathcal{A}_1 = (0, 0, a_1, a_2) \quad \text{and} \quad \mathcal{A}_n = (a_{k-2}, a_{k-1}, a_k, a_k).$$

Here we agree that $\mathcal{A}_1(x) = 1$ if $x \in [0, a_1]$ and $\mathcal{A}_n(x) = 1$ if $x \in [a_{k-1}, a_k]$. The other fuzzy intervals between $\mathcal{A}_1$ and $\mathcal{A}_k$ can be given in the form

$$\mathcal{A}_2 = (a_1, a_2, a_3, a_4),\ \mathcal{A}_3 = (a_3, a_4, a_5, a_6),\ \dots,\ \mathcal{A}_{n-1} = (a_{k-4}, a_{k-3}, a_{k-2}, a_{k-1}).$$

Note that the first divisional point on the interval $[0, p]$ is origo, and the last one is $a_k = p$. Hence, $\mathcal{A}_1(0) = 1$ and $\mathcal{A}_n(a_k) = \mathcal{A}_n(p) = 1$.

Especially, on x-axis, the divisional points for the intervals being parts of the supports of the fuzzy sets in T_n are as follows. The number of fuzzy sets in T_n is n and the number of the divisional points is $2n$. So, we have the divisional points $a_0, a_1, a_2, \dots, a_k$, where $k = 2n - 1$. And the first divisional point is $a_0 = 0$ and the last one $a_k = a_{2n-1} = p$.

The supports of all the fuzzy numbers form a cover to the interval $[0, p]$, such that the union of the covers of $\mathcal{A}_1$, $\mathcal{A}_2$ $\dots$, $\mathcal{A}_n$ is exactly the same as the interval itself, i.e.,

$$[0, p] = \bigcup_{i=1}^{n} \operatorname{supp} \mathcal{A}_i. \tag{18}$$

Example 1 Consider a collection of fuzzy intervals on $[0, p]$ where the fuzzy intervals are of the form (16) and (17) where $n = 7$. Hence, the set T_7 consists of the fuzzy intervals

$$\mathcal{A}_1 = (0, 0, a_1, a_2),\ \mathcal{A}_2 = (a_1, a_2, a_3, a_4),\ \mathcal{A}_3 = (a_3, a_4, a_5, a_6),$$
$$\mathcal{A}_4 = (a_5, a_6, a_7, a_8),\ \mathcal{A}_5 = (a_7, a_8, a_9, a_{10}),\ \mathcal{A}_6 = (a_9, a_{10}, a_{11}, a_{12}),$$
$$\text{and } \mathcal{A}_7 = (a_{11}, a_{12}, a_{13}, a_{13})$$

i.e.,

$$T_7 = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7\}$$

The divisional points are

$$a_0 = 0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13} = p.$$

The algebra $\mathcal{T}_7 = \langle T_7, \sqcup, \sqcap, \neg, \mathcal{A}_7 \rangle$ satisfies all the properties considered in Sect. 2.

4 An Application Example: Fuzzy Screening Systems

As a motivating example, we consider a case of a *fuzzy screening system*, a technique suggested by R. Yager (for example, see Yager [11]). These systems contain fuzzy data. The source material for this description about fuzzy screening systems in this section is taken from Robert Fullér's book [1] *Introduction to Neuro-Fuzzy Systems*.

A fuzzy screening system is a two stage process as follows:

- In the first stage, experts are asked to provide an evaluation of the alternatives. This evaluation consists of a rating for each alternative on each of the criteria.
- In the second stage, the methodology is used to aggregate the individual experts evaluations to obtain an overall linguistic value for each object.

The problem consists of three components.

(1) The first component is a collection

$$X = \{X_1, \ldots, X_p\}$$

of *alternative solutions* from amongst which we desire to select some subset to be investigated further.

(2) The second component is a group

$$A = \{A_1, \ldots, A_r\}$$

of *experts* whose opinion solicited in screening the alternatives.

Table 1 Scale of scores

Score name	Label	Score symbol
Outstanding	OU	S_7
Very high	VH	S_6
High	H	S_5
Medium	M	S_4
Low	L	S_3
Very low	VL	S_2
None	N	S_1

(3) The third component is a collection

$$C = \{C_1, \dots, C_m\}$$

of *criteria* which are considered relevant in the choice of the objects to be further considered.

For each alternative, each expert is required to provide his/her opinion. In particular, for each alternative an expert is asked to evaluate how well that alternative satisfies each of the criteria in the set C. These evaluations of alternative satisfaction to criteria will be given in terms of elements from the scale S in Table 1.

Based on intuition, the use of such a scale provides a natural ordering, $S_i > S_j$, if $i > j$, and the maximum and minimum of any two scores be defined by

$$\max\{S_i, S_j\} = S_i\,, \quad \text{if } S_i \geq S_j \tag{19}$$

$$\min\{S_i, S_j\} = S_j\,, \quad \text{if } S_i \leq S_j. \tag{20}$$

where max and min are fuzzy max and min defined in [1], i.e., these operations are the same as $\sqcup$ and $\sqcap$, respectively. Using our notation above, these conditions can be expressed in the form

$$S_i \sqcup S_j = S_j\,, \quad \text{if} \quad S_i \sqsubseteq S_j \tag{21}$$

$$S_i \sqcap S_j = S_i\,, \quad \text{if} \quad S_i \sqsubseteq S_j. \tag{22}$$

Thus for an alternative an expert provides a collection of n values

$$\{P_1, \dots, P_n\},$$

where P_j is the rating of the alternative on the jth criterion by the expert. Each P_j is an element in the set of allowable scores S,

$$S = \{S_7, S_6, S_5, S_4, S_3, S_2, S_1\}.$$

Assuming $n = 6$, an example of a typical scoring for an alternative from one expert would be

$$\{S_4, S_3, S_7, S_6, S_7, S_1\}$$

or, using the labels of the scores,

$$\{\mathrm{M, L, OU, VH, OU, N}\}.$$

Independent of this evaluation of alternative satisfaction to criteria, each expert must assign a measure of importance to each of the criteria. An expert uses the same scale, S, to provide the importance associated with the criteria.

The next step in the process is to find the overall evaluation for an alternative by a given expert. For this we use a methodology suggested by Yager [11]. A crucial aspect of this approach is the taking of the negation of the importances as

$$\mathrm{Neg}(S_i) = S_{q-i+1} \qquad (q \text{ is the number the scores in } S).$$

For the scale S the negation operation provides the following:

$$\mathrm{Neg(OU) = N, Neg(VH) = VL, Neg(H) = L, Neg(M) = M,}$$
$$\mathrm{Neg(L) = H, Neg(VL) = VH, Neg(N) = OU.}$$

Then the unit score of each alternative by each expert, denoted by U, is calculated as follows:

$$U = \min_j \{\mathrm{Neg}(I_j) \vee P_j\}, \tag{23}$$

where I_j denotes the importance of the jth criterion.

Because in our algebra $\neg I_j \sqcup P_j$ is the same as $\mathrm{Neg}(I_j) \vee P_j$ in Yager's system then, using the notation of our algebra, the unit score formula is

$$U = (\neg I_1 \sqcup P_1) \sqcap \ldots \sqcap (\neg I_m \sqcup P_m) \tag{24}$$

If we think that the operations of the screening systems are logical connectives, we note that the unit score formula (23) essentially is an *anding* of the criteria satisfactions modified by the importance of the criteria. The formula (23) can be seen as a measure of the degree to which an alternative satisfies the following statement:

All important criteria are satisfied.

The following example is considered the case where the experts $A_1, \ldots, A_r$ evaluates his/her opinion about the importance of each criterion C_i (i = 1, ..., 5) by using the scores from the scale S. Then the experts evaluate how well each alternative X_i $(i = 1 \ldots, p)$ satisfies each criterion. So, for example, an alternative X_i gets the evaluation given in Table 2 from an expert A_j.

Table 2 Evaluations for one alternative by one expert

Criterion	C_1	C_2	C_3	C_4	C_5
Importance	VH	VH	M	L	VL
Satisfaction score	M	L	OU	VH	OU

Example 2 Consider an alternative X_i with the following scores on five criteria in the following Table 2. An expert A_j gives his/her scores to the importance of each criterion and his/her scores to the alternative, how well the alternative meets each criterion.

Using this evaluation, we apply the truth value evaluation based on the Kleene algebra $\mathcal{K}_7 = \langle S, \sqcap, \sqcup, \neg, S_7 \rangle$, and the unit evaluation for the alternative X_i from the expert A_j is

$$\begin{aligned} U_{ij} &= (\mathrm{VL} \sqcup \mathrm{M}) \sqcap (\mathrm{VL} \sqcup \mathrm{L}) \sqcap (\mathrm{M} \sqcup \mathrm{OU}) \sqcap (\mathrm{H} \sqcup \mathrm{VH}) \sqcap (\mathrm{VH} \sqcup \mathrm{OU}) \\ &= \mathrm{M} \sqcap \mathrm{L} \sqcap \mathrm{OU} \sqcap \mathrm{VH} \sqcap \mathrm{OU} = \mathrm{L}. \end{aligned} \tag{25}$$

We note that comparing this result with that in the original example[1] with the same data we see that the results are identical.

The essential reason for the low performance of this objects is that it performed low on the second criterion which has a very high importance. Linguistically, Eq. (23) is saying that *If a criterion is important then an alternative should score well on it.*

The satisfaction scores S_i $(i = 1, \ldots, 7)$ are interpreted by fuzzy numbers or fuzzy intervals, for example, by the same fuzzy intervals as in Example 1 in Sect. 3, such that the scale of satisfaction scores

$$S = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\} \tag{26}$$

forms an associative Kleene algebra $\mathcal{K}_7 = \langle S, \sqcup, \sqcap, \neg, S_7 \rangle$. So, the calculation tools of these inferences are based on this algebra. Hence, fuzzy screening systems serve as a practical application example about fuzzy-valued Kleene algebras.

As a result of the first stage, we have for each alternative X_i a collection of evaluations

$$\{P_{i1}, \ldots, P_{ir}\}, \quad i = 1, \ldots, p \tag{27}$$

where $P_{ik} \in S$ is the unit evaluation of the ith alternative by the kth expert.

Example 2 shows how to aggregate the individual experts evaluations in order to get an overall linguistic value for each object after the evaluations made by each expert.

[1] See Robert Fullér [1], Ex. 1.18.1., p. 111–112.

5 Concluding Remarks

As a conclusion, we can answer in the affirmative to the questions we stated in the end of Sect. 1.

The construction of the above presented Kleene algebra is a mathematical basis for applications where fuzzy quantities are used as fuzzy scores in fuzzy inferences, like fuzzy screening systems. Hence, we found a mathematical model for fuzzy screening systems.

It must be noted that this kind of mathematical models work only if the linguistic scores correspond to the fuzzy quantities, i.e., fuzzy numbers or fuzzy intervals, in corresponding algebras. The linguistic interpretations of the fuzzy quantities are subjective, i.e., they are based on personal opinions. Hence, we cannot totally get rid of intuition. However, this thing is a strength in fuzzy systems, if we use it carefully.

Kleene algebras have a central role in fuzzy set theory. The author has shown that Prof. Zadeh's theory of standard fuzzy sets he presented in [12] is based on Kleene algebras. First, in symposium "Fuzziness in Finland", 2004, and in the paper [4] the author showed that Zadeh's theory in [12] forms a De Morgan algebra, and later on, for example, in [7] it is shown that standard fuzzy set theory forms a Kleene algebra. The author is calling this algebra by name *Zadeh algebra*.

Kleene algebras for fuzzy quantities serve an algebraic basis for *many-fuzzy-valued logics*. Truth values of this kind of logics are fuzzy numbers. The author has some ideas and sketches to create some fuzzy-many-valued logical systems. Some preliminaries are already considered in Mattila [5, 6].

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