Chapter 9
The Wave Equation

In this chapter, we present a short and even more far from exhaustive theoretical study of the wave equation. We establish the existence and uniqueness of the solution, as well as the energy estimates. We describe the qualitative behavior of solutions, which is very different from that of the heat equation. Again, we will mostly work in one dimension of space.

In the same chapter, we introduce finite difference methods for the numerical approximation of the wave equation. Here again, stability issues are prominent, and significantly more delicate than for the heat equation.

9.1 Regular Solutions of the Wave Equation

Recall that the general wave equation reads

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \text{ in } Q = \Omega \times]0, T[,$$

where Ω is an open subset of $\mathbb{R}^d$ and f is a given function on Q, complemented with boundary and initial conditions, see Chap. 1, Sects. 1.5 and 1.6. The propagation speed c is set to 1, which we can always assume after a change of time or length unit. There are two different settings depending on whether Ω is bounded or not. In the one-dimensional case, $d = 1$, we thus have either $\Omega =]a, b[$ or $\Omega = \mathbb{R}$ without loss of generality.[1]

[1] Admittedly, there is a third case, $\Omega = \mathbb{R}_+^*$, but we will not consider it here.

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Let us begin with the bounded case. We are thus looking for a function $u\colon [a,b]\times[0,T]\to\mathbb{R}$ which solves the initial-boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = f \text{ in } Q, \\ u(a,t) = u(b,t) = 0 \text{ for } t\in[0,T], \\ u(x,0) = u_0(x),\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \text{ for } x\in\,]a,b[, \end{cases} \tag{9.1}$$

with homogeneous Dirichlet boundary conditions for simplicity and two initial data, u_0 and u_1. If we think of the vibrating string interpretation, this means that the string is fixed at both ends, and that we are given its initial position and initial velocity. This is quite normal, since the equation is derived from Newton's law of motion and is of second order in time.

Definition 9.1 The quantity

$$E(t) = \frac{1}{2}\int_a^b \Big[\Big(\frac{\partial u}{\partial t}(x,t)\Big)^2 + \Big(\frac{\partial u}{\partial x}(x,t)\Big)^2\Big]\,dx$$

is called the *energy*.

Of course, we assume that the solution is regular enough for the above quantity to make sense. In the vibrating string interpretation, this is exactly the mechanical energy of the string at time t. The first term corresponds to the kinetic energy since it is half the square of the velocity at point x and time t, integrated along the string. The second term corresponds to the elastic energy, which can be seen by examining the work done by the exterior forces, based on the analysis in Chap. 1, Sect. 1.1. The initial energy is then

$$E(0) = \frac{1}{2}\int_a^b (u_1(x)^2 + u_0'(x)^2)\,dx.$$

The initial energy is finite for $u_0\in H^1(]a,b[)$ and $u_1\in L^2(a,b)$.

Proposition 9.1 *Let u be a smooth enough solution of problem (9.1), then we have*

$$\frac{dE}{dt}(t) = \int_a^b f(x,t)\frac{\partial u}{\partial t}(x,t)\,dx.$$

Proof By differentiation under the integral sign, we have

$$\frac{dE}{dt}(t) = \frac{1}{2}\int_a^b \Big[\frac{\partial}{\partial t}\Big(\Big(\frac{\partial u}{\partial t}\Big)^2\Big) + \frac{\partial}{\partial t}\Big(\Big(\frac{\partial u}{\partial x}\Big)^2\Big)\Big]\,dx$$
$$= \int_a^b \Big[\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\Big]\,dx.$$

We integrate the second term by parts

$$\int_a^b \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}\,dx = \Big[\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Big]_a^b - \int_a^b \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t}\,dx = -\int_a^b \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t}\,dx,$$

since $\frac{\partial u}{\partial t}(a,t) = \frac{\partial u}{\partial t}(b,t) = 0$ due to the Dirichlet boundary condition. Therefore,

$$\frac{dE}{dt}(t) = \int_a^b \Big[\frac{\partial u}{\partial t}\Big(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\Big)\Big]\,dx = \int_a^b f\frac{\partial u}{\partial t}\,dx,$$

and the proposition is proved. □

In the vibrating string interpretation, we thus find that the time derivative of the energy is the power of the applied forces, as is expected from physics.

Corollary 9.1 *If the right-hand side f in problem (9.1) vanishes, then the energy is constant*

$$E(t) = E(0).$$

Proof Indeed, in this case, $\frac{dE}{dt} = 0$. □

Remark 9.1 We note here a sharp contrast with the heat equation, for which the energy was exponentially decreasing for a zero right-hand side. The heat equation, which is a parabolic equation, dissipates the energy, whereas the wave equation—a hyperbolic equation—conserves the energy: a vibrating string keeps vibrating forever in the absence of dissipation. □

Corollary 9.2 *Problem (9.1) has at most one smooth solution.*

Proof Let u_1 and u_2 be solutions of problem (9.1), and $u = u_1 - u_2$. Then u is a solution of problem (9.1) with right-hand side $f = 0$, so that $E(t) = E(0)$, and zero initial data, so that $E(0) = 0$. It follows from Definition 9.1 that $u = 0$. □

In order to further exploit the energy, we need a general purpose result, known as *Gronwall's lemma* or *Gronwall's inequality*.

Theorem 9.1 (Gronwall's lemma) *Let α, β and γ be three continuous functions defined on* $[0, T]$ *such that α is differentiable on* $]0, T[$. *We assume that*

$$\alpha'(t) \le \beta(t)\alpha(t) + \gamma(t) \textit{ for all } t \in\,]0, T[.$$

Then, we have

$$\alpha(t) \le e^{\int_0^t \beta(s)\,ds}\alpha(0) + \int_0^t e^{\int_s^t \beta(u)\,du}\gamma(s)\,ds.$$

Proof Let $B(t) = \int_0^t \beta(s)\,ds$ and define $\delta(t) = e^{-B(t)}\alpha(t)$. Then δ is differentiable on $]0, T[$ and

$$\begin{aligned}\delta'(t) &= e^{-B(t)}\alpha'(t) - \beta(t)e^{-B(t)}\alpha(t) = e^{-B(t)}(\alpha'(t) - \beta(t)\alpha(t))\\ &\le e^{-B(t)}\gamma(t).\end{aligned}$$

Therefore, by the mean value inequality,

$$\delta(t) - \delta(0) \le \int_0^t e^{-B(s)}\gamma(s)\,ds$$

and we conclude by multiplying the above inequality by $e^{B(t)}$ and by noticing that $B(t) - B(s) = \int_s^t \beta(u)\,du$. □

Proposition 9.2 *We have the energy estimate*

$$\sup_{t\in[0,T]} E(t) \le e^T E(0) + \frac{1}{2}\int_0^T \int_a^b e^{T-s} f(x,s)^2\,dxds.$$

Proof It follows from Proposition 9.1 that

$$E'(t) \le \frac{1}{2}\int_a^b f(x,t)^2\,dx + \frac{1}{2}\int_a^b \Big(\frac{\partial u}{\partial t}(x,t)\Big)^2\,dx \le \frac{1}{2}\int_a^b f(x,t)^2\,dx + E(t).$$

Thus, by Gronwall's lemma,

$$E(t) \le e^t E(0) + \frac{1}{2}\int_0^t \int_a^b e^{t-s} f(x,s)^2\,dxds \le e^T E(0) + \frac{1}{2}\int_0^T \int_a^b e^{T-s} f(x,s)^2\,dxds,$$

for all $t \in [0, T]$. □

Remark 9.2 The energy estimate provides a stability result in the energy norm, in the sense of establishing the continuity of the solution with respect to the initial data and right-hand side. Indeed, if u_1 and u_2 are two solutions corresponding to right-hand sides f_1 and f_2 and initial data $u_{1,0}, u_{1,1}$ and $u_{2,0}, u_{2,1}$, applying the energy estimate to $u_1 - u_2$, we obtain

$$\begin{aligned}&\sup_{t\in[0,T]}\Big(\|u_1 - u_2\|^2_{H_0^1(]a,b[)} + \Big\|\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t}\Big\|^2_{L^2(a,b)}\Big)\\ &\qquad \le e^T(\|u_{1,0} - u_{2,0}\|^2_{H_0^1(]a,b[)} + \|u_{1,1} - u_{2,1}\|^2_{L^2(a,b)} + \|f_1 - f_2\|^2_{L^2(Q)}).\end{aligned}$$

□

We now use Fourier series to construct regular solutions of problem (9.1) when $f = 0$. For simplicity, we let $a = 0$, $b = 1$ and we assume that the initial data are compatible with the Dirichlet condition, i.e., $u_0(0) = u_0(1) = u_1(0) = u_1(1) = 0$. As in the case of the heat equation, we expand both functions in Fourier series

$$u_0(x) = \sum_{k=1}^{+\infty} b_k^0 \sin(k\pi x), \quad u_1(x) = \sum_{k=1}^{+\infty} b_k^1 \sin(k\pi x).$$

Theorem 9.2 *Let $u_0 \in C^4([0,1])$ and $u_1 \in C^3([0,1])$ be such that $u_0''(0) = u_0''(1) = u_1''(0) = u_1''(1) = 0$. Then the function defined by*

$$u(x,t) = \sum_{k=1}^{+\infty} \Big(b_k^0 \cos(k\pi t) + \frac{b_k^1}{k\pi} \sin(k\pi t) \Big) \sin(k\pi x) \tag{9.2}$$

belongs to $C^2([0,1] \times [0,+\infty[)$ and solves problem (9.1) with $f = 0$.

Proof Under the hypotheses made on u_0 and u_1, it is easy to see that $|b_k^0| \le Ck^{-4}$ and $|b_k^1| \le Ck^{-3}$ for some constant C. Then the series in formula (9.2) as well as the series of all first order and second order derivatives are normally convergent. Hence, u is of class C^2. Moreover, since the functions $(x,t) \mapsto e^{ik\pi(t \pm x)}$ are solutions of the wave equation with zero right-hand side, it is clear that the normal convergence of second derivatives implies that u is also a solution of the wave equation.

For $t = 0$, we have

$$u(x,0) = \sum_{k=1}^{+\infty} b_k^0 \sin(k\pi x) = u_0(x)$$

and

$$\frac{\partial u}{\partial t}(x,0) = \sum_{k=1}^{+\infty} b_k^1 \sin(k\pi x) = u_1(x),$$

hence the initial conditions are satisfied. Finally, the Dirichlet boundary conditions are also satisfied since $\sin(k\pi) = 0$. □

Remark 9.3 We find that the solution is a superposition of harmonics, see Chap. 1, Sect. 1.5. Which harmonics are excited depend on the initial conditions. For instance, for such a musical instrument as the piano, the strings are initially at rest, $u_0 = 0$, and are hit by a hammer, $u_1 \neq 0$. In the case of a guitar or a harpsichord, the strings are typically plucked, $u_0 \neq 0$, sometimes with no initial velocity, $u_1 = 0$. Note that other combinations are possible, all resulting in different sounds, see Figs. 9.1 and 9.2. □

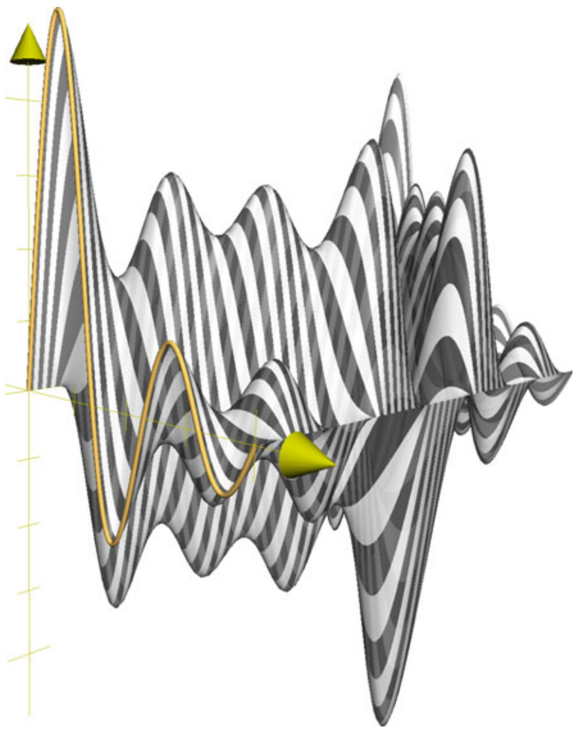

Fig. 9.1 A view of the evolution in the case of zero initial velocity u_1, u_0 has four nonzero harmonics

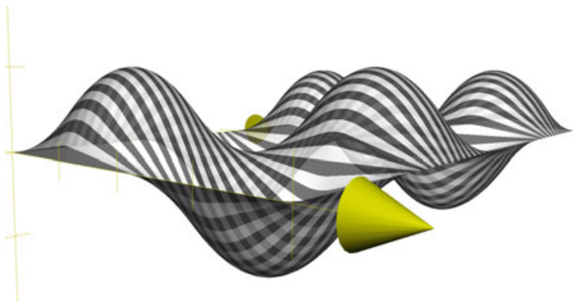

Fig. 9.2 A view of the evolution in the case of zero initial position, initial velocity $+1$ in $]0, \frac{1}{2}[$, -1 in $]\frac{1}{2}, 1[$, two hundred nonzero terms in the Fourier series

Remark 9.4 The regularity hypotheses made on u_0 and u_1 are just there to ensure easy convergence of the series of partial derivatives up to the second order. Indeed, if the series (9.2) converges in a much weaker sense, its sum is still going to be a solution of the wave equation in the sense of distributions at least, since differentiation is continuous in the sense of distributions. The difficulty lies in the meaning of the initial conditions, as some kind of continuity with respect to time is required for them to make sense. □

Remark 9.5 A fundamental difference with the heat equation is that the Fourier coefficients of $u(\cdot, t)$ are not rapidly damped by exponential terms for $t > 0$, which cause the solution of the heat equation to be smooth for $t > 0$, whatever the initial data. Here, the wave equation has no smoothing effect whatsoever. The regularity or lack thereof of the initial conditions is propagated in time without any gain. This is one of the main differences between parabolic and hyperbolic problems. $\square$

9.2 Variational Formulation and Existence of Weak Solutions

We now introduce a variational formulation for the wave equation in a manner that is quite similar to the one described in Sect. 7.6 for the heat equation.

Definition 9.2 The variational formulation of the wave equation (9.1) with homogeneous Dirichlet boundary condition, initial data $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and right-hand side $f \in L^2(Q)$ is: Find $u \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ such that, for all $v \in H_0^1(\Omega)$,

$$\begin{cases} \big((u|v)_{L^2(\Omega)}\big)'' + a(u,v) = (f|v)_{L^2(\Omega)} \text{ in the sense of } \mathscr{D}'(]0,T[), \\ (u(0)|v)_{L^2(\Omega)} = (u_0|v)_{L^2(\Omega)}, \\ (u'(0)|v)_{L^2(\Omega)} = (u_1|v)_{L^2(\Omega)}. \end{cases} \tag{9.3}$$

Remark 9.6 This definition clearly makes sense. The last two equations are a weak form of the initial conditions $u(0) = u_0$, $u'(0) = u_1$. $\square$

For simplicity, we work again on $\Omega =]0,1[$ with the minus Laplacian eigenfunctions $\phi_k(x) = \sqrt{2}\sin(k\pi x)$ and eigenvalues $\lambda_k = k^2\pi^2$, and we have $a(w,\phi_k) = \lambda_k (w|\phi_k)_{L^2(\Omega)}$ for all $w \in H_0^1(\Omega)$, see Eq. (7.4).

Theorem 9.3 *Let $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in L^2(Q)$. There exists a unique solution $u \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ of the initial-boundary value problem (9.3), which is given by*

$$u(t) = \sum_{k=1}^{+\infty} u_k(t)\phi_k, \tag{9.4}$$

where

$$u_k(t) = (u_0|\phi_k)_{L^2(\Omega)} \cos\big(\sqrt{\lambda_k}t\big) + \frac{(u_1|\phi_k)_{L^2(\Omega)}}{\sqrt{\lambda_k}} \sin\big(\sqrt{\lambda_k}t\big) \\ + \frac{1}{\sqrt{\lambda_k}} \int_0^t (f(s)|\phi_k)_{L^2(\Omega)} \sin\big(\sqrt{\lambda_k}(t-s)\big)\,ds. \tag{9.5}$$

Proof We start with the uniqueness. Let $u \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ be a solution of (9.3). We expand $u(t)$ on the Hilbert basis $(\phi_k)_{k\in\mathbb{N}^*}$ of $L^2(\Omega)$ so that, for all t,

$$u(t) = \sum_{k=1}^{+\infty} u_k(t)\phi_k$$

with

$$u_k(t) = (u(t)|\phi_k)_{L^2(\Omega)}$$

for all $k \in \mathbb{N}^*$ and the series converges in $L^2(\Omega)$. Likewise, we set $u_0 = \sum_{k=1}^{+\infty} u_{0,k}\phi_k$, $u_1 = \sum_{k=1}^{+\infty} u_{1,k}\phi_k$ and $f(t) = \sum_{k=1}^{+\infty} f_k(t)\phi_k$. Taking $\phi_k \in H_0^1(\Omega)$ as a test-function in problem (9.3), we obtain

$$\begin{cases} u_k''(t) + \lambda_k u_k(t) = f_k(t) \text{ in the sense of } \mathscr{D}'(]0,T[), \\ \qquad\qquad u_k(0) = u_{0,k}, \\ \qquad\qquad u_k'(0) = u_{1,k}, \end{cases}$$

for all $k \in \mathbb{N}^*$. For each k, this is a Cauchy problem for an ordinary differential equation which has the unique solution

$$u_k(t) = u_{0,k}\cos\big(\sqrt{\lambda_k}t\big) + \frac{u_{1,k}}{\sqrt{\lambda_k}}\sin\big(\sqrt{\lambda_k}t\big) + \frac{1}{\sqrt{\lambda_k}}\int_0^t f_k(s)\sin\big(\sqrt{\lambda_k}(t-s)\big)\,ds,$$

hence the uniqueness.

We now use the above series to prove existence. We have that $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ by hypothesis, therefore thanks to formula (7.7),

$$\|u_0\|^2_{H^1(\Omega)} = \sum_{k=1}^{+\infty}(1+\lambda_k)u_{0,k}^2,\ |u_0|^2_{H^1(\Omega)} = \sum_{k=1}^{+\infty}\lambda_k u_{0,k}^2 \text{ and } \|u_1\|^2_{L^2(\Omega)} = \sum_{k=1}^{+\infty} u_{1,k}^2. \tag{9.6}$$

Similarly, $f \in L^2(Q)$ and

$$\|f\|^2_{L^2(Q)} = \int_0^T \sum_{k=1}^{+\infty} f_k(t)^2\,dt. \tag{9.7}$$

As before, we consider the sequence of partial sums $U_n(t) = \sum_{k=1}^n u_k(t)\phi_k$ and show that it is Cauchy for both $C^0([0,T]; H_0^1(\Omega))$ and $C^1([0,T]; L^2(\Omega))$ norms. Let $p < q$ be two given integers and let us estimate $U_p - U_q$ in these various norms.

First of all, we have for all t

$$\begin{aligned}
|U_p(t) - U_q(t)|^2_{H_0^1(\Omega)} &= \sum_{k=p+1}^{q} \lambda_k u_k(t)^2 \\
&\leq 2 \sum_{k=p+1}^{q} \lambda_k \Big[u_{0,k}^2 + \frac{1}{\lambda_k} u_{1,k}^2 + \frac{1}{\lambda_k} \Big(\int_0^t |f_k(s)|\, ds \Big)^2 \Big] \\
&\leq 2 \sum_{k=p+1}^{q} \lambda_k u_{0,k}^2 + 2 \sum_{k=p+1}^{q} u_{1,k}^2 + 2T \sum_{k=p+1}^{q} \int_0^T f_k(s)^2\, ds,
\end{aligned}$$

since all the trigonometric terms are less than 1 in absolute value and by the Cauchy–Schwarz inequality. Therefore

$$\|U_p - U_q\|^2_{C^0([0,T];H_0^1(\Omega))} \leq 2 \sum_{k=p+1}^{q} \lambda_k u_{0,k}^2 + 2 \sum_{k=p+1}^{q} u_{1,k}^2 + 2T \sum_{k=p+1}^{q} \int_0^T f_k(s)^2\, ds$$

can be made as small as we wish by taking p large enough, due to the hypotheses on u_0, u_1 and f and formulas (9.6)–(9.7), and the sequence is consequently Cauchy in $C^0(0, T; H_0^1(\Omega))$.

It follows from the previous estimate and the Poincaré inequality that the sequence is also Cauchy in $C^0(0, T; L^2(\Omega))$. We need to look at its time derivative. Of course, $U_n'(t) = \sum_{k=1}^{n} u_k'(t)\phi_k$ with

$$u_k'(t) = -\sqrt{\lambda_k} u_{0,k} \sin\big(\sqrt{\lambda_k} t\big) + u_{1,k} \cos\big(\sqrt{\lambda_k} t\big) + \int_0^t f_k(s) \cos\big(\sqrt{\lambda_k}(t-s)\big)\, ds,$$

so that

$$\|U_p' - U_q'\|^2_{C^0([0,T];L^2(\Omega))} \leq 2 \sum_{k=p+1}^{q} \lambda_k u_{0,k}^2 + 2 \sum_{k=p+1}^{q} u_{1,k}^2 + 2T \sum_{k=p+1}^{q} \int_0^T f_k(s)^2\, ds$$

and the sequence U_n' is Cauchy in $C^0(0, T; L^2(\Omega))$, which completes the proof of the convergence of the series (9.4) in the above-mentioned spaces.

Regarding the wave equation itself, setting $F_n(t) = \sum_{k=1}^{n} f_k(t)\phi_k$, we have

$$\sum_{k=1}^{n} u_k''(t)\phi_k + \sum_{k=1}^{n} \lambda_k u_k(t)\phi_k = F_n(t).$$

For all test-functions $v \in H_0^1(\Omega)$, by taking the L^2 scalar product of the above formula with $v = \sum_{k=1}^{+\infty} v_k\phi_k$, we thus obtain

$$\sum_{k=1}^{n} u_k''(t)v_k + \sum_{k=1}^{n} \lambda_k u_k(t)v_k = (F_n(t)|v)_{L^2(\Omega)}.$$

Now $\sum_{k=1}^{n} u_k(t)v_k = (U_n(t)|v)_{L^2(\Omega)} \to (u(t)|v)_{L^2(\Omega)}$ in $C^0([0,T])$, so that

$$\sum_{k=1}^{n} u_k''(t)v_k = \big((U_n(t)|v)_{L^2(\Omega)}\big)'' \to \big((u(t)|v)_{L^2(\Omega)}\big)'' \text{ in the sense of } \mathscr{D}'(]0,T[)$$

when $n \to +\infty$. Similarly

$$\sum_{k=1}^{n} \lambda_k u_k(t)v_k = a(U_n(t), v) \to a(u(t), v) \text{ in } C^0([0,T]).$$

Finally, $F_n \to f$ in $L^2(0,T;L^2(\Omega))$ and therefore

$$(F_n(t)|v)_{L^2(\Omega)} \to (f(t)|v)_{L^2(\Omega)} \text{ in } L^2(0,T),$$

and we obtain the variational form of the wave equation in the limit $n \to +\infty$.

The initial conditions are obviously satisfied by construction. □

Remark 9.7 For this proof to work, we need the compatibility condition between the initial condition u_0 and the Dirichlet boundary condition, but no such condition is needed for the initial velocity u_1. Formulas (9.4)–(9.5) clearly generalizes the expansion obtained in Theorem 9.2. □

Remark 9.8 The d-dimensional wave equation can be solved along the exact same lines, see [5, 28]. However, here again, other approaches, such as semigroups, are possible. □

Remark 9.9 The series estimates above immediately imply stability in the energy norm for the weak solutions as well, in the sense that given two sets of data with corresponding solutions u_1 and u_2, we have

$$\begin{aligned}\|u_1 - u_2\|^2_{C^0([0,T];H_0^1(\Omega))} &+ \|u_1 - u_2\|^2_{C^1([0,T];L^2(\Omega))} \\ &\le C(|u_{1,0} - u_{2,0}|^2_{H_0^1(\Omega)} + \|u_{1,1} - u_{2,1}\|^2_{L^2(\Omega)} + \|f_1 - f_2\|^2_{L^2(Q)}).\end{aligned}$$

This is also a continuity result of the solution with respect to the initial conditions and right-hand side. □

As a consequence, the energy equality of Proposition 9.1 is still valid here, the energy being defined by $E(t) = \frac{1}{2}\big(\|u'(t)\|^2_{L^2(\Omega)} + |u(t)|^2_{H_0^1(\Omega)}\big)$ as before. More precisely,

Proposition 9.3 *Let u be the solution given by Theorem 9.3. Then we have $E \in H^1(]0,T[)$ with*

$$\frac{dE}{dt}(t) = \big(f(t)|u'(t)\big)_{L^2(\Omega)}.$$

Proof We approximate u_0, u_1 and f by smooth functions u_0^n, u_1^n and f^n, in their respective function spaces. By the stability estimate above, the corresponding solution u^n is such that $u^n \to u$ in $C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$. We can apply Proposition 9.1 to u^n so that[2]

$$\frac{dE^n}{dt}(t) = \big(f^n(t)|(u^n)'(t)\big)_{L^2(\Omega)},$$

where $E^n(t)$ is the energy of u^n. Clearly $E^n \to E$ in $C^0([0,T])$. Moreover,

$$\begin{aligned}
&\Big|\big(f^n(t)|(u^n)'(t)\big)_{L^2(\Omega)} - \big(f(t)|u'(t)\big)_{L^2(\Omega)}\Big| \\
&\quad\le \Big|\big(f^n(t) - f(t)|(u^n)'(t)\big)_{L^2(\Omega)}\Big| + \Big|\big(f(t)|(u^n)'(t) - u'(t)\big)_{L^2(\Omega)}\Big| \\
&\quad\le \|f^n(t) - f(t)\|_{L^2(\Omega)}\|(u^n)'(t)\|_{L^2(\Omega)} \\
&\qquad + \|f(t)\|_{L^2(\Omega)}\|(u^n)'(t) - u'(t)\|_{L^2(\Omega)},
\end{aligned}$$

so that squaring and integrating in time, we obtain

$$\Big\|\frac{dE^n}{dt} - (f|u')_{L^2(\Omega)}\Big\|^2_{L^2(0,T)} \le C\big(\|f^n - f\|^2_{L^2(0,T;L^2(\Omega))} + \|(u^n)' - u'\|^2_{C^0([0,T];L^2(\Omega))}\big).$$

It follows from this that $\frac{dE^n}{dt} \to (f|u')_{L^2(\Omega)}$ in $L^2(0,T)$, and since $\frac{dE^n}{dt} \to \frac{dE}{dt}$ in the sense of $\mathscr{D}'(]0,T[)$, that $\frac{dE}{dt} = (f|u')_{L^2(\Omega)}$ belongs to $L^2(0,T)$. □

Remark 9.10 In the case $f = 0$, we obtain that the energy is also conserved for weak solutions. □

9.3 The Wave Equation on $\mathbb{R}$

We now consider the case of the wave equation on $\mathbb{R}$ with $f = 0$. There is no boundary condition. In this case, there is an explicit formula for the solution, similar to that obtained for the transport equation and known as the d'Alembert formula, see [35].

Theorem 9.4 *Let $u_0 \in C^1(\mathbb{R})$ and $u_1 \in C^0(\mathbb{R})$. The solution of problem (9.1) on $\mathbb{R}$ with $f = 0$ is given by*

$$u(x,t) = \frac{1}{2}\Big(u_0(x+t) + u_0(x-t) + \int_{x-t}^{x+t} u_1(s)\,ds\Big). \tag{9.8}$$

[2] We admit here that both formulations coincide in the smooth case.

Proof The function given by formula (9.8) is continuous on $\mathbb{R} \times \mathbb{R}_+$, hence is a distribution on $\mathbb{R} \times \mathbb{R}_+^*$. It is in fact of class C^1 on $\mathbb{R} \times \mathbb{R}_+$, and we can write

$$u(x,t) = F(x+t) + G(x-t) \tag{9.9}$$

with $F(y) = \frac{1}{2}\big(u_0(y) + \int_0^y u_1(s)\,ds\big)$ and $G(y) = \frac{1}{2}\big(u_0(y) + \int_y^0 u_1(s)\,ds\big)$. Let us set $U(x,t) = F(x+t)$ and $V(x,t) = G(x-t)$. Of course, we have

$$\frac{\partial U}{\partial t}(x,t) = F'(x+t), \quad \frac{\partial U}{\partial x}(x,t) = F'(x+t) \tag{9.10}$$

and

$$\frac{\partial V}{\partial t}(x,t) = -G'(x-t), \quad \frac{\partial V}{\partial x}(x,t) = G'(x-t). \tag{9.11}$$

Let us compute $\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2}$ in the sense of distributions. We thus take $\varphi \in \mathscr{D}(\mathbb{R} \times \mathbb{R}_+^*)$ and consider the following duality bracket

$$\Big\langle \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2}, \varphi \Big\rangle = -\Big\langle \frac{\partial U}{\partial t} - \frac{\partial U}{\partial x}, \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x} \Big\rangle = 0,$$

by Eq. (9.10), and similarly $\frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} = 0$ in the sense of distributions by Eq. (9.11).

Concerning the initial conditions, of course

$$u(x,0) = \frac{1}{2}\Big(u_0(x) + u_0(x) + \int_x^x u_1(s)\,ds\Big) = u_0(x)$$

and since

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\big(u_0'(x+t) - u_0'(x-t) + u_1(x+t) + u_1(x-t)\big),$$

we have

$$\frac{\partial u}{\partial t}(x,0) = \frac{1}{2}\big(u_0'(x) - u_0'(x) + u_1(x) + u_1(x)\big) = u_1(x).$$

This proves the theorem. □

Remark 9.11 Formula (9.8) makes sense for much less regular data, for example u_0 and u_1 in $L^1_{\text{loc}}(\mathbb{R})$, and still gives rise to a solution of the wave equation in the sense of distributions. In fact, we may even take u_0 and u_1 in $\mathscr{D}'(\mathbb{R})$ by interpreting the integral as the sum of two primitives. The problem is thus to make sense of the initial condition in such a nonsmooth context, see Figs. 9.3 and 9.4. Some continuity with respect to time is needed, but we do not pursue in this direction.

Such an explicit formula as (9.8) is specific to the one-dimensional case. The solution is not so simple in higher dimensions of space. □

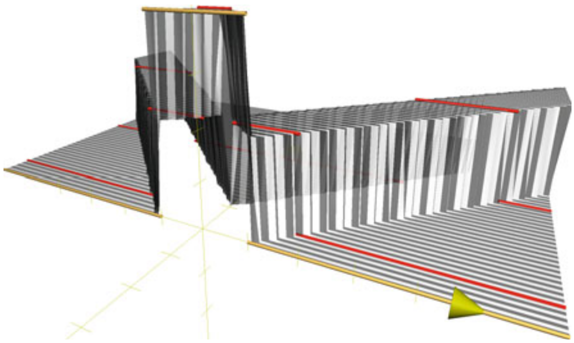

Fig. 9.3 A view of the evolution in the case of $u_0 = \mathbf{1}_{[-1/2,1/2]}$ and zero initial velocity, with the two issuing waves propagating right and left $\frac{1}{2}u_0(x-t)$ and $\frac{1}{2}u_0(x+t)$. Also pictured in *thicker red line* the solution at $t = \frac{1}{3}$ and $t = 2$

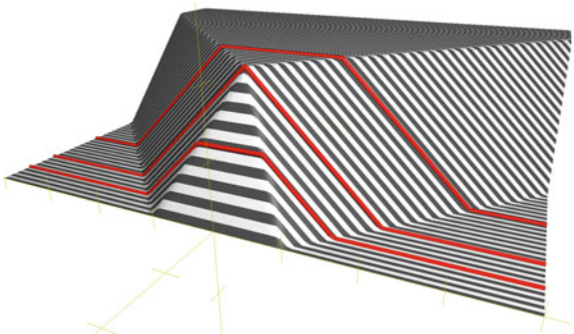

Fig. 9.4 A view of the evolution in the case of $u_1 = \mathbf{1}_{[-1,1]}$ and $u_0 = 0$. Also pictured in *thicker red line* the solution at $t = \frac{1}{2}$, $t = 1$ and $t = 2$. It is only Lipschitz in space and time

Remark 9.12 The solution pictured in Fig. 9.3 is discontinuous in space and time. It is thus meant to be understood as a solution of the wave equation in the sense of distributions. The interpretation of the initial conditions, in particular for the velocity, is admittedly a little more delicate. □

Remark 9.13 Independently of any considerations of initial data, it is easy to see that all solutions of the wave equation are of the form (9.9). Indeed, the change of variables $w = x + t$, $z = x - t$ leads to the equation $\frac{\partial^2 u}{\partial w \partial z} = 0$ whose solutions are clearly of the form $F(w) + G(z)$. The solution is thus seen as the superposition of two waves, one traveling to the left at speed -1 ($F(x+t)$) and the other traveling to the right at speed $+1$ ($G(x-t)$). In the general case, $c \neq 1$, the corresponding form is $u(x,t) = F(x+ct) + G(x-ct)$. □

Remark 9.14 We also see that the wave equation propagates waves at finite speed ($\pm c$), as opposed to the heat equation which has infinite speed of propagation. In particular, if the initial data are compactly supported in $[a,b]$, then the solution at time t is compactly supported in $[a-ct, b+ct]$. Another way of seeing this is to note that the value of the solution at point (x,t) only depends on what happens in its *backward cone of influence* $\{(y,s) \in \mathbb{R} \times \mathbb{R}_+; s \le t, |y-x| \le c(t-s)\}$, see Fig. 9.5. The information situated outside of the cone of influence does not have the time to propagate to point (x,t). □

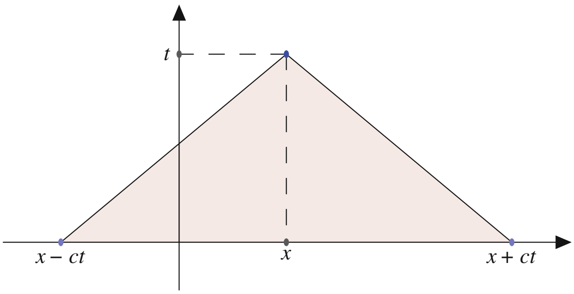

Fig. 9.5 The backward cone of influence of point (x, t)

Remark 9.15 If u_0 is compactly supported and $u_1 = 0$, an observer located at some point $x > 0$ initially outside of the support of u_0, sees a wave $\frac{1}{2}u_0(x - ct)$ reach him or her after some time, pass through, and then go back to exactly 0. This is a feature of the wave equation in odd dimensions of space. This explains why we see light and hear sounds as we do: a flash of light at some point in space-time results in a spherical wavefront expanding at the speed of light that an observer experiences as a single instantaneous flash when reached by the wavefront. The same goes for sound. This is not true in even dimensions. For example, if we throw a rock on a lake, the resulting wave on the surface of the lake expands as a circle traveling at the speed of waves on water, but never goes back to rest inside the disk, even though the solution is much smaller there. If we lived on the surface of the water, we would experience a flash followed by a never-ending afterglow… good thing we live in an odd-dimensional space. □

Remark 9.16 The wave equation is invariant under the change $t \to -t$. This means that time is reversible in the wave equation, which is another feature in sharp contrast with the heat equation. □

9.4 Finite Difference Schemes for the Wave Equation

The principle of finite difference methods for the wave equation is exactly the same as for the heat equation, and the notation is also the same. Since the wave equation is of second order in time, a natural idea is to consider two time steps finite difference schemes, even though we will see that this is not necessarily a good idea. We will assume the initial conditions U^0 and U^1 to be given in terms of u_0 and u_1. For instance, a simple choice could be

$$U^0 = S_h(u_0), \quad U^1 = U^0 + kS_h(u_1),$$

or higher order approximations for U^1.

The most obvious scheme consists in approximating the second time derivative by means of the usual central difference, which yields

$$\frac{u_n^{j+1}-2u_n^j+u_n^{j-1}}{k^2}-\frac{u_{n+1}^j-2u_n^j+u_{n-1}^j}{h^2}=f_n^j, \tag{9.12}$$

with the usual boundary conditions and initial data. This is obviously an explicit, two time steps scheme. In vector form, it reads

$$\frac{U^{j+1}-2U^j+U^{j-1}}{k^2}+A_hU^j=F^j,$$

where A_h is still given by formula (2.8) with $c=0$, p. 40 of Chap. 2. It should be quite clear that the scheme is consistent and of order 2 in space and time. Therefore, its convergence is solely a matter of stability.

We reformulate the above scheme as a single time step scheme by setting

$$V^j=\begin{pmatrix}U^j\\ U^{j-1}\end{pmatrix}\in\mathbb{R}^{2N},$$

and

$$\begin{aligned}\frac{1}{k^2}V^{j+1}&=\begin{pmatrix}\frac{2}{k^2}I-A_h & -\frac{1}{k^2}I\\ \frac{1}{k^2}I & 0\end{pmatrix}\begin{pmatrix}U^j\\ U^{j-1}\end{pmatrix}+\begin{pmatrix}F^j\\ 0\end{pmatrix}\\ &=\frac{1}{k^2}\mathscr{A}V^j+G^j\end{aligned}$$

with a $2N\times 2N$ amplification matrix $\mathscr{A}=\begin{pmatrix}C & -I\\ I & 0\end{pmatrix}$ with $C=2I-k^2A_h$, and $G^j\in\mathbb{R}^{2N}$. Unfortunately, the matrix $\mathscr{A}$ is not normal. Indeed

$$\mathscr{A}^T\mathscr{A}=\begin{pmatrix}C^2+I & -C\\ -C & I\end{pmatrix}\neq\begin{pmatrix}C^2+I & C\\ C & I\end{pmatrix}=\mathscr{A}\mathscr{A}^T.$$

Therefore, we only have $\rho(\mathscr{A})\leq |||\mathscr{A}|||_{2,h}$ and the condition $\rho(\mathscr{A})\leq 1+C(T)k$ is just a necessary condition for stability, see Remark 8.13 in Chap. 8, whereas we also would like to have a sufficient condition for stability. In order to have a necessary stability condition that is valid for all T, it is easier to require $\rho(\mathscr{A})\leq 1$. Let us see what we can say about the spectral radius of $\mathscr{A}$.

Lemma 9.1 *Let C be a $N\times N$ complex matrix and B the $2N\times 2N$ complex matrix defined by blocks as*

$$B=\begin{pmatrix}C & -I\\ I & 0\end{pmatrix}.$$

If $\lambda \in \mathbb{C}$ is an eigenvalue of B, then $\lambda \neq 0$ and $\lambda + \frac{1}{\lambda}$ is an eigenvalue of C. Conversely, if $\mu \in \mathbb{C}$ is an eigenvalue of C, then there exists an eigenvalue λ of B such that $\mu = \lambda + \frac{1}{\lambda}$.

Proof The proof is similar to that of Lemma 8.2 of Chap. 8. □

Proposition 9.4 *If $\frac{k}{h} \leq 1$, then the necessary stability condition for scheme (9.12) is satisfied.*

Proof Recall that stability is meant here in the sense of $\rho(\mathscr{A}) \leq 1$. So we need to find out when all the eigenvalues λ of $\mathscr{A}$ are such that $|\lambda| \leq 1$. According to Lemma 9.1, the eigenvalues in question are of the form $\lambda_{\pm} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$ where μ is an eigenvalue of $C = 2I - k^2 A_h$, hence is real. We thus see that there are two cases:

1. $|\mu| > 2$. In this case, the two eigenvalues $\lambda_{\pm}$ are real, distinct, and since their product is equal to 1, one of them is strictly larger than 1 in absolute value. Hence this is an unstable case.

2. $|\mu| \leq 2$. In this case, $\lambda_{\pm}$ are complex conjugate, and since their product is equal to 1, they are both of modulus 1. The necessary stability condition is thus satisfied.

Now we have $\mu = 2 - 4\frac{k^2}{h^2} \sin^2\bigl(\frac{p\pi}{2(N+1)}\bigr)$, $p = 1, \ldots, N$. Clearly, if $\frac{k}{h} \leq 1$, then we have $|\mu| \leq 2$. □

Remark 9.17 The condition $\frac{k}{h} \leq 1$ is called the *Courant–Friedrichs–Lewy* or *CFL* condition. In the general case, the CFL condition assumes the form $\frac{k}{h} \leq \frac{1}{c}$. In a sense, $\frac{h}{k}$ is the numerical velocity needed to reach the neighboring grid points in one time step starting from one spatial grid point, see Fig. 9.6. The CFL condition is that this numerical velocity must be larger than the propagation velocity.

In other words, the discrete backward cone of influence of a point (x_n, t_j) must contain its continuous backward cone of influence, in order for the scheme to have access to all the information needed to compute a relevant approximation at that point. Of course, this kind of requirement only applies to explicit schemes. □

Let us plot the result of the explicit scheme with + marks and the exact solution in solid line in Fig. 9.7. We take the same u_0 as for the heat equation, i.e., $u_0(x) = \sin(\pi x)/2 + \sin(2\pi x)$, and $u_1 = 0$. We have taken $U^1 = U^0$, which is actually a

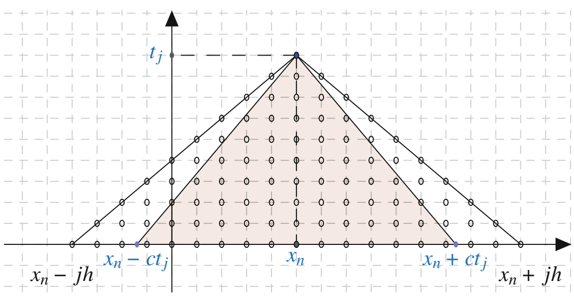

Fig. 9.6 Discrete cone of influence versus continuous cone of influence

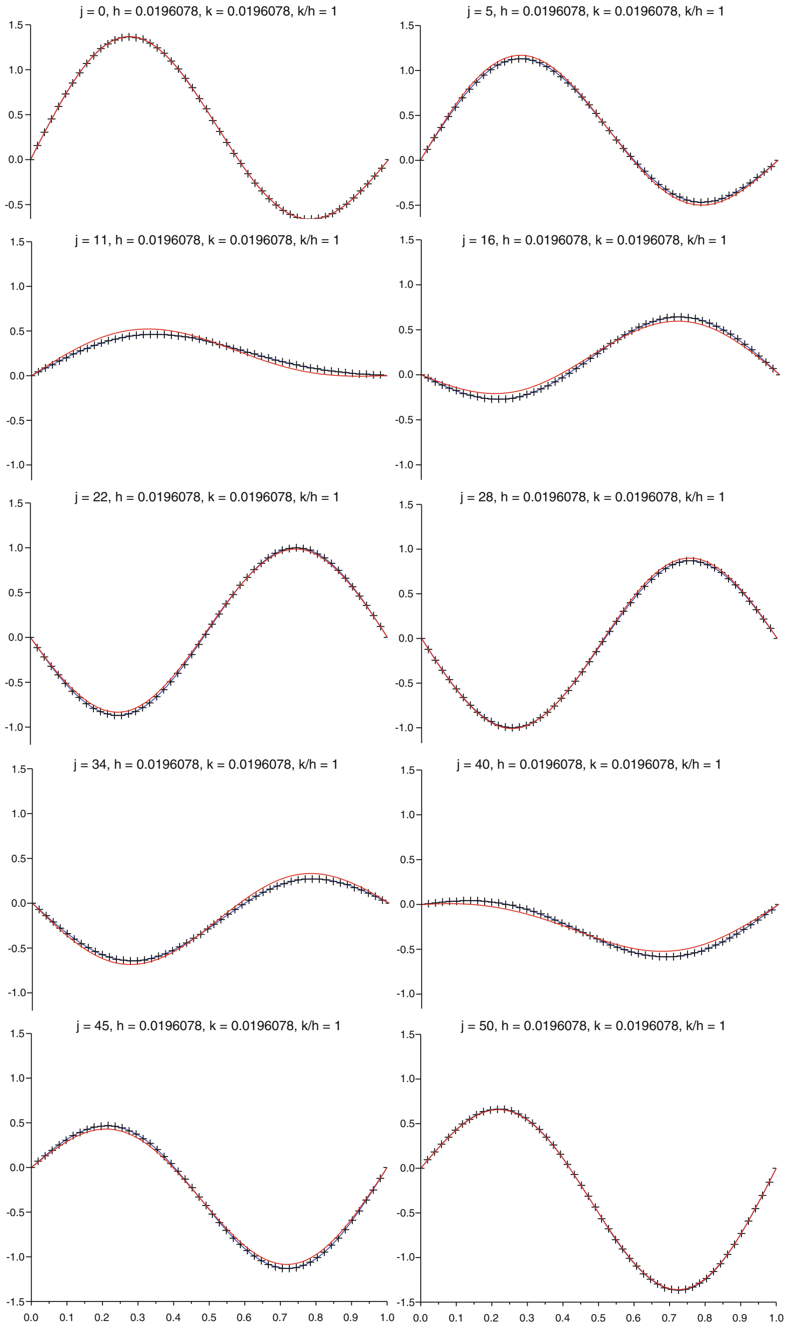

Fig. 9.7 Explicit scheme, $u_0(x) = \sin(\pi x)/2 + \sin(2\pi x)$, $u_1(x) = 0$

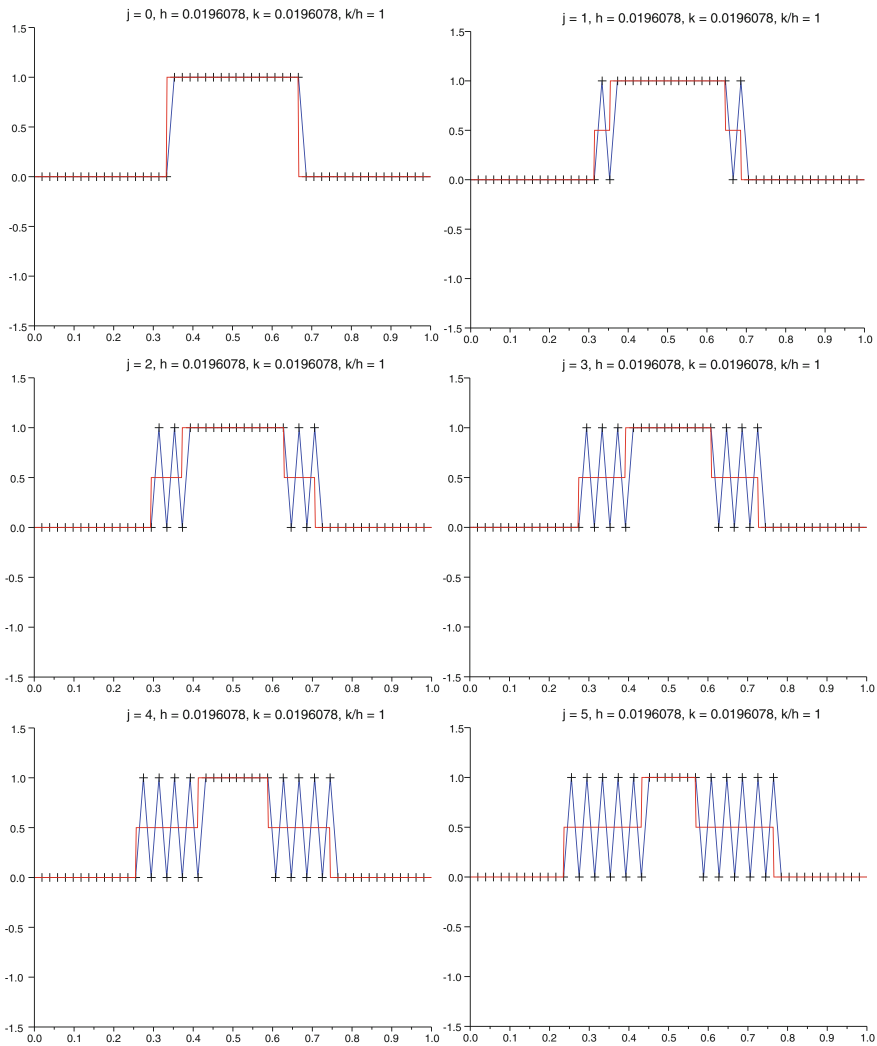

Fig. 9.8 Explicit scheme, $u_0 = \mathbf{1}_{[\frac{1}{3},\frac{2}{3}]}$, $u_1 = 0$

second order approximation of the condition $u_1 = 0$, hence the good global accuracy of the scheme in this particular case.

Of course, the initial condition is very smooth here. If we want to compute a discontinuous solution with this scheme, we run into trouble with severe unwanted oscillations, see Fig. 9.8. This kind of discontinuous solution is of physical interest in situations where *shock waves* occur. Devising numerical schemes capable of reliably capturing shocks thus requires skills that go beyond the scope of these notes.

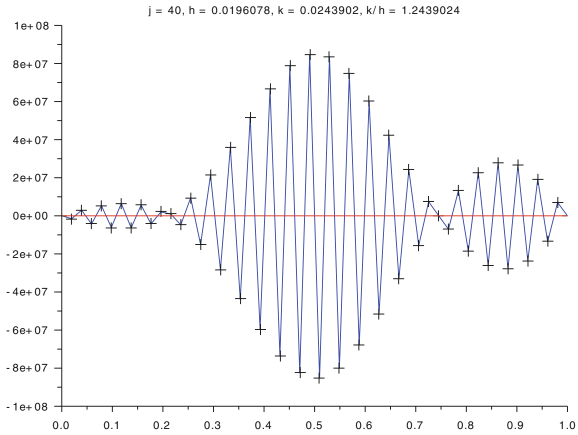

Fig. 9.9 Explicit scheme, $u_0(x) = \sin(\pi x)/2 + \sin(2\pi x)$, $u_1(x) = 0$, CFL condition not satisfied

For the record, Fig. 9.9 shows what happens after a few iterations when the CFL condition is violated, even with the very smooth initial condition used above.

The next obvious scheme is the implicit version of the former one

$$\frac{u_n^{j+1} - 2u_n^j + u_n^{j-1}}{k^2} - \frac{u_{n+1}^{j+1} - 2u_n^{j+1} + u_{n-1}^{j+1}}{h^2} = f_n^{j+1}, \tag{9.13}$$

with the usual boundary conditions and initial data. In vector form, it reads

$$\Big(\frac{1}{k^2}I + A_h\Big)U^{j+1} = \frac{2}{k^2}U^j - \frac{1}{k^2}U^{j-1} + F^{j+1}.$$

We rewrite it as a single time step scheme

$$V^{j+1} = \mathscr{A}V^j + G^j$$

with

$$\mathscr{A} = \begin{pmatrix} 2(I + k^2 A_h)^{-1} & -(I + k^2 A_h)^{-1} \\ I & 0 \end{pmatrix}.$$

Again, the matrix $\mathscr{A}$ is not normal, but we can look at its spectral radius. Setting $C = (I + k^2 A_h)^{-1}$, the same kind of arguments as before show that the eigenvalues λ of $\mathscr{A}$ are of the form $\lambda_\pm = \mu \pm \sqrt{\mu^2 - \mu}$ where μ is an eigenvalue of C. Now $\mu \in \,]0, 1[$, therefore $\mu^2 - \mu < 0$ and the eigenvalues λ_+ and λ_- are complex conjugate, of modulus $\sqrt{\mu}$. The necessary condition for stability is thus unconditionally satisfied.

The implicit scheme does a slightly better job of capturing shocks than the explicit scheme for the same discretization parameters, but it still has a lot of numerical diffusion that spreads out the shocks, see Fig. 9.10.

A third scheme is the θ-scheme for $\theta \in [0, \frac{1}{2}]$, written here for $f = 0$,

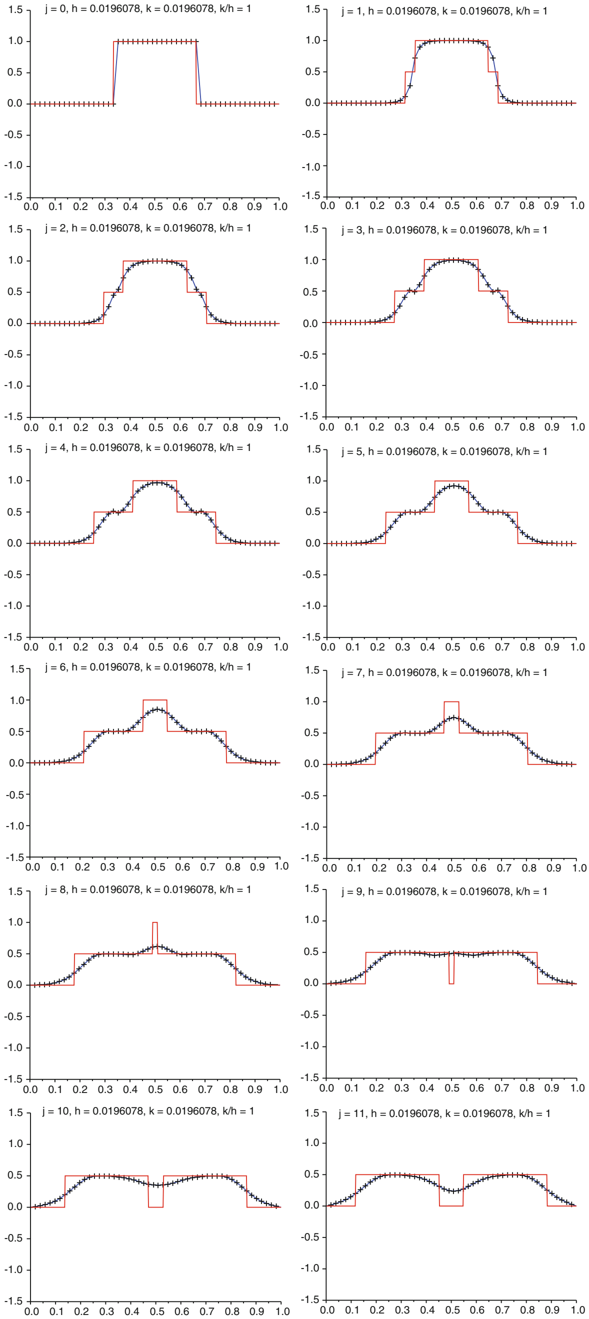

Fig. 9.10 Implicit scheme, $u_0 = \mathbf{1}_{[\frac{1}{3},\frac{2}{3}]}, u_1 = 0$

$$\frac{u_n^{j+1} - 2u_n^j + u_n^{j-1}}{k^2} - \theta \frac{u_{n+1}^{j+1} - 2u_n^{j+1} + u_{n-1}^{j+1}}{h^2} - (1-2\theta)\frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2} - \theta \frac{u_{n+1}^{j-1} - 2u_n^{j-1} + u_{n-1}^{j-1}}{h^2} = 0, \quad (9.14)$$

which reduces to the explicit scheme for $\theta = 0$ and is implicit for $\theta > 0$.

9.5 Stability via the Fourier Approach

So far we have only obtained necessary conditions for stability, because the matrices were not normal. As in the case of numerical schemes for the heat equation, we can also use the Fourier method to obtain sufficient conditions. Again, we work on the whole of $\mathbb{R}$. We take $f = 0$. As was already mentioned, there are no boundary conditions and let us forget for the moment that the solution is given by an explicit formula.

Concerning the relationship between discrete and semi-discrete schemes, every thing said for the heat equation holds true here. In addition, we note that, due to the finite speed of propagation, if the initial data is compactly supported in an interval, a finite difference scheme on the interval with boundary conditions will compute exactly the same values as the same scheme on $\mathbb{R}$ as long as the wave has not hit the ends of the interval. Therefore, so does the semi-discrete scheme, and the stability conditions obtained from the Fourier method actually apply to the scheme on an interval with boundary conditions (at least under the previous conditions).

Let us first consider the explicit scheme (9.12), for which we already have a necessary stability condition, but no sufficient condition. Both Fourier approaches are more complicated than for the heat equation, since the linear recurrence relations obtained are two-step relations, which are harder to analyze than the one-step relations in the heat equation case. Since the two Fourier approaches are very similar to each other, we first concentrate on the Fourier series point of view for brevity, see Sect. 8.7.

Since we are working on the whole space, the scheme (9.12) with zero right-hand side takes the form

$$\frac{u_n^{j+1} - 2u_n^j + u_n^{j-1}}{k^2} - \frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2} = 0, \text{ for } n \in \mathbb{Z}. \quad (9.15)$$

In Fourier space, the scheme reads

$$\frac{\mathscr{F}(u^{j+1})(s) - 2\mathscr{F}(u^j)(s) + \mathscr{F}(u^{j-1})(s)}{k^2} + \frac{4}{h^2}\sin^2\left(\frac{s}{2}\right)\mathscr{F}(u^j)(s) = 0, \quad (9.16)$$

for $s \in [0, 2\pi]$. As was mentioned above, this is a two-step linear recurrence relation, which we rewrite in vector form by introducing the $\mathbb{R}^2$-valued sequence

$Z_n^j = \begin{pmatrix} u_n^{j+1} \\ u_n^j \end{pmatrix}$, the Fourier series transform of which satisfies

$$\mathscr{F}(Z^{j+1})(s) = A(s)\mathscr{F}(Z^j)(s)$$

where

$$A(s) = \begin{pmatrix} 2 - a(s)^2 & -1 \\ 1 & 0 \end{pmatrix}$$

with $a(s) = \frac{2k}{h}\sin\left(\frac{s}{2}\right)$. The matrix A is called the *amplification matrix* of the scheme. We list without proof several properties of such schemes, since they are easy generalizations of former results.

Let $A \in C^0([0, 2\pi]; \mathbb{M}_2(\mathbb{C}))$ and consider the multiplier operator L defined on $L^2([0, 2\pi]; \mathbb{C}^2)$ with values in $L^2([0, 2\pi]; \mathbb{C}^2)$ by $(LY)(s) = A(s)Y(s)$, for almost all $s \in [0, 2\pi]$. The generalization of Eq. (8.7) in this case is

$$\|L\|_{\mathscr{L}(L^2([0,2\pi];\mathbb{C}^2))} = \max_{s\in[0,2\pi]} |||A(s)|||_2$$

(we use the standard hermitian norm on $\mathbb{C}^2$), see [67]. It follows that the scheme (9.16) is stable in L^2 if and only if there exists a constant $C(T)$ such that

$$\max_{s\in[0,2\pi]} |||A(s)^j|||_2 \leq C(T)$$

for all $j \leq T/k$. Since $\rho(A(s)) \leq |||A(s)^j|||_2^{1/j}$, a necessary condition of stability is that there exists a nonnegative constant C that does not depend on k and h such that $\rho(A(s)) \leq 1 + Ck$ for all s. We say that the scheme is *stable in the sense of von Neumann* if $\rho(A(s)) \leq 1$ for all s.

Proposition 9.5 *The scheme (9.16) is stable in the sense of von Neumann if and only if* $\frac{k}{h} \leq 1$.

Proof The characteristic polynomial of $A(s)$ is

$$P_{A(s)}(X) = X^2 - (2 - a(s)^2)X + 1.$$

The product of the two roots is 1, thus if they are both real and simple, one of them is strictly larger than 1 in absolute value. On the other hand, if they are complex conjugate, they are both of modulus 1. Consequently, von Neumann stability is equivalent to the discriminant being non positive.

We thus need to see under which condition $\Delta(s) = a(s)^2(a(s)^2 - 4) \leq 0$ for all s. Taking $s = \pi$, we see that a necessary condition is $\frac{k}{h} \leq 1$. Conversely, this condition is clearly sufficient. □

In the case of a normal amplification matrix, the former conditions are also sufficient for L^2-stability. Now the matrix $A(s)$ obtained above is not normal in general, except for $s = \pi$ if $\frac{k}{h} = 1$, thus we need a more general result. We work directly on the stability of the scheme expressed in Fourier space by the operator L above, since this is equivalent to L^2-stability for the discrete scheme, due to Parseval's formula.

Stability in L^2 is ensured if $|||A(s)^j|||_2 \le C(T)$ for all s and $j \le T/k$, where $C(T)$ does not depend on h or k. It should be noted that this condition is a priori much easier to check here than it was for (actual) finite difference schemes, since the matrix in question is always of the same size, i.e., 2×2, whereas the size of the matrix in finite difference schemes was $N \times N$ with $h = \frac{1}{N+1}$ and the norm also depended on h.

Proposition 9.6 *The discrete scheme (9.16) is not stable in L^2.*

Proof It is enough to look at what happens for $s = 0$. In this case, the amplification matrix has the double eigenvalue 1 and is clearly not diagonalizable. In effect,

$$A(0) = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} = P \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} P^{-1}, \text{ where } P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

from which we immediately deduce that

$$A(0)^j = \begin{pmatrix} j+1 & -j \\ j & 1-j \end{pmatrix}.$$

We thus have $|||A(0)^j|||_2 = \sqrt{2j^2 + 1 + 2j\sqrt{j^2+1}} \ge 2j$. It follows in particular that $|||A(0)^{T/k}|||_2 \ge \frac{2T}{k}$, hence the instability of the discrete scheme. □

Remark 9.18 This is an unsettling result, since we could have rightfully expected the very natural scheme (9.15) to be stable under the CFL condition $\frac{k}{h} \le 1$ or at least $\frac{k}{h} < 1$. This is not the case.

The reason for the natural scheme not to be stable is the following. If it was stable in ℓ^2, then having sequences of initial data $(u^0)_m$ and $(u^1)_m$ bounded in ℓ^2 would result in a sequence of solutions $(u^j)_m$ also bounded in ℓ^2, uniformly for $j \le T/k$. However, this boundedness could be achieved with $(u^0)_m$ and $(u^1)_m$ having strictly nothing to do with each other. Now we need to remember that $u_n^0 = u_0(nh)$ and u_n^1 is supposed to be some approximation of $u(nh, k) \approx u_0(nh) + k\frac{\partial u}{\partial t}(nh, 0)$. Thus both initial conditions of the wave equation $u(x, 0) = u_0(x)$ and $\frac{\partial u}{\partial t}(x, 0) = u_1(x)$ must somehow be taken into account in the discrete initial data u^0 and u^1 for the scheme to have any chance to converge. This is not the case if $(u^0)_m$ and $(u^1)_m$ can be chosen independently of each other.

Note that the scheme is unstable in spite of being von Neumann stable, an unhappy effect of terminology. □

Remark 9.19 We can see the instability of the scheme on the following example. Let us consider the initial data $u^0 = 0$ and u^1 given by $u_0^1 = h^{-1/2}$ and $u_n^1 = 0$ for $n \ne 0$. We have $\|u^1\|_{2,h} = 1$. Let us show that $\|u^j\|_{2,h}$ is not bounded for $j \le T/k$. In Fourier space, the recurrence relation reads

$$\mathscr{F}(u^{j+1})(s) = (2 - a^2(s))\mathscr{F}(u^j)(s) - \mathscr{F}(u^{j-1})(s).$$

The characteristic equation of this recurrence relation $X^2 - (2 - a^2(s))X + 1 = 0$ has two roots $r_\pm = e^{\pm i\theta(s)}$, $\theta(s) = \arccos(1 - \frac{a^2(s)}{2})$ for $s \neq 0$ and $s \neq 2\pi$, and a double root $r = 1$ for $s = 0$ or 2π.

We have $\mathscr{F}(u^0)(s) = 0$ and $\mathscr{F}(u^1)(s) = h^{-1/2}$, thus

$$\mathscr{F}(u^j)(s) = \begin{cases} h^{-1/2}\frac{\sin(j\theta(s))}{\sin(\theta(s))}, & \text{for } 0 < s < 2\pi, \\ jh^{-1/2}, & \text{for } s = 0 \text{ or } s = 2\pi. \end{cases}$$

We are interested in the L^2 norm of the above function. Clearly

$$\|\mathscr{F}(u^j)\|^2_{L^2(0,2\pi),h} \geq \int_0^{\theta^{-1}(\frac{\pi}{2j})} \frac{\sin^2(j\theta(s))}{\sin^2(\theta(s))}\,ds.$$

Now, on the interval $\left[0, \theta^{-1}\left(\frac{\pi}{2j}\right)\right]$, we have $\frac{\sin(j\theta(s))}{\sin(\theta(s))} \geq \frac{2j}{\pi}$. Consequently

$$\|\mathscr{F}(u^j)\|^2_{L^2(0,2\pi),h} \geq \frac{4j^2}{\pi^2}\theta^{-1}\left(\frac{\pi}{2j}\right).$$

After a little bit of computation, we find that $\theta^{-1}\left(\frac{\pi}{2j}\right) \sim \frac{\pi}{2\sqrt{2\lambda}j}$ when $j \to +\infty$. Therefore, for j large enough, we obtain

$$\|\mathscr{F}(u^j)\|^2_{L^2(0,2\pi),h} \geq \frac{j}{\pi\sqrt{2\lambda}} \to +\infty \text{ when } j \to +\infty.$$

Going back to the original discrete scheme, it follows that $\|u^j\|_{2,h} \to +\infty$ when $j \to +\infty$ for this particular sequence of bounded initial data. $\square$

In order to obtain sufficient stability conditions, we actually need to change the unknowns so as to appropriately take care of the wave equation initial conditions. First, we rewrite the wave equation: Find $u\colon\ Q = \mathbb{R} \times [0, T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0 \text{ in } Q, \\ u(x, 0) = u_0(x),\ \dfrac{\partial u}{\partial t}(x, 0) = u_1(x) \text{ for } x \in \mathbb{R}, \end{cases} \tag{9.17}$$

as a first order system.

Proposition 9.7 *Let $v = \frac{\partial u}{\partial t}$ and $w = \frac{\partial u}{\partial x}$. Problem (9.17) is equivalent to the system of first order PDEs*

$$\begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{\partial w}{\partial x} = 0 \text{ } in \text{ } Q, \\ \dfrac{\partial w}{\partial t} - \dfrac{\partial v}{\partial x} = 0 \text{ } in \text{ } Q, \\ v(x,0) = u_1(x), w(x,0) = u_0'(x) \text{ } for \text{ } x \in \mathbb{R}, \end{cases} \tag{9.18}$$

up to an additive constant.

Proof If u solves problem (9.17), then the first equation in (9.18) is just the wave equation, and the second equation is just $\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial t \partial x}$. The initial conditions are obvious.

Conversely, let (v, w) solve (9.18). Since Q is simply connected, the second equation implies that there exists $\widetilde{u}$ such that $v = \frac{\partial \widetilde{u}}{\partial t}$ and $w = \frac{\partial \widetilde{u}}{\partial x}$. The first equation is the wave equation for $\widetilde{u}$. By the second initial condition, there exists a constant c_0 such that $\widetilde{u}(x, 0) = u_0(x) + c_0$. Hence, $u = \widetilde{u} - c_0$ solves (9.17). □

Remark 9.20 Let us note that, in the case of a bounded interval with Dirichlet conditions, the energy estimate of Corollary 9.1 gives an L^2 bound on the variables v and w, and not directly on u. This explains the choice of these variables for an L^2 stability analysis. □

We perform a similar operation on the finite difference scheme. We use the notation $x_\tau = \tau h$ for $\tau \in \mathbb{R}$, which agrees with the former notation x_n when $\tau = n \in \mathbb{Z}$.

Proposition 9.8 *Let*

$$v_n^j = \frac{u_n^j - u_n^{j-1}}{k} \text{ } and \text{ } w_{n-1/2}^j = \frac{u_n^j - u_{n-1}^j}{h}, \tag{9.19}$$

for $n \in \mathbb{Z}$ and $j \in \mathbb{N}^*$. If u^j is a solution of the finite difference scheme (9.15), then (v^j, w^j) are solution of the finite difference scheme

$$\begin{cases} \dfrac{v_n^{j+1} - v_n^j}{k} - \dfrac{w_{n+1/2}^j - w_{n-1/2}^j}{h} = 0, \\ \dfrac{w_{n-1/2}^{j+1} - w_{n-1/2}^j}{k} - \dfrac{v_n^{j+1} - v_{n-1}^{j+1}}{h} = 0, \end{cases} \tag{9.20}$$

with v_n^1 and $w_{n-1/2}^1$ given by formula (9.19) in terms of the initial data u_n^0 and u_n^1 of (9.15).

Proof Replacing $v_n^j = \frac{u_n^j - u_n^{j-1}}{k}$ and $w_{n-1/2}^j = \frac{u_n^j - u_{n-1}^j}{h}$ into (9.20), we see that the second relation is satisfied by the very definition of v_n^j and $w_{n-\frac{1}{2}}^j$, and that the first relation reduces to the original finite difference scheme. Moreover, the initial conditions are satisfied by construction. □

Remark 9.21 We note that the scheme (9.20) is explicit. Indeed, assuming that v_n^j and $w_{n+1/2}^j$ are already known, the first relation gives $v_n^{j+1} = v_n^j + \frac{k}{h}(w_{n+1/2}^j - w_{n-1/2}^j)$ for all $n \in \mathbb{Z}$. After that, we see that $w_{n-1/2}^{j+1} = w_{n-1/2}^j + \frac{k}{h}(v_n^{j+1} - v_{n-1}^{j+1})$ for all $n \in \mathbb{Z}$ by the second relation. Hence, the solution of (9.20) with initial data v^1 and w^1 exists and is unique. $\square$

Remark 9.22 Clearly, v_n^j is intended to be an approximation of $v(x_n, t_j) = \frac{\partial u}{\partial t}(x_n, t_j)$ and $w_{n-1/2}^j$ an approximation of $w(x_{n-1/2}, t_j) = \frac{\partial u}{\partial x}(x_{n-1/2}, t_j)$.

In view of this, other initial data are reasonable for (9.20), for example $v_n^0 = u_1(x_n)$ and $w_{n-1/2}^0 = u_0'(x_{n-\frac{1}{2}})$, yielding a different approximation from which an approximation of u must be reconstructed. These initial conditions directly take into account the initial conditions of the wave equation. $\square$

This time, we choose to work in the continuous Fourier transform framework, see Sect. 8.8. We thus introduce the semi-discrete version of the scheme as

$$\begin{cases} \dfrac{v^{j+1}(x) - v^j(x)}{k} - \dfrac{w^j(x+h/2) - w^j(x-h/2)}{h} = 0, \\ \dfrac{w^{j+1}(x-h/2) - w^j(x-h/2)}{k} - \dfrac{v^{j+1}(x) - v^{j+1}(x-h)}{h} = 0, \end{cases} \tag{9.21}$$

for all $x \in \mathbb{R}$ and with appropriate initial data. Rewriting this in Fourier space, we obtain

$$\begin{cases} \dfrac{\widehat{v^{j+1}}(\xi) - \widehat{v^j}(\xi)}{k} - \dfrac{e^{i\frac{h\xi}{2}} - e^{-i\frac{h\xi}{2}}}{h}\widehat{w^j}(\xi) = 0, \\ e^{-i\frac{h\xi}{2}}\dfrac{\widehat{w^{j+1}}(\xi) - \widehat{w^j}(\xi)}{k} - \dfrac{1 - e^{-ih\xi}}{h}\widehat{v^{j+1}}(\xi) = 0, \end{cases}$$

for all $\xi \in \mathbb{R}$. Writing $Y^j(x) = \begin{pmatrix} v^j(x) \\ w^j(x) \end{pmatrix}$, we obtain

$$\widehat{Y^j}(\xi) = B(\xi)\widehat{Y^{j-1}}(\xi)$$

with the amplification matrix

$$B(\xi) = \begin{pmatrix} 1 & ia(\xi) \\ ia(\xi) & 1 - a(\xi)^2 \end{pmatrix}$$

where $a(\xi) = \frac{2k}{h}\sin\left(\frac{h\xi}{2}\right)$.

We see that amplification matrices are now complex matrices, we thus need to generalize the results of Chap. 8 to the complex case. First of all, when A is a complex $N \times N$ matrix, its induced matrix norm for the canonical Hermitian norm on $\mathbb{C}^N$ is defined as

$$\|A\|_{2,h} = \sup_{X \in \mathbb{C}^N, X \neq 0} \frac{\|AX\|_{2,h}}{\|X\|_{2,h}}.$$

A complex matrix A is said to be *normal* if $A^*A = AA^*$ where A^* is the adjoint matrix. The following results are proved along the same lines as the corresponding results in the real case.

Proposition 9.9 *Let A be a complex $N \times N$ matrix. We have*

$$|||A|||_{2,h} = \sqrt{\rho(A^*A)}.$$

In addition, if A is normal then $\rho(A) = \rho(A^*A)^{1/2} = |||A|||_{2,h}$.

We now return to the stability of scheme (9.21).

Proposition 9.10 *The scheme (9.21) is stable in the sense of von Neumann if and only if $\frac{k}{h} \le 1$.*

Proof The matrix $B(\xi)$ has the same characteristic polynomial as the matrix $A(s)$ of Proposition 9.5, therefore the proof is the same. □

The matrix $B(\xi)$ is not normal in general, thus von Neumann stability is not a priori sufficient for L^2 stability. The following simple matrix result is useful in this context. Let $M \in \mathbb{M}_2(\mathbb{C})$. Every complex matrix is triangularisable, thus we can write $M = PUP^{-1}$ with $P \in GL_2(\mathbb{C})$ and U upper-triangular.

Proposition 9.11 *For all diagonalizable matrices $M \in \mathbb{M}_2(\mathbb{C})$ such that $\rho(M) \le 1$, we have*

$$|||M^j|||_2 \le |||P|||_2 |||P^{-1}|||_2,$$

for all $j \in \mathbb{N}$.

Proof We can write $M = PUP^{-1}$ with $P \in GL_2(\mathbb{C})$ and U diagonal,

$$U = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where λ_1, λ_2 are the eigenvalues of M. We have $M^j = PU^jP^{-1}$. Therefore

$$|||M^j|||_2 \le |||P|||_2 |||U^j|||_2 |||P^{-1}|||_2.$$

Now $|||U^j|||_2 = \rho(M)^j$, which completes the proof. □

Remark 9.23 The constant $|||P|||_2 |||P^{-1}|||_2$ appearing in the estimate of M^j is nothing but the condition number (introduced in Remark 2.11 of Chap. 2) of the change of basis matrix P. □

Remark 9.24 If M is not diagonalizable, it has a double eigenvalue λ and we have $U = \Lambda + N$ with

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

by the Jordan decomposition theorem. The matrix N is nilpotent, $N^2 = 0$, and commutes with Λ. Therefore, by the binomial identity,

$$U^j = \Lambda^j + j\lambda^{j-1}N,$$

for all $j \in \mathbb{N}$. It follows that if $\rho(M) < 1$,

$$\|U^j\|_2 \leq C(\rho(M)),$$

for all $j \in \mathbb{N}$, where C is a function of the spectral radius, and if $\rho(M) = 1$ then

$$\|M^j\|_2 \to +\infty$$

when $j \to +\infty$. □

Let us now apply Proposition 9.11 to the study of the stability of the scheme (9.21) by applying the proposition to the matrix $M = B(\xi)$, for any $\xi \in \mathbb{R}$. We let

$$B(\xi) = P(\xi)U(\xi)P^{-1}(\xi)$$

where $P(\xi) \in GL_2(\mathbb{C})$ and $U(\xi) \in \mathbb{M}_2(\mathbb{C})$ is upper-triangular for all $\xi \in \mathbb{R}$. Of course, all these matrices are also functions of h and k.

Proposition 9.12 *Let $0 < \lambda_0 < 1$ and assume that $\frac{k}{h} \leq \lambda_0$. Then, the scheme (9.21) is stable in $L^2(\mathbb{R})$.*

Proof In the case $a(\xi) = 0$, then $B(\xi) = I = U(\xi) = P(\xi)$ and there is nothing to do. Let us assume that ξ is such that $a(\xi) \neq 0$. In this case, $\Delta(\xi) < 0$, there are two simple eigenvalues and $B(\xi)$ is diagonalizable. We already know that the eigenvalues of $B(\xi)$ are of modulus 1, so that $\rho(B(\xi)) = 1$. We have the estimate

$$\|B^j(\xi)\|_2 \leq \|P(\xi)\|_2 \|P^{-1}(\xi)\|_2,$$

by Proposition 9.11. Thus, we only need to bound the condition number of the matrix $P(\xi)$. Let us note for the record that $|a(\xi)| \leq 2\lambda_0 < 2$ for all ξ.

Computing the eigenvectors of $A(\xi)$, we find that the change of basis matrix

$$P(\xi) = \begin{pmatrix} ia(\xi) & ia(\xi) \\ \lambda_+(\xi) - 1 & \lambda_-(\xi) - 1 \end{pmatrix}$$

is uniformly bounded since $|a(\xi)| < 2$. More importantly, since $\lambda_\pm(\xi) - 1 = \dfrac{-a(\xi)^2 \pm ia(\xi)\sqrt{4 - a(\xi)^2}}{2}$, we clearly have

$$\|P(\xi)\|_2 \leq C|a(\xi)|.$$

Moreover,

$$P(\xi)^{-1} = \det(P(\xi))^{-1}\begin{pmatrix}\lambda_-(\xi)-1 & -ia(\xi)\\ 1-\lambda_+(\xi) & ia(\xi)\end{pmatrix} = \det(P(\xi))^{-1}Q(\xi).$$

We likewise have $|||Q(\xi)|||_2 \leq C|a(\xi)|$. Since

$$\det(P(\xi)) = ia(\xi)(\lambda_-(\xi)-\lambda_+(\xi)) = a(\xi)^2\sqrt{4-a(\xi)^2},$$

it follows that

$$|\det(P(\xi))| \geq 2\sqrt{1-\lambda_0^2}|a(\xi)|^2,$$

hence

$$|||P(\xi)^{-1}|||_2 \leq C|a(\xi)|^{-1},$$

and the result follows. □

Remark 9.25 We have now established the conditional ℓ^2 stability of scheme (9.20). This means that under the CFL condition, if the initial data v^1 and w^1 remain in a bounded set of ℓ^2, so do the corresponding solutions v^j and w^j for $j \leq T/k$. The unknowns v and w, which are in a sense quite natural for formulating the wave equation, are however not the initial unknown u, either continuous or discrete. The initial values v^1 and w^1 are supposed to be approximations of $\frac{\partial u}{\partial t}(x,0) = u_1(x)$ and $\frac{\partial u}{\partial x}(x,0) = u_0'(x)$ respectively. In this sense, they are independent from each other as opposed to what happened in the case of the natural scheme, cf. Remark 9.18.

A natural question now is to ask if the first order scheme provides some stability information for the original scheme. We encounter a difficulty here. Indeed, if we try to reconstruct u^j from w^j, i.e., perform a kind of discrete integration, we see that

$$u_n^j = u_0^j + h\sum_{l=0}^{n} w_{n-l-1/2}^j,$$

for $n \geq 0$. Now requiring that $u^j \in \ell^2(\mathbb{Z})$ implies that $u_n^j \to 0$ when $n \to +\infty$. So the partial sums on the right must converge and we must have

$$u_n^j = -h\sum_{l=n+1}^{\infty} w_{n-l-1/2}^j,$$

again for $n \geq 0$. This is not possible in general since $w^j \in \ell^2(\mathbb{Z}) \not\subset \ell^1(\mathbb{Z})$.

What we can say however, is that if we are given $u^0 \in \ell^2(\mathbb{Z})$ and $v^1 \in \ell^2(\mathbb{Z})$, and if we define $u^1 = u^0 + kv^1 \in \ell^2(\mathbb{Z})$ and set $w_{n-1/2}^1 = \frac{u_n^1-u_{n-1}^1}{h}$, that is to say if w^1 is the discrete derivative in some sense of an element of $\ell^2(\mathbb{Z})$, then for all j, so is w^j. Indeed, it is simply the discrete derivative of u^j obtained by the original scheme with the above initial conditions, by uniqueness. In this sense, we can say that the original scheme is stable in $\ell^2(\mathbb{Z})$ for such initial conditions, but with a stability measured in the norms $\|\frac{u_n^j-u_n^{j-1}}{k}\|_{2,h}+\|\frac{u_n^j-u_{n-1}^j}{h}\|_{2,h}$, which are more natural in view of Remark 9.20. □

Remark 9.26 The proof requires $\lambda_0<1$ to work. Indeed, if $\frac{k}{h}=1$, then for $\xi=\frac{\pi}{h}$,

$$B(\xi)=\begin{pmatrix}1 & 2i\\ 2i & -3\end{pmatrix}$$

has the double eigenvalue -1 and is not diagonalizable. Therefore $\|B(\xi)^{T/k}\|_2\sim Ck^{-1}$ with $C>0$ for k small for this value of ξ and the semi-discrete scheme is thus unstable in this case. Note that this tells us nothing about the stability of the discrete scheme when $k=h$, and this instability occurs even though the scheme is von Neumann stable. □

Remark 9.27 If we modify the second relation of the finite difference scheme as follows

$$\frac{w_{n-1/2}^{j+1}-w_{n-1/2}^{j}}{k}-\frac{v_n^j-v_{n-1}^j}{h}=0,$$

which may seem more natural than the scheme above, we get an amplification matrix

$$B(\xi)=\begin{pmatrix}1 & ia(\xi)\\ ia(\xi) & 1\end{pmatrix}.$$

This matrix is normal and its eigenvalues are $1\pm ia(\xi)$. They are of modulus strictly larger than 1 (except when $a(\xi)=0$). Hence this scheme is not stable in the sense of von Neumann, and since the matrix is normal and given the expression of $a(\xi)$, we see that it is not stable in L^2. This shows again that finite difference schemes must be chosen with care and that seemingly natural choices may very well fail. □

Let us now study the stability of the implicit scheme (9.13). The scheme is implicit and it is not obvious in the $\ell^2(\mathbb{Z})$ context that is even well-defined.

Proposition 9.13 *The implicit scheme (9.13) is well-defined with initial data in $\ell^2(\mathbb{Z})$.*

Proof We use the Fourier series argument. For $u\in\ell^2(\mathbb{Z})$, let $\mathscr{T}$ denote the continuous linear operator on $\ell^2(\mathbb{Z})$ defined by

$$(\mathscr{T}u)_n=-\lambda u_{n+1}+(1+2\lambda)u_n-\lambda u_{n-1},\ n\in\mathbb{Z},$$

and $\lambda=\frac{k^2}{h^2}$. We rewrite the implicit scheme as follows

$$\mathscr{T}u^{j+1}=v^j,$$

with $v_n^j = 2u_n^j - u_n^{j-1}$. The question is whether or not the operator $\mathscr{T}$ is an isomorphism.

Regarding uniqueness, we note that if $\mathscr{T}u = 0$, then u is of the form

$$u_n = C_1 r_1^n + C_2 r_2^n,\ n \in \mathbb{Z}, \tag{9.22}$$

for some C_1 and C_2 in $\mathbb{C}$, where r_1 and r_2 are the roots of the characteristic equation $-\lambda r^2 + (1+2\lambda)r - \lambda = 0$. These roots are real, both positive and their product is 1, therefore the only sequence of the form (9.22) that belongs to $\ell^2(\mathbb{Z})$ is such that $C_1 = C_2 = 0$.

We now consider existence. As before, for any $v \in \ell^2(\mathbb{Z};\mathbb{C})$ we let $\mathscr{F}v \in L^2(0, 2\pi;\mathbb{C})$ be defined by $\mathscr{F}v(s) = \sum_{n\in\mathbb{Z}} v_n e^{ins}$.

We have

$$\begin{aligned}
\mathscr{F}\mathscr{T}u(s) &= \sum_{n\in\mathbb{Z}} \big(-\lambda u_{n+1} + (1+2\lambda)u_n - \lambda u_{n-1}\big)e^{ins}\\
&= -\lambda \sum_{n\in\mathbb{Z}} u_{n+1}e^{ins} + (1+2\lambda)\sum_{n\in\mathbb{Z}} u_n e^{ins} - \lambda \sum_{n\in\mathbb{Z}} u_{n-1}e^{ins}\\
&= \big(-\lambda e^{-is} + (1+2\lambda) - \lambda e^{is}\big)\mathscr{F}u(s)\\
&= \Big(1 + 4\lambda \sin^2\Big(\frac{s}{2}\Big)\Big)\mathscr{F}u(s).
\end{aligned}$$

Now the function $s \mapsto \big(1 + 4\lambda \sin^2\big(\frac{s}{2}\big)\big)^{-1}$ is in $L^\infty(0, 2\pi)$, therefore

$$u = \mathscr{F}^{-1}\left(\frac{\mathscr{F}v(s)}{\big(1 + 4\lambda \sin^2\big(\frac{s}{2}\big)\big)}\right),$$

is a solution in $\ell^2(\mathbb{Z};\mathbb{C})$ of $\mathscr{T}u = v$. Of course, by uniqueness, when v is real-valued, so is u. □

Remark 9.28 We could not use the semi-discrete version of the scheme in Fourier space, because the equivalence of this scheme with the discrete scheme for piecewise constant initial data rests on the existence of the discrete scheme. The Fourier series approach does not suffer from this drawback. □

We can now switch to the semi-discrete point of view to study the stability. If we try to work on the initial formulation of the scheme in Fourier space

$$\frac{\widehat{u^{j+1}}(\xi) - 2\widehat{u^j}(\xi) + \widehat{u^{j-1}}(\xi)}{k^2} + \frac{4}{h^2}\sin^2\Big(\frac{h\xi}{2}\Big)\widehat{u^{j+1}}(\xi) = 0,$$

we encounter the same kind of difficulties as with the explicit scheme. Namely, we obtain an amplification matrix

$$A(\xi) = \begin{pmatrix} \frac{2}{1+a(\xi)^2} & -\frac{1}{1+a(\xi)^2} \\ 1 & 0 \end{pmatrix}.$$

This matrix is never normal. For $\xi = 0$, it is the same as in Proposition 9.6, therefore the semi-discrete scheme is not stable.

Once again, we must change the unknowns and use a first order system version of the scheme in order to be able to conclude. The first order scheme is simply

$$\begin{cases} \dfrac{v_n^{j+1} - v_n^j}{k} - \dfrac{w_{n+1/2}^{j+1} - w_{n-1/2}^{j+1}}{h} = 0, \\ \dfrac{w_{n-1/2}^{j+1} - w_{n-1/2}^j}{k} - \dfrac{v_n^{j+1} - v_{n-1}^{j+1}}{h} = 0. \end{cases} \tag{9.23}$$

Writing down the semi-discrete version of this last scheme, we obtain the following amplification matrix

$$B(\xi) = \frac{1}{1+a(\xi)^2} \begin{pmatrix} 1 & ia(\xi) \\ ia(\xi) & 1 \end{pmatrix}.$$

Now this matrix is normal. Its eigenvalues are $\lambda_\pm(\xi) = \frac{1 \pm ia(\xi)}{1+a(\xi)^2}$, so that

$$\rho(B(\xi)) = \frac{1}{\sqrt{1+a(\xi)^2}} \leq 1$$

for all $\xi \in \mathbb{R}$. We have thus shown

Proposition 9.14 *The implicit scheme (9.23) is unconditionally von Neumann stable and L^2 stable.*

Let us close this section by saying a few words about the stability of the θ-scheme (9.14). If we write the semi-discrete version of the scheme, apply the Fourier transform and rewrite the result in vector form, we obtain an amplification matrix

$$A(\xi) = \begin{pmatrix} -b(\xi) & -1 \\ 1 & 0 \end{pmatrix},$$

with

$$b(\xi) = \frac{(1-2\theta)a(\xi)^2 - 2}{1+\theta a(\xi)^2}.$$

This matrix is not normal. Its eigenvalues are the roots of the polynomial $P(X) = X^2 + b(\xi)X + 1$. The discriminant reads

$$\Delta(\xi) = \frac{a(\xi)^2\big((1-4\theta)a(\xi)^2 - 4\big)}{\big(1+\theta a(\xi)^2\big)^2}.$$

If the discriminant is positive for some value of ξ, we thus have two distinct real roots, the product of which is 1, hence von Neumann instability. If on the other hand, the discriminant is nonpositive for all ξ, we have two complex conjugate roots of modulus 1, hence von Neumann stability. Recalling that $a(\xi) = \frac{2k}{h}\sin\left(\frac{h\xi}{2}\right)$, we thus obtain the following proposition:

Proposition 9.15 *The θ-scheme is unconditionally von Neumann stable for $\theta \geq \frac{1}{4}$. For $\theta < \frac{1}{4}$, it is von Neumann stable under the condition $\frac{k}{h} \leq \frac{1}{\sqrt{1-4\theta}}$.*

Of course, in terms of L^2 stability, we have the exact same problem as before for $\xi = 0$, which implies L^2 instability of the semi-discrete θ-scheme. We can try to go around this difficulty by using again a system

$$\begin{cases} \dfrac{v_n^{j+1} - v_n^j}{k} - \theta \dfrac{w_{n+1/2}^{j+1} - w_{n-1/2}^{j+1}}{h} - (1-2\theta)\dfrac{w_{n+1/2}^{j} - w_{n-1/2}^{j}}{h} \\ \qquad\qquad\qquad\qquad\qquad\qquad - \theta \dfrac{w_{n+1/2}^{j-1} - w_{n-1/2}^{j-1}}{h} = 0, \\ \dfrac{w_{n-1/2}^{j+1} - w_{n-1/2}^{j}}{k} - \dfrac{v_n^{j+1} - v_{n-1}^{j+1}}{h} = 0. \end{cases} \tag{9.24}$$

Now on the surface, this scheme appears still to be a two time step scheme, hence nothing seems to be gained. We can however rewrite it as a one time step scheme as follows. We first apply the Fourier transform to the semi-discrete version of the scheme

$$\begin{cases} \widehat{v^{j+1}}(\xi) - \widehat{v^j}(\xi) - ia(\xi)\big(\theta\widehat{w^{j+1}}(\xi) + (1-2\theta)\widehat{w^j}(\xi) + \theta\widehat{w^{j-1}}(\xi)\big) = 0, \\ \widehat{w^{j+1}}(\xi) - \widehat{w^j}(\xi) - ia(\xi)\widehat{v^{j+1}}(\xi) = 0. \end{cases}$$

In addition to a formula for $\widehat{w^{j+1}}$ in terms of $\widehat{w^j}$ and $\widehat{v^{j+1}}$, the second equation also yields

$$\widehat{w^{j-1}}(\xi) = \widehat{w^j}(\xi) - ia(\xi)\widehat{v^j}(\xi).$$

We replace these expressions in the first equation

$$\begin{aligned} \widehat{v^{j+1}}(\xi) - \widehat{v^j}(\xi) - ia(\xi)\big(\theta(\widehat{w^j}(\xi) + ia(\xi)\widehat{v^{j+1}}(\xi)) \\ + (1-2\theta)\widehat{w^j}(\xi) + \theta(\widehat{w^j}(\xi) - ia(\xi)\widehat{v^j}(\xi))\big) = 0, \end{aligned}$$

or

$$(1+\theta a(\xi)^2)\big(\widehat{v^{j+1}}(\xi) - \widehat{v^j}(\xi)\big) - ia(\xi)\widehat{w^j}(\xi) = 0.$$

This scheme thus corresponds to the amplification matrix

$$B(\xi) = \begin{pmatrix} 1 & ia_\theta(\xi) \\ ia(\xi) & 1 - a(\xi)a_\theta(\xi) \end{pmatrix},$$

with

$$a_\theta(\xi) = \frac{a(\xi)}{1 + \theta a(\xi)^2}.$$

This matrix is not normal and has the same eigenvalues as the previous one, hence the same von Neumann stability. The case $a(\xi) = 0$ is not a problem anymore however, since the matrix is then the identity matrix.

Proposition 9.16 *The semi-discrete version of the θ-scheme (9.24) is stable in L^2 for $\theta \geq \frac{1}{4}$ under the condition $\frac{k}{h} \leq M$, for any given M. For $\theta < \frac{1}{4}$, given any $0 < \lambda_0 < \frac{1}{\sqrt{1-4\theta}}$, it is L^2 stable under the condition $\frac{k}{h} \leq \lambda_0$.*

Proof Let us consider the case $\theta \geq \frac{1}{4}$. First of all, at $\xi = \frac{2m\pi}{h}$, $m \in \mathbb{Z}$, we have $a(\xi) = 0$ so that nothing needs to be done for these values of ξ, as was already mentioned. For the other values of ξ, the matrix $B(\xi)$ is diagonalizable with two distinct, complex conjugate eigenvalues of modulus one, therefore no problem for the diagonal part either. We have the change of basis matrix

$$P(\xi) = \begin{pmatrix} -\frac{1}{2}\Big(\sqrt{\frac{4a_\theta(\xi)}{a(\xi)} - a_\theta(\xi)^2} + ia_\theta(\xi)\Big) & \frac{1}{2}\Big(\sqrt{\frac{4a_\theta(\xi)}{a(\xi)} - a_\theta(\xi)^2} - ia_\theta(\xi)\Big) \\ 1 & 1 \end{pmatrix},$$

with $\frac{4a_\theta(\xi)}{a(\xi)} - a_\theta(\xi)^2 \geq 0$ since $\theta \geq \frac{1}{4}$.

After a little bit of computer algebra aided manipulations, we obtain the following value for the condition number of $P(\xi)$:

$$\mathrm{cond}_2(P(\xi)) = \frac{\mathrm{sign}\,(a)(a+b) + \sqrt{a^2b^2 + (a-b)^2}}{\sqrt{4ab - a^2b^2}},$$

where $a = a(\xi)$ and $b = a_\theta(\xi)$ for brevity. Replacing b by its value as a function of a, we obtain

$$\begin{aligned} \mathrm{cond}_2(P(\xi)) &= \frac{2 + \theta a(\xi)^2 + |a(\xi)|\sqrt{1 + \theta^2 a(\xi)^2}}{\sqrt{(4\theta - 1)a(\xi)^2 + 4}} \\ &\leq 1 + \frac{|a(\xi)|}{2} + \frac{a(\xi)^2}{2} \\ &\leq 1 + M + 2M^2, \end{aligned}$$

hence the stability of the scheme. We leave the case $\theta < \frac{1}{4}$ as an exercise. □

Remark 9.29 Proposition 9.16 in the case $\theta \geq \frac{1}{4}$ is a bit of a disappointment. Indeed, in that case, the scheme is unconditionally von Neumann stable and we only obtain actual L^2 stability under the condition $\frac{k}{h} \leq M$ with M arbitrary. Now in practice, neither k nor h actually go to 0, and such a condition as $\frac{k}{h} \leq M$ with M arbitrary is not discernible from unconditional stability. □

Remark 9.30 Instead of using the Jordan decomposition of $B(\xi)$, we could think of using the Schur decomposition of $B(\xi)$, $B(\xi) = U(\xi)T(\xi)U(\xi)^*$, where $U(\xi)$ is unitary and $T(\xi)$ is upper triangular. The advantage of the Schur decomposition over the Jordan decomposition in this context, is that $|\!|\!|B(\xi)^j|\!|\!|_2 = |\!|\!|T(\xi)^j|\!|\!|_2$ for all j and we lose no information by passing from $B(\xi)$ to $T(\xi)$. Moreover, $T(\xi)^j$ is fairly easy to express explicitly. The disadvantage is that the expression of the upper right entry of $T(\xi)^j$ is even less user-friendly than $\text{cond}_2(P(\xi))$ when it comes to estimating it. We do not pursue this direction here. □

9.6 For a Few Schemes More

In the previous section, we rewrote the wave equation as the first order system (9.18). This system is of the form

$$\frac{\partial U}{\partial t} + \frac{\partial(f(U))}{\partial x} = 0$$

with

$$U(x,t) = \begin{pmatrix} v(x,t) \\ w(x,t) \end{pmatrix} \text{ and } f(U) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} U.$$

When U is $\mathbb{R}^p$-valued and f is a general nonlinear function from $\mathbb{R}^p$ to $\mathbb{R}^p$, satisfying certain conditions, this is a *(nonlinear) hyperbolic system*. Such systems are of paramount importance in many applications, for example in gas dynamics, and there is a very large body of numerical schemes that are adapted to the approximation of the solutions of such systems, see [42] for example.

We present a few of these schemes in the case of our simple $\mathbb{R}^2$-valued, linear hyperbolic system (9.18). We will also return to some of these schemes in the next chapter. Again, we work on the whole line[3] and on the usual (nh, jk) space-time finite difference grid for the approximations. We start with the *Lax–Friedrichs scheme*, which reads in general

$$\frac{U_n^{j+1} - \frac{1}{2}(U_{n+1}^j + U_{n-1}^j)}{k} + \frac{f(U_{n+1}^j) - f(U_{n-1}^j)}{2h} = 0,$$

[3]Boundary conditions are a delicate question for such systems.

and in our particular case

$$\begin{cases} v_n^{j+1} = \dfrac{1}{2}(v_{n+1}^j + v_{n-1}^j) + \dfrac{k}{2h}(w_{n+1}^j - w_{n-1}^j), \\ w_n^{j+1} = \dfrac{1}{2}(w_{n+1}^j + w_{n-1}^j) + \dfrac{k}{2h}(v_{n+1}^j - v_{n-1}^j). \end{cases}$$

The scheme is one-step, of order one and explicit. We write the usual semi-discrete version of the scheme, then apply the Fourier transform, and we obtain

$$\begin{cases} \widehat{v^{j+1}}(\xi) = \cos(h\xi)\widehat{v^j}(\xi) + i\dfrac{k}{h}\sin(h\xi)\widehat{w^j}(\xi), \\ \widehat{w^{j+1}}(\xi) = \cos(h\xi)\widehat{w^j}(\xi) + i\dfrac{k}{h}\sin(h\xi)\widehat{v^j}(\xi). \end{cases}$$

Therefore, the amplification matrix of the Lax–Friedrichs scheme is

$$B(\xi) = \begin{pmatrix} \cos(h\xi) & i\frac{k}{h}\sin(h\xi) \\ i\frac{k}{h}\sin(h\xi) & \cos(h\xi) \end{pmatrix}.$$

This matrix is normal and its spectral radius is $\rho(B(\xi)) = \big(\cos^2(h\xi) + \frac{k^2}{h^2}\sin^2(h\xi)\big)^{1/2}$. It clearly follows that

Proposition 9.17 *The Lax–Friedrichs scheme is von Neumann stable and L^2 stable under the condition $\frac{k}{h} \le 1$.*

We consider next the *Lax–Wendroff scheme*. In our particular case, the scheme reads

$$\begin{cases} v_n^{j+1} = v_n^j + \dfrac{k}{2h}(w_{n+1}^j - w_{n-1}^j) + \dfrac{k^2}{2h^2}(v_{n+1}^j - 2v_n^j + v_{n-1}^j), \\ w_n^{j+1} = w_n^j + \dfrac{k}{2h}(v_{n+1}^j - v_{n-1}^j) + \dfrac{k^2}{2h^2}(w_{n+1}^j - 2w_n^j + w_{n-1}^j). \end{cases}$$

The scheme is one-step, of order two and explicit. After Fourier transform, it becomes

$$\begin{cases} \widehat{v^{j+1}}(\xi) = \Big(1 - \dfrac{2k^2}{h^2}\sin^2\Big(\dfrac{h\xi}{2}\Big)\Big)\widehat{v^j}(\xi) + i\dfrac{k}{h}\sin(h\xi)\widehat{w^j}(\xi), \\ \widehat{w^{j+1}}(\xi) = \Big(1 - \dfrac{2k^2}{h^2}\sin^2\Big(\dfrac{h\xi}{2}\Big)\Big)\widehat{w^j}(\xi) + i\dfrac{k}{h}\sin(h\xi)\widehat{v^j}(\xi), \end{cases}$$

hence the amplification matrix

$$B(\xi) = \begin{pmatrix} \big(1 - \frac{2k^2}{h^2}\sin^2\big(\frac{h\xi}{2}\big)\big) & i\frac{k}{h}\sin(h\xi) \\ i\frac{k}{h}\sin(h\xi) & \big(1 - \frac{2k^2}{h^2}\sin^2\big(\frac{h\xi}{2}\big)\big) \end{pmatrix}.$$

This matrix is again normal and it follows from elementary computations that it has a spectral radius $\rho(B(\xi)) = \big(1 - 4\frac{k^2}{h^2}(1 - \frac{k^2}{h^2})\sin^4\big(\frac{h\xi}{2}\big)\big)^{1/2}$. Therefore, we see that

Proposition 9.18 *The Lax–Wendroff scheme is von Neumann stable and L^2 stable under the condition $\frac{k}{h} \le 1$.*

We can also revisit the *leapfrog scheme*, which reads here

$$\begin{cases} v_n^{j+1} = v_n^{j-1} + \dfrac{k}{h}(w_{n+1}^j - w_{n-1}^j), \\ w_n^{j+1} = w_n^{j-1} + \dfrac{k}{h}(v_{n+1}^j - v_{n-1}^j). \end{cases}$$

Note that it leapfrogs in time as well as in space. The scheme is two-step, of order two and explicit. To write an amplification matrix for it, we need to double the dimension and consider for example the vectors $\big(\widehat{v^{j+1}}(\xi), \widehat{w^{j+1}}(\xi), \widehat{v^j}(\xi), \widehat{w^j}(\xi)\big)^T$, a choice which yields the amplification matrix

$$B(\xi) = \begin{pmatrix} 0 & 2i\frac{k}{h}\sin(h\xi) & 1 & 0 \\ 2i\frac{k}{h}\sin(h\xi) & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

This matrix is not normal.

Proposition 9.19 *The leapfrog scheme is von Neumann stable under the condition $\frac{k}{h} \le 1$. It is L^2 stable under the condition $\frac{k}{h} \le \lambda_0 < 1$.*

Proof Let $b(\xi) = \frac{k}{h}\sin(h\xi)$. If we write $B(\xi)$ as a 2×2 block matrix of four 2×2 blocks, we see by Lemma 8.2 of Chap. 8 that its eigenvalues are given by $\lambda = \pm\sqrt{1-b^2} \pm ib$ when $\frac{k}{h} \le 1$, so that $\rho(B(\xi)) = 1$ for all ξ.

Concerning the L^2-stability, if $\frac{k}{h} \le \lambda_0 < 1$, we have four distinct eigenvalues of modulus 1, so we only need to estimate the condition number of the change of matrix basis

$$P(\xi) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{1-b^2} - ib & -\sqrt{1-b^2} - ib & \sqrt{1-b^2} + ib & -\sqrt{1-b^2} + ib \\ \sqrt{1-b^2} - ib & -\sqrt{1-b^2} - ib & -\sqrt{1-b^2} - ib & \sqrt{1-b^2} - ib \end{pmatrix}.$$

Using computer algebra again, we find that

$$\mathrm{cond}_2(P(\xi)) = \sqrt{\frac{1 + |b(\xi)|}{1 - |b(\xi)|}} \le \sqrt{\frac{2}{1 - \lambda_0}},$$

hence the stability of the scheme. □

Remark 9.31 The conditional stability of the leapfrog scheme is to be contrasted with the situation for the heat equation, where the leapfrog scheme is always unstable, see Proposition 8.10 in Chap. 8. □

Remark 9.32 We note that we cannot allow $\lambda_0 = 1$. Indeed, in this case, there are values of ξ for which $b(\xi) = \pm 1$. For these values of ξ, the matrix $B(\xi)$ has two double eigenvalues $\pm i$ and is not diagonalizable. Therefore $\|\!|B(\xi)^j|\!\|_2 \to +\infty$ as $j \to +\infty$, and the scheme is unstable in L^2. □

9.7 Concluding Remarks

To conclude this chapter, we note that there are other issues than just consistency and stability in the study of finite difference schemes for hyperbolic systems. Even though we have not mentioned them at all here, they are important in assessing the performance of a given scheme. Among these issues are *dissipativity*, i.e., the possible damping of wave amplitudes with time, and *dispersivity*, i.e., the possibility that numerically approximated waves of different frequencies could travel at different numerical speeds.

Naturally, there are other numerical methods applicable to the wave equation, for instance finite difference-finite element methods, see for example [65].

We have so far described and analyzed two major classes of numerical methods, finite difference methods and finite element methods, in the contexts of the three main classes of problems, elliptic, parabolic and hyperbolic. In the last chapter, we introduce a more recent method, the finite volume method, on a few elliptic and hyperbolic examples.

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