Chapter 2
Non-Archimedean Banach Spaces

In this chapter we gather some basic facts about non-archimedean Banach spaces, with a special emphasis on the so-called p-adic Hilbert space. Again the results here are well-known and will serve as background for the operator theory developed in later chapters.

Let $\mathbb{K}$ denote a *complete* non-archimedean valued field. The valuation on $\mathbb{K}$ will be denoted $|\cdot|$.

2.1 Non-Archimedean Norms

In this section we introduce and study basic properties of non-archimedean norms and non-archimedean normed spaces.

Definition 2.1. Let $\mathbb{E}$ be a vector space over $\mathbb{K}$. A *non-archimedean norm* on $\mathbb{E}$ is a map $\|\cdot\| : \mathbb{E} \to \mathbb{R}_+^*$ satisfying

(1) $\|x\| = 0$ if and only if $x = 0$;
(2) $\|\lambda x\| = |\lambda|\|x\|$ for any $x \in \mathbb{E}$ and any $\lambda \in \mathbb{K}$;
(3) $\|x + y\| \leq \max\{\|x\|.\|y\|\}$ for any $x, y \in \mathbb{E}$.

Property (3) of Definition 2.1 is referred to as the *ultrametric* or *strong triangle inequality*.

Definition 2.2. A non-archimedean normed space is a pair $(\mathbb{E}, \|\cdot\|)$ where $\mathbb{E}$ is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is a non-archimedean norm on $\mathbb{E}$.

Unless there is an explicit mention of the contrary, all norms that are considered in this chapter are non-archimedean. Here are some first examples of non-archimedean norms on some vector spaces.

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Example 2.3. The valuation on $\mathbb{K}$ itself is a non-archimedean norm.

Example 2.4. Consider the Cartesian product $\mathbb{K}^n$ with $n \in \mathbb{N}$ and define

$$\left\|(x_1, \ldots, x_n)\right\| = \max\left\{\left|x_i\right| : 1 \leq i \leq n\right\}.$$

Then this is a non-archimedean norm on $\mathbb{K}^n$.

Example 2.5. For each $i \in \mathbb{N}$ let $\mathbb{K}^{(i)} = \mathbb{K}$ and consider

$$P = \prod_{i=0}^{\infty} \mathbb{K}^{(i)}.$$

Then P is the direct product of a countable copies of $\mathbb{K}$. The set P is naturally a vector space over $\mathbb{K}$. Note that an element of P is just a sequence of elements of $\mathbb{K}$ of the form $(x_i)_{i\in\mathbb{N}}$.

Let

$$L^{\infty}(\mathbb{K}) = \left\{(x_i)_{i\in\mathbb{N}} \in P : \ (x_i)_{i\in\mathbb{N}} \text{ is bounded}\right\}.$$

Then $L^{\infty}(\mathbb{K})$ is a subspace of P. Define

$$\left\|(x_i)_i\right\|_{\infty} = \sup\left\{\left|x_i\right| : i \in \mathbb{N}\right\},$$

then, this is a non-archimedean norm on $L^{\infty}(\mathbb{K})$.

Example 2.6. With the notations of Example 2.5, consider

$$S = \sum_{i=0}^{\infty} \mathbb{K}^{(i)},$$

then S is the direct sum of a countable copies of $\mathbb{K}$ and it is a subspace of P. Note that an element of S is a sequence of the form $(x_i)_{i\in\mathbb{N}}$ such that $x_i \in \mathbb{K},\ x_i = 0$ for almost all (for all except for a finite number) $i \in \mathbb{N}$.

Define

$$\left\|(x_i)_{i\in\mathbb{N}}\right\| = \max\left\{\left\|x_i\right\| : i \in \mathbb{N}\right\},$$

then, this is a well-defined non-archimedean norm on S.

Example 2.7. Let $\mathbb{X}$ be a set and let $B(\mathbb{X}, \mathbb{K})$ be the set all *bounded* functions on $\mathbb{X}$ with values in $\mathbb{K}$, then with the operations

$$(f+g)(x) = f(x) + g(x),\ f, g \in B(\mathbb{X}, \mathbb{K}),\ \ x \in \mathbb{X}$$

$$(\lambda f)(x) = \lambda f(x),\ \ \lambda \in \mathbb{K},\ f \in B(\mathbb{X}, \mathbb{K}),\ \ x \in \mathbb{X}$$

Clearly, $B(\mathbb{X}, \mathbb{K})$ is a vector space over $\mathbb{K}$.

Define the sup-norm

$$\left\|f\right\|_{\infty} := \sup\left\{\left|f(x)\right| : x \in \mathbb{K}\right\},$$

then this is a non-archimedean norm on $B(\mathbb{X}, \mathbb{K})$.

Example 2.8. Let $(\mathbb{E}_i, \|\cdot\|_i),\ i = 1, 2$ be non-archimedean normed spaces over $\mathbb{K}$. Let $L(\mathbb{E}_1, \mathbb{E}_2)$ be the set of all $\mathbb{K}$-linear maps $A : \mathbb{E}_1 \to \mathbb{E}_2$, then it is naturally a vector space over $\mathbb{K}$. Let $B(\mathbb{E}_1, \mathbb{E}_2)$ be the set all $\mathbb{K}$-linear maps $A : \mathbb{E}_1 \to \mathbb{E}_2$ satisfying the following: there exists $C \geq 0$ such that for all $x \in \mathbb{E}_1$,

$$\left\|Ax\right\|_2 \leq C\left\|x\right\|_1.$$

Then $B(\mathbb{E}_1, \mathbb{E}_2)$ is a subspace of $L(\mathbb{E}_1, \mathbb{E}_2)$. Define for all $A \in B(\mathbb{E}_1, \mathbb{E}_2)$,

$$\left\|A\right\| = \sup_{x \in \mathbb{E}_1 \setminus \left\{0\right\}} \frac{\left\|A(x)\right\|_2}{\left\|x\right\|_1}.$$

Then, this is a non-archimedean norm on $B(\mathbb{E}_1, \mathbb{E}_2)$.

Proposition 2.9. *Let* $(\mathbb{E}, \|\cdot\|)$ *be a non-archimedean normed space. For* $x, y \in \mathbb{E}$,

$$\left\|x + y\right\| = \max\left\{\left\|x\right\|, \left\|y\right\|\right\}, \text{ if } \left\|x\right\| \neq \left\|y\right\|.$$

Proof. Suppose that $\|x\| < \|y\|$ so that $\max\{\|x\|, \|y\|\} = \|y\|$, then by Definition 1.1, $\|x + y\| \leq \|y\|$. Now

$$\left\|y\right\| = \left\|x + y - x\right\| \leq \max\left\{\left\|x + y\right\|, \left\|x\right\|\right\}.$$

But since $\|y\| > \|x\|$, we must have

$$\max\left\{\left\|x + y\right\|, \left\|x\right\|\right\} = \left\|x + y\right\|,$$

and therefore

$$\left\|y\right\| \leq \left\|x + y\right\|$$

and the conclusion follows.

Definition 2.10. Let $(\mathbb{E}, \|\cdot\|)$ be non-archimedean normed space and S be a non-empty subset of E. The set S is said to be *bounded* if the set of real numbers $\{\|x\| : x \in S\}$ is bounded.

Definition 2.11. A sequence $(x_i)_{i\in\mathbb{N}}$ in the normed space $(\mathbb{E}, \|\cdot\|)$ *converges* (strongly) to $x \in \mathbb{E}$ and we write

$$\lim_{i\to\infty} x_i = x$$

if the sequence of real numbers $(\|x_i - x\|)_{i\in\mathbb{N}}$ converges to 0.

Definition 2.12. A series $\sum_{i=0}^{\infty} x_i$ in $(\mathbb{E}, \|\cdot\|)$ *converges* to $x \in \mathbb{E}$ and we write

$$\sum_{i=0}^{\infty} x_i = x$$

if the sequence of partial sums $(s_n)_{n\in\mathbb{N}}$

$$s_n = \sum_{i=0}^{n} x_i, \quad n \in \mathbb{N}$$

converges to x.

Proposition 2.13. *Let $(\mathbb{E}, \|\cdot\|)$ be a non-archimedean normed space over $\mathbb{K}$. If the sequence $(x_i)_{i\in\mathbb{N}}$ converges in $\mathbb{E}$, then it is bounded.*

Proof. Suppose $(x_i)_{i\in\mathbb{N}}$ converges to x, then the sequence of real numbers $(\|x_i - x\|)_i$ converges in $\mathbb{R}$, therefore is bounded. It follows that the set $\{x_i : i \in \mathbb{N}\}$ is bounded as a subset of $\mathbb{E}$.

2.2 Non-Archimedean Banach Spaces

In this section we introduce non-archimedean Banach spaces, discuss their properties and illustrate with examples.

Let $(\mathbb{E}, \|\cdot\|)$ be a non-archimedean normed space, then a metric d can be defined on $\mathbb{E}$ to give it the topology of a metric space. This metric is defined by

$$x, y \in \mathbb{E}, \ \ d(x, y) := \Big\|x - y\Big\|.$$

Proposition 2.14. *The strong triangle inequality translates as follows:*

$$\text{for } x, y, z \in \mathbb{E}, \ \ d(x, y) \le \max\Big\{d(x, z), d(y, z)\Big\}.$$

Definition 2.15. A normed space $(\mathbb{E}, \|\cdot\|)$ is called a *Banach space* if it is complete with respect to the natural metric induced by the norm

$$d(x,y) = \Big\|x-y\Big\|, \quad x,y \in \mathbb{E}.$$

The spaces $\mathbb{K}$, $\mathbb{K}^n$, $\sum_{i=0}^{\infty} \mathbb{K}^{(i)}$, $L^{\infty}(\mathbb{K})$, $B(\mathbb{X},\mathbb{K})$, $B(\mathbb{E}_1,\mathbb{E}_2)$ with their respective norms are Banach spaces.

Proposition 2.16. *(1) a close subspace of a Banach space is a Banach space;*
(2) the direct sum of two Banach spaces is a Banach space.

Proof. (1) is clear. (2) the norm on the direct sum is defined by $\|(x,y)\| = \max\{\|x\|,\|y\|\}$. From there, the proof is also clear.

We next define the norm on the quotient space using the same notation for both the space and its quotient.

Definition 2.17. Let $\mathbb{E}$ be a Banach space and $\mathbb{V}$ a closed subspace of $\mathbb{E}$. Let $P : \mathbb{E} \to \mathbb{E}/\mathbb{V}$ be the quotient map. Define

$$\Big\|Px\Big\| = d(x,\mathbb{V}), \quad x \in \mathbb{E},$$

where

$$d(x,B) = \inf\Big\{d(x,z) : z \in \mathbb{V}\Big\} = \inf\Big\{\Big\|x-z\Big\| : z \in \mathbb{V}\Big\}$$

is the distance from x to $\mathbb{V}$.

Remark 2.18. This norm is well defined because $Px = Py$ if and only if $x - y \in \mathbb{V}$, moreover $\|Px\| \leq \|x\|$ for any $x \in \mathbb{E}$.

Proposition 2.19. *The norm in Definition 2.17 is a non-archimedean norm on* $\mathbb{E}\backslash\mathbb{V}$.

Proof. (1) First $\|0\| = \|P(0)\| = 0$ since $0 \in \mathbb{V}$. Next, if $\|Px\| = 0$ then $d(x,V) = 0$ hence $x \in \mathbb{V}$, and $Px = 0$.
(2) For any $\lambda \in \mathbb{K}^*$,

$$\begin{aligned}
\Big\|\lambda Px\Big\| &= \Big\|P(\lambda x)\Big\| \\
&= \inf\Big\{\Big\|\lambda x - z\Big\| : z \in \mathbb{V}\Big\} \\
&= \Big|\lambda\Big| \inf\Big\{\Big\|x - \frac{z}{\lambda}\Big\| : z \in \mathbb{V}\Big\} \\
&= \Big|\lambda\Big| \inf\Big\{\Big\|x-y\Big\| : y \in V\Big\} \\
&= \Big|\lambda\Big|\Big\|Px\Big\|.
\end{aligned}$$

(3) For $x, y \in \mathbb{E}$, since $\mathbb{V}$ is closed, there exist $z_1, z_2, z_3 \in \mathbb{V}$ such that

$$\left\|Px\right\| = \left\|x - z_1\right\|, \quad \left\|Py\right\| = \left\|y - z_2\right\|, \quad \left\|P(x+y)\right\| = \left\|x + y - z_3\right\|$$

$$\begin{aligned}
\left\|P(x) + P(y)\right\| &= \left\|P(x+y)\right\| \\
&= \left\|x + y - z_3\right\| \\
&\leq \left\|(x+y) - (z_1 + z_2)\right\| \quad (\text{because } (z_1 + z_2) \in \mathbb{V}) \\
&= \left\|(x - z_1) + (y - z_2)\right\| \\
&\leq \max\left\{\left\|x - z_1\right\|, \left\|y - z_2\right\|\right\} \\
&= \max\left\{\left\|Px\right\|, \left\|Py\right\|\right\}.
\end{aligned}$$

There will be more examples in the later parts of the book.

An example that plays a very important role in the theory of non-archimedean Banach space, is the following:

Example 2.20. Let $c_0(\mathbb{K})$ denote the set of all sequences $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{K}$ such that

$$\lim_{i\to\infty} \left|x_i\right| = 0.$$

Then, $c_0(\mathbb{K})$ is a vector space over $\mathbb{K}$ and

$$\left\|(x_i)_{i\in\mathbb{N}}\right\| = \sup_{i\in\mathbb{N}} \left|x_i\right|$$

is a non-archimedean norm for which $(c_0(\mathbb{K}), \|\cdot\|)$ is a Banach space.

Another important example which will play a central role in the book is now defined. It is a modified version of $c_0(\mathbb{K})$.

Example 2.21. Let $\omega = (\omega_i)_{i\in\mathbb{N}}$ be a sequence of *non-zero* elements in $\mathbb{K}$. We define the space $\mathbb{E}_\omega$, by

$$\mathbb{E}_\omega = \left\{x = (x_i)_{i\in\mathbb{N}} : \forall i, x_i \in \mathbb{K} \text{ and } \lim_{i\to\infty} (\left|\omega_i\right|^{1/2}\left|x_i\right|) = 0\right\}.$$

On $\mathbb{E}_\omega$, we define

$$x = (x_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega, \quad \left\|x\right\| = \sup_{i\in\mathbb{N}} \left(\left|\omega_i\right|^{1/2}\left|x_i\right|\right).$$

Then, $(\mathbb{E}_\omega, \|\cdot\|)$ is a non-archimedean Banach space.

Definition 2.22. The Banach space $\mathbb{E}_\omega$ of Example 2.21, equipped with its norm, is called a *p-adic Hilbert space*.

The space $\mathbb{E}_\omega$ will play a central role in the book and will be the subject of further studies in this chapter and in later chapters.

Example 2.23. In reference to Example 2.8, if we take $\mathbb{E}_1 = \mathbb{E}_2 = \mathbb{E}$, the non-archimedean normed space $B(\mathbb{E},\mathbb{E})$ is denoted $B(\mathbb{E})$ and consists of all $\mathbb{K}$-linear maps $A : \mathbb{E} \to \mathbb{E}$ (also called "*linear operators*") satisfying

$$\exists C \geq 0 \text{ such that } \forall x \in \mathbb{E}, \ \left\|Ax\right\| \leq C\left\|x\right\|.$$

Recall that the norm on $B(\mathbb{E})$ is

$$\left\|A\right\| = \sup_{x\in E\setminus\{0\}} \frac{\left\|Ax\right\|}{\left\|x\right\|}.$$

Then $B(\mathbb{E})$ is also a very important non-archimedean Banach space and will be thoroughly discussed in Chap. 3. It is called the space of *bounded* or *continuous* linear operators on $\mathbb{E}$.

Example 2.24. Again in reference to Example 2.8, if we take $\mathbb{E}_1 = \mathbb{E}$ a Banach space over $\mathbb{K}$ and $\mathbb{E}_2 = \mathbb{K}$, then the Banach space $B(\mathbb{E},\mathbb{K})$ is called the *dual* of $\mathbb{E}$ and denoted $\mathbb{E}^*$. The dual $\mathbb{E}^*$, then, is the space of bounded linear functionals on $\mathbb{E}$. If $\xi \in \mathbb{E}^*$, then

$$\left\|\xi\right\|_* = \sup_{x\in\mathbb{E}\setminus\{0\}} \frac{\left|\langle\xi, x\rangle\right|}{\left\|x\right\|}.$$

With this norm $\mathbb{E}^*$ is a Banach space.

Let $\mathbb{E}^{**}$ be the dual of $\mathbb{E}^*$, then, there is a natural $\mathbb{K}$-linear map:

$$j_{\mathbb{E}} : \mathbb{E} \to \mathbb{E}^{**} \text{ such that } \forall x \in \mathbb{E}, \ j_{\mathbb{E}}(x)(\xi) = \langle\xi, x\rangle, \quad \forall\xi \in \mathbb{E}^*.$$

We now consider some properties of Banach spaces that will be useful later.

Proposition 2.25. *Let* $(\mathbb{E}, \|\cdot\|)$ *be a Banach space. The series* $\sum_{i=0}^{\infty} x_i$ *converges in* $\mathbb{E}$ *if and only if the sequence of general terms* $(x_i)_{i\in\mathbb{N}}$ *converges to* 0.

Proof. Suppose that the series converges, then it is clear that the general term converges to 0. Conversely, suppose that

$$\lim_{i\to\infty} x_i = 0.$$

This means that for any $\varepsilon > 0$ there exists N such that for $i > N$, $\|x_i\| < \varepsilon$.

Consider the sequence of partial sums $(s_k)_{k\in\mathbb{N}}$ where

$$s_k = \sum_{i=0}^{k} x_i.$$

Then for $n > m > N$

$$\begin{aligned}\|s_n - s_m\| &= \|x_{m+1} + \ldots + x_n\|\\ &\leq \max\left\{\left\|x_j\right\| : m+1 \leq j \leq n\right\}\\ &< \varepsilon.\end{aligned}$$

The sequence of partial sums $(s_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathbb{E}$ hence it converges.

As in the classical case, we have the following definition.

Definition 2.26. Let $\mathbb{E}$ be a vector space over $\mathbb{K}$ and $\|\cdot\|_1$ and $\|\cdot\|_2$ two non-archimedean norms on $\mathbb{E}$ for each of which $\mathbb{E}$ is a Banach space. The two norms are said to be *equivalent* if there exist positive constants c_1 and c_2 such that for any $x \in \mathbb{E}$,

$$c_1\left\|x\right\|_1 \leq \left\|x\right\|_2 \leq c_2\left\|x\right\|_1.$$

Proposition 2.27. *On a finite dimensional Banach space over $\mathbb{K}$, all non-archimedean norms are equivalent.*

Proof. We use induction on the dimension n. If $n = 1$, let $\|x\|_0 = |x|$ be the norm determined by the absolute value. Now let $\|\cdot\|$ be any norm on $\mathbb{K}$, then for any $x \in \mathbb{K}$,

$$\left\|x\right\| = \left|x\right|\left\|1\right\| = c\left\|x\right\|_0, \text{ with } c = \left\|1\right\|$$

which implies that $\|\cdot\|$ is equivalent to $\|\cdot\|_0$. Suppose that the proposition is true for a space of dimension $(n-1)$. Let $\mathbb{E}$ be of dimension n and let $\{e_1, \ldots, e_n\}$ be a basis for $\mathbb{E}$. First we have the natural norm on $\mathbb{E}$ which is

$$x \in \mathbb{E},\ x = \sum_{i=1}^{n} x_i e_i,\ \left\|x\right\|_0 = \max\left\{\left|x_i\right| : 1 \leq i \leq n\right\}.$$

Let $\|\cdot\|$ be any norm on $\mathbb{E}$. We want to show that $\|\cdot\|$ is equivalent to $\|\cdot\|_0$.

For any $x = \sum_{i=1}^{n} x_i e_i$ we have

$$\left\|x\right\| = \left\|\sum_{i=1}^{n} x_i e_i\right\| \leq \max\left\{\left|x_i\right|\left\|e_i\right\| : 1 \leq i \leq n\right\} \leq C\left\|x\right\|_0$$

where $C = \max\{\|e_i\| : 1 \leq i \leq n\}$ and we find

$$\left\|x\right\| \leq C\left\|x\right\|_0.$$

To obtain the other inequality which will complete the equivalence, we let $\mathbb{V}$ be the subspace of $\mathbb{E}$ generated by $\{e_1, \ldots, e_{n-1}\}$, then

$$x = y + x_n e_n$$

where $y = \sum_{i=1}^{n-1} x_i e_i \in \mathbb{V}$. We note that $\mathbb{V}$ is a closed subspace of $\mathbb{E}$, being the set of all vectors in E whose n-th component is zero. Therefore, it follows that

$$a = \inf\left\{\left\|z + e_n\right\| : z \in \mathbb{V}\right\} > 0$$

then

$$\left\|x_n^{-1} y + e_n\right\| \geq a > 0.$$

Put

$$b = a\left\|e_n\right\|^{-1} \text{so that } b \leq 1.$$

Suppose first that $x_n \neq 0$, then

$$\left\|e_n\right\|^{-1}\left\|x_n^{-1} y + e_n\right\| \geq b.$$

Now

$$\left\|x\right\| = \left|x_n\right|\left\|e_n\right\|\left(\left\|e_n\right\|^{-1}\left\|x_n^{-1} y + e_n\right\|\right) \geq b\left\|x_n e_n\right\|$$

and we find

$$\left\|x\right\| \geq b\left\|x_n e_n\right\|.$$

Lemma 2.28. $\|x\| \geq b\|y\|$.

Proof. Suppose that $\|x\| < b\|y\|$ hence $\|y + x_n e_n\| < b\|y\|$ and since $b \leq 1$ we find that

$$\|y + x_n e_n\| < \|y\|$$

which implies that

$$\left\|x_n e_n\right\| = \left\|(y + x_n e_n) - y\right\| = \left\|y\right\|$$

and since $\|y + x_n e_n\| \geq b\|x_n e_n\|$ we get a contradiction.

Now we have

$$\left\|x\right\| \geq b\left|x_n\right|\left\|e_n\right\| \text{ and } \left\|x\right\| \geq b\left\|y\right\|.$$

By induction, there exist constants b' and b'' such that

$$\left\|x\right\| \geq bb'\left|x_n\right| \text{ and } \left\|x\right\| \geq bb'' \max\left\{\left|x_i\right| : 1 \leq i \leq (n-1)\right\}.$$

Let $C = \min\{bb', bb''\}$. Then,

$$\left\|x\right\| \geq C \max\left\{\left|x_i\right| : 1 \leq i \leq n\right\} = C\left\|x\right\|_0.$$

Suppose next that $x_n = 0$. In this case, we still have

$$\left\|x\right\| \geq b\left\|y\right\|$$

and the same argument carries on, hence, $\|\cdot\|$ is equivalent to $\|\cdot\|_0$.

2.3 Free Banach Spaces

In this section we define and discuss properties of Banach spaces which have bases. Let $\mathbb{E}$ be a Banach space over $\mathbb{K}$.

Definition 2.29. A family $(v_i)_{i\in I}$ of vectors in $\mathbb{E}$ indexed by a set I converges to 0 and we write

$$\lim_{i\in I} v_i = 0$$

if

$$\forall \varepsilon > 0, \left\{ i \in I : \left\| v_i \right\| \geq \varepsilon \right\} \text{ is finite.}$$

Definition 2.30. Let $v \in E$ and let $(v_i)_{i \in I}$ be a family of elements of $\mathbb{E}$ indexed by the set I. We say that v is the sum of the family $(v_i)_{i \in I}$ and we write

$$\sum_{i \in I} v_i = v$$

if $\forall \varepsilon > 0$, there exists a finite subset $J_0 \subset I$ such that for any finite $J \subset I$, $J \supseteq J_0$

$$\left\| \sum_{i \in J} v_i - v \right\| \leq \varepsilon.$$

In this situation, we also say that the family $(v_i)_{i \in I}$ is summable and its sum is v.

Proposition 2.31. *Let the family $(v_i)_{i \in I}$ be summable in $\mathbb{E}$ with sum $v \in \mathbb{E}$, then*

$$\lim_{i \in I} v_i = 0.$$

Proof. Given $\varepsilon > 0$, let $H = \{i \in I : \|v_i\| \geq \varepsilon\}$. Since the family $(v_i)_{i \in I}$ is summable with sum v, there exists a finite subset J_0 of I such that for any finite subset J of I containing J_0,

$$\left\| \sum_{i \in J} v_i - v \right\| \leq \varepsilon.$$

Let $j \in I \setminus J_0$ and consider $J = J_0 \cup \{j\}$ then

$$\left\| \sum_{i \in J} v_i - v \right\| \leq \varepsilon.$$

Since

$$\left\| \sum_{i \in J_0} v_i - v \right\| \leq \varepsilon$$

it follows that

$$\max \left\{ \left\| \sum_{i \in J} v_i - v \right\|, \left\| \sum_{i \in J_0} v_i - v \right\| \right\} \leq \varepsilon$$

which implies that

$$\left\| v_j \right\| \leq \varepsilon.$$

Since this holds for any $j \notin J_0$, we conclude that

$$H \subset J_0$$

hence H is finite and therefore $\lim_{i \in I} v_i = 0$.

Definition 2.32. A *basis* for $\mathbb{E}$ is a family of elements of $\mathbb{E}$, $\{e_i : i \in I\}$ indexed by a set I such that for every $x \in \mathbb{E}$ there exists a unique family $(x_i)_{i \in I}$ of elements in $\mathbb{K}$ such that

$$\sum_{i \in I} x_i e_i = x.$$

In this situation, in view of Proposition 2.31, $\lim_{i \in I} x_i e_i = 0$.

Example 2.33. In Example 2.20 we introduced the Banach space $c_0(\mathbb{K})$. It has the following basis $\left\{e_i : i \in \mathbb{N}\right\}$, where $e_0 = (1, 0, 0, \ldots), e_1 = (0, 1, 0, 0, \ldots) \ldots$ in other words, e_i is the sequence all whose terms are 0 except the i-th term which is equal to 1. If $x = (x_i)_{i \in \mathbb{N}} \in c_0(\mathbb{K})$ then

$$x = \sum_{i=0}^{\infty} x_i e_i$$

and

$$\lim_{i \to \infty} \left| x_i \right| = 0.$$

Let p be a prime, and suppose $\mathbb{K}$ is a finite extension of $\mathbb{Q}_p$, the field of p-adic numbers. We let $\mathbb{Z}_p$ be the ring of p-adic integers.

Definition 2.34. For each $i \in \mathbb{N}$ we define the *Mahler function* M_i to be the function

$$M_i : \mathbb{Z}_p \to \mathbb{K}, \quad M_0(x) = 1 \text{ and for } i > 0,\ M_i(x) = \binom{x}{i} = \frac{x(x-1)\ldots(x-i+1)}{i!}, \quad x \in \mathbb{Z}_p.$$

The function M_i satisfies the following:

(1) $M_i(j) = 0$ if j is an integer with $j < n$;
(2) $M_i(i) = 1$;
(3) $M_i(x)$ is a polynomial function of degree i.

Let $C(\mathbb{Z}_p, \mathbb{K})$ be the $\mathbb{K}$-vector space of continuous functions from the compact set $\mathbb{Z}_p$ to $\mathbb{K}$, equipped with the sup-norm

$$\left\|f\right\|_\infty := \sup_{z \in \mathbb{Z}_p} \left|f(z)\right|.$$

Example 2.35. The space $C(\mathbb{Z}_p, \mathbb{K})$ is a free Banach space because of the following classic theorem of Mahler for whose proof we refer to [46] or [53].

Theorem 2.36. *The following hold:*

(1) For each $i \in \mathbb{N}$, $M_i \in C(\mathbb{Z}_p, \mathbb{K})$ *and* $\|M_i\| = 1$*;*
(2) For each $f \in C(\mathbb{Z}_p, \mathbb{K})$ *there exists a unique sequence* $(a_i)_{i \in \mathbb{N}} \subset \mathbb{K}$ *such that*

$$f(x) = \sum_{i=0}^{\infty} a_i M_i(x), \quad x \in \mathbb{Z}_p.$$

The series converges uniformly and

$$\left\|f\right\|_\infty = \max\left\{\left|a_i\right| : i \in \mathbb{N}\right\};$$

(3) If $(a_i)_i \in \mathbb{N} \in c_0(\mathbb{K})$ *then, the function*

$$f(x) = \sum_{i=0}^{\infty} a_i M_i, \ \ x \in \mathbb{Z}_p$$

defines an element of $C(\mathbb{Z}_p, \mathbb{K})$.

We now introduce the notion of orthogonality:

Definition 2.37. We say that $x, y \in \mathbb{E}$ are orthogonal to each other if

$$\left\|ax + by\right\| = \max\left\{\left\|ax\right\|, \left\|by\right\|\right\}, \quad \text{for any } \ a, b \in \mathbb{K}.$$

This definition is clearly symmetric and generalizes as follows:

Definition 2.38. Let $(v_i)_{i \in I}$ be a family of vectors in $\mathbb{E}$. We say that the family is *orthogonal* if for any $J \subset I$ and for any family $(a_i)_{i \in J}$ of elements of $\mathbb{K}$ such that $\lim_{i \in J} a_i v_i = 0$,

$$\left\|\sum_{i \in J} a_i x_i\right\| = \max\left\{\left\|a_i x_i\right\| : i \in J\right\}.$$

Definition 2.39. An *orthogonal basis* for the Banach space $\mathbb{E}$ is a base which is an orthogonal family.

This means, then, that a family $\{e_i : i \in I\}$ is an orthogonal basis if and only if

(1) For every $x \in E$, there exists a unique family $(x_i)_{i\in I} \subset \mathbb{K}$ such that $x = \sum_{i\in I} x_i e_i$;
(2) $\left\|x\right\| = \max\left\{\left\|x_i e_i\right\| : i \in I\right\}$;

The orthogonal basis $\{e_i : i \in I\}$ is called an *orthonormal* basis if $\|e_i\| = 1$ for all $i \in I$.

Remark 2.40. The space $c_0(\mathbb{K})$ has a natural orthonormal base, namely, $e_i : i = 0, 1, \ldots$ where the sequence $e_i = (\delta_{i,j})_j \in \mathbb{N}$ and $\delta_{i,j}$ is the Kronecker symbol.

Remark 2.41. The sequence of Mahler functions $\{M_i : i = 0, 1, \ldots\}$ forms an orthonormal basis of $C(\mathbb{Z}_p, \mathbb{K})$.

2.4 The *p*-adic Hilbert Space $\mathbb{E}_\omega$

In this section we discuss properties of the so-called p-adic Hilbert space $\mathbb{E}_\omega$ which will be the focus of operator theory in the later chapters. We follow Diarra [20] closely.
Recall from Example 2.21 and Definition 2.22 that given $\omega = (\omega_i)_{i\in\mathbb{N}} \subset \mathbb{K}^*$,

$$\mathbb{E}_\omega = \left\{x = (x_i)_{i\in\mathbb{N}} : x_i \in \mathbb{K},\ \ \forall i \in \mathbb{N},\ \ \lim_{i\to\infty} \left|\omega_i\right|^{1/2}\left|x_i\right| = 0\right\}.$$

The space $\mathbb{E}_\omega$ is equipped with the norm

$$x = (x_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega, \qquad \left\|x\right\| := \sup_{i\in\mathbb{N}} \left|\omega_i\right|^{1/2}\left|x_i\right|.$$

Another characterization of $\mathbb{E}_\omega$ is the following:

Proposition 2.42. $x = (x_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega$ *if and only if* $\lim_{i\to\infty} x_i^2\omega_i = 0$.

Proposition 2.43. *The normed space* $(\mathbb{E}_\omega, \|\cdot\|)$ *is a free Banach space with orthogonal basis* $\{e_i : i = 0, 1, \ldots\}$ *where* $e_i = (\delta_{i,j})_{j\in\mathbb{N}}$ *and* $\delta_{i,j}$ *is the Kronecker symbol. For each* $i \in \mathbb{N}$

$$\left\|e_i\right\| = \left|\omega_i\right|^{1/2}.$$

Remark 2.44. The orthogonal basis $\{e_i : i = 0, 1, 2, \ldots\}$ is called the *canonical basis* of $\mathbb{E}_\omega$.

Proposition 2.45. *Let* $\langle \cdot, \cdot \rangle : \mathbb{E}_\omega \times \mathbb{E}_\omega \to \mathbb{K}$ *be defined as follows: for* $x = (x_i)_{i\in\mathbb{N}},\ y = (y_i)_{i\in\mathbb{N}}$

$$\langle x, y\rangle = \sum_{i=0}^{\infty} x_i y_i \omega_i.$$

Then

(1) $\langle x, y\rangle$ *is well-defined, i.e., the series converges in* $\mathbb{K}$*;*
(2) $\langle \cdot, \cdot \rangle$ *is symmetric, bilinear form on* $\mathbb{E}_\omega$*;*
(3) $\langle \cdot, \cdot \rangle$ *satisfies the Cauchy–Schwarz inequality, namely,*

$$\left|\langle x, y\rangle\right| \le \left\|x\right\| . \left\|y\right\|;$$

(4) $\langle \cdot, \cdot \rangle$ *is continuous.*

Proof. (2) is clear and (4) follows from (3).
(1)

$$\begin{aligned}\lim_{i\to\infty} \left|x_i\right|\left|y_i\right|\left|\omega_i\right| &= \lim_{i\to\infty} \left|x_i\right|\left|\omega_i\right|^{1/2}\left|y_i\right|\left|\omega_i\right|^{1/2}\\ &= \Big(\lim_{i\to\infty} \left|x_i\right|\left|\omega_i\right|^{1/2}\Big) . \Big(\lim_{i\to\infty} \left|y_i\right|\left|\omega_i\right|^{1/2}\Big)\\ &= 0 \ \text{ since } x, y \in \mathbb{E}_\omega.\end{aligned}$$

(3)

$$\begin{aligned}\left|\Big\langle x, y\Big\rangle\right| &= \Big|\sum_{i=0}^{\infty} x_i y_i \omega_i\Big|\\ &\le \sup_{i\in\mathbb{N}} \left|x_i\right|\left|\omega_i\right|^{1/2}\left|y_i\right|\left|\omega_i\right|^{1/2}\\ &\le \sup_{i\in\mathbb{N}} \left|x_i\right|\left|\omega_i\right|^{1/2} . \sup_{i\in\mathbb{N}} \left|y_i\right|\left|\omega_i\right|^{1/2}\\ &= \left\|x\right\| . \left\|y\right\|.\end{aligned}$$

Moreover, we have:

Proposition 2.46. *The following hold:*

(1) $\langle \cdot, \cdot \rangle$ *is non-degenerate;*

(2) $\langle x, x\rangle = \sum_{i=0}^{\infty} x_i^2 \omega_i$*;*

(3) $\langle e_i, e_j\rangle = \delta_{i,j}\omega_i$ for $i, j \in \mathbb{N}$;
(4) $\langle x, e_k\rangle = x_k\omega_k$.

Proof. We prove only (1). Suppose $\langle x, y\rangle = 0$ for all $y \in \mathbb{E}_\omega$. Then, in particular, for any k, if $y = e_k$, we find that $x_k\omega_k = 0$ which implies that $x_k = 0$ as $\omega_k \neq 0$. Since this holds for all k, $x = 0$.

The space $\mathbb{E}_\omega$ endowed with the above-mentioned norm and inner product, is called a p-adic (or non-archimedean) Hilbert space. In contrast with classical Hilbert spaces, the norm on $\mathbb{E}_\omega$ does not stem from the inner product. Further, the space $\mathbb{E}_\omega$ contains isotropic vectors, that is, vectors $x \in \mathbb{E}_\omega$ such that $\langle x, x\rangle = 0$ while $x \neq 0$. Let us construct one of those isotropic vectors of $\mathbb{E}_\omega$. To simply things, suppose $\mathbb{K} = \mathbb{Q}_p$ where p is a prime satisfying $p \equiv 1$ (mod) 4 and let $\omega = (\omega_i)_{i\in\mathbb{N}}$, where $\omega_0 = 1,\ \omega_1 = 1,\ \omega_i = p^i$ for $i \geq 2$. If we consider the nonzero vector $x = (x_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega$, given by, $x_0 = 1,\ x_1 = \sqrt{-1} \in \mathbb{Q}_p,\ x_i = 0,\ i \geq 2$, it easily follows that $\langle x, x\rangle = 0$ while $\|x\| = 1$.

In the case of the p-adic Hilbert space $\mathbb{E}_\omega$, the definition of an orthogonal basis becomes the following:

Definition 2.47. $\{h_i : i = 0, 1, \ldots\} \subset \mathbb{E}_\omega$ is an orthogonal basis for $\mathbb{E}_\omega$ if

(1) For every $x \in \mathbb{E}_\omega$ there exists a unique sequence $(x_i)_{i\in\mathbb{N}}$ such that $x = \sum_{i=0}^{\infty} x_i h_i$; and
(2) $\Big\|x\Big\| = \sup_{i\in\mathbb{N}} \Big|x_i\Big|\Big\|h_i\Big\|$.

An example of an orthogonal basis is the canonical basis $\{e_i : i = 0, 1, \ldots\}$ of Proposition 2.43.

We next consider perturbations of orthogonal bases in $\mathbb{E}_\omega$. Namely, if $\{h_i : i = 0, 1, \ldots\}$ is an orthogonal basis and if $\{f_i : i = 0, 1, \ldots\}$ is another sequence of vector of $\mathbb{E}_\omega$, not necessarily an orthogonal basis, such that the difference $f_i - h_i$ is small in a certain sense; we investigate conditions under which the family $\{f_i : i = 0, 1, \ldots\}$ is a basis of $\mathbb{E}_\omega$.

We refer to Chap. 3 for deeper results where invertible operators play crucial roles, however, we have the following:

Proposition 2.48. *Let* $\{f_i : i = 0, 1, \ldots\}$ *be an orthogonal basis in E and let* $\{h_j : j = 0, 1, \ldots\}$ *be a basis in E such that*

$$\Big\|h_i - f_i\Big\| < \Big\|f_i\Big\|$$

then $\{h_j : j = 0, 1, \ldots\}$ *is an orthogonal basis.*

Proof. Let $\{a_i : i = 0, 1, \ldots\}$ be a sequence in $\mathbb{K}$ such that $\lim_{i\to\infty} a_i h_i = 0$, and let $x = \sum_{i=0}^{\infty} a_i h_i$. We observe that the assumption $\Big\|h_i - f_i\Big\| < \Big\|f_i\Big\|$ implies that $\Big\|h_i\Big\| = \Big\|f_i\Big\|$ for all i. This, in turn, implies that $\lim_{i\to\infty} a_i f_i = 0$. Let $y = \sum_{i=0}^{\infty} a_i f_i$, then,

$$\begin{aligned}
\left\|x-y\right\| &= \left\|\sum_{i=0}^{\infty} a_i(h_i - f_i)\right\| \\
&\le \sup_{i\in\mathbb{N}} \left|a_i\right| \left\|h_i - f_i\right\| \\
&< \sup_{i\in\mathbb{N}} \left|a_i\right| \left\|f_i\right\| \\
&= \left\|\sum_{i\in\mathbb{N}} a_i f_i\right\| \\
&= \left\|x\right\|,
\end{aligned}$$

and therefore, $\left\|x-y\right\| < \left\|x\right\|$.
Now this implies that $\left\|y\right\| = \left\|(y-x)+x\right\| = \left\|x\right\|$ and therefore

$$\left\|x\right\| = \sup_{i\in\mathbb{N}} \left|a_i\right| \left\|h_i\right\|$$

and hence $\{h_i : i = 0, 1, \ldots\}$ is an orthogonal basis.

We conclude this background chapter with a structure theorem for $\mathbb{E}_\omega$.

Proposition 2.49. *Let $\pi \in \mathbb{K}$ such that $|\pi| < 1$. There exists a sequence*

$$\{f_i : i = 0, 1, \ldots\} \subset \mathbb{E}_\omega$$

satisfying the following

(1) For each $i \in \mathbb{N}$, f_i is a scalar multiple of e_i such that

$$\left|\pi\right| \le \left\|f_i\right\| \le 1;$$

(2) $\{f_i : i = 0, 1, \ldots\}$ is an orthogonal basis for $\mathbb{E}_\omega$ in the sense of Definition 2.47.

Proof. (1) follows from the following

Lemma 2.50. *For every $x \in \mathbb{E}_\omega$, $x \neq 0$, there exists $x' \in \mathbb{E}_\omega$ a scalar multiple of x such that*

$$\left|\pi\right| \le \left\|x\right\| \le 1.$$

Proof. Since $|\pi| < 1$, there exists an integer n such that

$$\left|\pi\right|^{n+1} \leq \left\|x\right\| \leq \left|\pi\right|^{n}.$$

Dividing through by $\left|\pi\right|^{n}$ gives the result.

(2) From (1) we can write for each $i \in \mathbb{N}$, $f_i = \lambda_i e_i$ where $\lambda_i \in \mathbb{K}$ and $\lambda_i \neq 0$. Let $x \in \mathbb{E}_\omega$, then, there exists $(x_i)_{i\in\mathbb{N}} \subset \mathbb{K}$ such that

$$\begin{aligned} x = \sum_{i=0}^{\infty} x_i e_i &= \sum_{i=0}^{\infty} \frac{x_i}{\lambda_i} f_i \\ &= \sum_{i=0}^{\infty} y_i f_i, \quad \text{with } y_i = \frac{x_i}{\lambda_i} \in \mathbb{K}. \end{aligned}$$

Next suppose

$$x = \sum_{i=0}^{\infty} y_i f_i = \sum_{i=0}^{\infty} z_i f_i, \quad y_i, z_i \in \mathbb{K}, \quad \forall i.$$

Then

$$x = \sum_{i=0}^{\infty} y_i \lambda_i e_i = \sum_{i=0}^{\infty} z_i \lambda_i e_i$$

which implies $\lambda_i y_i = \lambda_i z_i$ for all i, and hence $y_i = z_i$ for all i. Moreover, for $x \in \mathbb{E}_\omega, \quad x = \sum_{i=0}^{\infty} y_i f_i$

$$\begin{aligned} \left\|x\right\| &= \left\| \sum_{i=0}^{\infty} y_i f_i \right\| \\ &= \left\| \sum_{i=0}^{\infty} y_i \lambda_i e_i \right\| \\ &= \sup_{i\in\mathbb{N}} \left|y_i\right| \left|\lambda_i\right| \left\|e_i\right\| \\ &= \sup_{i\in\mathbb{N}} \left|y_i\right| \left\|f_i\right\| \quad \text{as} \left|\lambda_i\right| \left\|e_i\right\| = \left\|f_i\right\| \end{aligned}$$

and hence $\{f_i : i = 0, 1, \ldots\}$ is an orthogonal basis in the sense of Definition 2.47.

Theorem 2.51. *The p-adic Hilbert space* $\mathbb{E}_\omega$ *is bicontinuously isomorphic to* $c_0(\mathbb{K})$.

Proof. We show that there exists a continuous, linear, bijection $\Phi : c_0(\mathbb{K}) \to \mathbb{E}_\omega$ whose inverse $\Psi : \mathbb{E}_\omega \to c_0(\mathbb{K})$ is also continuous.

We use the orthogonal basis $\{f_i : i = 0, 1, \ldots\}$ of Proposition 2.49 for $\mathbb{E}_\omega$. Let $\Phi : c_0(\mathbb{K}) \to \mathbb{E}_\omega$ be defined as follows, $x = (x_i)_{i\in\mathbb{N}} \in c_0(\mathbb{K})$, $\lim_i \left|x_i\right| = 0$,

$$\Phi(x) = \sum_{i=0}^{\infty} x_i f_i.$$

Since $\left\|f_i\right\| \leq 1$, then

$$\lim_{i\to\infty} \left|x_i\right| \left\|f_i\right\| \leq \lim_{i\to\infty} \left|x_i\right| = 0$$

hence Φ is well-defined and is clearly linear.

$$\begin{aligned}
\left\|\Phi(x)\right\| &= \left\| \sum_{i=0}^{\infty} x_i f_i \right\| \\
&= \sup_{i\in\mathbb{N}} \left|x_i\right| \left\|f_i\right\| \\
&\leq \sup_{i\in\mathbb{N}} \left|x_i\right|, \quad \text{as } \left\|f_i\right\| \leq 1 \\
&= \left\|x\right\|
\end{aligned}$$

hence Φ is continuous.

Let $\Psi : \mathbb{E}_\omega \to c_0(\mathbb{K})$ be defined as follows, $y = \sum_{i=0}^{\infty} y_i f_i$, $\lim_{i\to\infty} \left|y_i\right| \left\|f_i\right\| = 0$, and

$$\Psi(y) = (y_i)_{i\in\mathbb{N}}.$$

Since $\left\|f_i\right\| \geq \left|\pi\right|$, then

$$0 = \lim_{i\to\infty} \left|y_i\right| \left\|f_i\right\| \geq \left|\pi\right| \lim_{i\to\infty} \left|\pi y_i\right|$$

hence Ψ is well-defined and is clearly linear.

$$\begin{aligned}
\left\|\Psi(y)\right\| &= \left\|(y_i)_{i\in\mathbb{N}}\right\| \\
&= \sup_{i\in\mathbb{N}} \left|y_i\right| \\
&\leq \frac{1}{\left|\pi\right|} \sup_{i\in\mathbb{N}} \left\|f_i\right\| \left|y_i\right|, \quad \text{as } \left|\pi\right| \leq \left\|f_i\right\| \\
&= \frac{1}{\left|\pi\right|} \left\|y\right\|
\end{aligned}$$

hence Ψ is continuous. From their definitions, it is clear that Φ and Ψ are inverses of each other. This concludes the proof.

2.5 Bibliographical Notes

The material in this chapter mostly comes from the following sources: Diarra [20], Diagana [13], Schikhof [46], and van Rooij [53].

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