Chapter 2
Mumford Curves

The incentive to create the theory of holomorphic functions over a non-Archimedean field was Tate's elliptic curve. By means of rigid geometry one can explain Tate's elliptic curve from the geometric point of view, whereas Tate originally formulated it in terms of function fields; cf. Sect. 2.1.

In the following sections we study Mumford's generalization of Tate's curve to curves of higher genus in the context of rigid geometry. We introduce discontinuous actions of certain subgroups Γ of $\mathrm{PGL}(2, K)$ on the projective line in the style of Schottky. The structure of these groups Γ was found by Ihara; cf. Sect. 2.2.

Mumford curves will be introduced as orbit spaces $\Gamma\backslash\Omega$, where $\Omega \subset \mathbb{P}^1_K$ is the largest subdomain of $\mathbb{P}^1_K$ on which Γ acts in an ordinary way. The construction of the quotient $\Gamma\backslash\Omega$ can be carried out in the framework of classical rigid geometry. Note that Mumford achieves much more general results in his ground braking article [75] which deals exclusively with formal schemes. The concept here follows geometric constructions in order to explain the ideas behind Mumford's construction.

Chapter 2 is somehow a counterpart of Riemann surfaces and their Jacobians. We provide the full picture of Mumford curves and their Jacobians which are rigid analytic tori. We show the duality theory of rigid analytic tori, Riemann's period relations and, moreover, Riemann's vanishing theorem.

Our approach is a refined version of the work of Drinfeld and Manin [64] and [65], where they work over a p-adic field; i.e., a finite extension of $\mathbb{Q}_p$. Here we consider a general non-Archimedean field as defined in Definition 1.1.1; notably we mention the work of Gerritzen [31–33].

2.1 Tate's Elliptic Curve

The following statements can be found in [85, §3, VI and VII] or [93], where they are stated for non-Archimedean fields; cf. Definition 1.1.1.

Theorem 2.1.1 (Tate). *Let K be a non-Archimedean field of arbitrary characteristic and let $q \in K^\times$ with $0 < |q| < 1$. Then the field of meromorphic q-periodic*

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functions on $\mathbb{G}_{m,K}$ *is an elliptic function field* $F(q)$; *i.e.,* $F(q)$ *is finitely generated field of transcendence degree* 1 *over* K *and of genus* 1. *More precisely,* $F(q) = K(\wp, \tilde{\wp})$, *where*

$$\wp(\xi) = \sum_{n\in\mathbb{Z}} \frac{q^n\xi}{(1-q^n\xi)^2} - 2\cdot s_1,$$
$$\tilde{\wp}(\xi) = \sum_{n\in\mathbb{Z}} \frac{q^{2n}\xi^2}{(1-q^n\xi)^3} + s_1$$

with

$$s_\ell := \sum_{m\geq 1} \frac{m^\ell q^m}{1-q^m} \quad \textit{for } \ell \in \mathbb{N}.$$

The associated projective curve $E(q)$ *is given by the inhomogeneous equation*

$$\tilde{\wp}^2 + \wp\cdot\tilde{\wp} = \wp^3 + B\cdot\wp + C$$

for $B := -5\cdot s_3$, $C := \frac{1}{12}(5\cdot s_3 + 7\cdot s_5)$, *which actually lie in* $q\mathbb{Z}[[q]]$.

Its j*-invariant is*

$$j(q) = \frac{(1-48\cdot B)^3}{\Delta} = \frac{1}{q} + R(q),$$

where

$$R(q) = 744 + 196884\cdot q + \cdots \in \mathbb{Z}[[q]],$$
$$\Delta(q) = B^2 - C - 64\cdot B^3 + 72\cdot BC - 432\cdot C^2 = q\cdot\prod_{n\geq 1}\left(1-q^n\right)^{24} \in \mathbb{Z}[[q]].$$

For every element $j \in K$ *with* $|j| > 1$ *there exists a unique* $q \in K$ *with* $0 < |q| < 1$ *such that* $j = j(q)$.

In the above statement, the ring $\mathbb{Z}[[q]]$ is viewed as a subring of K. Actually, the Tate curve can also be defined over the power series ring $\mathbb{Z}[[Q]]$, where Q is a variable.

Note that the theorem is entirely stated in terms of function fields. The associated elliptic curve $E(q)$ of the function field $F(q)$ is defined via the equivalence of categories between the category of function fields and the one of normal projective curves. In terms of rigid geometry $E(q)$ is a 1-dimensional rigid analytic torus $\mathbb{G}_{m,K}/q^{\mathbb{Z}}$.

Up to a finite separable extension of the ground field there are two types of elliptic curves over a non-Archimedean field; cf. [15, §1.5]:

Theorem 2.1.2. *Let* E *be an elliptic curve over a non-Archimedean field* K. *After a suitable finite separable extension of the ground field there are two possibilities*:

(i) *If* $|j(E)| \le 1$, *then* E *has good reduction.*
(ii) *If* $|j(E)| > 1$, *then* E *is isomorphic to the rigid analytic torus* $\mathbb{G}_{m,K}/q^{\mathbb{Z}}$ *for a unique* $q \in K^{\times}$ *with* $0 < |q| < 1$. *The* j*-invariant bijectively depends on* q *by a series*

$$j(q) = \frac{1}{q} + f(q)$$

with $f(q) \in R[[q]] \subset K$. *Thus,* $j(q)$ *converges on the open punctured disc and* j *gives rise to a biholomorphic map*

$$j : \{q \in K^{\times}; 0 < |q| < 1\} \xrightarrow{\sim} \{j \in K^{\times}; |j| > 1\}.$$

Over an algebraically closed field, an elliptic curve is uniquely determined up to isomorphism by its j-invariant. If the characteristic of the ground field is unequal 2, every elliptic curve E can be defined by a Legendre equation

$$E \cong E_{\lambda} := V\big(Y^2 \cdot Z - X \cdot (X - Z) \cdot (X - \lambda Z)\big) \subset \mathbb{P}^2_K$$

in the projective plane with $\lambda \in K - \{0, 1\}$. Its j-invariant is

$$j(E) = 2^8 \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2(1 - \lambda)^2}$$

which is invariant under the substitution

$$\lambda \longmapsto \lambda, \quad 1 - \lambda, \quad \frac{1}{\lambda}, \quad \frac{1}{1 - \lambda}, \quad \frac{\lambda - 1}{\lambda}, \quad \frac{\lambda}{\lambda - 1}.$$

This reflects the isomorphism $E_{\lambda} \cong E_{\lambda'}$ for λ' equal to one of these values. In view of these isomorphisms, we may assume that $|\lambda| \le 1$, when K is a non-Archimedean field. Then

E_{λ} *has good reduction if and only if* $|\lambda| = 1$ *and* $|1 - \lambda| = 1$.

E_{λ} *has multiplicative reduction if and only if* $0 < |\lambda| < 1$ *or* $|1 - \lambda| < 1$.

Good reduction means that the polynomial $P(X) := X(X - 1)(X - \lambda)$ has three distinct roots in the reduction, whereas multiplicative reduction means that two roots of $P(X)$ collapse in the reduction.

The relationship between the modulus q and the Legendre parameter $\lambda \in K$ with $0 < |\lambda| < 1$ is the following:

$$q(\lambda) = c_2\lambda^2 + c_3\lambda^3 + \cdots$$

with $|c_2| = 1$ and $|c_i| \le 1$ for $i \in \mathbb{N}$. There are always exactly two values λ_1, λ_2 such that $|\lambda_i| < 1$ for $i = 1, 2$ with $j(\lambda_1) = j(\lambda_2)$; cf. [10, §9.7].

Using rigid geometry, one can construct Tate's curve in a geometric way. Consider an element $q \in K$ of a non-Archimedean field K with absolute value

$0 < |q| < 1$. Then $M := \{q^n; n \in \mathbb{Z}\}$ is a multiplicative lattice in the multiplicative group $\mathbb{G}_{m,K}$ in the sense of Sect. 2.7 and hence one can construct the quotient

$$E(q) := \mathbb{G}_{m,K}/M,$$

which is a proper smooth rigid analytic curve. We know from Theorem 1.8.1 that $E(q)$ is the analytification of a smooth projective curve. It is easy to show that $E(q)$ is an elliptic curve. Moreover, in the situation of Theorem 2.1.1, the field of rational functions on $E(q)$ is the field $F(q)$.

In the next section we will study more general group actions than Tate's action $M \times \mathbb{P}^1_K \to \mathbb{P}^1_K$; $(q, z) \longmapsto q \cdot z$, on the projective line. The group M is only a special case of a Schottky group; cf. Example 2.2.3, and so Tate's curves are a special case of Mumford curves; cf. Theorem 2.3.1. In the following sections we will present much more general results.

2.2 Schottky Groups

From now on, let K be a non-Archimedean field as defined in Definition 1.1.1. In this section we will study the structure of those finitely generated subgroups of the projective linear group $\mathrm{PGL}(2, K)$, which are free of torsion and act discontinuously on a non-empty open subdomain of the projective line.

For this, we cannot make use of the tree presentation of p-adic numbers as Drinfeld and Manin do. Instead we follow the classical method of isometric circles as invented by Ford [28]; see also the article of Gerritzen [31], which was slightly generalized by Kotissek [57].

In the following we consider the projective line $\mathbb{P}^1_K$ and equip the set $\mathbb{P}^1_K(K)$ of its K-rational points with the topology induced by the absolute value of K. The points in $\mathbb{P}^1_K(K) = K \cup \{\infty\}$ can be written in the form $[x, y]$ for $(x, y) \in K^2 - \{0\}$ if we want to mention their homogeneous coordinates. Hereby two symbols $[x, y]$ and $[x', y']$ are identified if there exists a $\lambda \in K^\times$ with $(x', y') = \lambda(x, y)$. A point $z \in K$ corresponds to $[z, 1]$ and ∞ to $[1, 0]$.

Each matrix $A := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, K)$ gives rise to an automorphism

$$\gamma_A : \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K,\ z = [x, y] \longmapsto \frac{az+b}{cz+d} := [ax + by, cx + dy],$$

with the usual convention for ∞:

$$\frac{az+b}{cz+d} := \begin{cases} \dfrac{az+b}{cz+d} & \text{if } cz + d \neq 0 \text{ and } z \neq \infty, \\ \infty & \text{if } cz + d = 0 \text{ and } z \neq \infty, \\ a/c & \text{if } z = \infty \text{ and } c \neq 0, \\ \infty & \text{if } z = \infty \text{ and } c = 0. \end{cases}$$

Such a map is called a *Möbius transformation*. The map γ_A equals id if and only if $A = \lambda \cdot I_2$ for some $\lambda \in K^\times$. For all $A, B \in \mathrm{GL}(2, K)$ we have that $\gamma_{A \circ B} = \gamma_A \circ \gamma_B$. The group

$$\mathrm{PGL}(2, K) := \mathrm{GL}(2, K)/K^\times$$

is called the *projective linear group*. It is easy to see that $\mathrm{PGL}(2, K)$ is the group of K-rational automorphisms $\mathrm{Aut}(\mathbb{P}^1_K)$ of the projective line. It is generated by the elements $(z \mapsto z + b, z \mapsto a \cdot z, z \mapsto 1/z)$, because

$$\frac{az+b}{cz+d} = \frac{bc-ad}{c^2}\left(z + \frac{d}{c}\right)^{-1} + \frac{a}{c}$$

for $c \neq 0$.

Definition 2.2.1. Let $z_1, z_2, z_3, z_4 \in \mathbb{P}^1_K(K)$ with $\{z_1, z_2\} \cap \{z_3, z_4\} = \emptyset$, then

$$\mathrm{CR}(z_1, z_2, z_3, z_4) := \begin{cases} \dfrac{z_1 - z_3}{z_1 - z_4} \cdot \dfrac{z_2 - z_4}{z_2 - z_3} & \text{for } z_1, z_2, z_3, z_4 \in K \\ \dfrac{z_1 - z_3}{z_2 - z_3} & \text{for } z_1, z_2, z_3 \neq \infty,\ z_4 = \infty \\ \dfrac{z_2 - z_4}{z_1 - z_4} & \text{for } z_1, z_2, z_4 \neq \infty,\ z_3 = \infty \\ \dfrac{z_1 - z_3}{z_1 - z_4} & \text{for } z_1, z_3, z_4 \neq \infty,\ z_2 = \infty \\ \dfrac{z_2 - z_4}{z_2 - z_3} & \text{for } z_2, z_3, z_4 \neq \infty,\ z_1 = \infty \\ 1 & \text{for } z_1 = z_2 = \infty \text{ oder } z_3 = z_4 = \infty \end{cases}$$

is called the *cross ratio* of z_1, z_2, z_3, z_4. The cross ratio is invariant under Möbius transformations.

Each matrix $A \in \mathrm{GL}(2, K)$ has two eigenvalues $\lambda_1, \lambda_2 \in L^\times$, where L/K is a field extension of degree $[L : K] \leq 2$. If $\lambda_1 \neq \lambda_2$, the map γ_A has two fixed points $z_1, z_2 \in \mathbb{P}^1_K(L)$. One can choose the coordinate in such a way that $z_1 = 0$ and $z_2 = \infty$. Then the map γ_A is just the multiplication by $q := \lambda_1/\lambda_2$. If $|\lambda_1| < |\lambda_2|$, then z_1 is the *attractive fixed point* and ∞ is the *repelling fixed point*. This means that, for every $w \in \mathbb{P}^1_K - \{0, \infty\}$, the sequence $(\gamma_A^n(w); n \in \mathbb{N})$ converges to z_1. In particular, z_1 and z_2 are K-rational in this case as K is complete. Such transformations are called *hyperbolic transformations*. The element $q_\gamma := q$ is called the *multiplier* of the hyperbolic transformation γ; it is uniquely determined by γ due to the requirement that $|q_\gamma| < 1$. If $|\lambda_1| = |\lambda_2|$, then the transformation is called *elliptic*. In this case, the transformation is of type $z \mapsto a \cdot z$ with $|a| = 1$.

If $\lambda_1 = \lambda_2$, then the transformation is called *parabolic*. In this case, the transformation looks like $z \mapsto z + b$. Thus, we have the following:

$$\gamma_A \text{ elliptic or parabolic} \iff \left|\frac{(a+d)^2}{ad-bc}\right| \leq 1,$$

$$\gamma_A \text{ hyperbolic} \iff \left|\frac{(a+d)^2}{ad-bc}\right| > 1.$$

Consider a subgroup $\Gamma \subset \mathrm{PGL}(2, K)$ and a K-rational point w of $\mathbb{P}^1_K(K)$. Put

$$L_\Gamma(w) := \left\{ z \in \mathbb{P}^1_K(K); \begin{array}{l} \text{there exists pairwise distinct } \gamma_n \in \Gamma \\ \text{for } n \in \mathbb{N} \text{ with } \gamma_n(w) \to z \end{array} \right\}.$$

Even, if one allows limit points z with values in some field extension, then they are K-rational, because K is complete and w is K-rational. Put

$$L_\Gamma := \bigcup_{w \in \mathbb{P}^1_K(K)} L_\Gamma(w), \quad \text{the set of } \textit{limit points of } \Gamma,$$

$$\Omega_\Gamma := \mathbb{P}^1_K - L_\Gamma, \quad \text{the set of } \textit{ordinary points of } \Gamma.$$

These group are named *discontinuous* if $L_\Gamma \neq \mathbb{P}^1_K(K)$ and if for every point w of $\mathbb{P}^1_K(K)$ the topological closure $\overline{\Gamma w}$ of the orbit of w is compact with respect to the metric topology of $\mathbb{P}^1_K(K)$. If K is locally compact, the latter hypothesis is always fulfilled.

Definition 2.2.2. A subgroup $\Gamma \subset \mathrm{PGL}(2, K)$ is called a *Schottky group* if Γ is finitely generated, free of torsion and discontinuous.

The group is named after Friedrich Schottky (1851–1935) who worked with similar group actions in complex analysis [86].

Example 2.2.3. If $\gamma \in \mathrm{PGL}(2, K)$ is hyperbolic, then $\Gamma := \langle\gamma\rangle$ is a Schottky group. The set L_Γ of the limit points of Γ consists of the two fixed points of γ.

Proof. We may assume that 0 and ∞ are the fixed points. Then γ acts via multiplication with an element $\lambda \in K^\times$ with $|\lambda| \neq 1$. Then it is clear that Γ is free of torsion and $\overline{\Gamma w}$ is compact for all $w \in \mathbb{P}^1_K(K)$ and $L_\Gamma = \{0, \infty\}$. □

This example corresponds to Tate's elliptic curve. The most general example is presented in Example 2.2.13. In the following let Γ be a Schottky group. For a $\gamma = \gamma_A$ with $c \neq 0$ set

$$r_\gamma := \frac{\sqrt{|ad-bc|}}{|c|}.$$

In the following we collect some properties of Schottky groups.

Proposition 2.2.4. *If $\Gamma \subset \mathrm{PGL}(2, K)$ is a Schottky group, then we have:*

(a) *Γ is discrete.*
(b) *Every $\gamma \in \Gamma - \{\mathrm{id}\}$ is hyperbolic.*
(c) *If $\infty \notin L_\Gamma$, then for $\gamma = \gamma_A \in \Gamma$ with $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, K)$ we have:*

$$\gamma = \mathrm{id} \quad \Longleftrightarrow \quad \gamma(\infty) = \infty \quad \Longleftrightarrow \quad c = 0.$$

(d) *If $\infty \notin L_\Gamma$, then $R(\Gamma, \delta) := \{\gamma \in \Gamma - \{\mathrm{id}\}; r_\gamma \geq \delta\}$ is finite for every $\delta > 0$.*
(e) *$L_\Gamma(w)$ is closed for every $w \in \mathbb{P}^1_K(K)$.*
(f) *$L_\Gamma = L_\Gamma(\infty)$ if $\infty \notin L_\Gamma$.*

Proof. (a) The unit element of Γ is not an accumulation point of Γ, because otherwise $L_\Gamma = \mathbb{P}^1_K(K)$.

(b) Assume that $\gamma \in \Gamma - \{\mathrm{id}\}$ is elliptic or parabolic. For a suitable choice of the coordinate γ is given by a matrix $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or by $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, respectively. Since γ is not of finite order, the elements $(\gamma^n; n \in \mathbb{N})$ are pairwise distinct. Since the closure of the orbit $\Gamma[1, 1]$ or of $\Gamma[1, 0]$, respectively, is compact, we may assume that the limit $\lim\limits_{n\to\infty} \gamma^n[1, 1]$ or $\lim\limits_{n\to\infty} \gamma^n[0, 1]$, respectively, exists and is a limit point. Then it is easy to see that the identity is an accumulation point of Γ. This contradicts (a).

(c) Only the assertion that $c = 0$ implies $\gamma = \mathrm{id}$ requires a proof. In this case γ is of type $z \mapsto az + d$. By (b) the map γ equals id or is hyperbolic. The latter means that $|a| \neq 1$. Then we may assume $|a| > 1$ and hence ∞ is the limit of the sequence $\gamma^n[1, 1]$. Since $[1, 0] = \infty \notin L_\Gamma$, we get $\gamma = \mathrm{id}$.

(d) By (c) the value r_γ is defined for every $\gamma \in \Gamma - \{\mathrm{id}\}$. If $R(\Gamma, \delta)$ has infinitely many elements, then there exist matrices $A_n := \left(\begin{smallmatrix} a_n & b_n \\ 1 & d_n \end{smallmatrix}\right)$ in $\mathrm{PGL}(2, K)$ such that $(\gamma_n := \gamma_{A_n} \in R(\Gamma, \delta); n \in \mathbb{N})$ are pairwise distinct. Since Γ is discontinuous, we may assume that the sequences $a_n = \gamma_n(\infty)$, $d_n = -\gamma_n^{-1}(\infty)$ and $b_n = -\gamma_n(0) \cdot \gamma_n^{-1}(\infty)$ converge to elements a, b, c of $\mathbb{P}^1_K(K)$. Since $\infty \notin L_\Gamma$, the points a, b, c lie in K, and hence

$$\lim_{n\to\infty} \begin{pmatrix} a_n & b_n \\ 1 & d_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix} =: A \in M(2 \times 2, K).$$

The determinant of A is $\lim\limits_{n\to\infty} |a_n d_n - b_n| = \lim\limits_{n\to\infty} r^2_{\gamma_n} \geq \delta^2$. Thus, the limit $\lim\limits_{n\to\infty} \gamma_n = \gamma$ belongs to $\mathrm{PGL}(2, K)$. This contradicts (a).

(e) We may assume that $w \in K$ by choosing a suitable coordinate of $\mathbb{P}^1_K$. Let $z_m \in L_\Gamma(w)$ for $m \in \mathbb{N}$ with $z_0 := \lim\limits_{m\to\infty} z_m$. There exists an element $\gamma_m \in \Gamma$ such that $|\gamma_m(w) - z_m| < 1/m$ and $\gamma_m \notin \{\gamma_1, \ldots, \gamma_{m-1}\}$. Then $\lim\limits_{m\to\infty} \gamma_m(w) = z_0$ and, hence, $z_0 \in L_\Gamma(w)$.

(f) Consider an element $z \in L_\Gamma(w)$ for some $w \in \mathbb{P}^1_K(K)$ and let z be the limit $\lim\limits_{n\to\infty} \gamma_n(w)$. If there exists a $c > 0$ with $|w - \gamma_n^{-1}(\infty)| \geq c$ for all $n \in \mathbb{N}$, then

$$\left|\gamma_n(\infty) - \gamma_n(w)\right| = \left|\frac{a_n}{c_n} - \frac{a_n w + b_n}{c_n w + d_n}\right| = \frac{r^2_{\gamma_n}}{|w - \gamma_n^{-1}(\infty)|} \leq \frac{r^2_{\gamma_n}}{c},$$

which converges to 0 by (d). If for every $c > 0$ there exists an $n \in \mathbb{N}$ such that $|w - \gamma_n^{-1}(\infty)| \le c$, then $\lim_{n\to\infty} \gamma_n(w) \in L_\Gamma(\infty)$. Since $L_\Gamma(\infty)$ is closed by (e), it follows that $z \in L_\Gamma(\infty)$. □

Notation 2.2.5. For later use we provide some explicit calculations when $\infty \notin L_\Gamma$. Consider a Möbius transformation γ given by an invertible matrix $A := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}(2, K)$. The inverse of A is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Let Γ be a Schottky group with $\infty \notin L_\Gamma$. For $\gamma := \gamma_A \in \Gamma - \{\mathrm{id}\}$ put

$$\begin{aligned}
r_\gamma &:= \frac{\sqrt{|ad - bc|}}{|c|}, & r_{\gamma^{-1}} &= r_\gamma, \\
m_\gamma &:= \gamma^{-1}(\infty) = -d/c, & m_{\gamma^{-1}} &= \gamma(\infty) = a/c, \\
v_\gamma(z) &:= \gamma'(z) = \frac{ad - bc}{(cz + d)^2}, & v_{\gamma^{-1}}(z) &= \left(\gamma^{-1}\right)'(z) = \frac{ad - bc}{(-cz + a)^2}, \\
\left|v_\gamma(z)\right| &= \frac{r_\gamma^2}{|z - m_\gamma|^2}, \\
V_\gamma^- &:= \left\{z \in \mathbb{P}_K^1;\, \left|v_\gamma(z)\right| > 1\right\} = \left\{z \in \mathbb{P}_K^1;\, |m_\gamma - z| < r_\gamma\right\}, \\
V_\gamma^+ &:= \left\{z \in \mathbb{P}_K^1;\, \left|v_\gamma(z)\right| \ge 1\right\} = \left\{z \in \mathbb{P}_K^1;\, |m_\gamma - z| \le r_\gamma\right\}.
\end{aligned}$$

Note that $c \neq 0$ due to Proposition 2.2.4(c). For $\gamma = \mathrm{id}$ set $v_{\mathrm{id}} = 1$.

The domains of type V_γ^- and V_γ^+ are called *open rational discs* and *closed rational discs*, respectively; cf. Definition 2.4.1 below. They are isomorphic to the open unit disc and to the closed unit disc, respectively.

As an example, consider a matrix $A := \left(\begin{smallmatrix} 0 & q^2 \\ 1 & 1 \end{smallmatrix}\right)$ with $0 < |q| < 1$. Thus, we have that $\gamma z = q^2/(z + 1)$ for $\gamma := \gamma_A$. Then $m_\gamma = -1$, $m_{\gamma^{-1}} = 0$ and $r_\gamma = |q|$. In this case we have

$$\begin{aligned}
V_\gamma^+ &= \left\{z \in \mathbb{P}_K^1;\, |z + 1| \le |q|\right\}, \\
V_{\gamma^{-1}}^+ &= \left\{z \in \mathbb{P}_K^1;\, |z| \le |q|\right\}.
\end{aligned}$$

In particular, one computes $\gamma(\mathbb{P}_K^1 - V_\gamma^-) = V_{\gamma^{-1}}^+$ and $V_\gamma^+ \cap V_{\gamma^{-1}}^+ = \emptyset$.

Lemma 2.2.6. *Let $\Gamma \subset \mathrm{PGL}(2, K)$ be a Schottky group with $\infty \notin L_\Gamma$. Then with the above notations we have*:

(a) $v_{\alpha\beta}(z) = v_\alpha(\beta z) \cdot v_\beta(z)$ *for* $\alpha, \beta \in \Gamma$.
(b) $\gamma(\mathbb{P}_K^1 - V_\gamma^-) = V_{\gamma^{-1}}^+$ *and* $\gamma(\mathbb{P}_K^1 - V_\gamma^+) = V_{\gamma^{-1}}^-$ *for* $\gamma \in \Gamma - \{\mathrm{id}\}$.
(c) $|m_\alpha - m_\beta| = \dfrac{r_\alpha \cdot r_\beta}{r_{\alpha\beta^{-1}}}$ *for elements* $\alpha, \beta \in \Gamma$ *with* $\alpha \neq \beta$.
(d) $V_\gamma^+ \cap V_{\gamma^{-1}}^+ = \emptyset$ *for* $\gamma \in \Gamma - \{\mathrm{id}\}$.

Proof. (a) follows from the chain rule for derivatives.

(b) follows from $v_{\gamma^{-1}}(\gamma z) = 1/v_\gamma(z)$ as was shown in (a).

(c) follows by a direct computation.

(d) follows from (c). Indeed, every element $\gamma \in \Gamma - \{\mathrm{id}\}$ is hyperbolic and $\gamma \neq \gamma^{-1}$. Therefore, $|m_\gamma - m_{\gamma^{-1}}| = \frac{|a+d|}{\sqrt{|ad-bc|}} \cdot r_\gamma > r_\gamma$. □

Definition 2.2.7. Let $\Gamma \subset \mathrm{PGL}(2, K)$ be a Schottky group. A *fundamental domain* for Γ is a complement

$$E := \mathbb{P}^1_K - \left(B_1^- \cup \cdots \cup B_n^-\right)$$

of a finite union of open discs satisfying the following properties:

(i) $\bigcup_{\gamma \in \Gamma} \gamma(E)$ equals the set Ω_Γ of ordinary points of Γ,
(ii) $E \cap \gamma(E) = \emptyset$ for almost all $\gamma \in \Gamma$,
(iii) $E \cap \gamma(E^-) = \emptyset$ for all $\gamma \in \Gamma - \{\mathrm{id}\}$ where

$$E^- := \mathbb{P}^1_K - \left(B_1^+ \cup \cdots \cup B_n^+\right)$$

is the complement of the union of the associated closed discs.

In this context a disc is a rational disc; i.e., it is given by a center which is a point in K and a radius belonging to $|K^\times|$ so that B^- and B^+ are defined.

Lemma 2.2.8. *In the situation of Lemma 2.2.6, let V_γ^- be defined as in Notation 2.2.5 and let $R(\Gamma, \delta) \subset \Gamma$ be defined as in Proposition 2.2.4. Let $\delta > 0$ be small enough such that $R(\Gamma, \delta)$ is a system of generators of Γ. If $\alpha = \gamma_1 \cdot \ldots \cdot \gamma_n$ is a minimal representation with elements $\gamma_i \in R(\Gamma, \delta)$, then $V_\alpha^- \subset V_{\gamma_n}^-$.*

Proof. We proceed by induction on n. For $n = 1$ there is nothing to prove. Thus, let $n \geq 2$, and set $\beta := \gamma_2 \cdot \ldots \cdot \gamma_n$. Due to the induction hypothesis $V_\beta^- \subset V_{\gamma_n}^-$. Since $\alpha \notin R(\Gamma, \delta)$, we have $r_\alpha < \delta \leq r_{\gamma_1}$. From Lemma 2.2.6 it follows that $|m_\alpha - m_\beta| = r_\alpha r_\beta / r_{\gamma_1} < r_\beta$, and hence $V_\alpha^- \subset V_\beta^- \subset V_{\gamma_n}^-$ due to the ultrametric inequality. □

Lemma 2.2.9. *In the situation of Lemma 2.2.6, set $\mathfrak{V}(\Gamma) := \{V_\gamma^-; \gamma \in \Gamma - \{\mathrm{id}\}\}$ and*

$$R(\Gamma) := \left\{\gamma \in \Gamma;\ V_\gamma^- \textit{ maximal in } \mathfrak{V}(\Gamma)\right\}.$$

Then $R(\Gamma)$ is finite and $R(\Gamma)$ is stable under the map $\gamma \longmapsto \gamma^{-1}$.

Proof. The finiteness follows from Lemma 2.2.8 and Proposition 2.2.4(d). The second claim follows from

$$\begin{aligned} V_\alpha^- \subsetneqq V_\beta^- \quad &\Longleftrightarrow \quad r_\beta > |m_\alpha - m_\beta| = \frac{r_\alpha r_\beta}{r_{\alpha\beta^{-1}}} \text{ and } r_\alpha < r_\beta \\ &\Longleftrightarrow \quad r_\alpha < r_{\alpha\beta^{-1}} \text{ and } r_\alpha < r_\beta \\ &\Longleftrightarrow \quad r_{\alpha^{-1}} < r_{\beta\alpha^{-1}} \text{ and } r_{\alpha^{-1}} < r_{\beta^{-1}} \quad \Longleftrightarrow \quad V_{\alpha^{-1}}^- \subsetneqq V_{\beta^{-1}}^-. \end{aligned}$$ □

Proposition 2.2.10. *In the situation of Lemma 2.2.6, we have:*

(a) $E := \mathbb{P}^1_K - \bigcup_{\alpha \in R(\Gamma)} V_\alpha^-$ *is a fundamental domain for* Γ.
(b) $R(\Gamma)$ *generates* Γ.

Proof. (a) From Lemma 2.2.8 it follows

$$E = \left\{ z \in \mathbb{P}^1_K;\ \left| v_\gamma(z) \right| \le 1 \text{ for all } \gamma \in \Gamma - \{\mathrm{id}\} \right\},$$
$$E^- = \left\{ z \in \mathbb{P}^1_K;\ \left| v_\gamma(z) \right| < 1 \text{ for all } \gamma \in \Gamma - \{\mathrm{id}\} \right\}.$$

$R(\Gamma)$ is finite due to Lemma 2.2.9, so E is an affinoid domain of $\mathbb{P}^1_K$. Due to Lemma 2.2.8 the set E contains only ordinary points and $E \cap \gamma(E) \ne \emptyset$ only when $\gamma \in R(\Gamma) \cup \{\mathrm{id}\}$, which is a finite set. Thus, $\gamma(E)$ contains only ordinary points, and hence $\bigcup_{\gamma \in \Gamma} \gamma(E) \subset \Omega_\Gamma$.

To show the converse inclusion, consider $z \in \Omega_\Gamma$. Such a point is contained only in a finite number of discs $V \in \mathfrak{V}(\Gamma)$. Indeed, assuming the contrary, there would exist an infinite number of $\gamma_n \in \Gamma$ such that $z \in V_{\gamma_n}^-$. Since $r_{\gamma_n} \to 0$ due to Proposition 2.2.4(d), the point z would be the limit of the points $m_{\gamma_n} = \gamma_n^{-1}(\infty)$ and hence z would belong to $L_\Gamma(\infty)$. That would be a contradiction to $z \in \Omega_\Gamma$.

For $z \in \Omega_\Gamma - E$ put

$$a := \sup_{\gamma \in \Gamma} \left| v_\gamma(z) \right| > 1.$$

Since $|v_\gamma(z)| > 1$ for only finitely many $\gamma \in \Gamma$, the supremum is attained, say $a = |v_\alpha(z)|$. Then we see by Lemma 2.2.6(a) that

$$\left| v_\gamma\big(\alpha(z)\big) \right| = \frac{|v_{\gamma\alpha}(z)|}{|v_\alpha(z)|} \le 1 \quad \text{for all } \gamma \in \Gamma,$$

and hence $\alpha(z) \in E$. Thus, we obtain $\Omega_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma(E)$.

It remains to show that $E^- \cap \gamma(E) = \emptyset$. For every point $z \in E^-$ we have $|v_\gamma(z)| < 1$. Since

$$1 = v_{\mathrm{id}}(z) = v_{\gamma^{-1}\gamma}(z) = v_{\gamma^{-1}}\big(\gamma(z)\big) \cdot v_\gamma(z),$$

it follows that $|v_{\gamma^{-1}}(\gamma(z))| > 1$, and hence $\gamma(z) \notin E$ for all $\gamma \in \Gamma - \{\mathrm{id}\}$.

(b) Let $\Gamma' := \langle R(\Gamma) \rangle \subset \Gamma$ be the subgroup generated by $R(\Gamma)$. Then Γ and Γ' have the same fundamental domain, as follows from (a). Thus, $\mathbb{P}^1_K - \bigcup_{\alpha \in R(\Gamma)} V_\alpha^-$ is a fundamental domain for Γ' as well. From (a) it follows $\Omega_{\Gamma'} \subset \Omega_\Gamma$. Obviously we have that $L_{\Gamma'} \subset L_\Gamma$, and hence $\Omega_{\Gamma'} = \Omega_\Gamma$. Consider now some $\gamma \in \Gamma$. Then $\gamma(\infty) \in \Omega_{\Gamma'}$. Thus, there exists $\gamma' \in \Gamma'$ such that $\gamma'\gamma(\infty) \in E$. Since $\infty \in E^-$ and $\gamma'\gamma(\infty) \in E$, we see that $\gamma'\gamma = \mathrm{id}$ by (a) and thus $\gamma \in \Gamma'$. □

Later on, we want to construct the quotient $X_\Gamma := \Gamma \backslash \Omega_\Gamma$ explicitly. For this construction, it is useful to have a suitable fundamental domain. However the fundamental domain of Proposition 2.2.10(a) can be quite complicated. For example, the system $(V_\alpha^- \alpha \in R(\Gamma))$ has the disadvantage that it can happen that $V_\alpha^- = V_\beta^-$ and $V_{\alpha^{-1}}^- \neq V_{\beta^{-1}}^-$ or $V_\alpha^- \subsetneqq V_\beta^+ - V_\beta^-$. In the following we want to improve the system by varying the radii which are chosen equal to 1 for the above system. We will do it in a constructive way.

Notation 2.2.11. Let $\Gamma \subset \mathrm{PGL}(2, K)$ be a Schottky group with $\infty \notin L_\Gamma$ and let $\rho : \Gamma \to \overline{K}^\times$ be a group homomorphism into the multiplicative group of the complete algebraic closure $\overline{K}$ of K. For $\gamma \in \Gamma - \{\mathrm{id}\}$ set

$$
\begin{aligned}
&w_\gamma(z) := \rho(\gamma) \cdot v_\gamma(z) = \rho(\gamma) \cdot \gamma'(z) \quad \text{so that } w_{\alpha\beta}(z) = w_\alpha\big(\beta(z)\big) \cdot w_\beta(z),\\
&W_\gamma^- := \big\{z \in \mathbb{P}_K^1; \big|w_\gamma(z)\big| > 1\big\} = \big\{z \in \mathbb{P}_K^1; |z - m_\gamma| < \sqrt{\big|\rho(\gamma)\big|} \cdot r_\gamma\big\},\\
&W_\gamma^+ := \big\{z \in \mathbb{P}_K^1; \big|w_\gamma(z)\big| \geq 1\big\} = \big\{z \in \mathbb{P}_K^1; |z - m_\gamma| \leq \sqrt{\big|\rho(\gamma)\big|} \cdot r_\gamma\big\}.
\end{aligned}
$$

Note that $w_\gamma(z)$ is the derivative of $\overline{\gamma}(z) := \rho(\gamma) \cdot \gamma(z)$ and $m_{\overline{\gamma}} = m_\gamma$ and $r_{\overline{\gamma}} = \sqrt{|\rho(\gamma)|} \cdot r_\gamma$. If $|\rho(\gamma)| < 1$ then $W_\gamma^+ \subset V_\gamma^-$ and $V_{\gamma^{-1}}^+ \subset W_{\gamma^{-1}}^-$.

The product formula implies $\gamma(\mathbb{P}_K^1 - W_\gamma^\pm) = W_{\gamma^{-1}}^\mp$ as in Lemma 2.2.6(b). Put

$$
\begin{aligned}
F &:= \mathbb{P}_K^1 - \bigcup_{\gamma \in \Gamma - \{\mathrm{id}\}} W_\gamma^- = \big\{z \in \mathbb{P}_K^1; \big|w_\gamma(z)\big| \leq 1 \text{ for all } \gamma \in \Gamma - \{\mathrm{id}\}\big\},\\
F^- &:= \mathbb{P}_K^1 - \bigcup_{\gamma \in \Gamma - \{\mathrm{id}\}} W_\gamma^+ = \big\{z \in \mathbb{P}_K^1; \big|w_\gamma(z)\big| < 1 \text{ for all } \gamma \in \Gamma - \{\mathrm{id}\}\big\}.
\end{aligned}
$$

ρ is called *separating* if there exists a finite system of generators $\alpha_1, \ldots, \alpha_g$ of Γ such that the closed discs $W_{\alpha_1}^+, \ldots, W_{\alpha_g}^+, W_{\alpha_1^{-1}}^+, \ldots, W_{\alpha_g^{-1}}^+$ are pairwise disjoint. In this case, we use the numbering $\alpha_{g+j} := \alpha_j^{-1}$ for $j = 1, \ldots, g$.

Proposition 2.2.12. *In the situation of Notation 2.2.11, assume that the homomorphism $\rho : \Gamma \to \overline{K}^\times$ is separating. Then we have the following*:

(a) *There exists an element $q \in \sqrt{|K^\times|}$, $q < 1$ such $|w_{\alpha_i}(z)| < q$ for all $z \in W_{\alpha_j}^+$ and $j \in \{1, \ldots, 2g\}$ with $j \neq i$.*

(b) *Let $\gamma = \alpha_{j(1)} \cdot \ldots \cdot \alpha_{j(n)}$ be a reduced representation with $\alpha_{j(\nu)}$ in $\{\alpha_1, \ldots, \alpha_{2g}\}$ and $n \geq 1$. Consider a point $z \in \mathbb{P}_K^1 - W_{\alpha_{j(n)}}^-$. Then $\gamma(z)$ belongs to $W_{\alpha_{j(1)}^{-1}}^+$ and $|w_\gamma(z)| < q^{n-1}$; in particular, this is true for $z \in F$.*

If $n \geq 2$, then we even have that $\gamma(z) \in W_{\alpha_{j(1)}^{-1}}^-$.

(c) *We have that $W_\gamma^- \subset W_{\alpha_{j(n)}}^-$ for every $\gamma \in \Gamma - \{\mathrm{id}\}$ as in* (b). *In particular,*

$$
F = \mathbb{P}_K^1 - \bigcup_{i=1}^{2g} W_{\alpha_i}^- \quad \textit{and} \quad F^- = \mathbb{P}_K^1 - \bigcup_{i=1}^{2g} W_{\alpha_i}^+.
$$

(d) *The system $(W_{\alpha_1}^-, \ldots, W_{\alpha_{2g}}^-)$ of discs has the properties*:

(d.1) $\gamma(F) \cap F \neq \emptyset \Longleftrightarrow \gamma \in \{\mathrm{id}, \alpha_1, \ldots, \alpha_{2g}\}$,
(d.2) $\alpha_i(F) \cap F = W^+_{\alpha_i^{-1}} - W^-_{\alpha_i^{-1}}$.

(e) *Γ is a free group and $(\alpha_1, \ldots, \alpha_g)$ is a free system of generators.*

Proof. (a) This follows from the maximum principle, but it can also be seen by explicit calculations as follows. Since $W^+_{\alpha_i} \cap W^+_{\alpha_j} = \emptyset$ for $i \neq j$, it follows that

$$|m_{\alpha_i} - m_{\alpha_j}| > \max\Big\{r_{\alpha_i} \cdot \sqrt{\big|\rho(\alpha_i)\big|}, r_{\alpha_j} \cdot \sqrt{\big|\rho(\alpha_j)\big|}\Big\}.$$

For $z \in W^+_{\alpha_j}$ and $i \neq j$ it follows that $|z - m_{\alpha_i}| = |m_{\alpha_j} - m_{\alpha_i}|$ by the ultrametric inequality, and hence

$$\big|w_{\alpha_i}(z)\big| = \frac{|\rho(\alpha_i)| \cdot r^2_{\alpha_i}}{|z - m_{\alpha_i}|^2} = \frac{|\rho(\alpha_i)| \cdot r^2_{\alpha_i}}{|m_{\alpha_j} - m_{\alpha_i}|^2} < 1;$$

cf. Notation 2.2.5. Choose $q \in \sqrt{|K^\times|}$ with

$$1 > q > \max\left\{\frac{|\rho(\alpha_i)| \cdot r^2_{\alpha_i}}{|m_{\alpha_j} - m_{\alpha_i}|^2}; 1 \leq i, j \leq 2g, i \neq j\right\}.$$

Then q satisfies the assertion.

(b) We proceed by induction on n. For $n = 1$ is $\gamma = \alpha_{j(1)}$. For every z in $\mathbb{P}^1_K - W^-_{\alpha_{j(1)}}$ we have $\alpha_{j(1)}(z) \in W^+_{\alpha^{-1}_{j(1)}} \subset \mathbb{P}^1_K - W^+_{\alpha_{j(1)}}$, because $W^+_{\alpha_1}, \ldots, W^+_{\alpha_g}$, $W^+_{\alpha_1^{-1}}, \ldots, W^+_{\alpha_g^{-1}}$ are pairwise disjoint. Thus, we obtain for the absolute value $|w_{\alpha_{j(1)}}(z)| < 1 = q^0$.

Now assume $n \geq 2$. Set $\beta := \alpha_{j(2)} \cdot \ldots \cdot \alpha_{j(n)}$. From the induction hypothesis we obtain that $|w_\beta(z)| < q^{n-2}$ and $\beta(z) \in W^+_{\alpha^{-1}_{j(2)}} \subset \mathbb{P}^1_K - W^+_{\alpha_{j(1)}}$ and $|w_{\alpha_{j(1)}}(\beta(z))| < q$ for $z \in \mathbb{P}^1_K - W^-_{\alpha_{j(n)}}$, because $\alpha_{j(1)} \neq \alpha^{-1}_{j(2)}$. Then it follows that $\gamma(z) = \alpha_{j(1)}(\beta(z)) \in W^-_{\alpha^{-1}_{j(1)}}$. The inequality follows from the chain rule

$$\big|w_\gamma(z)\big| = \big|w_{\alpha_{j(1)}}\big(\beta(z)\big) \cdot w_\beta(z)\big| = \big|w_{\alpha_{j(1)}}\big(\beta(z)\big)\big| \cdot \big|w_\beta(z)\big| < q \cdot q^{n-2} = q^{n-1}.$$

(c) One has $m_\gamma = \alpha^{-1}_{j(n)} \cdot \ldots \cdot \alpha^{-1}_{j(1)}(\infty) \in W^-_{\alpha_{j(n)}}$ and $|w_\gamma(z)| < q^{n-1}$ for z in $\mathbb{P}^1_K - W^-_{\alpha_{j(n)}}$ due to (b). Thus, we see that $W^-_\gamma \subset W^-_{\alpha_{j(n)}}$ and, hence, $W^+_\gamma \subset W^+_{\alpha_{j(n)}}$. The latter implies the assertion on F and F^-.

(d.1) The implication "$\rightarrow$" follows from (b) and "$\leftarrow$" follows from (d.2).

(d.2) For $\alpha \in \{\alpha_1, \ldots, \alpha_{2g}\}$ we have that $\alpha(\mathbb{P}^1_K - W^\pm_\alpha) = W^\mp_{\alpha^{-1}}$ by the chain rule.

(e) If Γ were not free, then there would be a relation $\mathrm{id} = \alpha_{j(1)} \cdot \ldots \cdot \alpha_{j(n)}$ with $n \geq 1$. So, one obtains $z \in W^-_{\alpha^{-1}_{j(1)}}$ for all $z \in F$ due to (b). This is impossible. $\square$

Using Proposition 2.2.12 one can construct examples of Schottky groups.

Example 2.2.13. Let $g \geq 1$ be an integer. Consider g pairs (B_i^-, B_{g+i}^-) of open discs and let (B_i^+, B_{g+i}^+) be the associated closed affinoid discs in $\mathbb{P}_K^1$ for $i = 1, \ldots, g$. Assume that the $2g$ closed discs are pairwise disjoint and that ∞ does not belong to any of the closed discs.

Let $\alpha_1, \ldots, \alpha_g$ be Möbius transformations in $\mathrm{PGL}(2, K)$ such that $\alpha_i(\mathbb{P}_K^1 - B_i^+) = B_{g+i}^-$ for $i = 1, \ldots, g$. Then $\Gamma := \langle \alpha_1, \ldots, \alpha_g \rangle$ is a Schottky group and $(\alpha_1, \ldots, \alpha_g)$ is a free system of generators.

A fundamental domain for Γ is given by $F := \mathbb{P}_K^1 - \bigcup_{i=1}^{2g} B_i^-$ and the set of its associated ordinary points is given by $\Omega_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma(F)$.

Such transformations $\alpha_i \in \mathrm{PGL}(2, K)$ with $\alpha_i(\mathbb{P}_K^1 - B_i^+) = B_{g+i}^-$ exist for $i = 1, \ldots, g$ if the centers and the radii of $B_1, \ldots, B_{2g}$ are K-rational.

Proof. Using the geometric configuration, one verifies that all the statements of Proposition 2.2.12 hold. First one shows that Γ is free and that $(\alpha_1, \ldots, \alpha_g)$ is a free system of generators. In fact, put $\alpha_{g+i} := \alpha_i^{-1}$ for $i = 1, \ldots, g$, as before. If $\gamma = \alpha_{j(1)} \ldots \alpha_{j(n)} \in \Gamma$ is a reduced representation with $j(i) \in \{1, \ldots, 2g\}$, then $\gamma(\infty) \in B^-_{\alpha_{j(1)}^{-1}}$. Therefore $\gamma \neq \mathrm{id}$. Then one can construct a group homomorphism $\rho : \Gamma \to \overline{K}^\times$ by choosing the images of ρ on the generating system $(\alpha_1, \ldots, \alpha_g)$ arbitrarily. So one can define ρ such that $B_i^\pm = W_{\alpha_i}^\pm$ for $i = 1, \ldots, 2g$. As in Proposition 2.2.12 all assertions on F resp. Ω_Γ follow.

It remains to show that Γ acts discontinuously. We only have to explain that the closure of the orbit Γw is compact for every $w \in \mathbb{P}_K^1(K)$. Since $\infty \notin L_\Gamma$, it suffices to show that every sequence $(\gamma_i(w); i \in \mathbb{N})$ with pairwise distinct $\gamma_i \in \Gamma$ admits a convergent subsequence. Put

$$F(n) := \bigcup_{\ell(\gamma) \leq n} \gamma(F) = \mathbb{P}_K^1 - (B_{n,1} \cup \cdots \cup B_{n,r(n)}),$$

where $B_{n,j}$ are open discs contained in a large disc $B_0 := \{z \in \mathbb{A}_K^1; |z| \leq c\}$ because of $\infty \in F$. The symbol $\ell(\gamma)$ indicates the number of elements used in a reduced representation of γ as a product of the α_i. Note that there are only finitely many $\gamma \in \Gamma$ with $\ell(\gamma) \leq n$. Since F is a fundamental domain for Γ, almost all $(\gamma_i(w); i \in \mathbb{N})$ are contained in $\mathbb{P}_K^1 - F_n$. So almost all $(\gamma_i(w); i \in \mathbb{N})$ are contained in $(B_{n,1} \cup \cdots \cup B_{n,r(n)})$. Then there exists a sequence $(B_{n,k(n)}; n \in \mathbb{N})$ such that infinitely many of the elements $(\gamma_i(w); i \in \mathbb{N})$ are contained in $B_{n,k(n)}$. Moreover, we can arrange the sequence in such a way that

$$B_{n+1,k(n+1)} \subset B_{n,k(n)} \quad \text{for all } n \in \mathbb{N}.$$

The radii $\rho(n)$ of $B_{n,k(n)}$ tend to 0 for $n \to \infty$. In fact, the set of the heights (cf. Definition 1.3.3) of the annuli $B_{n,k(n)} - B_{n+1,k(n+1)}$ is finite, because they are related under certain elements of Γ. Now we can choose elements $i(n) \in \mathbb{N}$ such that $\gamma_{i(n)}(w) \in B_{n,k(n)}$ for all $n \in \mathbb{N}$. Since the radii $\rho(n)$ tend to 0, the sequence $(\gamma_{i(n)}(w); n \in \mathbb{N})$ is a Cauchy sequence. Since K is complete, the sequence $(\gamma_{i(n)}(w); n \in \mathbb{N})$ converges. □

Now we come to the main theorem of this section. The key result here is that every Schottky group can be obtained by the method of Example 2.2.13.

Theorem 2.2.14. *Let Γ be a Schottky group with $\infty \notin L_\Gamma$. Let $R(\Gamma)$ be the subset of Γ indexing the maximal discs V_γ^-; cf. Lemma 2.2.9.*

Then for every $q \in \mathbb{R}$ with $q < 1$ there exists a group homomorphism $\rho : \Gamma \to \overline{K}^\times$ and a separating system of generators $\alpha_1, \ldots, \alpha_g$ of Γ with respect to ρ with the following properties:

(i) $\alpha_i \in R := R(\Gamma)$ *for* $i = 1, \ldots, g$.
(ii) $q < \sqrt{|\rho(\alpha_i)|} < 1$ *for* $i = 1, \ldots, g$.

In particular, $(\alpha_1, \ldots, \alpha_g)$ is a free system of generators of Γ.

Proof. We proceed by induction on the number of elements of $R := R(\Gamma)$.

If R consists of ≤ 2 elements, then $\Gamma = \langle \gamma \rangle$ is the free group generated by one element; cf. Proposition 2.2.10.

The induction step will be done in several steps depending on the geometry of the fundamental domain given by the discs $(V_\alpha^-; \alpha \in R)$ defined in Lemma 2.2.9. Now we define

$$\begin{aligned}
t &:= \min\{r_\alpha; \alpha \in R\},\\
R' &:= \{\alpha \in R; r_\alpha > t\},\\
\Gamma' &:= \langle R' \rangle,\\
q' &:= \max\left\{q, \frac{r_\alpha}{|m_\alpha - m_\beta|}; \alpha, \beta \in R, |m_\alpha - m_\beta| > r_\alpha\right\}.
\end{aligned}$$

By the induction hypothesis we may assume that there exist a group homomorphism $\rho' : \Gamma' \to \overline{K}^\times$ and a separating system $\alpha_1, \ldots, \alpha_g$ for the subgroup $\Gamma' := \langle R' \rangle \subset \Gamma$ with respect to ρ', where $q' < \sqrt{|\rho'(\alpha_i)|} < 1$ for $i = 1, \ldots, g$. Here q' is chosen in such a way that for the enlarged discs $W^+_{\alpha_i^{-1}}$ we have $V_\gamma^- \cap W^+_{\alpha_i^{-1}} = \emptyset$ if $|m_\gamma - m_{\alpha_i^{-1}}| > r_{\alpha_i^{-1}} = r_{\alpha_i}$ and $\gamma \in R - R'$ Due to Proposition 2.2.12(c), the group Γ' has the fundamental domain

$$F' := \mathbb{P}^1_K - \bigcup_{i=1}^{2g} W^-_{\alpha_i} = \mathbb{P}^1_K - \bigcup_{\gamma \in \Gamma' - \{\mathrm{id}\}} W^-_\gamma .$$

Now we have to consider the elements $\gamma \in R$ with $V_\gamma^- \subset F'$. This set is given by

$$T' := \left\{\alpha \in R; r_\alpha = t, m_\alpha \in F'\right\} \subset R - R'.$$

Moreover, we know that $R(\Gamma') = R'$. In fact, the inclusion $R' \subset R(\Gamma')$ is trivial because of $R' \subset R(\Gamma)$. Conversely, by Lemma 2.2.8 for every $\alpha \in R(\Gamma')$ there exists an element $\beta \in R'$ with $V_\alpha^- \subset V_\beta^-$, because R' generates Γ'. Since V_α^- is maximal and $\beta \in \Gamma'$, we obtain that $V_\alpha^- = V_\beta^-$ and hence $\alpha \in R'$ as R' is a subset of $R(\Gamma)$. In particular, we see that $\alpha_1, \ldots, \alpha_g \in R'$.

Due to the induction principle, we may assume that $\langle \Gamma', T'\rangle = \Gamma$. The problem now is how to vary the radii of the discs V_α^- for $\alpha \in R - R'$ via a suitable extension $\rho : \Gamma \to \overline{K}^\times$ of $\rho' : \Gamma' \to \overline{K}^\times$ in order to obtain a separating system for Γ. For this we have to analyze the configuration of the discs $(V_\alpha^-; \alpha \in T')$.

Let us first deduce some properties of T':

(1) If $\alpha \in T'$ and $\beta \in \Gamma$ with $r_\beta = t$ and $|m_\alpha - m_\beta| = t$, then $\beta \in T'$.

Since V_α^- is maximal and $V_\beta^+ = V_\alpha^+ \subset F'$, the disc V_β^- is maximal and $m_\beta \in F'$.

(2) For every $\gamma \in R - R'$ with $m_\gamma \notin F'$ there exists $\alpha \in \{\alpha_1, \ldots, \alpha_g\}$ such that $|m_\gamma - m_{\alpha^{-1}}| = r_\alpha$.

Since $m_\gamma \notin F'$, it follows that $m_\gamma \in W_\alpha^-$ for some $\alpha \in \{\alpha_1, \ldots, \alpha_{2g}\}$; i.e.

$$|m_\gamma - m_\alpha| < \sqrt{\left|\rho'(\alpha)\right|} \cdot r_\alpha.$$

Since V_γ^- is maximal, we have that $|m_\gamma - m_\alpha| \geq r_\alpha$, and so $\sqrt{|\rho'(\alpha)|} > 1$. Thus, it follows that $\alpha^{-1} \in \{\alpha_1, \ldots, \alpha_g\}$. If $|m_\gamma - m_\alpha| > r_\alpha$ then, due to the choice of q', we have that

$$r_\alpha \leq q' \cdot |m_\gamma - m_\alpha| < q' \cdot \sqrt{\left|\rho'(\alpha)\right|} \cdot r_\alpha < r_\alpha,$$

since $q'\sqrt{|\rho'(\alpha)|} < 1$. This is impossible, and hence we see $|m_\gamma - m_\alpha| = r_\alpha$.

(3) For $\gamma \in R - R'$ there exists a unique $\beta \in R$ with $|m_\gamma - m_\beta| < t$, $r_\beta = t$ and $\beta^{-1} \in T'$. In particular, $\beta\gamma^{-1} \in \Gamma'$. If $\gamma \in T'$, then $\beta \in T'$.

For every $\gamma \in R - R'$ we have $r_\gamma = t$. If $\gamma^{-1} \in T'$, then we choose $\beta := \gamma$. Otherwise, we have that $m_{\gamma^{-1}} \notin F'$ because of the very definition of T'. So due to (2) there exists $\alpha \in \{\alpha_1, \ldots, \alpha_g\}$ such that $|m_{\gamma^{-1}} - m_{\alpha^{-1}}| = r_\alpha$. Then set $\gamma_1 := \alpha^{-1}\gamma$. In particular, by Lemma 2.2.6(c) we have

$$r_\alpha = |m_{\gamma^{-1}} - m_{\alpha^{-1}}| = \frac{r_\alpha \cdot r_\gamma}{r_{\alpha^{-1}\gamma}} = \frac{r_\alpha \cdot r_\gamma}{r_{\gamma_1}},$$

and hence $r_{\gamma_1} = r_\gamma = t$. So we obtain that $\gamma_1 \in R - R'$. Moreover, it follows that

$$|m_\gamma - m_{\gamma_1}| = \frac{r_\gamma \cdot r_{\gamma_1}}{r_{\gamma\gamma_1^{-1}}} = \frac{t^2}{r_\alpha} < t.$$

If $m_{\gamma_1^{-1}} \notin F'$, then repeat the procedure with γ_1 instead of γ. Thus, we can construct a sequence $\gamma_n = \alpha_{j(n)}^{-1} \cdot \ldots \cdot \alpha_{j(1)}^{-1} \cdot \gamma$ with $j(\nu) \in \{1, \ldots, g\}$ and $r_{\gamma_n} = t$ such that

$$|m_{\gamma_{n-1}} - m_{\gamma_n}| = \frac{r_{\gamma_n} \cdot r_{\gamma_{n-1}}}{r_{\alpha_{j(n)}}} < t \quad \text{and also} \quad |m_\gamma - m_{\gamma_n}| < t,$$

because of the ultrametric inequality. In particular, we have $\gamma_n \in R$. Since R has only finitely many elements, the procedure stops after finitely many steps. Indeed, if not, then we would arrive at a non-trivial relation $\mathrm{id} = \alpha_{j(n)} \cdot \ldots \cdot \alpha_{j(m+1)}$, which is impossible as $(\alpha_1, \ldots, \alpha_g)$ is a free system of generators. Thus, we finally arrive at the situation that $\beta = \gamma_n \in R$ with $m_{\beta^{-1}} \in F'$; i.e., $\beta^{-1} \in T'$ and $\beta\gamma^{-1} = \alpha_{j(1)}^{-1} \ldots \alpha_{j(n)}^{-1} \in \Gamma'$.

If $\gamma \in T'$, then $m_\gamma \in F'$. Since $|m_\gamma - m_\beta| < t$, we have $m_\beta \in F'$, and then $\beta \in T'$ because of $r_\beta = t$.

Next we turn to the proof of the uniqueness. Assume that there are α, β in R with the asserted properties. Now put $\delta := \alpha\beta^{-1} \in \Gamma'$ and assume $\delta \neq \mathrm{id}$. Then we have that

$$\delta(m_{\beta^{-1}}) = \alpha\beta^{-1}\big(\beta(\infty)\big) = \alpha(\infty) = m_{\alpha^{-1}};$$

cf. Notation 2.2.5. Because $\alpha^{-1}, \beta^{-1} \in T'$, we have $m_{\alpha^{-1}}, m_{\beta^{-1}} \in F'$ and hence $F' \cap \delta(F') \neq \emptyset$. Thus, we see that $\delta \in \{\alpha_1, \ldots, \alpha_{2g}\}$ due to Proposition 2.2.12(d.1), and hence $|\rho'(\delta)| \neq 1$. Moreover, we have that

$$|m_{\delta^{-1}} - m_{\alpha^{-1}}| = \frac{r_\delta \cdot r_\alpha}{r_{\delta^{-1}\alpha}} = \frac{r_\delta \cdot r_\alpha}{r_\beta} = r_\delta,$$
$$|m_\delta - m_{\beta^{-1}}| = \frac{r_\delta \cdot r_\beta}{r_{\delta\beta}} = \frac{r_\delta \cdot r_\beta}{r_\alpha} = r_\delta,$$

because $r_\alpha = t = r_\beta$, and hence $m_{\beta^{-1}} \in V_\delta^+$ and $m_{\alpha^{-1}} \in V_{\delta^{-1}}^+$. Since $|\rho'(\delta)| \neq 1$, we have that $V_{\delta^{-1}}^+ \subset W_{\delta^{-1}}^-$ or $V_\delta^+ \subset W_\delta^-$ and so $m_{\alpha^{-1}} \in W_{\delta^{-1}}^-$ or $m_{\beta^{-1}} \in W_\delta^-$. This contradicts the fact that $m_{\beta^{-1}} \in F'$ and $m_{\alpha^{-1}} \in F'$ because of $\alpha^{-1}, \beta^{-1} \in T'$.

(4) $\Gamma = \langle \Gamma', T\rangle$ where $T := \{\alpha \in T'; \alpha^{-1} \in T'\}$.

It suffices to show that $\gamma \in \langle \Gamma', T\rangle$ for all $\gamma \in R - R'$. From (3) it follows that $\gamma \in \langle \Gamma', T'\rangle$, and also $T' \subset \langle \Gamma', T\rangle$. In fact, for $\alpha \in T'$ there exists $\beta \in R$ with $\beta\alpha^{-1} = \gamma \in \Gamma'$ and $\beta^{-1} \in T'$ by (3). Since $\alpha \in T'$, we also have that $\beta \in T'$ by (3). Thus, β and β^{-1} belong to T', and hence $\beta \in T$. Thus, we obtain that $\alpha = \beta\gamma^{-1}$ lies in $\langle \Gamma', T\rangle$.

For $\alpha \in T$ put

$$T_\alpha := \big\{\beta \in T; |m_\alpha - m_\beta| \leq t\big\}.$$

Because of the uniqueness in (3), we even have

$$T_\alpha := \big\{\beta \in T; |m_\alpha - m_\beta| = t\big\} \cup \{\alpha\}.$$

(5) If $\alpha \in T$ and $\beta \in T_\alpha$, then $T_\alpha = T_\beta$.

This follows from the ultrametric inequality.

(6) If $\alpha \in T$, then $T_{\alpha^{-1}} = \{\beta\alpha^{-1}; \beta \in T_\alpha, \beta \neq \alpha\} \cup \{\alpha^{-1}\}$.

Indeed, if $\beta \in T_\alpha$ and $\alpha \neq \beta$, then

$$t = |m_\alpha - m_\beta| = \frac{r_\alpha \cdot r_\beta}{r_{\beta\alpha^{-1}}} = \frac{t \cdot t}{r_{\beta\alpha^{-1}}},$$

and hence $r_{\beta\alpha^{-1}} = t$. Thus, by the same formula $|m_{\beta\alpha^{-1}} - m_{\alpha^{-1}}| = t$ and so $\beta\alpha^{-1} \in T'$ due to (1) because of $\alpha^{-1} \in T'$. Similarly, one shows that $\alpha\beta^{-1} \in T'$, because one can use the fact that $T_\beta = T_\alpha$; cf. (5). Thus, we see $\beta\alpha^{-1} \in T$. The computation showed $\beta\alpha^{-1} \in T_{\alpha^{-1}}$. Conversely, consider an element $\beta \in T_{\alpha^{-1}}$ with $\beta \neq \alpha^{-1}$. From what we have proved above, it follows that $\beta(\alpha^{-1})^{-1} \in T_\alpha$. Obviously we have that $\beta(\alpha^{-1})^{-1} \neq \alpha$ because of $\beta \neq \mathrm{id}$, and hence the element $\beta = (\beta(\alpha^{-1})^{-1})\alpha^{-1}$ belongs to the right-hand side.

(7) If $\alpha \in T$ and $\beta \in T_\alpha$ with $\alpha \neq \beta$, then $T_{\alpha^{-1}} \cap T_{\beta^{-1}} = \emptyset$.

Indeed, if $T_{\alpha^{-1}} \cap T_{\beta^{-1}} \neq \emptyset$, then $T_{\alpha^{-1}} = T_{\beta^{-1}}$ by (5), and hence $T_\alpha = T_\beta$ as follows from Lemma 2.2.6(c). From (6) we obtain that $\beta^{-1} = \delta\alpha^{-1}$ for some $\delta \in T_\alpha$. Similarly one has that $\alpha^{-1} = \delta^{-1}\beta^{-1} \in T_{\beta^{-1}}$ with $\delta^{-1} \in T_\beta$ by (6). Since $T_\alpha = T_\beta$, we see $\delta, \delta^{-1} \in T_\alpha$. But this is impossible due to Lemma 2.2.6(d), since $|m_\delta - m_{\delta^{-1}}| > r_\delta = t$.

For $\alpha \in T$ put

$$S_\alpha := T_\alpha \cup \bigcup_{\beta \in T_\alpha} T_{\beta^{-1}} \quad \text{and} \quad \Gamma_\alpha := \langle S_\alpha \rangle.$$

The union over $\beta \in T_\alpha$ is disjoint due to (7), because by (5) we know that $T_\alpha = T_\beta = T_\gamma$ for $\beta, \gamma \in T_\alpha$. Furthermore, $T_\alpha \cap T_{\beta^{-1}} = T_\beta \cap T_{\beta^{-1}} = \emptyset$ due to Lemma 2.2.6(c). Then (6) implies that

$$\text{(8)} \quad \begin{aligned} S_\alpha &= T_\alpha \cup T_{\alpha^{-1}} \cup \textstyle\bigcup_{\beta \in T_\alpha} \{\delta\beta^{-1}; \delta \in T_\alpha, \delta \neq \beta\} \\ &= T_\alpha \cup T_{\alpha^{-1}} \cup \{\delta\varepsilon^{-1}; \delta, \varepsilon \in T_\alpha, \delta \neq \varepsilon\}. \end{aligned}$$

In particular, S_α is invariant under the inverse map $\gamma \mapsto \gamma^{-1}$. In a special case the arrangement of the discs is shown in Fig. 2.1.

If $\gamma \in T_\alpha$, then $T_\gamma = T_\alpha$ by (5) and hence $S_\gamma = S_\alpha$.

If $\gamma \in T_{\beta^{-1}}$ for some $\beta \in T_\alpha$, then $T_\alpha = T_\beta$ and $T_\gamma = T_{\beta^{-1}}$ by (5). Thus, we see $T_\gamma = T_{\alpha^{-1}}$. Therefore, we may assume $\gamma = \alpha^{-1}$. Since S_α is invariant under the inverse map, we obtain $S_\alpha = S_{\alpha^{-1}}$. Thus, (6) implies:

(9) We have that $S_\alpha = S_\gamma$ for every $\gamma \in S_\alpha$. Therefore,

$$T = S_{\tau_1} \dot{\cup} \cdots \dot{\cup} S_{\tau_r}$$

is a disjoint union of sets S_{τ_i} for suitable $\tau_1, \ldots, \tau_r \in T$.

(10) T_α is a free system of generators of Γ_α and is separating with respect to a suitable group homomorphism $\rho_\alpha : \Gamma_\alpha \to \overline{K}^\times$ which can be chosen arbitrarily close to 1.

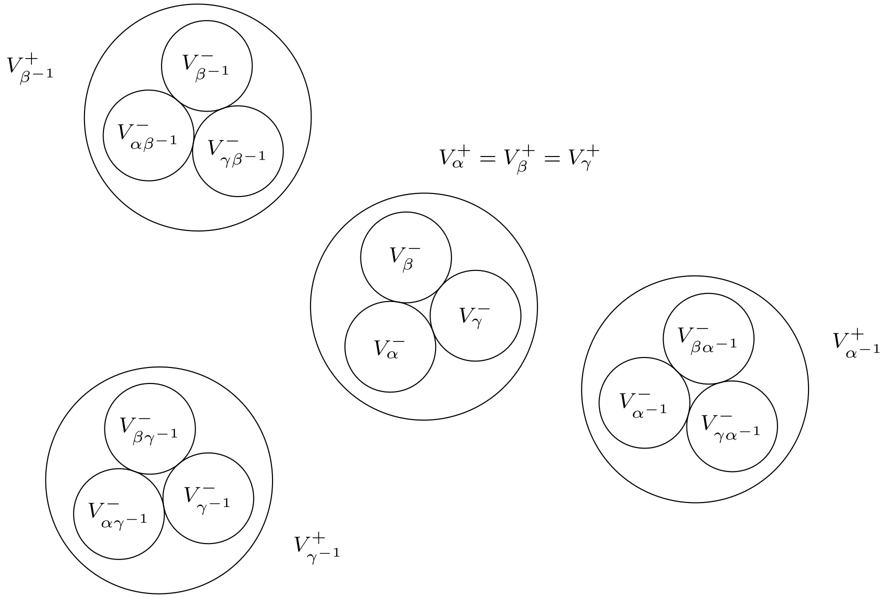

Fig. 2.1 Position of the discs S_α for $T_\alpha = \{\alpha, \beta, \gamma\}$

Indeed, from (8) it follows $\langle T_\alpha \rangle = \Gamma_\alpha$ and from (7) that $V^+_{\beta_1^{-1}} \cap V^+_{\beta_2^{-1}} = \emptyset$ for $\beta_1, \beta_2 \in T_\alpha$ with $\beta_1 \neq \beta_2$. For $q_\alpha \in |\overline{K}^\times|$ with $q_\alpha < 1$ put

$$\begin{aligned} B^+_\beta &:= \{z \in \mathbb{P}^1_K;\, |m_\beta - z| \leq q_\alpha r_\beta\}, \\ B^+_{\beta^{-1}} &:= \{z \in \mathbb{P}^1_K;\, |m_{\beta^{-1}} - z| \leq r_\beta / q_\alpha\}, \end{aligned}$$

and define B^-_β and $B^-_{\beta^{-1}}$ similarly. In this way we shrink $V^\pm_\beta$ and enlarge $V^\pm_{\beta^{-1}}$ for $\beta \in T_\alpha$. If we choose q_α close to 1, then the discs B^+_β for $\beta \in T_\alpha \cup T_\alpha^{-1}$ are pairwise disjoint. Furthermore, $\beta(\mathbb{P}^1_K - B^\pm_\beta) = B^\mp_{\beta^{-1}}$ for $\beta \in T_\alpha$. Thus, we arrive at the situation of Example 2.2.13, and hence T_α is a free system of generators of Γ_α.

Then we can define $\rho_\alpha : \Gamma_\alpha \to \overline{K}^\times$ by choosing values $\rho_\alpha(\beta)$ for $\beta \in T_\alpha$ with $q' < \sqrt{|\rho_\alpha(\beta)|} < 1$ and extending the map by linearity. This means that we shrink the discs V^-_β for $\beta \in T_\alpha$. Due to (7) we can do this in such a way that the intersection $W^+_{\beta_1^{-1}} \cap W^+_{\beta_2^{-1}} = \emptyset$ is empty for $\beta_1, \beta_2 \in T_\alpha$ with $\beta_1 \neq \beta_2$.

(11) $R(\Gamma_\alpha) = S_\alpha$.

By definition we have that $S_\alpha \subset R(\Gamma_\alpha)$. Any $\gamma \in R(\Gamma_\alpha)$ has a unique representation as a finite product by elements of $T_\alpha \cup T_\alpha^{-1}$. Thus, for every $\gamma \in R(\Gamma_\alpha)$ we have by Proposition 2.2.12(c) that $W^-_\gamma \subset W^-_\beta$ for some $\beta \in T_\alpha \cup T_\alpha^{-1}$. Since we can choose ρ_α such that $|\rho_\alpha|$ is arbitrarily close to 1, we obtain that $V^+_\gamma \subset V^+_\beta$ for some β in

$T_\alpha \cup T_\alpha^{-1}$. Thus, we see that $|m_\gamma - m_\beta| \le t$ and $r_\gamma = t$, because t is the maximal possible radius. If $|m_\gamma - m_\beta| < t$, then $\gamma = \beta$ due to (3). If $|m_\gamma - m_\beta| = t$, then $\gamma \in T_\beta \subset S_\alpha$, and hence the assertion follows.

In (4) we defined the set T and showed that $\Gamma = \langle \Gamma', T \rangle$. In (9) we saw that T decomposes into a disjoint union

$$T = S_{\tau_1} \dot\cup \cdots \dot\cup S_{\tau_r}.$$

Put $T_i := T_{\tau_i}$ and $\Gamma_i := \Gamma_{\tau_i} := \langle S_{\tau_i} \rangle = \langle T_{\tau_i} \rangle$ for $i = 1, \ldots, r$. In (10) we saw that Γ_i is a free group and, moreover, we constructed a separating group homomorphism $\rho_i : \Gamma_i \to \overline{K}^\times$. From the induction hypothesis we know that the subgroup $\Gamma_0 := \Gamma'$ is free and that there exists a separating group homomorphism $\rho_0 := \rho'$. Then let $T_0 := \{\alpha_1, \ldots, \alpha_g\}$ be the separating basis of Γ_0. Then set

$$\mathcal{T} = T_0 \cup T_1 \cup \cdots \cup T_r.$$

Due to our construction we have $T_i \cap T_j = \emptyset$ for $i \ne j$, and hence $V_\alpha^+ \cap V_\beta^+ = \emptyset$ for $\alpha^{\pm 1} \in T_i$ and $\beta^{\pm 1} \in T_j$ for $i \ne j$. With respect to $\rho_0, \ldots, \rho_r$ we define the subdomains $W_{\alpha^{\pm 1}}^{\pm}$ for $\alpha \in \mathcal{T}$ as in Notation 2.2.11. We can choose the ρ_i in such a way that $W_\alpha^+ \cap W_\beta^+ = \emptyset$ for all $\alpha, \beta \in \mathcal{T} \cup \mathcal{T}^{-1}$ with $\alpha \ne \beta$. Then we claim

(12) $\Gamma = \Gamma_0 \amalg \cdots \amalg \Gamma_r$

is a coproduct in the category of groups. Let $\rho : \Gamma \to \overline{K}^\times$ be the group homomorphism induced by $\rho_0, \ldots, \rho_r$ given on the factors.

In fact, due to (4) the group Γ is generated by Γ_0 and T. Since T is contained in $\langle \Gamma_1, \ldots, \Gamma_r \rangle$, we have that $\Gamma \subset \langle \Gamma', \Gamma_1, \ldots, \Gamma_r \rangle$, and hence that Γ coincides with $\langle \Gamma', \Gamma_1, \ldots, \Gamma_r \rangle$.

The direct decomposition $\coprod_{i=0}^r \Gamma_i$ of Γ follows as in (10). Indeed, if there is a reduced product $\gamma := \delta_{j(1)}^{\pm 1} \cdot \ldots \cdot \delta_{j(n)}^{\pm 1}$ with $\delta_{j(\nu)} \in \mathcal{T}$, then $\gamma(z) \in W_{\delta_{j(1)}^{-1}}^+$ or $\gamma(z) \in W_{\delta_{j(1)}}^+$ for every point z in $F := \mathbb{P}_K^1 - \bigcup_{\delta^{\pm 1} \in \mathcal{T}} W_\delta^-$. Thus, we see $\gamma \ne \mathrm{id}$. So, the coproduct is direct and ρ is a separating morphism.

Then $(\mathcal{T}, \rho)$ satisfies the assertion of the theorem. Indeed, each factor in (12) is a free group due to the induction hypothesis and because of (10).

This finishes the proof of Theorem 2.2.14. $\square$

Remark 2.2.15. The radii r_{α_i} of the discs V_{α_i} belong to $\sqrt[2]{|K^\times|}$.

If the valuation of K is not discrete, the homomorphism ρ can be chosen in such a way that $\rho \in \mathrm{Hom}(\Gamma, \overline{K}^\times)$ and the radii $\sqrt[2]{|\rho(\alpha_i)|} \cdot r_{\alpha_i}$ of the discs W_{α_i} belong to $|K^\times|$.

If the valuation is discrete and π is a uniformizer of the valuation, then

$$\max\left\{ \frac{r_\alpha}{|m_\alpha - m_\beta|}; \alpha, \beta \in R(\Gamma), r_\alpha < |m_\alpha - m_\beta| \right\} \le \sqrt[2]{|\pi|}.$$

If one puts $q = \sqrt[2]{|\pi|}$, then there exists a separating homomorphism ρ in $\mathrm{Hom}(\Gamma, \overline{K}^{\times})$ with $|\pi| < |\rho(\alpha_i)| < 1$ for $i = 1, \ldots, g$. Without loss of generality one can choose $|\rho(\alpha_i)| = \sqrt[2]{|\pi|}$ for $i = 1, \ldots, g$. Then the radii of the discs W_{α_i} belong to $\sqrt[2]{|K^{\times}|}$. This is the best possibility.

Corollary 2.2.16 (Ihara). *Schottky groups are free.*

This result was shown by Ihara in the case of a discrete valuation; cf. [48]. See also the book of Serre [90, II, §1.5].

For later use we add a result on the geometry of the fundamental domain.

Corollary 2.2.17. *Let $\Gamma \subset \mathrm{PGL}(2, K)$ be a non-trivial Schottky group with $\infty \notin L_\Gamma$. Then there exists a separating system of generators $(\alpha_1, \ldots, \alpha_g)$ with respect to a suitable homomorphism $\rho : \Gamma \to \overline{K}^{\times}$ which can be chosen in such a way that $|\rho(\alpha_i)| < 1$ is arbitrarily close to 1 for $i = 1, \ldots, g$. Put $\alpha_{g+i} = \alpha_i^{-1}$ for $i = 1, \ldots, g$, and put*

$$E^\circ := \mathbb{P}^1_K - \left[V^-_{\alpha_1} \cup \cdots \cup V^-_{\alpha_g} \cup V^+_{\alpha_{g+1}} \cup \cdots \cup V^+_{\alpha_{2g}}\right].$$

Then E° is a complete system of representative of $\Gamma\backslash\Omega_\Gamma$.

Let $z_1, z_2 \in \Omega_\Gamma$. Assume that $z_1 \in E^\circ$ and $\beta z_2 \in E^\circ$ for some $\beta \in \Gamma$. Then, for $\gamma \in \Gamma$, the following is true:

If $\beta = \mathrm{id}$, then $|\gamma z_1 - \gamma z_2| \le r_\gamma$.

If $\beta \neq \mathrm{id}$, then $|\gamma z_1 - \gamma z_2| \le \max\{\frac{r_\gamma \cdot r_{\gamma\beta^{-1}}}{r_\beta}, r_\gamma, r_{\beta\gamma^{-1}}\}$.

If $\ell(\beta)$ is bounded, then the distance $|\gamma z_1 - \gamma z_2| \to 0$ converges uniformly to 0 if $\ell(\gamma) \to \infty$ tends to ∞.

If $V \subset \Omega_\Gamma$ is an affinoid subdomain, there exists an $N \in \mathbb{N}$ such that the intersection $V \cap \gamma(V) = \emptyset$ is empty for all $\gamma \in \Gamma$ with $\ell(\gamma) \ge N$.

Proof. The first assertion follows from Theorem 2.2.14. If we approach $\rho(\alpha_i)$ to 1 from below for $i = 1, \ldots, g$, then we see that the discs $V^+_{\alpha_1}, \ldots, V^+_{\alpha_{2g}}$ are the maximal closed discs in the family $(V^+_\gamma; \gamma \in \Gamma)$. Due to Lemma 2.2.6(b) we know that $\gamma z_1 \in V^+_{\gamma^{-1}}$ and $\gamma z_2 = \gamma\beta^{-1}(\beta z_2) \in V^+_{\beta\gamma^{-1}}$.

If $\beta = \mathrm{id}$, then $\gamma z_1, \gamma z_2 \in V^+_{\gamma^{-1}}$, and hence, $|\gamma z_1 - \gamma z_2| \le r_{\gamma^{-1}} = r_\gamma$.

If $\beta \neq \mathrm{id}$, then $|\gamma z_1 - m_{\gamma^{-1}}| \le r_\gamma$ and $|\gamma z_2 - m_{\beta\gamma^{-1}}| \le r_{\beta\gamma^{-1}}$; cf. Notation 2.2.5. Then it follows from Lemma 2.2.6(c) that

$$|m_{\gamma^{-1}} - m_{\beta\gamma^{-1}}| = \frac{r_{\gamma^{-1}} \cdot r_{\beta\gamma^{-1}}}{r_\beta}.$$

Then the asserted estimate follows by the ultrametric inequality.

The bound tends to 0 if $\ell(\gamma)$ tends to ∞ due to Proposition 2.2.4(d).

Let $E \subset \Omega_\Gamma$ be a fundamental domain. If $V \subset \Omega$ is affinoid, then there exists finitely many $\beta_1, \ldots, \beta_r$ with $V \subset \beta_1(E) \cup \cdots \cup \beta_r(E)$. Therefore, it suffices to show that $E \cap \gamma(E) \neq \emptyset$ for only finitely many $\gamma \in \Gamma$. This is true due to the very definition of a fundamental domain. Since Γ is a free group, there exists an $N \in \mathbb{N}$ such that, for a finite subset $\Gamma_0 \subset \Gamma$, every $\gamma \in \Gamma$ with $\ell(\gamma) \geq N$ does not belong to Γ_0. $\square$

2.3 Definition and Properties

For the following we keep the notations and hypotheses of Notations 2.2.5 and 2.2.11. Let $\Gamma \subset \mathrm{PGL}(2, K)$ be a non-trivial Schottky group and assume $\infty \notin L_\Gamma$.

Due to Theorem 2.2.14 the group Γ is free with g generators $\alpha_1, \ldots, \alpha_g$; we set $\alpha_{g+i} := \alpha_i^{-1}$ for $i = 1, \ldots, g$. Let

$$F := \mathbb{P}_K^1 - \bigcup_{i=1}^{2g} W_{\alpha_i}^- = \left\{ z \in \mathbb{P}_K^1; \left| w_{\alpha_i}(z) \right| \leq 1 \text{ for } i = 1, \ldots, 2g \right\}$$

be the fundamental domain as constructed in Proposition 2.2.12 by Theorem 2.2.14. Then

$$\Omega_\Gamma := \bigcup_{\gamma \in \Gamma} \gamma(F) \subset \mathbb{P}_K^1$$

is the maximal open set where Γ acts discontinuously. In the following we will equip the orbit space $X_\Gamma := \Gamma \backslash \Omega_\Gamma$ with a rigid analytic structure in a canonical way such that the residue map $p : \Omega_\Gamma \to X_\Gamma$ is a rigid analytic morphism. Moreover, we will see that X_Γ is a smooth proper rigid analytic space of dimension 1 and that p is an unramified covering in the topological sense; i.e., there is an admissible covering $\{V_i; i \in I\}$ of X_Γ such that each $p^{-1}(V_i) = \bigcup_{j \in I_i} U_{i,j}$ is a disjoint union and the restriction $p|_{U_{i,j}} : U_{i,j} \xrightarrow{\sim} V_i$ of p to each $U_{i,j}$ is an isomorphism.

Theorem 2.3.1. *In the above situation we have*:

(a) *There is a unique structure of a rigid-analytic variety on X_Γ such that the residue map $p : \Omega_\Gamma \to X_\Gamma$ is an unramified covering in the topological sense.*
(b) *X_Γ is a smooth proper rigid analytic curve of genus g.*
(c) *X_Γ is the analytification of a smooth projective algebraic curve X_Γ^{alg}.*
(d) *Every meromorphic function on X_Γ is a rational function on X_Γ^{alg}.*

The curves X_Γ as defined above are called Mumford curves.

Proof. (a) Any $\gamma \in \Gamma$ has a unique reduced representation

$$\gamma = \alpha_{j(1)} \cdot \ldots \cdot \alpha_{j(n)} \quad \text{with } j(i) \in \{\alpha_1, \ldots, \alpha_{2g}\}.$$

Let $\ell(\gamma) := n$ be the length of γ. Then

$$\Omega_\Gamma(n) := \bigcup_{\gamma\in\Gamma,\ell(\gamma)\le n} \gamma(F)$$

is the complement of finitely many open discs in $\mathbb{P}^1_K$ and hence carries a unique structure of a smooth affinoid domain of dimension 1. The inclusion map $\Omega_\Gamma(m) \hookrightarrow \Omega_\Gamma(n)$ for $m \le n$ is an open immersion of affinoid domains. Thus, $\Omega_\Gamma = \bigcup_{n\in\mathbb{N}} \Omega_\Gamma(n)$ inherits a unique structure of a smooth rigid analytic domain such that all $\Omega_\Gamma(n)$ are open subdomains. Since all $\Omega_\Gamma(n)$ are separated as affinoid domains, Ω_Γ is separated.

As a set of points we define X_Γ as the orbit space of the action of Γ on Ω_Γ. The rigid analytic structure of X_Γ will be defined as a geometric quotient of Ω_Γ; i.e., a set $V \subset X_\Gamma$ and a covering $\mathfrak{V}$ of $V \subset X_\Gamma$, respectively, is admissible if $p^{-1}(V) \subset \Omega_\Gamma$ and $p^*\mathfrak{V}$, respectively, are admissible. Due to Proposition 2.2.12(a) there exists $q \in \sqrt{|K^\times|}$ with $q < 1$ such that $|w_{\alpha_i}(z)| < q$ for all $z \in W^+_{\alpha_j}$, for $j = 1,\dots,2g$ with $j \ne i$. Now choose $q' \in \sqrt{|K^\times|}$ with $q < q' < 1$, and put

$$U_i := \left\{z \in \mathbb{P}^1_K;\, q' \le \left|w_{\alpha_i}(z)\right| \le 1 \text{ for } i = 1,\dots,2g\right\},$$
$$U_0 := \left\{z \in \mathbb{P}^1_K;\, \left|w_{\alpha_i}(z)\right| \le q' \text{ for } i = 1,\dots,2g\right\}.$$

Then $\{U_0,\dots,U_{2g}\}$ is an admissible covering of F with the properties

$$U_0 \cap U_i = \left\{z \in \mathbb{P}^1_K;\, \left|w_{\alpha_i}(z)\right| = q'\right\} \quad \text{for } i = 1,\dots,2g,$$
$$U_i \cap U_j = \emptyset \quad \text{for } 1 \le i < j \le 2g.$$

The map $p : U_i \to X_\Gamma$ is injective. We will endow X_Γ with the structure of a rigid analytic space in the following way. We view $p : U_i \to X_\Gamma$ as an open immersion and we equip the image $V_i := p(U_i)$ with the holomorphic structure given by U_i. The family

$$\left\{V_i := p(U_i);\, i = 0,\dots,2g\right\}$$

is regarded as an admissible covering of X_Γ. Thus, we obtain the structure of a rigid analytic variety on X_Γ which is smooth and 1-dimensional.

The map $p : \Omega_\Gamma \to X_\Gamma$ is a covering in the topological sense and is a quotient in the categorical sense; i.e., every Γ-invariant morphism $\Omega_\Gamma \to Z$ factorizes through $p : \Omega_\Gamma \to X_\Gamma$.

(b) X_Γ is separated, because Ω_Γ is separated; cf. [10, 9.6.1/5]. To show that X_Γ is proper, we have to construct a further covering $\{V'_0,\dots,V'_{2g}\}$ by affinoid domains of X_Γ such that $V_i \Subset V'_i$ for $i = 0,\dots,2g$; cf. Definition 1.6.3. Thus, we choose two absolute values $q_1, q_2 \in \sqrt{|K^\times|}$ with $q < q_1 < q' < q_2 < 1$, where q_1 is close to q and q_2 is close to 1. Put

$$U'_i := \left\{z \in \mathbb{P}^1_K;\, q_1 \le \left|w_{\alpha_i}(z)\right| \le 1/q_2\right\} \quad \text{for } i = 1,\dots,2g,$$
$$U'_0 := \left\{z \in \mathbb{P}^1_K;\, \left|w_{\alpha_i}(z)\right| \le q_2 \text{ for } i = 1,\dots,2g\right\}.$$

Then $p : U_i' \to X_\Gamma$ is an open immersion for $i = 0, \dots, 2g$. Hence, by setting $V_i' := p(U_i')$ for $i = 0, \dots, 2g$, the covering $\{V_0', \dots, V_{2g}'\}$ satisfies the requirement of Definition 1.6.3.

Next we want to determine the genus of X_Γ. For this we will construct a non-trivial differential form ω on X_Γ and calculate the degree of its divisor. Let $w_\gamma(z)$ be as defined in Notation 2.2.11. Then look at the formal series

$$g_1(z) := \sum_{\gamma \in \Gamma} w_\gamma(z) \quad \text{and} \quad g_2(z) := \sum_{\gamma \in \Gamma} \rho(\gamma) w_\gamma(z).$$

We may assume here without loss of generality that $(\alpha_1, \dots, \alpha_g)$ is separating for ρ and ρ^2. By Proposition 2.2.12(b) these series converge on F. By the product formula $w_{\gamma\alpha}(z) = w_\gamma(\alpha(z)) \cdot w_\alpha(z)$, the series also converges on $\alpha(F)$ for $\alpha \in \Gamma$, and hence on Ω_Γ. Moreover, we have

$$w_\alpha(z) \cdot g_1(\alpha z) = g_1(z) \quad \text{and} \quad \rho(\alpha) \cdot w_\alpha(z) \cdot g_2(\alpha z) = g_2(z).$$

Now look at the meromorphic differential form on Ω_Γ

$$\omega := \frac{g_1^2(z)}{g_2(z)} dz.$$

Since $\rho(\alpha) \cdot \alpha'(z) = w_\alpha(z)$, we obtain

$$\alpha^* \omega = \frac{g_1^2(\alpha(z))}{g_2(\alpha(z))} \cdot \alpha'(z) \cdot dz = \frac{g_1^2(z) \cdot \rho(\alpha) w_\alpha(z)}{w_\alpha(z)^2 \cdot g_2(z)} \frac{w_\alpha(z)}{\rho(\alpha)} \cdot dz = \omega.$$

Thus, ω is invariant under Γ, and so it induces a differential form on X_Γ.

It remains to determine the degree of $\operatorname{div}(\omega)$. Equivalently, we can consider the divisor associated to g_1 and g_2 on Ω_Γ which consists of Γ-orbits and count the number of the orbits with multiplicity. It suffices to do it for g_1, because the arguments for g_2 are analogous.

By Proposition 2.2.12(b) we know that $|w_\gamma(z)| < 1$, for all $z \in F^-$ and all transformations $\gamma \in \Gamma - \{\mathrm{id}\}$. Since $w_{\mathrm{id}}(z) = 1$, the ultrametric inequality yields $|g_1(z)| = 1$ for all $z \in F^- - \{\infty\}$.

Next we compute the degree of $g_1 | (W_{\alpha_i}^+ - W_{\alpha_i}^-)$. Here we have two dominating terms, namely $w_{\mathrm{id}}(z)$ and $w_{\alpha_i}(z)$. Their absolute values are equal to 1, whereas those of all the others terms are less than 1. Consider the Laurent series with respect to the coordinate $\zeta := (z - m_{\alpha_i})/\pi_i$, where $\pi_i \in \overline{K}^\times$ is a constant to normalize the coordinate to absolute value equal to 1. On $(W_{\alpha_i}^+ - W_{\alpha_i}^-)$ it is given by $1 + \zeta^2$ up to terms of absolute value less than 1. Now it is an elementary fact about such functions that their number of zeros is 2. Thus, we see that each $i \in \{1, \dots, g\}$ gives rise to a Γ-orbit of degree 2. Furthermore, there is exactly one orbit for the poles of dz; namely $\{m_\gamma; \gamma \in \Gamma\} = \Gamma\infty$ of order 2. Thus, we see that the degree of ω is $2g - 2$, and hence the genus of X_Γ is g.

(c) and (d) Follow from Theorem 1.8.1. □

2.4 Skeletons

In their article [64] Manin and Drinfeld make fundamental use of the tree representation of the p-adic numbers. In order to deal with the case of arbitrary non-Archimedean valuations, we have to generalize the approach slightly.

Let us start with some definitions. For the following, we fix a non-Archimedean field K.

Definition 2.4.1. The *closed unit disc* $\mathbb{D}_K$ is the affinoid space $\operatorname{Sp} K\langle\xi\rangle$. The *open unit disc* is the rigid analytic space $\mathbb{D}_K^- := \{z \in \mathbb{D}_K;\, |\xi(z)| < 1\}$. A *closed rational disc* or an *open rational disc* is a rigid analytic space isomorphic to $\mathbb{D}_K$ or $\mathbb{D}_K^-$, respectively.

A *closed rational annulus* or an *open rational annuls* is a rigid analytic space isomorphic to

$$A(r,1)^+ = \big\{z \in \mathbb{D}_K;\, r \le \big|\xi(z)\big| \le 1\big\},$$

respectively to

$$A(r,1)^- = \big\{z \in \mathbb{D}_K;\, r < \big|\xi(z)\big| < 1\big\},$$

for an element r of the value group $|K^\times|$. The number $r \le 1$ is called the *height* of the annulus $A(r,1)^\pm$; cf. Definition 1.3.3.

A subdomain Ω of the projective line $\mathbb{P}_K^1$ is a closed rational disc if and only if there exists a coordinate function ξ of $\mathbb{P}_K^1$ with a zero in $\mathbb{P}_K^1(K)$ and a number $r \in |K^\times|$ such that $\Omega = \{z \in \mathbb{P}_K^1;\, |\xi(z)| \le r\}$.

A subdomain Ω of the projective line $\mathbb{P}_K^1$ is a closed rational annulus if and only if Ω is the complement of a closed rational disc by an open rational disc. In fact, there exists a coordinate function ξ on $\mathbb{P}_K^1$ which has a zero in the open disc and a pole outside the closed disc. Then ξ yields the description of the complement as an annulus. This can easily be seen by the description of the invertible functions on a disc in Proposition 1.2.1(b).

Definition 2.4.2. The *standard reduction map* $\rho : \mathbb{P}_K^1 \to \mathbb{P}_k^1$ of the projective line is associated to the choice of a coordinate function on $\mathbb{P}_K^1$ which also serves as a coordinate function on $\mathbb{P}_R^1$. Then ρ is the specialization map $\mathbb{P}_K^1 \to \mathbb{P}_k^1$ on $\mathbb{P}_R^1$.

The *canonical reduction map* $\rho : \mathbb{D}_K \to \mathbb{A}_k^1$ of the unit disc $\mathbb{D}_K$ is the map which, as above, associates to a K'-rational point x of $\mathbb{D}_K$ its reduction $\tilde{x}$ which is defined as the closed point $x_R \otimes_R k$ of the extension $x_R : \operatorname{Spec}(R') \to \mathbb{A}_R^1$ of x.

If $D \subset \mathbb{D}_K$ is the unit disc punctured by finitely many maximal open discs $D(a_1)^-, \ldots, D(a_n)^-$ of $\mathbb{D}_K$, then the reduction map $\rho : \mathbb{D}_K \to \mathbb{A}_k^1$ restricts to a reduction map $\rho : D \to \mathbb{A}_k^1 - \{\rho(a_1), \ldots, \rho(a_n)\}$. We refer to this as *canonical reduction* as well.

In the following we make use of some notion about graphs; these are explained in Sect. A.1. Note that we here identify an edge e of a graph with its inverse $\overline{e}$; i.e., we consider only "geometric edges". In Definition 2.4.3 we do not need an orientation on the graph. Such graphs are called *geometric*.

Definition 2.4.3. Let Z be a rigid analytic space which is geometrically connected and locally planar. The latter means that, locally with respect to the holomorphic topology, Z is isomorphic to affinoid subdomains of the projective line.

A *semi-stable skeleton* of Z is a surjective map $\rho : Z \to S$ from Z to a geometric graph S with the following properties:

(i) The inverse image $\rho^{-1}(v)$ of a vertex $v \in V(S)$ is either the whole $\mathbb{P}^1_K$ or a domain in $\mathbb{P}^1_K$ which is isomorphic to the closed unit disc $\mathbb{D}_K$ punctured by finitely many maximal open discs $D_1^- \cup \cdots \cup D_n^-$ of $\mathbb{D}_K$.
(ii) The inverse image $\rho^{-1}(e)$ of an edge $e \in E(S)$ is isomorphic to an open rational annulus $A(\varepsilon(e), 1)^-$ of a certain height $\varepsilon(e) \in |K^\times|$.
(iii) ρ is continuous; i.e., the inverse image $\rho^{-1}(\{v_1, e, v_2\})$ of an edge e with its two extremities v_1, v_2 is an affinoid subdomain of Z or the whole Z.

Since the reduction of $\rho^{-1}(v)$ for a vertex $v \in S$ is isomorphic to a projective line minus finitely many closed points, it is irreducible and, hence, the sup-norm is multiplicative on $\rho^{-1}(v)$; cf. Remark 1.4.6.

A *semi-stable skeleton* $\rho : Z \to S$ of Z is said to *separate the points* $a_1, \ldots, a_n$ of Z if these points are mapped to vertices such that for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ either the points a_i, a_j are mapped to different vertices of S or, if mapped to the same vertex $v \in V(S)$, the points a_i, a_j have different reductions under the canonical reduction map.

A semi-stable skeleton of Z is called *stable with respect to given points* $a_1, \ldots, a_n$ for $n \geq 3$ if it separates the points and if, for each vertex v, the sum of the number of points of $\rho(a_1), \ldots, \rho(a_n)$ equal to v and of the index of v is at least 3; cf. Definition A.1.7.

Note that, for the definition of the index, in Definition A.1.7 one distinguishes between e and its inverse $\overline{e}$. One can also define the index of a vertex by the number of geometric edges which have v as an extremity.

Example 2.4.4. Let $r \in |K^\times|$ with $r < 1$. Consider the rational annulus $A(r, 1) := \{x \in \mathbb{D}_K; |r| \leq |\xi(x)|\}$, where ξ is a coordinate of the disc $\mathbb{D}_K$. We define a skeleton $\rho : A(r, 1) \to S$ in the following way. The image S consists of two vertices v_r, v_1 which are connected by one edge e. The map $\rho : A(r, 1) \to S$ sends the subset where ξ takes absolute values r to v_r, and the subset where ξ takes absolute values 1 to v_1, and the subset where ξ takes absolute values $r < |\xi(x)| < 1$ to e.

More precisely, the reduction of $A(r, 1)$ consists of two lines $\tilde{L}_r \cup \tilde{L}_1$, where $\tilde{L}_r = \mathbb{P}^1_k - \{0\}$ with coordinate $\widetilde{\xi/c}$, $c \in K$ with $|c| = r$, and $\tilde{L}_1 = \mathbb{P}^1_k - \{\infty\}$ with coordinate $\tilde{\xi}$. The point $\infty \in \tilde{L}_r$ is identified with the point $0 \in \tilde{L}_1$.

If f is a holomorphic function on $A(r, 1)$, then $f|_{A(r,r)}$ has a sup-norm $|c_r|$ and on $A(1, 1)$ a sup-norm $|c_1|$. Via the reduction one gets two functions $\tilde{f}_r := \widetilde{f/c_r}$ on L_r and $\tilde{f}_1 := \widetilde{f/c_1}$ on L_1. Both functions have a certain order m_r of $\tilde{f}_r$ at $\infty \in \tilde{L}_r$ and m_1 at $0 \in \tilde{L}_1$. Then one easily shows that $m_1 + m_r$ equals the number of zeros of f on the open annulus $A(r,1)^-$. If there are no zeros, then $|c_r| = r^{\pm m_1} \cdot |c_1|$. Such a situation will be discussed in Sect. 4.3 in a more general context.

Lemma 2.4.5. *Let $n \geq 3$ and let $a_1, \dots, a_n \in \mathbb{P}^1_K(K)$ be pairwise distinct K-rational points. Then there exists a stable skeleton of $\mathbb{P}^1_K$ which separates the points $a_1, \dots, a_n$. Furthermore, the graph of this skeleton is a tree.*

A stable skeleton separating $a_1, \dots, a_n$ is uniquely determined by these points.

Proof. To prove the existence, we proceed by induction on n. For $n = 3$ we can choose a coordinate such that the points are $0, 1, \infty$. Then the standard reduction separates these points. In this case the skeleton consists of a single vertex without any edges.

Now let $n \geq 3$ and assume that we have already constructed a stable skeleton $\rho : \mathbb{P}^1_K \to S$ which separates the points $a_1, \dots, a_n$. Consider an additional point $b \in \mathbb{P}^1_K(K)$ which is not equal to any of the $a_1, \dots, a_n$.

If $\rho(b) = v$ is a vertex, then we have to distinguish two cases. If the reduction of b under the canonical reduction map of the vertex is different from the reduction of the points $a_1, \dots, a_n$ which are also mapped to v under ρ, then it is not necessary to change anything. On the other hand, if this is not the case, then there is a point a_i with $\rho(a_i) = v$ which has the same canonical reduction as b. Then we introduce a new vertex v' and a new geometric edge e' which connects v with v'. The map $\rho' : \mathbb{P}^1_K \to S'$ is defined as follows. Let ζ be a coordinate function on the maximal open disc D^- inside $\rho^{-1}(v)$ which contains a_i and normalize ζ by $\zeta(a_i) = 0$ and by the condition that sup-norm of $|\zeta|_{D^-} = 1$. Then set $\rho'(z) = v'$ for all $z \in D^-$ with $|\zeta(z)| \leq |\zeta(b)|$ and $\rho'(z) = e'$ for all $z \in D^-$ with $|\zeta(z)| > |\zeta(b)|$ and $\rho'(z) = \rho(z)$ for all points $z \in \mathbb{P}^1_K - D^-$. So $\rho'(b)$ and $\rho'(a_i)$ are equal, but b and a_i have distinct reductions. Furthermore, v' is connected to v by e'. Then it is clear that ρ' is stable.

If $\rho(b) = e$ is an edge, then $A^- := \rho^{-1}(e)$ is an open annulus of height $\varepsilon \in |K^\times|$, since A^- is rational. Let ζ be a coordinate on $A^- := \rho^{-1}(e)$ which is normalized by having sup-norm 1. We introduce a new vertex v' and two new geometric edges e'_0, e'_1 which connect v' to the extremities of e in S. Then we define $\rho' : \mathbb{P}^1_K \to S'$ by the following formula. If $z \in A^-$, then put

$$\rho'(z) := \begin{cases} e'_1 & \text{if } |\zeta(z)| > |\zeta(b)|, \\ v' & \text{if } |\zeta(z)| = |\zeta(b)|, \\ e'_0 & \text{if } |\zeta(z)| < |\zeta(b)|. \end{cases}$$

If $z \in \mathbb{P}^1_K - A^-$, then put $\rho'(z) = \rho(z)$. Thus, $v' = \rho'(b)$ is connected to the remaining part by two edges and hence ρ' is stable.

To prove the uniqueness, we proceed by induction on n. Consider first the case $n = 3$. If S consists of a single vertex, it is the standard one with $a_1 = 0$, $a_2 = 1$ and $a_3 = \infty$. Otherwise, S has at least two terminal vertices in Definition A.1.7, since it is a finite tree. Then there are at least four points due to the very definition of stability. This is a contradiction.

Now let us consider the case $n \geq 4$. If S consists of single point, then it is easy to see that S is unique. Thus, we may assume that there is more than one vertex; in particular, there exists a terminal vertex v of S. Then v supports at least two points; say a_1 and a_2. Choose a coordinate function ζ on $\mathbb{P}^1_K$ such that $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$ and $|a_\nu| \geq 1$ for $\nu = 4, \ldots, n$. Then $\rho^{-1}(v)$ is uniquely determined by the position of the points and is independent of the stable skeleton. Indeed, we have that

$$\rho^{-1}(v) = D^+ - \left[D_1^- \cup \cdots \cup D_m^-\right],$$

where D^+ is the unit disc at 0 and D_μ^- are maximal open discs in D^+. Furthermore, in each D_μ^- there exists at least one of the given points.

If v supports more than 2 points, then we can remove one and thereby we do not destroy the stability, and hence we are done by the induction hypothesis. Consider now the case, where v supports exactly 2 points.

If v has only one neighbor w supporting also a point, then we contract the vertex and the edge e leading from v to w and remove a_1. In this way we get again a stable skeleton separating the remaining points as w supports now two points. Note that in this case $\rho^{-1}\{v, e\}$ is also uniquely determined. Therefore, we are also done by the induction hypothesis. If w does not support any of the points, then the index of w is at least 3. Then, by removing v, the edge e and a_1, one obtains a skeleton with vertex w which supports one point and its index is at least 2. So, we end up with a stable skeleton and are done by the induction hypothesis. □

Proposition 2.4.6. *Let Ω be a connected affinoid subdomain of $\mathbb{P}^1_K$. Then the following assertions are equivalent*:

(a) *Ω admits a semi-stable skeleton.*
(b) *Ω is the projective line punctured by finitely many open rational discs.*

If, in the representation $\Omega := \mathbb{P}^1_K - (D_1^- \cup \cdots \cup D_n^-)$, even the closed discs $D_1^+, \ldots, D_n^+$ are pairwise disjoint, the graph of the skeleton can be chosen in such a way that its terminal vertices correspond to $D_j^+ - D_j^-$ for $j = 1, \ldots, n$.

Proof. (a) → (b): Since Ω is affinoid, it is quasi-compact and not equal to the whole projective line. Thus, the skeleton is finite and there are terminal vertices. Here, a vertex v is called *terminal* if the reduction of $\rho^{-1}(v)$ is a projective line $\mathbb{P}^1_k$ punctured by more points than the index of v. All these points correspond to open rational discs in the complement of Ω in $\mathbb{P}^1_K$.

(b) → (a): Let $\Omega := \mathbb{P}^1_K - (D_1^- \cup \cdots \cup D_n^-)$ be a complement of $\mathbb{P}^1_K$ of the union of open rational discs $D_1^-, \ldots, D_n^-$ which are pairwise disjoint. In particular, there are K-rational points $a_i \in D_i^-$ for $i = 1, \ldots, n$.

If $n \leq 2$, then Ω is an annulus, and the assertion was explained in Example 2.4.4.

If $n \geq 3$, then there exists a stable skeleton $\rho : \mathbb{P}^1_K \to S$, which separates the points $\{a_1, \ldots, a_n\}$ by Lemma 2.4.5. First we refine the skeleton in the following way. Each a_j is supported by a vertex v_j. Then, for each $j = 1, \ldots, n$, we introduce a new vertex v'_j, which is associated to a small closed rational disc $D'_j \subset D_j^-$

around a_j, and a new edge e_j which is associated to the open rational annulus $D_j^- - D_j'$. Now, there exists a coordinate function ξ on $\mathbb{P}_K^1$ with a zero at a_1 and a pole at a_2. Since D_1^- is a rational disc, we can adjust ξ such that the sup-norm of ξ on D_1^- is equal to 1. By Proposition 1.3.4 we have that the absolute value function $|\xi|$ behaves like the one of a power of the coordinate functions on the annuli associated to the edges of this skeleton. Then we subdivide every edge at the subannulus of height 1 where $|\xi|$ takes the value 1. Thus, we obtain a skeleton such that ρ restricts to a skeleton $\rho_1 : \mathbb{P}_K^1 - D_1^- \to S_1$, where S_1 is obtained from S by removing all the vertices and edges where $|\xi|$ takes values less than 1. Likewise we proceed with all the other discs, and hence we obtain a semi-stable skeleton $\rho_n : \Omega \to S_n$ as required.

The additional assertion follows easily from the proof of "b $\to$ a" as well, because the subset $\{z; |\xi(z)| = 1\}$ is exactly $D_1^+ - D_1^-$ and $|\xi|$ takes only values greater than 1 on all the other closed discs D_j^+ for $j \geq 2$. Eventually one has to contract the part where $|\xi|$ is equal to 1. □

Corollary 2.4.7. *Let K be algebraically closed. If $\Omega \subset \mathbb{P}_K^1$ is a connected affinoid subdomain, then Ω has a semi-stable skeleton, and hence Ω is a closed rational disc punctured by finitely many open rational discs.*

Proof. Since Ω is affinoid, it is strictly contained in $\mathbb{P}_K^1$. So, there exists a coordinate function ξ on $\mathbb{P}_K^1$ such that ξ has its pole a_0 outside Ω and its zero inside Ω. Moreover, one can adjust ξ such that the sup-norm of $\xi|_\Omega$ is equal to 1. Thus, Ω is contained in the closed unit disc $\mathbb{D}_K$. By Theorem 1.3.7 we know that Ω is a union of finitely many rational domains. It suffices to consider the case $\Omega = X(f_1/f_0, \dots, f_N/f_0)$, where $f_0, \dots, f_N$ are holomorphic functions on $X := \mathbb{D}_K$ without common zeros. Due to Theorem 1.2.5 we may assume that $f_0, \dots, f_N$ are polynomials, because invertible functions on $\mathbb{D}_K$ have constant absolute value functions by Proposition 1.2.1. Since $f_0, \dots, f_N$ have no common zeros, there exists some $r' \in |K^\times|$ such that Ω is contained in $\Omega' := \{x \in \mathbb{D}_K; |f_0(x)| \geq r'\}$. Note that Ω' is equal to $\mathbb{D}_K$ minus finitely many open discs around the zeros of f_0.

Now it suffices to analyze the structure of $\Omega'(f_i/f_0)$. It follows from Proposition 2.4.6 that there exists a semi-stable skeleton of Ω'. Then it is an easy combinatorial game to show how to obtain a skeleton of $\Omega'(f_i/f_0)$ from the skeleton of Ω'. In more detail, the absolute value function $|f_0|$ of f_0 is constant on the pre-image of vertices and behaves like a power of the absolute value of the coordinate on the pre-image of an edge by Proposition 1.3.4. The function $|f_1|$ behaves similarly after removing small discs around the zeros of f_1. Thus, by subdividing some annuli associated to the skeleton of Ω', we construct a new skeleton such that $\Omega'(f_i/f_0)$ can be viewed as the preimage of a subgraph of this new skeleton. □

The number $n + 1$ of holes used in the representation of Ω as a subset of $\mathbb{P}_K^1$ can be characterized in terms of the structure of the group of invertible holomorphic functions on Ω. There is the following result.

Proposition 2.4.8. *Let $\Omega \subset \mathbb{P}^1_K$ be a closed rational disc which is punctured by finitely many open rational discs*

$$\Omega := \mathbb{P}^1_K - \big(D(a_0, r_0)^- \cup \cdots \cup D(a_n, r_n)^-\big).$$

Let ξ be a coordinate function on $\mathbb{P}^1_K$ with a pole at a_0 and a zero outside of $D(a_0, r_0)^-$ such that $D(a_0, r_0)^- := \{x \in \mathbb{P}^1; |\xi(x)| > 1\}$.

Put $\xi_\nu := c_\nu/(\xi - \xi(a_\nu))$, where $c_\nu \in K^\times$ has absolute value $|c_\nu|$ equal to the sup-norm of $\xi - \xi(a_\nu)$ on the disc $D(a_\nu, r_\nu)^-$. Then we have:

(a) *Every holomorphic function on Ω has a unique representation*

$$f = \sum_{i=0}^{\infty} c_{0,i}\xi^i + \sum_{\nu=1}^{n}\sum_{i=1}^{\infty} c_{\nu,i}\xi_\nu^i$$

with coefficients $c_{\nu,i} \in K$. For the sup-norm we have that

$$|f|_\Omega = \max\big\{|c_{\nu,i}|; i \in \mathbb{N}, \nu = 0, \ldots, n\big\}.$$

(b) *If f has no zeros on Ω, then f has a unique representation*

$$f = c \cdot (\xi - a_1)^{m_1} \ldots (\xi - a_n)^{m_n} \cdot (1 + h),$$

where $c \in K^\times$ is a unit, h is a holomorphic function on Ω with sup-norm $|h|_\Omega < 1$, and where $m_1, \ldots, m_n$ are integers.

Proof. (a) Since f can be approximated by rational functions which have poles only in $\{a_0, \ldots, a_n\}$, it suffices to verify the assertion for such rational functions. In that case we have a unique partial fraction decomposition. Such a decomposition is of the same form as in the assertion, but there are only finitely many coefficients unequal 0.

To verify the assertion on the sup-norm, we first assume that the closed discs $D(a_\nu, r_\nu)^+$ are pairwise disjoint. In that case the assertion follows by the ultrametric inequality, because the sup-norm of ξ_μ on $D(a_\nu, r_\nu)^+$ is less than 1 for all μ, ν with $\mu \neq \nu$. By a limit argument (enlarging the discs) and the maximum principle this implies the assertion in the general case.

The assertion about the uniqueness follows from the formula for $|f|_\Omega$.

(b) As in the proof of (a) it suffices to verify the assertion for rational functions which have poles only in $\{a_0, \ldots, a_n\}$. Moreover, we may assume that the closed discs $D(a_\nu, r_\nu)^+$ are pairwise disjoint. By Proposition 1.3.4 we know that the restriction of f to the annulus $D(a_\nu, r_\nu)^+ - D(a_\nu, r_\nu)^-$ can be written as $c_\nu \xi_\nu^{m_\nu} \cdot (1 + h)$, where the sup-norm of h is less than 1. After replacing f by $f \cdot \prod_{\nu=1}^n \xi_\nu^{-m_\nu}$ we can assume that all the exponents m_ν are zero for $\nu = 1, \ldots, n$. In that case we will verify that m_0 is also zero, and that f equals $c \cdot (1 + h)$ with a constant $c \in K^\times$ and with a holomorphic function h on Ω with sup-norm less that 1.

Indeed, by (a) we may assume that $|f|_\Omega = 1$ and that $|f|$ takes a maximum on an annulus $D(a_\nu, r_\nu)^+ - D(a_\nu, r_\nu)^-$ for some $\nu \in \{1, \ldots, n\}$; otherwise consider $1/f$.

Then by (a) it follows $|c_{0,0}| = 1$ and $|c_{\nu,i}| \leq 1$ for all ν, i. Since f has no zeros on $D(a_0, r_0)^+ - D(a_0, r_0)^-$, by Proposition 1.3.4 we see that $m_0 = 0$. Thus, the assertion is proved. □

Remark 2.4.9. In the situation of Proposition 2.4.8 the group of invertible holomorphic functions on Ω can be represented in the following way:

$$\mathcal{O}_\Omega(\Omega)^\times \cong K^\times \times \mathbb{Z}^n \times \left\{1 + h, h \in \mathcal{O}_\Omega(\Omega) \text{ with } |h|_\Omega < 1\right\}.$$

Here $n + 1$ is the number of holes of Ω in $\mathbb{P}^1_K$.

Proof. Put $H := \{m \in \mathbb{Z}^{n+1}; m_0 + \cdots + m_n = 0\} \cong \mathbb{Z}^n$. The map

$$\varphi : K \longrightarrow \mathcal{O}_\Omega(\Omega)^\times,\ (m_0, \ldots, m_n) \longmapsto \xi^{m_0} \cdot (\xi - a_1)^{m_1} \cdot \ldots \cdot (\xi - a_n)^{m_n}$$

is injective, and its image is a direct summand by Proposition 2.4.8. Thus, we see that the assertion is true. □

Proposition 2.4.10. *In the situation of Proposition 2.4.8 consider a meromorphic function f on Ω which is not identically zero. Assume, in addition, that the annulus $A_\nu := D_\nu^+ - D_\nu^-$ belongs to Ω, and that $f|_{A_\nu}$ has neither zeros nor poles for $\nu = 1, \ldots, n$. Then the degree of the divisor of f on Ω is given by the formula*

$$\deg \operatorname{div}(f) = -\sum_{\nu=0}^{n} \operatorname{ord}_{A_\nu} f.$$

Here $\operatorname{ord}_{A_\nu} f$ is the exponent of the dominating term in the Laurent expansion of $f|_{A_\nu}$ with respect to the coordinate function ξ_ν on D_ν^+; cf. Proposition 1.3.4.

In Proposition 4.3.1 there is a more general formula than the given one.

Proof. The support of the divisor of f is finite. Thus there exists a rational function g on $\mathbb{P}^1_K$ such that $\operatorname{div}(g|_\Omega) = \operatorname{div}(f)$. It we put $u := f/g$, then u is an invertible holomorphic function on Ω. Using an approximation as in the proof of Proposition 2.4.8 we can also assume that $u = 1 + h$ with a holomorphic function h on Ω with $|h|_\Omega < 1$. Thus, we can replace f by g, because f and g have the same order on A_ν. Since $\deg \operatorname{div} g = 0$, it remains to see that $\operatorname{ord}_{A_\nu} g = \deg \operatorname{div}(g|_{D_\nu^-})$ for $\nu = 0, \ldots, n$. The latter follows easily from Theorem 1.2.5, because the degree of a Weierstraß polynomial equals the number of its zeros in the unit disc. □

Now let us return to the Schottky groups.

Proposition 2.4.11. *Let Γ be a Schottky group. Consider the situation of Notation 2.2.11 with respect to a separating homomorphism $\rho : \Gamma \to \overline{K}^\times$. Assume that the radii $\sqrt{|\rho(\gamma)|} \cdot r_\gamma$ belong to $|K^\times|$ for all $\gamma \in \Gamma - \{\mathrm{id}\}$. Let $F \subset \Omega_\Gamma$ be the fundamental domain of Γ as in Notation 2.2.11. Then we have:*

(a) *There exists a semi-stable skeleton* $\rho_F : F \to S_F$ *of* F *with terminal vertices* $v_1, \ldots, v_{2g}$ *such that* $\rho_F^{-1}(v_i) = W_i^+ - W_i^-$ *for all* $i = 1, \ldots, 2g$, *where* $W_i^\pm := W_{\alpha_i}^\pm$ *for the system* $(\alpha_1, \ldots, \alpha_{2g})$. *In particular,* S_F *is a tree.*
(b) *There exists a semi-stable skeleton* $\rho_\Omega : \Omega_\Gamma \to S_\Omega$ *of* Ω_Γ *which extends* ρ_F *such that* Γ *acts on* S_Ω *canonically. In particular,* S_Ω *is a tree.*
(c) *There exists a semi-stable skeleton* $\rho_X : X_\Gamma \to S_X$ *of* X_Γ *which is the quotient of* ρ_Ω *with respect to the action of* Γ. *In particular, the quotient map* $p_S : S_\Omega \to S_X$ *is the universal covering in the category of graphs. There is a commutative diagram*

$$\begin{tikzcd} \Omega_\Gamma \arrow[r, "p_X"] \arrow[d, "\rho_\Omega"] & X_\Gamma \arrow[d, "\rho_X"] \\ S_\Omega \arrow[r, "p_S"] & S_X. \end{tikzcd}$$

Proof. (a) Consider the fundamental domain

$$F = \mathbb{P}_K^1 - \bigcup_{i=1}^{2g} W_{\alpha_i}^-$$

associated to a separating basis $\alpha_1, \ldots, \alpha_g$ of Γ. Set $\alpha_{i+g} := \alpha_i^{-1}$ for $i = 1, \ldots, g$. By Proposition 2.4.6 there exists a semi-stable skeleton $\rho_F : F \to S_F$. It is clear that S_F satisfies the claim.

(b) The skeleton $S_0 := S_F$ constructed in (a) has $2g$ terminal vertices $v_1, \ldots, v_{2g}$. Then each α_i maps the domains $\rho^{-1}(v_i)$ bijectively to $\rho^{-1}(v_{i+g})$ for all $i = 1, \ldots, g$. Thus we obtain a skeleton

$$S_1 := S_0 \cup \bigcup_{i=1}^{2g} \alpha_i(S_0)$$

by gluing the skeleton S_0 with the skeleton $\alpha_i(S_0)$ of $\alpha_i(F)$ along v_{i+g} for $i = 1, \ldots, 2g$. Then S_1 is a skeleton of

$$\Omega_\Gamma(n) := \bigcup_{\gamma \in \Gamma; \ell(\gamma) \leq n} \gamma(F)$$

for $n = 1$. Continuing in this way, one obtains skeletons S_n of $\Omega_\Gamma(n)$ for all $n \in \mathbb{N}$ and hence in the limit a skeleton S_Ω of Ω_Γ. The group Γ acts on S_Ω by translation and is compatible with action on Ω_Γ.

(c) The skeleton S_X is obtained from S_F by identifying the terminal vertices v_i and v_{i+g} for $i = 1, \ldots, g$. This is compatible with the group action of Γ on Ω_Γ as was explained in (a). $\square$

Corollary 2.4.12. *In the situation of Proposition 2.4.11, let e be an edge of the skeleton S_Ω. If $\gamma \in \Gamma$ fixes e, then $\gamma = \mathrm{id}$.*

Proof. The skeleton S_Ω is the universal covering of S_X. Thus, Γ is canonically isomorphic to the deck transformation group of S_Ω / S_X. If a deck transformation fixes one point, then it is equal to the identity. □

Remark 2.4.13. A *ray* $(e_i; i \in \mathbb{N})$ in S_Ω is an infinite path without backtracking; cf. Definition A.1.3. Note that a ray has an origin and no target in S_Ω. Two rays are called *equivalent* if they are equal after removing finite parts at their origins.

An *axis* $(e_i; i \in \mathbb{Z})$ in S_Ω is an infinite path without backtracking. Note that an axis has neither an origin nor a target in S_Ω.

However we have:

(a) Each ray in S_Ω defines a unique point in the set L_Γ of the limit points of Γ.
(b) The equivalence classes of rays correspond bijectively to the limits points of Γ.
(c) If $\alpha \in \Gamma - \{\mathrm{id}\}$, then let $x_\alpha \subset S_\Omega$ be the axis leading from z_α^- to z_α^+, where z_α^+ is the attractive fixed point and z_α^- is the repelling fixed point of α. Then α acts on x_α by shifting.

Proof. (a) We choose the coordinate function on $\mathbb{P}_K^1$ such that $\infty \in \rho^{-1}(v_0)$. Thus, for $i \in \mathbb{N}$ we have

$$\rho^{-1}(v_i) = D(a_{v_i}, r_i)^+ - \left[D(a_{v_i,1}, r_i)^- \cup \cdots \cup D(a_{v_i,k_i}, r_i)^-\right],$$

where $D(a_{v_i,j}, r_i)^-$ are maximal open rational discs in the closed rational disc $D(a_{v_i}, r_i)^+$. Let $(e_i; i \in \mathbb{N})$ be a ray. Then the edge e_i satisfies

$$\rho^{-1}(e_i) = D(a_{v_i,j}, r_i)^- - D(a_{v_{i+1}}, r_{i+1})^+$$

for a suitable $j \in \{1, \ldots, k_i\}$. Only finitely many heights r_{i+1}/r_i can occur, because they are related under Γ. Thus, the limit $\lim a_i$ exists and is a limit point of Γ; cf. the proof of Example 2.2.13.

(b) If two rays define the same limit point, then they are equivalent, because a ray induces a filter of neighborhoods of the limit point. Conversely, every limit point induces a ray, because $L_\Gamma = L_\Gamma(\infty)$ by Proposition 2.2.4(f).

(c) The subtree x_α of S_Ω contains the vertices associated to the sets $W_{\alpha^n}^+ - W_{\alpha^n}^-$ for $n \in \mathbb{Z}$. Then it is clear that α acts by shifting. □

Definition 2.4.14. Let $\rho : Z \to S$ be a semi-stable skeleton.

(a) A *path* $c := e_1 + \cdots + e_n$ in Z is a path $(e_1, \ldots, e_n)$ in S; cf. Definition A.1.3. The *length of a path* $c := e_1 + \cdots + e_n$ in Z is defined by

$$\ell(c) := -\sum_{\nu=1}^{n} \log \varepsilon(e_\nu),$$

where $\varepsilon(e_\nu)$ is the height of the annulus $\rho^{-1}(e_\nu)$.

(b) For two paths $p = \sum_{\mu=1}^{m} e_\mu$ and $p' = \sum_{\nu=1}^{n} e'_\nu$ in Z their *pairing* is defined by

$$[p, p'] := \sum_{\mu,\nu} [e_\mu, e'_\nu],$$

where

$$[e_1, e_2] = \begin{cases} -\log \varepsilon(e) & \text{if } e_1 = e_2 = e, \\ \log \varepsilon(e) & \text{if } e_1 = -e_2 = e, \\ 0 & \text{if } e_1 \neq \pm e_2, \end{cases}$$

for edges $e_1, e_2 \in E(S)$, where $\varepsilon(e)$ is the height of the annulus $\rho_\Omega^{-1}(e)$. Here $-e$ means the edge e with the opposite orientation.

Lemma 2.4.15. *In the situation of Definition 2.4.14, let $\rho' : Z \to S'$ be a second semi-stable skeleton. Assume that there is a map $\varrho : S' \to S$ which contracts subtrees satisfying $\varrho \circ \rho' = \rho$. Thus, one has two notions of length according to the chosen skeletons. Let $v'_1, v'_2 \in S'$ be two vertices which are mapped to vertices v_1, v_2 in S, respectively. Then for each path c' leading from v'_1 to v'_2, the image $\varrho(c')$ has the same length as the path c leading from v_1 to v_2.*

Proof. We may assume that v_1 and v_2 are connected by a single edge e. Let $c := (e'_1, e'_2, \ldots, e'_n)$ be the path in S' leading from v'_1 to v'_2. Then $\rho'^{-1}(c) = \rho^{-1}(\{e\})$. Since the height of an annulus $A(r_1 r_2, 1)$ is the product of the heights of $A(r_1, 1)$ and $A(r_2, 1)$, the assertion follows. □

Notation 2.4.16. Let Γ be a Schottky group.

For every $\alpha \in \Gamma$ with $\overline{\alpha} \neq 1$ in $H := \Gamma/[\Gamma, \Gamma] = \Gamma_{\mathrm{ab}}$ there is an axis x_α in S_Ω which we orient from the attractive fixed point z_α^+ to the repulsive z_α^-. Let c_α be the part of x_α which belongs to a fundamental domain of $\alpha^{\mathbb{Z}}$. Note that c_α is a finite path in S_Ω. Let $e_1, \ldots, e_r$ be the consecutive edges with the induced orientation of c_α. Then, with the notation of Proposition 2.4.11,

$$\overline{c}_\alpha := \overline{p_S(c_\alpha)} := \sum_{i=1}^{r} p_S(e_i) \in Z_1(S_X, \mathbb{Z})$$

is a 1-chain. Its homology class in $H_1(S_X, \mathbb{Z})$ can be identified with the image of $\alpha \in \Gamma$ in the maximal abelian quotient

$$\pi_1(S_X) = \Gamma \longrightarrow H := \Gamma/[\Gamma, \Gamma].$$

More precisely, here one has to consider the realization $\mathrm{real}(S_X)$ of the graph S_X; cf. Definition A.1.2.

Remark 2.4.17. The pairing of Definition 2.4.14 induces a scalar product on $H_1(S_X, \mathbb{Z})$. This bilinear form is symmetric and positive definite.

The notion of skeletons is useful for the interpretation of $H_1(X, \mathbb{Z})$.

Remark 2.4.18. Affinoid spaces with smooth reduction are simply connected in the sense that they admit only trivial coverings in the topological sense, as easily follows from Proposition 3.1.12. Furthermore, every affinoid subdomain Ω of the projective line $\mathbb{P}^1_K$ is also simply connected. Indeed, one can assume that K is algebraically closed. Thus, Ω admits a semi-stable skeleton which is tree due to Corollary 2.4.7, and so it is simply connected.

If a rigid analytic space X admits a semi-stable skeleton S_X, then every finite topological covering $Y \to X$ inherits a semi-stable skeleton S_Y such that $S_Y \to S_X$ is a topological covering as well and is compatible with the map $Y \to X$. Therefore, one can view $H_1(S_X, \mathbb{Z})$ as a replacement of "$H_1(X, \mathbb{Z})$".

2.5 Automorphic Functions

In this section we return to Mumford curves. Thus, let us consider a Schottky group $\Gamma \subset \mathrm{PGL}(2, K)$ of rank $g \geq 1$; i.e., it is a free group over g generators. Let $\Omega := \Omega_\Gamma \subset \mathbb{P}^1_K$ be the set of ordinary points of Γ and let X_Γ be the associated Mumford curve. As was explained in Sect. 2.3, the canonical map $p : \Omega_\Gamma \to X := X_\Gamma$ can be viewed as the universal covering in the rigid analytic sense. Equivalently, the universal covering can be identified with the associated skeletons $p_S : S_\Omega \to S_X$. Moreover, the rigid analytic deck transformation group

$$\Gamma \cong \pi_1(X) \cong \pi_1(S_X)$$

can be identified with the deck transformation group of the skeleton S_X. Its maximal abelian quotient

$$H := \Gamma_{\mathrm{ab}} := \Gamma/[\Gamma, \Gamma] \cong H_1(X, \mathbb{Z}) \cong H_1(S_X, \mathbb{Z})$$

is the group of closed cycles by Notation 2.4.16. In the following we review some results taken from [64] and adapt them to the case of a non-Archimedean field with valuation which is not necessarily discrete.

Definition 2.5.1. A *K-divisor* on Ω is a function $\Omega(\overline{K}) \to \mathbb{Z}$, $n \mapsto n_z$, with the following properties:

(i) $n_{z_1} = n_{z_2}$ if z_1 and z_2 are conjugate over K.
(ii) There is a finite extension L/K such that every $z \in \Omega(\overline{K})$ with $n_z \neq 0$ belongs to $\Omega(L)$.
(iii) The set $\{z; n_z \neq 0\}$ has no accumulation points in Ω with respect to the holomorphic topology.

We denote by $\mathcal{D}$ the set of K-divisors. The group Γ acts on $\mathcal{D}$.

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