Preface

Projective algebraic curves or abelian varieties are defined as the vanishing locus of finite families of homogeneous polynomials in a projective space fulfilling certain conditions. Except for elliptic curves or hyperelliptic curves, it is difficult to pin down equations which give rise to curves or abelian varieties.

Over the complex numbers one has analytic tools to construct and to uniformize such objects. For example, every smooth curve of genus $g \geq 2$ has a representation $\Gamma\backslash\mathbb{H}$, where $\mathbb{H}$ is the upper half-plane and $\Gamma \subset \mathrm{Aut}(\mathbb{H})$ is a group acting on $\mathbb{H}$. Similarly, every compact complex Lie group is of type $\mathbb{C}^n/\Lambda$, where Λ is a lattice in $\mathbb{C}^n$; the abelian varieties among the compact complex Lie groups can be characterized via polarizations. Moreover, one can construct curves and abelian varieties in this way via algebraization of the analytic quotients. Thus, the geometry and the construction of such objects are completely clarified.

Over a complete field K with respect to a non-Archimedean valuation, one can expect similar tools as in the complex case once a good theory of holomorphic functions has been established.

Historically, the theory started with the simplest case of an elliptic curve over K. One can define the elliptic curve by a minimal Weierstraß equation with integral coefficients. If this equation reduces to an elliptic curve over the residue field, we say that the given elliptic curve has good reduction. In this case there is no uniformization at all; such curves can be regarded as liftings of elliptic curves defined over the residue field. On the other hand, if the Weierstraß equation reduces to a cubic with an ordinary double point, then the situation looks better from the viewpoint of uniformization. As an abstract group its K-rational points are represented by a quotient $K^\times/q^\mathbb{Z}$ for some non-integral $q \in K^\times$ without any further structure. Originally Tate wanted to construct "analytic" quotients $\mathbb{G}_{m,K}/q^\mathbb{Z}$ of the multiplicative group of a non-Archimedean field K by the lattice $q^\mathbb{Z}$; a construction which cannot be carried out in the category of ordinary schemes directly.

Thus, there was the desire to create a theory of "analytic spaces" over a non-Archimedean field which allows such constructions. This was exactly the incentive of Tate to understand elliptic curves with multiplicative reduction by "analytic" means. In 1961 Tate gave a seminar at Harvard where he developed a theory of rigid analytic spaces; cf. [92].

Later on, using methods from formal algebraic geometry, Mumford generalized the construction of Tate's elliptic curve to curves of higher genus [75] – nowadays called Mumford curves – as well as to abelian varieties with split torus reduction [76]. Moreover, Mumford's constructions even work over complete Noetherian rings of higher dimension.

The relationship between formal algebraic geometry and rigid geometry was clarified by Raynaud in [80]. As a sort of reverse, Raynaud worked on the rigid analytic uniformization of abelian varieties and their duals over non-Archimedean fields [79].

The ideas of Mumford and Raynaud were picked up by Chai and Faltings and generalized to abelian varieties with semi-abelian reductions over fields of fractions of complete Noetherian normal rings of higher dimension. Whereas in the rigid analytic context, the periods of the uniformization enter the scene quite naturally even in the absence of a polarization, Chai and Faltings made the observation that the periods are encoded in the coefficients of the theta function associated to a principal polarization, in analogy to the complex case. So, for them it was not necessary to invoke rigid geometry.

Nevertheless, rigid geometry is a means to unfold the geometric ideas behind the formal constructions used by Mumford, Chai and Faltings. The results on uniformization and construction provide a method to parameterize polarized abelian varieties and their semi-abelian degeneration in a universal way. So, they became the essential ingredients for the construction of a toroidal compactification of the moduli space of polarized abelian varieties by Chai and Faltings; cf. [27].

This book thoroughly treats the main results on rigid geometry and their applications as they grew out of the notes of Tate. The focus of this book lies on the arithmetic geometry of curves and their Jacobians over non-Archimedean fields.

After an introduction to rigid geometry in Chap. 1, we directly concentrate on the main topic. Following ideas of Drinfeld and Manin [64], Mumford curves are treated in Chap. 2 via classical Schottky uniformization. Their Jacobians are rigid analytic tori which are constructed by automorphic functions. This is explained on an elementary level. Thus, we achieve the rigid analytic counterpart of the fascinating theory of Riemann surfaces and their Jacobians. The remainder of the book (Chaps. 3 to 7) deals with smooth rigid analytic curves and their semi-stable reductions or with proper smooth rigid analytic group varieties and their semi-abelian reductions. The intention here is to comprehensively present the rigid analytic uniformization and construction of curves and their Jacobians or of abelian varieties over non-Archimedean fields. Moreover, the structure of abeloid varieties, which are the counterparts of compact complex Lie groups, is presented in details.

The reader is assumed to be familiar with basic algebraic geometry in the style of Grothendieck and with standard facts about abelian varieties. The reader can consult [15, Chaps. 2 and 9], [60] and [74].

Since there are several books which deal with the foundations of rigid geometry, cf. [1, 9, 10], there is no need to develop it again. Therefore, the prerequisites on classical rigid geometry are only surveyed in Chap. 1 without giving proofs. In the same way the basic results on the relation between formal and rigid geometry are handled in Chap. 3, as they are presented in [14] and were revisited a few years ago in [1]. For the basic theory of formal and rigid geometry the reader may also consult [9] where it is carefully explained. There are other foundations of non-Archimedean analysis by Berkovich [6] and Huber [47], but these are not involved in this book. So, we concentrate on the main applications which are not touched or only partially studied in other books; cf. [30] and [35]. Compared to the existing literature, many proofs have been substantially improved and some new results have been added.

It is a pleasure for me to express my gratitude to my students Sophie Schmieg and Alex Morozov for proofreading and comments. Also I would like to thank colleagues, including Siegfried Bosch, Barry Green, Urs Hartl, Dino Lorenzini, Florian Pop, Stefan Wewers, for discussions and valuable suggestions. I am especially indebted to Ernst Kani, who helped me to edit the manuscript.

In particular I am glad to acknowledge here the extraordinary help from Michel Raynaud, who contributed many ideas to this book.

Münster, Germany
September 2015

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Springer

http://www.springer.com/978-3-319-27369-3

Rigid Geometry of Curves and Their Jacobians

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2016, XVIII, 386 p. 1 illus., Hardcover

ISBN: 978-3-319-27369-3