Chapter 2
Hyperbolic Systems of Balance Laws

In this chapter, we introduce the class of dynamical systems to be considered in this monograph. We here restrict ourselves to the systems described by linear (or linearized) PDEs of hyperbolic type with one spatial variable, which naturally arise in many practical problems such as the study of gas flow in ducts, the flow of irrigation water in the canals or the current flow in the electrical transmission lines (Bounit 2003; Miano and Maffucci 2001; Rojek 2002). Section 2.1 introduces the mathematical representation of the considered DPSs in the form of the linear, strongly coupled PDEs of hyperbolic type, as well as its decoupled representation in terms of the characteristic variables. Section 2.2 deals with the general initial and boundary conditions for the above-mentioned equations. In Sect. 2.3, two typical examples of 2×2 systems of balance laws—double-pipe heat exchanger and transportation pipeline—are introduced and analyzed in the hyperbolic PDE framework. Motivated by the practical examples from Sect. 2.3, two typical configurations of boundary inputs are considered in Sect. 2.4, which will be analyzed throughout the rest of the book. Finally, Sect. 2.5 summarizes the results of this chapter.

2.1 Linear PDE Representation

Many physical systems with an engineering interest mentioned in Chap. 1 can be described, after possible linearization in a given operating point, by the system of one-dimensional linear hyperbolic PDEs of balance laws. Therefore, we are concerned with $n \times n$ systems, i.e., systems of n equations with n dependent variables, given by the following equation (see Bartecki 2013c; Bastin and Coron 2011; Chentouf and Wang 2009; Christofides and Daoutidis 1998b; Diagne et al. 2012; Evans 1998; Mattheij et al. 2005; Strikwerda 2004):

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$$E\frac{\partial x'(l,t)}{\partial t} + F\frac{\partial x'(l,t)}{\partial l} = Hx'(l,t)\,, \tag{2.1}$$

with $x'(l,t)\colon \Omega \times [0,+\infty) \to \mathbb{R}^n$ being a vector function representing the spatiotemporal distribution of the n physical state-variable functions (in short, state variables)

$$x'(l,t) = \left[x_1'(l,t)\;\; x_2'(l,t)\;\; \ldots\;\; x_n'(l,t)\right]^T, \tag{2.2}$$

where $\Omega = [0, L] \subset \mathbb{R}$ is the domain of the spatial variable l; $[0, +\infty) \subset \mathbb{R}$ is the domain of the time variable t; and $E, F, H \in \mathbb{R}^{n\times n}$ are matrices of constant coefficients. The matrix F is usually obtained as a result of the local linearization of the so-called flux function, whereas the matrix H represents here the local linearization of the *source term* (Bereux and Sainsaulieu 1997; Perthame and Simeoni 2003).

Remark 2.1 The systems described by Eq. (2.1) are commonly known as *systems of balance laws*. In the special case when there is no "production", i.e., for $H = 0$, the system is usually called *system of conservation laws*. However, for the first case some authors favor the term *system of conservation laws with source* (see Christie et al. 1991; Dafermos 2010; Domański 2006).

Remark 2.2 Each equation of the system (2.1) can contain both temporal and spatial derivatives of different state variables $x_i'(l,t)$, for $i = 1, 2, \ldots, n$. Therefore, this PDE representation is known as a *strongly coupled* one.

Assuming that for $\det(E) \neq 0$ and $\det(F) \neq 0$ there exists a non-singular transformation matrix $S \in \mathbb{R}^{n\times n}$ such that the following equation holds (Bartecki 2013c):

$$S^{-1}FE^{-1}S = \Lambda, \tag{2.3}$$

where $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix, pre-multiplying both sides of Eq. (2.1) by S^{-1} and using the following identity

$$E^{-1}SS^{-1}E \equiv I, \tag{2.4}$$

we transform Eq. (2.1) into the following form:

$$S^{-1}E\frac{\partial x'(l,t)}{\partial t} + S^{-1}FE^{-1}SS^{-1}E\frac{\partial x'(l,t)}{\partial l} = S^{-1}Hx'(l,t)\,. \tag{2.5}$$

Then, taking into account Eqs. (2.3) and (2.4) and introducing the vector of characteristic state-variables $x(l,t)$ given by

$$x(l,t) = S^{-1}Ex'(l,t), \tag{2.6}$$

we can write Eq. (2.1) in the following form:

$$\frac{\partial x(l,t)}{\partial t} + \Lambda \frac{\partial x(l,t)}{\partial l} = Kx(l,t), \tag{2.7}$$

with

$$K = S^{-1} H E^{-1} S. \tag{2.8}$$

Remark 2.3 Owing to the diagonal form of the matrix Λ, each equation of the system (2.7) contains both temporal and spatial derivatives of the same characteristic state variable $x_i(l,t)$, for $i = 1, 2, \ldots, n$. Therefore, this system is commonly referred to as *decoupled* or *weakly coupled*, i.e., coupled only through the terms that do not contain derivatives.

Remark 2.4 In order to distinguish between the two different types of state-variable functions appearing in Eqs. (2.1) and (2.7), respectively, the elements of $x'(l,t)$ will be throughout the rest of this monograph called "physical" or "original" state variables, whereas those of $x(l,t)$ will be referred to as "characteristic" ones.

Definition 2.5 (Courant and Hilbert 1989; Lax 1957). The system (2.1) is said to be *hyperbolic* if there exists a diagonal matrix Λ with real entries and a non-singular matrix S such that Eq. (2.3) holds. Additionally, if all diagonal entries of Λ are distinct, then Eq. (2.1) is said to be *strictly hyperbolic*.

Therefore, strict hyperbolicity of the system means that the matrix Λ in Eq. (2.7) takes the following form:

$$\Lambda = \operatorname{diag}\left(\lambda_1, \ldots, \lambda_p, \lambda_{p+1}, \ldots, \lambda_n\right), \tag{2.9}$$

where $\lambda_i \in \mathbb{R} \setminus 0$ for $i = 1, 2, \ldots, n$, represents the eigenvalues of the matrix FE^{-1} arranged in descending order

$$\lambda_1 > \cdots > \lambda_p > 0 > \lambda_{p+1} > \cdots > \lambda_n, \tag{2.10}$$

with $p \leq n$ representing the number of positive eigenvalues.

The matrix S in Eq. (2.3) can be presented as follows:

$$S = \left[s_1 \ldots s_p \; s_{p+1} \ldots s_n\right], \tag{2.11}$$

where $s_i \in \mathbb{R}^n$ for $i = 1, 2, \ldots, n$, denotes the corresponding column eigenvectors of the matrix FE^{-1}.

Remark 2.6 As is generally known, every real symmetric matrix is Hermitian, and therefore all its eigenvalues are real. Consequently, if the matrix FE^{-1} is symmetric, then the system (2.1) is hyperbolic.

Remark 2.7 In the case of the hyperbolic PDEs describing the considered systems of balance laws, the diagonal entries of Λ represent the wave propagation velocities or mass and energy transport rates.

2.2 Initial and Boundary Conditions

In order to obtain a unique solution of Eq. (2.1), the appropriate *initial* and *boundary* conditions must be specified. The initial conditions represent the initial (i.e., determined for $t = 0$) distribution of all n physical state variables for the whole set Ω

$$x'(l,0) = x'_0(l), \tag{2.12}$$

where $x'_0(l)\colon \Omega \to \mathbb{R}^n$ is a given vector function.

By virtue of the transformation given by Eq. (2.6), the initial conditions (2.12) can be expressed in terms of the characteristic state variables as

$$x(l,0) = x_0(l) = S^{-1}Ex'_0(l). \tag{2.13}$$

On the other hand, the boundary conditions represent the requirements to be met by the solution $x'(l,t)$ of Eq. (2.1) at the boundary points of Ω. They can express, e.g., the *boundary feedbacks* and *reflections*, as well as the external *boundary inputs* to the system. In general, these conditions may take the form of a linear combination of the Dirichlet and Neumann boundary conditions, as the so-called boundary conditions of the third kind (Ancona and Coclite 2005; Dooge and Napiorkowski 1987).

As can be shown, based on the method of characteristics, the number of boundary conditions specified in terms of the characteristic state variables at $l = 0$ should be equal to the number p of positive eigenvalues of the system, whereas the number of boundary conditions specified for $l = L$ should equal the number $(n - p)$ of its negative eigenvalues (Evans 1998; Holderith and Réti 1981; Thompson 1987). Therefore, introducing the following partition of the vector of the characteristic state variables

$$x^+ = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}, \quad x^- = \begin{bmatrix} x_{p+1} \\ \vdots \\ x_n \end{bmatrix}, \tag{2.14}$$

the boundary conditions of Dirichlet type which are often encountered for the considered class of systems can be written in the following compact way (see, e.g., Bastin et al. 2009; Diagne et al. 2012; Gunzburger and Plemmons 1979; Xu and Sallet 2002; Ziółko 2000):

$$\begin{aligned} \begin{bmatrix} x^+(0,t) \\ x^-(L,t) \end{bmatrix} &= P \begin{bmatrix} x^+(L,t) \\ x^-(0,t) \end{bmatrix} + Ru(t) \\ &= \begin{bmatrix} P_{0L} & P_{00} \\ P_{LL} & P_{L0} \end{bmatrix} \begin{bmatrix} x^+(L,t) \\ x^-(0,t) \end{bmatrix} + \begin{bmatrix} R_0 \\ R_L \end{bmatrix} u(t). \end{aligned} \tag{2.15}$$

The vector function $u(t)\colon [0, +\infty) \to \mathbb{R}^r$ in Eq. (2.15) expresses the inhomogeneity of the boundary conditions which can be identified with r external "characteristic" inputs to the system, including both control signals and external disturbances. The constant matrices $P_{0L} \in \mathbb{R}^{p\times p}$ and $P_{L0} \in \mathbb{R}^{(n-p)\times(n-p)}$ express the feedbacks from the boundary $l = L$ to the boundary $l = 0$ and from $l = 0$ to $l = L$, respectively. The matrices $P_{00} \in \mathbb{R}^{p\times(n-p)}$ and $P_{LL} \in \mathbb{R}^{(n-p)\times p}$ express boundary reflections for $l = 0$ and $l = L$, respectively. Finally, $R_0 \in \mathbb{R}^{p\times r}$ and $R_L \in \mathbb{R}^{(n-p)\times r}$ represent the effect of the characteristic inputs $u(t)$ on the boundary conditions $x^+(0, t)$ and $x^-(L, t)$, respectively. The notion of the characteristic boundary inputs will be discussed in more detail in Sect. 2.4.

The important question is, what should the boundary conditions be in terms of the original physical state variables and not in terms of the characteristic ones, i.e., how to determine the correct type and number of boundary conditions providing a necessary condition between the characteristic variables in $x(l, t)$ and the original variables in $x'(l, t)$ for the given boundary. The above problem has been thoroughly examined in a paper by Guaily and Epstein (2013). According to their approach, one should construct the following Jacobian matrix

$$J = \frac{\partial x'(l, t)}{\partial x(l, t)} = \left(E^{-1}S\right)^T \tag{2.16}$$

and then analyze its partial determinants which cannot be zero for the given set of the original state variables represented by the columns of J.

Assumption 2.8 Further results will be generally based on the assumption that neither boundary feedback nor reflection is present in the system, i.e., $P_{00} = P_{0L} = P_{L0} = P_{LL} = 0$ in Eq. (2.15). In this case the boundary conditions express solely the boundary inputs to the system.

Remark 2.9 As shown by Ziółko in (1989), (1990) and (2000), the Hurwitz stability of the matrix K from Eq. (2.7) ensures the internal stability of the considered hyperbolic systems whereas the Schur stability of the matrix P from Eq. (2.15) provides the boundary stability, both representing the stability criteria for the initial-boundary value problem given by Eqs. (2.7), (2.13) and (2.15). Therefore, by Assumption 2.8 we focus here on the internal stability which, in turn, implies the bounded-input bounded-output (BIBO) stability. This problem will be discussed more comprehensively in Sect. 3.1.1.

2.3 Examples of 2 × 2 Systems

An important class of the considered DPSs is constituted by the systems which can be described, under certain assumptions, by the system of *two* equations of hyperbolic type with *two* conserved state variables. The following are some typical examples:

- Thin-walled double-pipe heat exchanger with distributed temperatures $\vartheta_1(l, t)$ and $\vartheta_2(l, t)$ of the heating and the heated fluid (Abu-Hamdeh 2002; Ansari and Mortazavi 2006; Arbaoui et al. 2007; Bagui et al. 2004; Bartecki 2007; Bunce and Kandlikar 1995; Das and Dan 1996; Delnero et al. 2004; Gvozdenac 1990; Gvozdenac 2012; Jäschke and Skogestad 2014; Łach and Pieczka 1985; Maidi et al. 2010; Malinowski and Bielski 2004; Roetzel and Xuan 1999; Skoglund et al. 2006; Taler 2011; Yin and Jensen 2003; Zavala-Río et al. 2009),

- Transportation pipeline with distributed pressure $p(l, t)$ and flow $q(l, t)$ of the transported medium (Bartecki 2009b; Blažič et al. 2004; Covas et al. 2005; Kowalczuk and Gunawickrama 2004; Krichel and Sawodny 2014, Lee et al. 2006; Lopes dos Santos et al. 2010; Matko et al. 2006; Oldenburger and Goodson 1963; Reddy et al. 2011; Rojek 2002; Ułanowicz 2009; Zavala 2014; Zecchin 2010; Ziółko 2000),
- Unidirectional open channel flow in navigable rivers and irrigation channels described by the linearized Saint-Venant equations with the discharge $q(l, t)$ and water depth $h(l, t)$ (Bastin et al. 2009; Bounit 2003; Diagne and Sène 2013; Martins et al. 2012; Litrico and Fromion 2009b),
- Electrical transmission line with distributed voltage $u(l, t)$ and current $i(l, t)$ (Górecki et al. 1989; Miano and Maffucci 2001; Mitkowski 2014; Wang et al. 2005; Ziółko 2000).

Considering the motivation behind this book (see Chap. 1), only the two first examples will be discussed in the following subsections.

2.3.1 Double-Pipe Heat Exchanger

The considered heat exchanger consists of two concentric pipes (tubes) containing fluids flowing from the inlet of each tube toward its outlet (Fig. 2.1). In order to avoid ambiguity, the external tube will be referred to as *shell* and the internal simply as *tube*. Heat is transferred from one fluid to the other through the *wall* of the tube, either from tube side to shell side or vice versa. Depending on the flow arrangement, the fluids can enter the shell and the tube from the same or from the opposite ends of the exchanger. The first configuration is commonly known as *parallel-flow*, while the second is usually referred to as *counter-flow*.

In order to develop the mathematical model of the exchanger based on the energy balance equations, the following simplifying assumptions are made:

- Exchanger is perfectly insulated from the environment.
- There are no internal thermal energy sources.
- The flows are sufficiently turbulent to cause effective heat transfer.
- Only forced heat convection is considered (i.e., longitudinal heat conduction within the fluids and wall is neglected).
- Heat accumulation in the tube is neglected (thin-walled heat exchanger).[1]
- Pressure drops of fluids along the shell and the tube are negligible.

[1] Analysis of the thick-walled double-pipe heat exchanger has been performed, e.g., in (Bartecki 2015c).

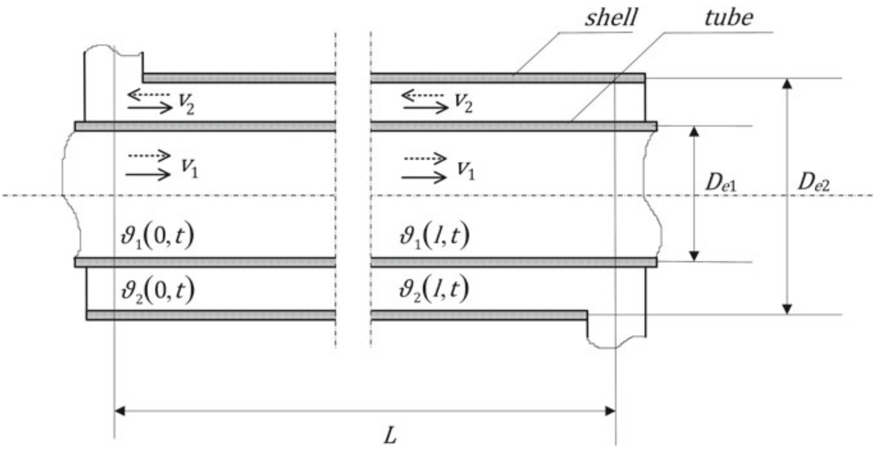

Fig. 2.1 Schematic of a double-pipe heat exchanger. v_1, v_2—tube- and shell-side fluid velocities; ϑ_1, ϑ_2—tube- and shell-side fluid temperatures; L—heat exchanger length; D_{e1}, D_{e2}—tube and shell diameters. Solid arrows show flow directions for the parallel-flow mode, whereas dotted ones—for the counter-flow mode

- The densities and heat capacities of the shell, tube and fluids are time and space invariant.
- The convective heat transfer coefficients are constant and uniform over each surface.

According to the above assumptions, double-pipe heat exchanger depicted in Fig. 2.1 is governed, based on the thermal energy balance equations, by the following PDE system (Bartecki 2007; Delnero et al. 2004; Gvozdenac 1990; Maidi et al. 2010; Zavala-Río et al. 2009):

$$\frac{\partial \vartheta_1(l,t)}{\partial t} + v_1 \frac{\partial \vartheta_1(l,t)}{\partial l} = \alpha_1\big(\vartheta_2(l,t) - \vartheta_1(l,t)\big), \tag{2.17}$$
$$\frac{\partial \vartheta_2(l,t)}{\partial t} + v_2 \frac{\partial \vartheta_2(l,t)}{\partial l} = \alpha_2\big(\vartheta_1(l,t) - \vartheta_2(l,t)\big), \tag{2.18}$$

where α_1 and α_2 represent generalized parameters including heat transfer coefficients, fluid densities, specific heats, and geometric dimensions of the exchanger. As can be seen, Eqs. (2.17), (2.18) are given directly in the weakly coupled hyperbolic form of Eq. (2.7).

Comparing these equations to Eqs. (2.1), (2.7), one obtains the following vector of the state variables:

$$x'(l,t) = x(l,t) = \begin{bmatrix} \vartheta_1(l,t) \\ \vartheta_2(l,t) \end{bmatrix}, \tag{2.19}$$

and the following matrices of constant coefficients:

$$E=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad F=\Lambda=\begin{bmatrix}v_1 & 0\\ 0 & v_2\end{bmatrix}, \quad H=K=\begin{bmatrix}-\alpha_1 & \alpha_1\\ \alpha_2 & -\alpha_2\end{bmatrix}. \tag{2.20}$$

It is apparent that the system eigenvalues, i.e., the characteristic speeds λ_1 and λ_2 are given here directly by the fluid velocities v_1 and v_2, as mentioned in Remark 2.7. Depending on the flow configuration, we obtain $p = n = 2$ in Eq. (2.9) for the parallel-flow, which corresponds to $\lambda_1 = v_1 > 0$, $\lambda_2 = v_2 > 0$ and $p = 1$, $n = 2$ for the counter-flow configuration with $\lambda_1 = v_1 > 0$, $\lambda_2 = v_2 < 0$. Initial conditions in Eq. (2.12) here take the following form:

$$\vartheta_1 (l, 0) = \vartheta_{10} (l)\,, \tag{2.21}$$

$$\vartheta_2 (l, 0) = \vartheta_{20} (l)\,, \tag{2.22}$$

where $\vartheta_{10}(l)$, $\vartheta_{20}(l)$: $\Omega = [0, L] \rightarrow \mathbb{R}$ are functions representing initial temperature profiles along the spatial axis of the exchanger.

On the other hand, the form of the boundary conditions in Eq. (2.15) depends on the flow arrangement of the heat exchanger. Assuming the lack of the boundary feedbacks and reflections, i.e., $P_{00} = P_{0L} = P_{L0} = P_{LL} = 0$, for the case of the parallel-flow, one obtains

$$\vartheta_1 (0, t) = \vartheta_{1i} (t)\,, \tag{2.23}$$

$$\vartheta_2 (0, t) = \vartheta_{2i} (t)\,, \tag{2.24}$$

whereas for the counter-flow configuration

$$\vartheta_1 (0, t) = \vartheta_{1i} (t)\,, \tag{2.25}$$

$$\vartheta_2 (L, t) = \vartheta_{2i} (t)\,, \tag{2.26}$$

with $\vartheta_{1i}(t)$, $\vartheta_{2i}(t)$: $[0, +\infty) \rightarrow \mathbb{R}$ representing time-varying inlet temperatures of the tube-side and shell-side fluids. These temperatures can be taken as the input signals $u'_1(t)$ and $u'_2(t)$ constituting the vector $u'(t) = u(t)$ in Eq. (2.15). For the parallel-flow configuration we obtain $R_0 = I_2$ and for the counter-flow, $R_0 = [1 \;\; 0\,]$ and $R_L = [0 \;\; 1]$.

2.3.2 Transportation Pipeline

A transportation pipeline of length L and diameter D_p with a reservoir in the upstream section and valve in the downstream section is depicted in Fig. 2.2. Pressure value in the inlet point ($l = 0$) of the pipeline is influenced by the variable fluid level $h(t)$ and equals $p_i(t)$. Similarly, the fluid flow $q_o(t)$ at the pipeline outlet ($l = L$) can vary due to the changes in the valve position.

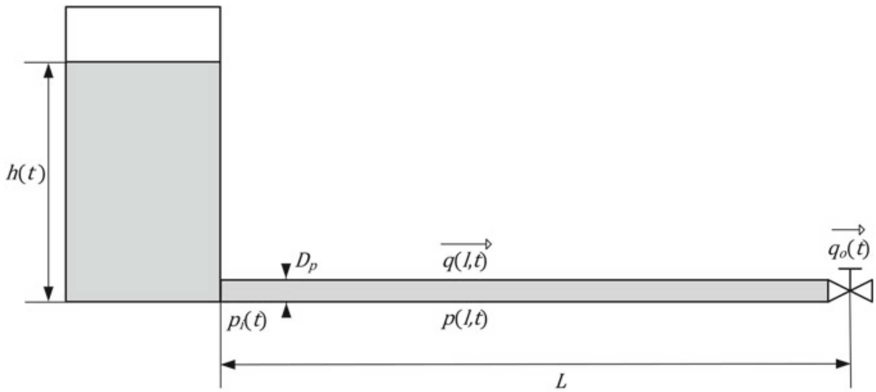

Fig. 2.2 Schematic of a transportation pipeline. $p(l, t)$—fluid pressure; $q(l, t)$—fluid flow; $p_i(t) = p(0, t)$—inlet fluid pressure; $q_o(t) = q(L, t)$—outlet fluid flow; L—pipeline length; D_p—pipeline diameter

In order to develop a mathematical model of the fluid flow in the pipeline, the following assumptions are made:

- The flow is one-dimensional—mass flow q and pressure p depend only on time t and on the geometrical variable l.
- Pipeline is laid horizontally and completely filled up with the transported fluid.
- The fluid is compressible, viscous, homogenous and of constant density ρ.
- The flow is adiabatic and isothermal (i.e., no transfer of thermal energy between fluid and pipeline will be considered, pressure changes of the fluid do not affect its temperature, also temperature changes due to the friction are neglected).
- Friction effects are described by the Darcy–Weisbach equation, with constant value of linear friction coefficient f.

According to the above assumptions, the mathematical model of the flow in the considered pipeline can be described, based on the appropriate momentum and continuity equations, by the following set of PDEs (Bartecki 2009b; Blažič et al. 2004; Covas et al. 2005; Izquierdo and Iglesias 2002; Kowalczuk and Gunawickrama 2004; Oldenburger and Goodson 1963; Ziółko 2000):

$$\frac{A_p}{c^2}\frac{\partial p\,(l,t)}{\partial t}+\frac{\partial q\,(l,t)}{\partial l}=0, \tag{2.27}$$

$$\frac{1}{A_p}\frac{\partial q\,(l,t)}{\partial t}+\frac{\partial p\,(l,t)}{\partial l}=-\frac{f}{2\rho D_p A_p^2}q\,(l,t)\left|q\,(l,t)\right|, \tag{2.28}$$

where $A_p = \pi D_p^2/4$ is the cross-sectional area of the pipeline and c is the speed of sound waves in the transported fluid. The above equations are to be completed by the initial conditions, describing the initial pressure and flow profiles along the pipeline

$$p\left(l,0\right)=p_0\left(l\right), \tag{2.29}$$

$$q\left(l,0\right)=q_0\left(l\right), \tag{2.30}$$

with $p_0(l), q_0(l)$: $\Omega = [0, L] \to \mathbb{R}$. The form of the possible boundary conditions will be discussed at the end of this subsection.

As can be seen, Eqs. (2.27) and (2.28) are strongly coupled due to the presence of both temporal and spatial derivatives of $q(l,t)$ and $p(l,t)$ in each of the equations. Moreover, Eq. (2.28) is semi-linear due to the friction term on its right-hand side. In order to transform these equations into the linear form of Eq. (2.1), they need to be linearized around the steady-state solutions $\bar{p}(l)$ and $\bar{q}(l)$. These solutions can be determined by setting to zero all partial derivatives with respect to t in Eqs. (2.27) and (2.28). Assuming $\bar{q}(l) > 0$, we obtain the following set of ordinary differential equations (ODEs):

$$\frac{d\bar{q}\left(l\right)}{dl}=0, \tag{2.31}$$

$$\frac{d\bar{p}\left(l\right)}{dl}=-\frac{f}{2\rho D_p A_p^2}\bar{q}^2\left(l\right), \tag{2.32}$$

whose solution, assuming constant boundary conditions $\bar{p}(0) = p_i$ and $\bar{q}(L) = q_o$, takes the following form:

$$\bar{q}\left(l\right)=q_o, \tag{2.33}$$

$$\bar{p}\left(l\right)=p_i-\frac{f q_o^2 l}{2\rho D_p A_p^2}, \tag{2.34}$$

representing the steady-state pressure and flow profiles along the pipeline.

By introducing new variables $\tilde{p}(l,t)$ and $\tilde{q}(l,t)$ representing the pressure and flow deviations from the steady state, respectively,

$$\tilde{p}\left(l,t\right)=p\left(l,t\right)-\bar{p}\left(l\right), \tag{2.35}$$

$$\tilde{q}\left(l,t\right)=q\left(l,t\right)-\bar{q}\left(l\right), \tag{2.36}$$

Equations (2.27) and (2.28) can be written in the following linearized form:

$$C_h\frac{\partial\tilde{p}\left(l,t\right)}{\partial t}+\frac{\partial\tilde{q}\left(l,t\right)}{\partial l}=0, \tag{2.37}$$

$$L_h\frac{\partial\tilde{q}\left(l,t\right)}{\partial t}+\frac{\partial\tilde{p}\left(l,t\right)}{\partial l}=-R_h\tilde{q}\left(l,t\right), \tag{2.38}$$

where R_h stands for *hydraulic resistance*, L_h represents *hydraulic inductivity* (inertance), both related to the mass flow rate, and C_h denotes *hydraulic capacitance* divided by c^2

$$R_h = \frac{f q_o}{\rho D_p A_p^2}, \qquad L_h = \frac{1}{A_p}, \qquad C_h = \frac{A_p}{c^2}. \tag{2.39}$$

The initial conditions for the linearized Eqs. (2.37) and (2.38) take, based on Eqs. (2.29) and (2.30) and Eqs. (2.35) and (2.36), the following form:

$$\tilde{p}\,(l, 0) = \tilde{p}_0\,(l) = p_0\,(l) - \bar{p}\,(l)\,, \tag{2.40}$$

$$\tilde{q}\,(l, 0) = \tilde{q}_0\,(l) = q_0\,(l) - \bar{q}\,(l)\,. \tag{2.41}$$

Comparing Eqs. (2.37) and (2.38) to the general strongly coupled Eq. (2.1), one obtains the following vector of the physical state variables:

$$x'(l, t) = \begin{bmatrix} \tilde{p}(l, t) \\ \tilde{q}(l, t) \end{bmatrix}, \tag{2.42}$$

and the matrices of constant coefficients

$$E = \begin{bmatrix} C_h & 0 \\ 0 & L_h \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ 0 & -R_h \end{bmatrix}. \tag{2.43}$$

In order to obtain the linearized equations of the pipeline in the weakly coupled form of Eq. (2.7), the decoupling procedure given by Eqs. (2.3)–(2.8) needs to be applied. According to Eq. (2.3), the eigendecomposition of the following matrix

$$FE^{-1} = \begin{bmatrix} 0 & \dfrac{1}{L_h} \\ \dfrac{1}{C_h} & 0 \end{bmatrix} \tag{2.44}$$

results in the diagonal matrix Λ of its eigenvalues

$$\Lambda = \operatorname{diag}\,(\lambda_1, \lambda_2) = \operatorname{diag}\left(\frac{1}{\sqrt{L_h C_h}}, -\frac{1}{\sqrt{L_h C_h}}\right), \tag{2.45}$$

and in the following matrix S of its eigenvectors:

$$S = \begin{bmatrix} \sqrt{\dfrac{C_h}{L_h}} & -\sqrt{\dfrac{C_h}{L_h}} \\ 1 & 1 \end{bmatrix}, \tag{2.46}$$

with its inverse equal to

$$S^{-1} = \begin{bmatrix} \dfrac{1}{2}\sqrt{\dfrac{L_h}{C_h}} & \dfrac{1}{2} \\ -\dfrac{1}{2}\sqrt{\dfrac{L_h}{C_h}} & \dfrac{1}{2} \end{bmatrix}. \tag{2.47}$$

It should be noted, according to Remark 2.7, that the diagonal entries of the matrix Λ in Eq. (2.45) represent the so-called propagation speeds (also known as phase velocities) of the sound waves travelling in the fluid transported along the pipeline. The first element λ_1 represents a wave traveling in a positive direction, i.e., from $l = 0$ to $l = L$, whereas λ_2 represents a wave traveling in the opposite direction.

By introducing, according to Eq. (2.6), the characteristic state variables

$$x_1(l,t) = \frac{1}{2}\sqrt{L_h C_h}\tilde{p}(l,t) + \frac{1}{2}L_h\tilde{q}(l,t), \tag{2.48}$$

$$x_2(l,t) = -\frac{1}{2}\sqrt{L_h C_h}\tilde{p}(l,t) + \frac{1}{2}L_h\tilde{q}(l,t), \tag{2.49}$$

and determining, based on Eq. (2.8), the matrix K in the following form

$$K = S^{-1}HE^{-1}S = \begin{bmatrix} -\frac{R_h}{2L_h} & -\frac{R_h}{2L_h} \\ -\frac{R_h}{2L_h} & -\frac{R_h}{2L_h} \end{bmatrix}, \tag{2.50}$$

one obtains the weakly coupled equations of the pipeline expressed in terms of its characteristic state variables

$$\frac{\partial x_1(l,t)}{\partial t} + \frac{1}{\sqrt{L_h C_h}}\frac{\partial x_1(l,t)}{\partial l} = -\frac{R_h}{2L_h}x_1(l,t) - \frac{R_h}{2L_h}x_2(l,t), \tag{2.51}$$

$$\frac{\partial x_2(l,t)}{\partial t} - \frac{1}{\sqrt{L_h C_h}}\frac{\partial x_2(l,t)}{\partial l} = -\frac{R_h}{2L_h}x_1(l,t) - \frac{R_h}{2L_h}x_2(l,t), \tag{2.52}$$

with the following initial conditions:

$$x_1(l,0) = x_{10}(l) = \frac{1}{2}\sqrt{L_h C_h}\tilde{p}_0(l) + \frac{1}{2}L_h\tilde{q}_0(l), \tag{2.53}$$

$$x_2(l,0) = x_{20}(l) = -\frac{1}{2}\sqrt{L_h C_h}\tilde{p}_0(l) + \frac{1}{2}L_h\tilde{q}_0(l), \tag{2.54}$$

where $\tilde{p}_0(l)$ and $\tilde{q}_0(l)$ represent the initial pressure and flow deviations given by Eqs. (2.40) and (2.41), respectively.

From the analysis in Sect. 2.2, we have that due to the opposite signs of λ_1 and λ_2 in Eq. (2.45), the boundary conditions expressed in terms of the characteristic state variables x_1 and x_2 should be given for $l = 0$ and $l = L$, respectively, giving as a result

$$x_1(0,t) = \frac{1}{2}\sqrt{L_h C_h}\tilde{p}(0,t) + \frac{1}{2}L_h\tilde{q}(0,t), \tag{2.55}$$

$$x_2(L,t) = -\frac{1}{2}\sqrt{L_h C_h}\tilde{p}(L,t) + \frac{1}{2}L_h\tilde{q}(L,t). \tag{2.56}$$

Furthermore, in order to analyze the possible *physical* boundary conditions, i.e., conditions given in terms of the primary state variables $x_1'(l,t) = \tilde{p}(l,t)$ and $x_2'(l,t) = \tilde{q}(l,t)$, we construct, based on Eq. (2.16), the following Jacobian matrix

$$J = \left(E^{-1}S\right)^T = \begin{bmatrix} \dfrac{1}{\sqrt{L_h C_h}} & \dfrac{1}{L_h} \\ -\dfrac{1}{\sqrt{L_h C_h}} & \dfrac{1}{L_h} \end{bmatrix} \tag{2.57}$$

whose rows refer to the characteristic variables, $x_1(l,t)$ with the boundary condition at $l = 0$ and $x_2(l,t)$ with the boundary condition given for $l = L$. Furthermore, the columns of J refer to the physical state variables $x_1'(l,t) = \tilde{p}(l,t)$ and $x_2'(l,t) = \tilde{q}(l,t)$, respectively. Since the first row of J does not contain any zero entries, it is possible to take as the boundary conditions for $l = 0$ any physical variable, $\tilde{p}(0,t)$ or $\tilde{q}(0,t)$. The same concerns the second row of J which corresponds to the boundary condition imposed for $l = L$, with physical variables $\tilde{p}(L,t)$ or $\tilde{q}(L,t)$. For example, the boundary conditions can take the following form:

$$\tilde{p}\,(0,t) = \tilde{p}_i\,(t)\,, \tag{2.58}$$

$$\tilde{q}\,(L,t) = \tilde{q}_o\,(t)\,, \tag{2.59}$$

which is in accordance with the physical assumptions made at the beginning of this subsection (see Fig. 2.2).

From the above analysis, it follows that the boundary conditions given in terms of the characteristic state variables may contain the boundary values of the original (physical) state variables which may not be specified by the physical boundary conditions. For example, Eqs. (2.55) and (2.56) contain $\tilde{q}(L,t)$ and $\tilde{p}(0,t)$ which do not appear in Eqs. (2.58) and (2.59). The problem will be considered in Sect. 4.5.

2.4 Boundary Input Configurations for 2 × 2 Systems

Motivated by the practical examples discussed in Sect. 2.3, this section introduces two different configurations of Dirichlet boundary inputs for the considered class of 2×2 systems, assuming the typical situation where the physical input vector $u'(t)$ refers to the two boundary values of the state variables. For the first configuration, both boundary inputs, $u_1'(t)$ and $u_2'(t)$, are given for the same edge ($l = 0$) of the spatial domain Ω and for the second, the input functions act on the two different edges, $l = 0$ and $l = L$, respectively. The considerations are based on Assumption 2.8 which states that neither boundary feedback nor reflection is present in the system. Therefore, two definitions are given below in order to distinguish between the two above-mentioned configurations.

Definition 2.10 The physical boundary inputs of the system (2.1) with $n = 2$ state variables and $r = 2$ boundary inputs will be referred to as *collocated* physical boundary inputs of Dirichlet type assuming the following form of the input vector:

$$u'^{+}(t) = \begin{bmatrix} u_1'^{+}(t) \\ u_2'^{+}(t) \end{bmatrix} = \begin{bmatrix} x_1'(0, t) \\ x_2'(0, t) \end{bmatrix}. \tag{2.60}$$

Definition 2.11 The physical boundary inputs of the system (2.1) with $n = 2$ state variables and $r = 2$ boundary inputs will be referred to as *anti-collocated* physical boundary inputs of Dirichlet type assuming the following form of the input vector:

$$u'^{\pm}(t) = \begin{bmatrix} u_1'^{+}(t) \\ u_2'^{-}(t) \end{bmatrix} = \begin{bmatrix} x_1'(0, t) \\ x_2'(L, t) \end{bmatrix}. \tag{2.61}$$

Remark 2.12 In the above two definitions, the partitioning of the input vector $u'(t)$ is introduced following the notation of Eqs. (2.14) and (2.15), depending on whether the given boundary condition refers to $l = 0$ $(+)$ or $l = L$ $(-)$.

Similarly, these two configurations can be formulated in terms of the characteristic state variables of Eq. (2.7). Therefore, two more definitions are introduced below.

Definition 2.13 The boundary inputs of the decoupled system (2.7) with $n = 2$ characteristic state variables and $r = 2$ boundary inputs will be referred to as *collocated* characteristic boundary inputs of Dirichlet type assuming the following form of the characteristic input vector:

$$u^{+}(t) = \begin{bmatrix} u_1^{+}(t) \\ u_2^{+}(t) \end{bmatrix} = \begin{bmatrix} x_1(0, t) \\ x_2(0, t) \end{bmatrix}. \tag{2.62}$$

Definition 2.14 The boundary inputs of the decoupled system (2.7) with $n = 2$ characteristic state variables and $r = 2$ boundary inputs will be referred to as *anti-collocated* characteristic boundary inputs of Dirichlet type assuming the following form of the characteristic input vector:

$$u^{\pm}(t) = \begin{bmatrix} u_1^{+}(t) \\ u_2^{-}(t) \end{bmatrix} = \begin{bmatrix} x_1(0, t) \\ x_2(L, t) \end{bmatrix}. \tag{2.63}$$

Remark 2.15 Taking into account the results of Sect. 2.2, it is evident that the collocated characteristic inputs will be imposed for $\lambda_1 > 0$ and $\lambda_2 > 0$, whereas the anti-collocated ones for $\lambda_1 > 0$ and $\lambda_2 < 0$.

Remark 2.16 It is clear that the boundary input configurations introduced by Definitions 2.13 and 2.14 can be seen as two special cases of the general boundary conditions given by Eq. (2.15), with $r = n = 2$ and $P_{00} = P_{0L} = P_{L0} = P_{LL} = 0$. For the collocated inputs we have $p = 2$ and $R_0 = I_2$, whereas for the anti-collocated ones, $p = 1$, $R_0 = [1 \;\; 0]$ and $R_L = [0 \;\; 1]$.

As mentioned at the beginning of this subsection, the above assumptions about the form of the possible boundary input configurations have some practical motivation which have been presented in Sect. 2.3. For example, in the case of the double-pipe heat exchanger operating in the *parallel-flow* mode, the inlet temperatures of the heated and the heating media are given for the same geometric point of the exchanger, $l = 0$ (see Fig. 2.1). In turn, the temperatures of the fluids flowing into the exchanger operating in the *counter-flow* mode are specified for its two opposite sides, $l = 0$ and $l = L$. Moreover, since the physical equations of the exchanger have the weakly coupled form of Eq. (2.7), its characteristic state variables are equal to the physical ones and the same applies to the boundary input variables: $u_1(t) = u_1'(t) = \vartheta_{1i}(t)$, $u_2(t) = u_2'(t) = \vartheta_{2i}(t)$.

On the other hand, for the case of the strongly coupled system, the relationships between the physical and characteristic boundary inputs can be found based on Eqs. (2.6), (2.60) and (2.62), as shown by the following results.

Result 2.17 For the case of the collocated boundary inputs, the following relationships between the physical and characteristic input vectors hold:

$$u^{+}(t) = S^{-1}Eu'^{+}(t), \tag{2.64}$$

$$u'^{+}(t) = E^{-1}Su^{+}(t), \tag{2.65}$$

assuming $\det(S) \neq 0$ and $\det(E) \neq 0$.

For $\lambda_1 > 0$ and $\lambda_2 < 0$ the appropriate relationships are slightly more complex, as shown below.

Result 2.18 For the anti-collocated boundary inputs, the relationships between the physical and characteristic boundary input vector are as follows:

$$u^{\pm}(t) = \left(S^{-1}E\right)_d u'^{\pm}(t) + \left(S^{-1}E\right)_a x'_{L0}(t) \tag{2.66}$$

with

$$x'_{L0}(t) = \begin{bmatrix} x_1'(L, t) \\ x_2'(0, t) \end{bmatrix}, \tag{2.67}$$

and

$$u'^{\pm}(t) = \left(E^{-1}S\right)_d u^{\pm}(t) + \left(E^{-1}S\right)_a x_{L0}(t) \tag{2.68}$$

with

$$x_{L0}(t) = \begin{bmatrix} x_1(L, t) \\ x_1(0, t) \end{bmatrix}, \tag{2.69}$$

where the subscripts "d" and "a" stand for the diagonal and the antidiagonal parts of the matrices, respectively.

It results from Eqs. (2.66) and (2.67) that for the anti-collocated configuration, the elements of the characteristic input vector $u^{\pm}(t)$ partially depend on the state variables $x'_{L0}(t)$ which are not specified by the physical boundary input vector $u'^{\pm}(t)$. The converse is also true, as shown in Eqs. (2.68) and (2.69). This problem has already been demonstrated on the example of the transportation pipeline with the characteristic boundary Eqs. (2.55) and (2.56) and physical boundary Eqs. (2.58) and (2.59).

2.5 Summary

This chapter started with Sect. 2.1 introducing the mathematical description of the considered class of DPSs in the form of the linear, strongly coupled PDEs of hyperbolic type, as well as its decoupled representation expressed in terms of the characteristic variables. It has been complemented by Sect. 2.2 dealing with the general initial and boundary conditions for the above-mentioned equations. Motivated by the author's research, Sect. 2.3 has introduced two typical examples of 2×2 hyperbolic systems of balance laws—double-pipe heat exchanger and transportation pipeline—whose properties will be analyzed in detail throughout the rest of the monograph. Finally, two typical configurations of boundary inputs have been presented in Sect. 2.4, both in terms of physical and characteristic state variables. The next chapter deals with the state-space representation of the considered hyperbolic systems which is naturally based on the PDE models introduced here.

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