

Chapter 2

Spaces of Analytic Functions

Abstract We present here spaces of analytic functions $\mathbf{A}_{\alpha,d}^n \subset \mathbf{S}^n$ as well as spaces, $\mathbf{A}_{\alpha,d,T}^n \subset \mathbf{S}_T^n$, $n = 1, 2, 3$. In this chapter, we shall study the properties of these spaces, we shall prove in Chap. 3 that if the components of the initial condition vector \mathbf{u}^0 belong to $\mathbf{A}_{\alpha,d}^3$ then each component of $\mathbf{N}\mathbf{u}$ of (1.23) belongs to $\mathbf{A}_{\alpha,d,T}^3$, and we shall furthermore prove in Chap. 4 that the solution to (1.23) belongs to $\mathbf{A}_{\alpha,d,T}^3$, for all T sufficiently small. These spaces are in fact special cases of the spaces \mathbf{S}^n and \mathbf{S}_T^n introduced in Sect. 1.2. They provide several conveniences, such as enabling sharper error bounds and yielding exponential convergence of our approximate solution which we obtain in Chap. 5.

2.1 The Spaces $\mathbf{A}_{\alpha,d}^n$ and $\mathbf{A}_{\alpha,d,T}^n$

Let n denote a positive integer, and let us now define vector spaces of functions $\mathbf{A}_{\alpha,d}^n$ and $\mathbf{A}_{\alpha,d,T}^n$.

Definition 2.1.1. Set $\bar{r}^* = \bar{r} + i\bar{\rho}$, with $\bar{r} \in \mathbb{R}^n$, $\bar{\rho} \in \mathbb{R}^n$, set $r = |\bar{r}|$, and $\rho = |\bar{\rho}|$.

(a) Corresponding to some positive numbers α and d , let $\mathbf{A}_{\alpha,d}$ denote the family of all functions f with the following properties:

(i) *Analyticity property.* There exists a positive number $d' > d$, such that each f is analytic in the domain

$$\mathcal{D}_{d'}^n = \{\bar{r}^* = \bar{r} + i\bar{\rho} \in \mathbb{C}^n : \rho < d'\}; \quad (2.1)$$

and

(ii) *Asymptotic property.* There exist positive numbers $C = C(f, d')$ and $\alpha' > \alpha$, such that for all $\bar{r}^* \in \mathcal{D}_{d'}^n$,

$$|f(\bar{r}^*)| < C \exp(-\alpha' r). \quad (2.2)$$

Notice that if f and g belong either to $\mathbf{A}_{\alpha,d}$ or to $\mathbf{A}_{\alpha,d,T}$, then so does h , where for any constants a , b , and c , $h = af + bg$, or $h = cf$.

- (b) We also define the space $\mathbf{A}_{\alpha,d,T}^n$ of functions $f = f(\bar{r}^*, t^*)$ such that $f(\cdot, t) \in \mathbf{A}_{\alpha,d}^n$ for each fixed $t \in [0, T]$, and such that $f(\bar{r}, \cdot)$ is an analytic and uniformly bounded function of $t = t^*$ in the “eye-shaped” region

$$\mathcal{D}_{d',T} = \{t^* \in \mathbb{C} : |\arg(t^*/(T - t^*))| < d'\}. \quad (2.3)$$

- (c) The spaces $\mathbf{A}_{\alpha,d}^n$ and $\mathbf{A}_{\alpha,d,T}^n$, are normed for $p \in [1, \infty)$ by $\|\cdot\|_p$, i.e.,

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(\bar{r})|^p d\bar{r} \right)^{1/p}, \text{ if } p \in [1, \infty) \text{ and} \quad (2.4)$$

$$\|f\|_\infty = \sup_{\bar{r} \in \mathbb{R}^n} |f(\bar{r})|.$$

and similarly,

$$\|f\|_{p,T} = \left(\int_{\mathbb{R}^n} \int_0^T |f(\bar{r}, t)|^p d\bar{r} dt \right)^{1/p}, \text{ if } p \in [1, \infty) \text{ and} \quad (2.5)$$

$$\|f\|_{\infty,T} = \sup_{\bar{r} \in \mathbb{R}^n, t \in (0,T)} |f(\bar{r}, t)|.$$

The following theorem describes an important and beautiful property of the class of functions $\mathbf{A}_{\alpha,d}^n$.

Theorem 2.1.1. *In the notation of Definition 2.1.1, let $f \in \mathbf{A}_{\alpha,d}^n$. Let \hat{f} denote the Fourier transform of \hat{f} , i.e.,*

$$\hat{f}(\bar{\Lambda}) = \int_{\mathbb{R}^n} f(\bar{r}') \exp(i \bar{\Lambda} \cdot \bar{r}') d\bar{r}'. \quad (2.6)$$

Then $\hat{f} \in \mathbf{A}_{d,\alpha}^n$.

Proof. The one-dimensional version of this result is found in Theorem 26 of Sect. 1.27 of [3]. The proof of the three-dimensional case is similar, and we omit it.

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Upon recalling the inverse Fourier transform formula for \mathbb{R}^3 [see, e.g., (1.21)] we also have by Parseval's theorem that

$$\|\hat{f}\|_2 = (2\pi)^{3/2} \|f\|_2, \quad (2.7)$$

and similarly for $\|\hat{f}\|_{2,T}$.

Definition 2.1.2. Let n denote a positive integer, and set $\mathbf{b} = (b_1, \dots, b_n)$, with b_j nonnegative integers and with $|\mathbf{b}| = \sum_{j=1}^n b_j$, and define

$$D^{\mathbf{b}} f = \frac{\partial^{|\mathbf{b}|} f}{\partial (x^1)^{b_1}, \dots, \partial (x^n)^{b_n}}. \quad (2.8)$$

Theorem 2.1.2. Let d, d', α , and α' defined as in Definition 2.1.1 be given.

- (i) If $f \in A_{\alpha,d}^n$, then $D^{\mathbf{b}} f \in A_{\alpha,d}^n$;
- (ii) If f is analytic in the domain $\mathcal{D}_{d'}^n$ and if for all $\bar{r}^* = \bar{r} + i \bar{\rho}$ in $\mathcal{D}_{d'}^n$ and constants $C > 0$, $m \geq 0$ and $\alpha'' \in (\alpha, \alpha')$, we have $f(\bar{r}^*) \leq C r^m \exp(-\alpha'' r)$, then $f \in A_{\alpha,d}^n$.

Proof. Part (i): In the notation of Definition 2.1.1, taking $d'' \in (d, d')$ and $\alpha'' \in (\alpha, \alpha')$, we have for any $\bar{r}^* = (x^1, x^2, x^3) \in \mathcal{D}_{d''}^n$ and any $\varepsilon \in \min(0, d' - d'', (\alpha' - \alpha''))$, that

$$\frac{\partial f}{\partial x^j} = \frac{k!}{2\pi i} \int_{|\bar{r}^* - \bar{\rho}^*| = \varepsilon} \frac{f(\bar{\rho}^*) d\xi^j}{(\xi^j - x^j)^k}. \quad (2.9)$$

Hence by our assumption that for all $\bar{r}^* \in \mathcal{D}_{\alpha',d'}^n$ we have $|f(\bar{r}^*)| \leq C \exp(-\alpha' r)$, it follows that $|f(\bar{\rho}^*)| \leq C \exp(-\alpha' \rho + \varepsilon) < C \exp(-\alpha'' \rho)$, and so

$$\left| \frac{\partial f}{\partial x^j} \right| \leq \frac{C}{2\pi \varepsilon^k} \exp(-\alpha'' r), \quad (2.10)$$

This proves Theorem 2.1.2 for the case of $f = f(\bar{r}^*)$ when taking one derivative with respect to x^j . The proof for the case of $D^{\mathbf{b}} f$ is similar, just by repeating the one-dimensional argument. The proof for the case of $f = f(\bar{r}^*, t)$ with $t \in [0, T]$ is also similar, and we omit it.

Part (ii). Let us select $\varepsilon > 0$, such that $\alpha'' - \varepsilon > \alpha$, and let us then select $R > 0$, such that if $r \geq R$, then $r^m \leq \exp(\varepsilon r)$. Then we have $C r^m e^{\alpha'' r} < C e^{\beta r}$, for $r \geq R$, where $\beta = \alpha'' - \varepsilon > \alpha$. We now select $C' > 0$ such that $C' e^{\varepsilon R} = C R^m$. Then $C' e^{\beta r} \leq C r^m e^{\alpha'' r}$ for $r \in (0, \infty)$.

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2.2 Denseness of $A_{\alpha,d}^n$ in S^n

As we already mentioned, our preference is to work with the spaces $A_{\alpha,d}^n$, not only for computing the solution of the N-S equations efficiently and accurately, but also for obtaining a simpler proof of existence of the solution to the equations. We first show in this section that the Sinc spaces are dense in the spaces S^n . We then use this result in Sect. 2.2.2 to show that the spaces $A_{\alpha,d}^n$ defined above are dense in the spaces S^n .

Now, let $h > 0$, let $k \in \mathbb{Z}$, let $\xi \in \mathbb{C}$, let $\mathbf{k} \in \mathbb{Z}^n$, and let $\bar{r} \in \mathbb{R}^n$. Let us define the Sinc function $S(k, h)(\xi)$ in one dimension and a product $\mathcal{S}(\mathbf{k}, h)(\bar{r}^*)$, in n dimensions, where now $\bar{r}^* \in \mathbb{C}^n$.

$$S(k, h)(x) = \frac{\sin\left(\frac{\pi}{h}(x - kh)\right)}{\frac{\pi}{h}(x - kh)}, \quad (2.11)$$

$$\mathcal{S}(\mathbf{k}, h)(\bar{r}^*) = \prod_{j=1}^n S(k_j, h)(\xi^j), \quad \xi^j \in \mathbb{C}.$$

2.2.1 Denseness of Sinc Approximation in S^n

We shall prove the denseness of Sinc approximation only with respect to the “sup” norm; we omit the proofs of \mathbf{L}^2 —approximation at this time, since such proofs follow almost verbatim from the \mathbf{L}^∞ ones of this section. We thus prove only the following theorem in the remainder of this section.

Theorem 2.2.1. *Let $\bar{r} = (x^1, \dots, x^n) \in \mathbb{R}^n$, let $g \in \mathbf{S}^n$, let $h > 0$, let N be a positive integer, and set*

$$g_{N,h}(\bar{r}) = \sum_{\mathbf{k} \in \mathbb{Z}_N^n} g(\mathbf{k}h) \mathcal{S}(\mathbf{k}, h)(\bar{r}), \quad (2.12)$$

where

$$\mathbb{Z}_N^n = \{\mathbf{k} = (k^1, \dots, k^n) \in \mathbb{Z}^n : -N \leq k^j \leq N, j = 1, \dots, n\}. \quad (2.13)$$

Given any positive number ε , we can select $h > 0$ and N such that

$$\|g - g_{N,h}\|^\infty = \sup_{\bar{r} \in \mathbb{R}^n} |g(\bar{r}) - g_{N,h}(\bar{r})| < \varepsilon. \quad (2.14)$$

We split the proof of this theorem into the proofs of some lemmas.

We omit the proof of the following lemma since the (a)-Part of it is well known ([2], Theorem 1.2.1), and since the (b)-Part follows directly from the (a)-Part.

Lemma 2.2.1.

(a) *If $(x, y) \in \mathbb{C} \times (-\pi/h, \pi/h)$, then*

$$e^{ixy} = \sum_{j \in \mathbb{Z}} S(j, h)(x) e^{ijhy}. \quad (2.15)$$

(b) Let Q^n be defined by

$$Q^n = \{\bar{\rho} = (\xi^1, \dots, \xi^n) \in \mathbb{R}^n : |\xi^j| < \pi/h, j = 1, \dots, n\}. \quad (2.16)$$

Then we have the identity

$$\exp(i \bar{r} \cdot \bar{\rho}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathcal{S}(\mathbf{k}, h)(\bar{r}) \exp(i j h \bar{\rho} \cdot \mathbf{k}) \quad (2.17)$$

for all $(\bar{r}, \bar{\rho}) \in \mathbb{C}^n \times Q^n$.

Remark 2.2.1. The function on the right-hand side of (2.15) can be extended as a function of y to the real line \mathbb{R} , where for arbitrary integer m it is a periodic copy of $e^{i x y}$ on $(-\pi/h, \pi/h)$ to the interval $((2m-1)\pi/h, (2m+1)\pi/h)$. The infinite series on the right-hand side of (2.15) is discontinuous at each of the points $(2m+1)\pi/h$ where it takes on the value $\cos(\pi x)$. In particular, both functions, that on the left-hand side of (2.15) and that on the right-hand side, are bounded by 1 on $\mathbb{R} \times \mathbb{R}$. Similarly, both sides of (2.17) are identically equal on $\mathbb{C} \times Q^n$, and also the right-hand side of (2.17) has periodic extension to all of \mathbb{R}^n , similar to that of (2.15) for the $n = 1$ case, and so both sides of (2.17) are bounded by 1 on $\mathbb{R}^n \times \mathbb{R}^n$.

Let $g_{N,h}$ be defined as in (2.12) and set

$$g_h(\bar{r}) = \lim_{N \rightarrow \infty} g_{N,h}(\bar{r}). \quad (2.18)$$

Lemma 2.2.2. Let $g \in S^n$, and let g_h be defined as in (2.18). Given $\varepsilon > 0$ there exists $h > 0$ such that

$$\|g - g_h\|^\infty = \sup_{\bar{r} \in \mathbb{R}^n} |g(\bar{r}) - g_h(\bar{r})| < \frac{\varepsilon}{2}. \quad (2.19)$$

Proof. Let \hat{g} denote the n -dimensional Fourier transform of g , i.e., with \bar{r} and $\bar{\rho}$ in \mathbb{R}^n ,

$$\hat{g}(\bar{r}) = \int_{\mathbb{R}^n} \exp(i \bar{r} \cdot \bar{\rho}) g(\bar{\rho}) d\bar{\rho}. \quad (2.20)$$

It then follows immediately, upon using (2.17) and applying the inverse of the Fourier transform formula of (2.20), that

$$\begin{aligned} & g(\bar{r}) - g_h(\bar{r}) \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\exp(-i \bar{r} \cdot \bar{\rho}) - \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathcal{S}(\mathbf{k}, h)(\bar{r}) \exp(-i h \mathbf{k} \cdot \bar{\rho}) \right) \\ & \quad \cdot \hat{g}(\bar{\rho}) d\bar{\rho}. \end{aligned} \quad (2.21)$$

It is well known ([1], Corollary 3.2.1) that if $g \in \mathbf{S}^n$, then its Fourier transform, $\hat{g} \in \mathbf{S}^n$. Hence there exist constants $C > 0$ and $\beta > 0$, such that

$$|\hat{g}(\rho)| \leq C (1 + \rho)^{-n-\beta}. \quad (2.22)$$

Since the difference between each side of (2.21) is zero on Q^n and is bounded by 2 on $\mathbb{R}^n \setminus \overline{Q^n}$, and since $0 \leq \rho^{n-1} < (1 + \rho)^{n-1}$, we have, with Ω_n denoting¹ the “surface area” of the unit ball in \mathbb{R}^n , that

$$\begin{aligned} \|g - g_h\|^\infty &\leq \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n \setminus Q^n} g(\bar{\rho}) d\bar{\rho} \right| \\ &\leq \frac{2C}{(2\pi)^n} \int_{\mathbb{R}^n \setminus Q^n} (1 + \rho)^{-n-\beta} d\bar{\rho} \\ &< \frac{2C}{(2\pi)^n} \int_{\pi/h}^\infty (1 + \rho)^{-n-\beta} d\bar{\rho} \\ &= \frac{2C \Omega_n}{(2\pi)^n} \int_{\pi/h}^\infty \rho^{n-1} (1 + \rho)^{-n-\beta} d\rho \\ &< \frac{2C \Omega_n}{(2\pi)^n} \int_{\pi/h}^\infty (1 + \rho)^{-\beta-1} d\rho \\ &= \frac{2C \Omega_n}{(2\pi)^n \beta (1 + \pi/h)^\beta}. \end{aligned} \quad (2.23)$$

Under our assumption that $\beta > 0$, the right-hand side of (2.23) clearly approaches 0 as $h \rightarrow 0$. Hence, given any $\varepsilon > 0$, we can determine $h > 0$, such that $\|g(\bar{r}) - g_h(\bar{r})\|^\infty < \varepsilon/2$.

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Lemma 2.2.3. *Let $a > 0$, $\gamma > 0$, set $w_\gamma(a, t) = (1 + (a^2 + t^2)^{1/2})^{-\gamma}$, assume that $h > 0$, and that N is a positive integer. Then,*

(i)

$$\sum_{|j| > N} w_{\gamma+1}(a, jh) \leq \frac{2}{\gamma h} w_\gamma(a, Nh). \quad (2.24)$$

¹ $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$.

(ii) If $\gamma \leq 2(1 + a)$, then

$$\sum_{j \in \mathbb{Z}} w_{\gamma+1}(a, jh) \leq \frac{4}{h\gamma} w_{\gamma}(a, 0). \quad (2.25)$$

Proof. Part (i). Note that $w_{\gamma+1}(a, t)$ is an even function of t which is monotonically decreasing on $(0, \infty)$. Since $|t|/(a^2 + t^2)^{1/2} < 1$,

$$\begin{aligned} \sum_{|j| > N} w_{\gamma+1}(a, jh) &\leq \frac{2}{h} \int_{Nh}^{\infty} w_{\gamma+1}(a, t) dt \\ &\leq \int_{Nh}^{\infty} \frac{t}{(a^2 + t^2)^{1/2}} w_{\gamma+1}(a, t) dt \\ &= \frac{2}{h\gamma} w_{\gamma}(a, Nh). \end{aligned} \quad (2.26)$$

Part (ii). Set

$$\begin{aligned} I(w) &= \frac{1}{h} \int_{\mathbb{R}} w_{\gamma+1}(a, t) dt \\ J(w) &= \sum_{j \in \mathbb{Z}} w_{\gamma+1}(a, jh). \end{aligned} \quad (2.27)$$

Then $J(w) \leq I(w) + |J(w) - I(w)|$.

Let us first bound $I(w)$. Inserting the factor $|t|/(a^2 + t^2)^{1/2} < 1$ into the integral in (2.27) yields

$$\begin{aligned} I(w) &\leq \int_{\mathbb{R}} \frac{|t|}{(a^2 + t^2)^{1/2}} (1 + (a^2 + t^2)^{1/2})^{-1-\gamma} dt \\ &= 2 \int_0^{\infty} \frac{t}{(a^2 + t^2)^{1/2}} (1 + (a^2 + t^2)^{1/2})^{-1-\gamma} dt \\ &= \frac{2}{\gamma} w_{\gamma}(a, 0) = \frac{2}{\gamma(1+a)^{\gamma}}. \end{aligned} \quad (2.28)$$

Next, for any $\psi \in C^1[0, h]$ we have the easily verified identity

$$\frac{h}{2} (\psi(0) + \psi(h)) = \int_0^h E_h(t) \psi'(t) dt, \quad (2.29)$$

where $E_h(t)$ is defined on $(0, h)$ by $E_h(t) = t - h/2$. We can extend the definition of $E_h(t)$ to all of \mathbb{R} by setting $E_h(t) = E_h(t - mh)$ on $(mh, (m+1)h)$, with m

an arbitrary integer. We furthermore set $E_h(mh) = 0$, $m \in \mathbb{Z}$, and we then have $\sup_{t \in \mathbb{R}} |E_h(t)| \leq h/2$.

Notice now that $w'_{\gamma+1}(t) > 0$ if $t < 0$, and $w'_{\gamma+1}(t) < 0$ if $t > 0$, and it follows, thus, from the definition of $J(w)$ given in (2.27) and our remarks following (2.29) that

$$|I(w) - J(w)| \leq -2 \int_0^\infty w'_{\gamma+1}(a, t) dt = 2 w_{\gamma+1}(a, 0). \quad (2.30)$$

If we now add the right-hand sides of (2.28) and (2.30) we find, under the assumptions made in the statement of Lemma 2.2.2 (ii), that

$$\begin{aligned} J(w) &\leq I(w) + |I(w) - J(w)| \leq \frac{2}{h \gamma} w_\gamma(a, 0) + 2 w_{\gamma+1}(a, 0) \\ &\leq \frac{4}{h \gamma} w_\gamma(a, 0). \end{aligned} \quad (2.31)$$

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Lemma 2.2.4. *Let g_h and $g_{N,h}$ be defined as in (2.12). If \mathbb{Z}_N^n is defined as in Theorem 2.2.1 and if h is selected as in Lemma 2.2.2, then we can select $N > 0$ such that*

$$\|g_h - g_{N,h}\|^\infty \leq \frac{\varepsilon}{2}. \quad (2.32)$$

Proof. We shall again use our above definition of $w_{\gamma+1}$ i.e., taking $r = |\bar{r}|$, we set $w_{\gamma+1}(r) = (1+r)^{-\gamma-1}$, since it is now convenient to take $a = 0$ and $\gamma + 1 = n + \beta$, with $\beta > 0$. We can then write the difference $g_h - g_{N,h}$ as a telescoping sum,

$$\begin{aligned} \|g_h - g_{N,h}\|^\infty &= \sup_{\bar{r} \in \mathbb{R}^n} \left| \sum_{\mathbf{k} \in \mathbb{Z}^n \setminus \mathbb{Z}_N^n} g(h\mathbf{k}) \mathcal{S}(\mathbf{k}, h)(\bar{r}) \right| \\ &\leq C \sum_{\mathbf{k}^n \in \mathbb{Z}^n \setminus \mathbb{Z}_N^n} w_{n+\beta}(h|\mathbf{k}^n|) \\ &\leq C \sum_{\ell=1}^n \sigma^{(\ell)} \sum_{|k^\ell| > N} w_{n+\beta}(h|\mathbf{k}^{(\ell)}|). \end{aligned} \quad (2.33)$$

In this notation, the operator $\sigma^{(\ell)}$ denotes a product of $n - 1$ sums, with $\ell - 1$ of them taken over $\mathbb{Z}_N^{\ell-1}$ and the other $n - \ell$ of them taken over $\mathbb{Z}^n \setminus \mathbb{Z}_N^\ell$, where \mathbb{Z}_N^{n-1} is defined as in Theorem 2.2.1. More specifically, we can write $\sigma^\ell = \sum_{\mathbb{Z}_N^{\ell-1}} \sum_{\mathbb{Z}^n \setminus \mathbb{Z}^\ell}$ (with these sums being replaced by “1” when $\ell - 1 = 0$ and when $n - \ell = 0$).

Since $w_{n+\beta}(r) > 0$, we increase the right-hand side of (2.33) by replacing each truncated sum $\sum_{\mathbf{k}^\ell \in \mathbb{Z}_N^\ell}$ with $\sum_{\mathbf{k}^\ell \in \mathbb{Z}^\ell}$, $\ell = 1, \dots, n-1$. Similarly (note, at the symmetry and positivity of $w_{n+\beta}(r)$ enable us to replace each double sum $\sum_{k^1 \in \mathbb{Z}} \sum_{|k_\ell| > N}$ occurring in the sum line of $\sigma^{(\ell)}$ with $\sum_{\ell \in \mathbb{Z}} \sum_{|k^1| > N}$, for $\ell = 2, \dots, n$. Upon doing this, and taking $\mathbf{k}^{n-1} = \{(k^2, \dots, k^n) \in \mathbb{Z}^{n-1}\}$, each ℓ th row of sums becomes the same, i.e., we get n such sums, so that

$$\|g_h - g_{N,h}\|^\infty \leq C n \sum_{\mathbf{k}^{n-1} \in \mathbb{Z}^{n-1}} \sum_{|k^1| > N} w_{n+\beta}(h |k^1, \mathbf{k}^{n-1}|). \quad (2.34)$$

Applying Lemma 2.2.3 (i) to bound the sum with respect to k^1 on the right of (2.34), we get

$$\|g_h - g_{N,h}\|^\infty \leq \frac{2 C n}{(n + \beta - 1) h} \sum_{\mathbf{k}^{n-1} \in \mathbb{Z}^{n-1}} w_{n+\beta-1}(h |N, \mathbf{k}^{n-1}|). \quad (2.35)$$

We next successively apply Lemma 2.2.3 (ii) to each single sum in the product of $n-1$ sums of (2.35), one at a time, under the assumption that $(1 + N h) > \beta$. After applying Lemma 2.2.3 (ii) to the first sum, the resulting (2.35) becomes

$$\begin{aligned} & \|g_h - g_{N,h}\|^\infty \\ & \leq \frac{4 (2 C n)}{(n + \beta - 1)(n + \beta - 2) h^2} \sum_{\mathbf{k}^{n-2} \in \mathbb{Z}^{n-2}} w_{n+\beta-2}(h |N, 0, \mathbf{k}^{n-2}|). \end{aligned} \quad (2.36)$$

Hence repeating this process another $n-2$ times, we get our final result,

$$\|g_h - g_{N,h}\|^\infty \leq \frac{4^{n-1} (2 C n)}{h^n (\beta)_n (1 + N h)^\beta}, \quad (2.37)$$

in which $(\beta)_n = \beta(\beta+1)\dots(\beta+n-1)$.

Since h is fixed, it is clear that the right-hand side of (2.37) approaches zero as $N \rightarrow \infty$, and we can thus select N so that $2(1 + N h) > \beta$, and also, so that the right-hand side of (2.37) is less than $\varepsilon/2$.

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By combining the results of Lemmas 2.2.2 and 2.2.4 we get $\|g - g_{N,h}\|^\infty \leq \|g - g_h\|^\infty + \|g_h - g_{N,h}\|^\infty < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

This completes the proof of Theorem 2.2.1. ■

2.2.2 Denseness of the Space $\mathbf{A}_{\alpha,d}^n$ in \mathbf{S}^n

Let \bar{r} and $\bar{\rho}$ belong to \mathbb{Z}_N^n , let $\mathcal{S}(\mathbf{k}, h)(\bar{r})$ be defined as in the above subsection, let $g \in \mathbf{S}^n$, let \mathbb{Z}_N^n be defined as in (2.13), and for arbitrary fixed $\bar{\rho} \in \mathbb{R}^n$, let us consider the Sinc approximation of $\kappa(g, \bar{\rho}, \bar{r}) \equiv \exp(-|\bar{r} - \bar{\rho}|^2) g(\bar{r})$, i.e., let us examining the difference

$$E_{N,h}(\kappa) = \sup_{\bar{r} \in \mathbb{R}^n} \left| \kappa(g, \bar{\rho}, \bar{r}) - \sum_{\mathbf{k} \in \mathbb{Z}_N^n} \kappa(g, \bar{r}, h \mathbf{k}) \mathcal{S}(\mathbf{k}, h)(\bar{r}) \right|. \quad (2.38)$$

Evidently, given any $\bar{\rho} \in \mathbb{R}^n$, $\kappa \in \mathbf{S}^n$ whenever $g \in \mathbf{S}^n$. Hence, it follows that Theorem 2.2.1 applies to this function κ uniformly, for $\bar{\rho} \in \mathbb{R}^n$. We thus take $\bar{\rho} = \bar{r}$ in (2.38), to get $\kappa(g, \bar{r}, \bar{r}) = g(\bar{r})$ enabling us to state the following result:

Theorem 2.2.2. *Given any $g \in \mathbf{S}^n$, and given any positive number ε , there exists a positive integer N , constants $c_{\mathbf{k}}$, and functions $w_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}_N^n$ with each of the $w_{\mathbf{k}}$ belonging to $\mathbf{A}_{\alpha,d}^n$, such that*

$$\left\| g - \sum_{\mathbf{k} \in \mathbb{Z}_N^n} c_{\mathbf{k}} w_{\mathbf{k}} \right\|^\infty < \varepsilon. \quad (2.39)$$

Proof. The above description of κ taken together with Theorem 2.2.1 enables us to state that given any $\varepsilon > 0$, there exists a number $h > 0$ and a positive integer N such that

$$\sup_{\bar{r} \in \mathbb{R}^n} \left| g(\bar{r}) - \sum_{\mathbf{k} \in \mathbb{Z}_N^n} g(h \mathbf{k}) \exp(-|\bar{r} - h \mathbf{k}|^2) \mathcal{S}(\mathbf{k}, h)(\bar{r}) \right| < \varepsilon. \quad (2.40)$$

The proof is completed upon noting that each of the functions

$$w_{\mathbf{k}}(\bar{r}) = \exp(-|\bar{r} - h \mathbf{k}|^2) \mathcal{S}(\mathbf{k}, h)(\bar{r}) \quad (2.41)$$

belongs to $\mathbf{A}_{\alpha,d}^n$ for all positive numbers α and d .

■

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<http://www.springer.com/978-3-319-27524-6>

Navier–Stokes Equations on $\mathbb{R}^3 \times [0, T]$

Stenger, F.; Tucker, D.; Baumann, G.

2016, X, 226 p. 25 illus. in color., Hardcover

ISBN: 978-3-319-27524-6