Preface

In this monograph we study the properties of solutions of the Navier–Stokes (N–S) partial differential equations (PDE) on $(x, y, z, t) \in \mathbb{R}^3 \times (0, T)$. Initially we convert the PDE to a system of integral equations (IE). We then describe spaces $\mathbf{A}$ of analytic functions that house solutions of this equation, and we show that these spaces of analytic functions are dense in the spaces $\mathbf{S}$ of rapidly decreasing and infinitely differentiable functions. These spaces are defined more explicitly later in this monograph. Some reasons for doing this are the following:

1. The functions of $\mathbf{S}$ are nearly always conceptual rather than explicit, i.e., relatively few such explicit functions are known, and except in concept, they differ from functions of calculus, which are generally analytic.
2. Initial and boundary conditions of solutions of PDE are usually given by scientists of applications, and as such, they are nearly always piecewise analytic, and in this case the solutions have the same properties.
3. When methods of approximation are applied to functions of $\mathbf{A}$, they converge at an exponential rate, whereas methods of approximation applied to the functions of $\mathbf{S}$ converge only at a polynomial rate.
4. The space $\mathbf{A}$ also provides other conveniences, such as enabling sharper bounds on the solution, enabling easier existence proofs, and enabling a more accurate and more efficient method of solution including accurate error bounds—all of which are included in this monograph.

Following our proofs of denseness, we prove the existence of a solution of the IE in the space of functions $\mathbf{A} \cap \mathbb{R}^3 \times (0, T)$, and we provide an explicit novel algorithm based on Sinc approximation and Picard-like iteration for computing the solution.

We also provide an explicit *Mathematica* program for computing the solution based on our approximation procedure, given the initial divergence-free velocity, and we provide explicit illustrations of our computed solution.

More specifically, the problem which we shall analyze and solve numerically in this monograph is the PDE problem as described by Fefferman [1] for the space $\mathbf{S}$, i.e.,

$$\frac{\partial u^j}{\partial t} - \varepsilon \,\Delta\, u^j = -\sum_{k=1}^{3} u^k \,\frac{\partial u^j}{\partial x^k} - \frac{\partial p}{\partial x^j}, \quad (\mathbf{r}, t) \in \mathbb{R}^3 \times \mathbb{R}_+. \tag{1}$$

This problem is to be solved subject to the divergence-free condition

$$\operatorname{div} \mathbf{u} = \sum_{k=1}^{3} \frac{\partial u^j}{\partial x^j} = 0, \quad (\bar{r}, t) \in \mathbb{R}^3 \times \mathbb{R}_+, \tag{2}$$

and subject to initial conditions

$$\mathbf{u}^0(\bar{r}) = \mathbf{u}(\bar{r}, 0) \quad \mathbf{r} \in \mathbb{R}^3. \tag{3}$$

Here, $\mathbf{u} = \left(u^1, u^2, u^3\right)$ denotes a velocity vector of flow, p denotes the pressure, $\mathbf{u}^0 = \left(u^{0,1}, u^{0,2}, u^{0,3}\right)$ is a given divergence-free velocity field on $\mathbb{R}^3$, ε is a positive coefficient (the viscosity), and Δ denotes the Laplacian, $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{(\partial x^i)^2}$.

This monograph deals mainly with the solution of the above PDE using its corresponding integral equation formulation, due to recent developments enabling much more efficient and much more rapidly convergent solutions, making it possible for us to obtain solutions over infinite domains [2]. We thus first derive the following integral equation (IE), which can be written in the operator form

$$\mathbf{u} = \mathbf{v} + \mathbf{N}\,\mathbf{u}, \tag{4}$$

where the terms on the right-hand side are defined as follows:

$$\begin{aligned} \mathbf{v}(\bar{r}, t) &= \int_{\mathbb{R}^3} \mathscr{G}(\bar{r} - \bar{r}', t)\, \mathbf{u}^0(\bar{r})\, d\bar{r}' \\ \mathbf{N}\,\mathbf{u}(\bar{r}, t) &= \int_0^t \int_{\mathbb{R}^3} \left\{ \left(\left(\nabla' \mathscr{G}\right) \cdot \mathbf{u}(\bar{r}', t') \right) \mathbf{u}(\bar{r}', t') \right. \\ &\quad \left. + \left(\nabla' \mathscr{G}\right)\; p(\bar{r}', t') \right\} d\bar{r}'\, dt'. \end{aligned} \tag{5}$$

In (5) ∇' indicates the gradient taken with respect to $\bar{r}'$, and we have written $\mathscr{G}$ for $\mathscr{G}(\bar{r} - \bar{r}', t - t')$.

According to [1], the initial condition $\mathbf{u}^0$ must belong to a class of functions $\mathbf{S}^3$ which are infinitely differentiable with respect to all variables and which are rapidly decreasing with respect to each spacial variable on the real line. This class is described in more detail in Sect. 1.2 below. We also introduce in Chap. 2 a class of analytic functions $\mathbf{A}$ which is a subclass of $\mathbf{S}$, which we prove to be dense in the class $\mathbf{S}$, and to which the solutions of (4) belong to whenever $\mathbf{u}^0 \in \mathbf{A}$.

After proving the denseness of the class $\mathbf{A}$ in $\mathbf{S}$, we prove in Chap. 3 that the integral part of the above IE maps functions of $\mathbf{A}$ back into $\mathbf{A}$; in Chap. 4 we prove that if the initial condition vector $\mathbf{u}^0$ belongs to $\mathbf{A}$ and is divergence-free, then the above (IE) has a solution in $\mathbf{A}$ for all T sufficiently small; in Chap. 5 we introduce an iterative method of solution of the IE; and in Chap. 6 we provide two explicit examples and its numerical solution. Appendix A provides a detailed step-by-step description of our method of solution including an explicit *Mathematica* program based on our explicit algorithmic procedure. Appendix B contains for demonstration purposes an explicit example of a result data file generated by our algorithm.

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