

# Preface

The Laplacian, like few other objects, is nearly ubiquitous in mathematics. The aim of this book is to provide an introduction to the study of the spectrum of the Laplacian on *hyperbolic surfaces*. These are the Riemannian surfaces of constant negative curvature  $-1$ .

In the more familiar context of the Euclidean circle the Laplacian is given simply by the second derivative, and the analysis of its spectrum is the subject of classical Fourier theory. Besides its intrinsic interest, Fourier analysis has long been one of the principal tools in analytic number theory. Indeed, one of the proofs of the analytic continuation of the Riemann zeta function is based on the Poisson summation formula. In the first chapter of this book, in order to get things rolling, we will recall this and other basic Fourier analytic facts.

In 1956 the Norwegian mathematician Atle Selberg proposed a vast generalization of the Poisson summation formula, now referred to as the *Selberg trace formula*. This formula, which we prove in Chap. 5, is reminiscent of the so-called explicit formulae in the analytic theory of  $L$ -functions. In the powerful analogy that the Selberg trace formula evokes, the closed geodesics on a compact hyperbolic surface  $S$  play the role of prime numbers. The trace formula establishes a relation between these closed geodesics and the Laplacian spectrum of  $S$ . These results were, upon their publication, immediately recognized as a new perspective on the Riemann hypothesis. While the latter remains an enigma, the trace formula has, since the late 1960s, played an increasingly central role in the ambitious program of Robert Langlands. The latter aims to link number theory with harmonic analysis on *locally symmetric spaces*, chief among which are the hyperbolic surfaces.

There are several textbooks already dedicated to the extension of the classical Fourier theory to hyperbolic surfaces; we mention in particular the now classic text of Iwaniec [63] (geared towards applications to analytic number theory) as well as the book by Buser [24], which has a more geometric outlook. In French, the closest

work to the present one is probably the book by Kowalski [71] where a large part of the basic theory is presented.<sup>1</sup>

## Why Another Book?

For well over 30 years now the Langlands program has made enormous advances, and spectral theory and number theory alike have harvested its fruits. But it has become difficult to penetrate such a larger and larger mathematical landscape. We wanted to write a book on the Langlands program in a more classical, and hopefully more accessible, language. This inclination has naturally led us to explore in more detail compact arithmetic hyperbolic surfaces. In particular, we present a proof of the first striking result in the Langlands program: the *Jacquet-Langlands correspondence*. The proof we give – due to Bolte and Johansson – should be more readily comprehensible to a reader unfamiliar with the language of adeles. With any luck, this expository simplification will encourage the reader to dive headlong into the original work of Jacquet and Langlands.

An added motivation for an updating of the literature was given by three other recent results: the lower bound on the Laplacian eigenvalues of arithmetic hyperbolic surfaces by the method Luo-Rudnick-Sarnak, the lovely proof of the existence of cusp forms for congruence subgroups of  $SL(2, \mathbf{Z})$  due to Lindenstrauss and Venkatesh, and finally Lindenstrauss' proof, by purely ergodic theoretic arguments, of the *arithmetic* quantum unique ergodicity conjecture of Rudnick and Sarnak in the compact setting. We describe these in detail in the introduction, and we later give complete proofs of the first two results. The last chapter contains an introduction to the work of Lindenstrauss on the quantum unique ergodicity conjecture. Via this last chapter in particular, we hope to lead the reader to the heart of current research.

## To Whom Is This Book Addressed?

We presuppose a basic knowledge of differential geometry and functional analysis. Our desire has been to make the entire text comprehensible to an ambitious first year graduate student – or possibly a colleague whose speciality lies elsewhere but whose curiosity is piqued by the subject. Each chapter has its own level of difficulty, however. In the first chapters, for example, we develop the spectral theory of the Laplacian on hyperbolic surfaces using only the basic algebraic and analytic tools of a first year graduate course (integration, Fourier analysis, Hilbert spaces). By contrast, the three last chapters, being more directly plugged into current research, would more naturally find their place in an advanced graduate level course.

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<sup>1</sup>Strictly speaking, the closest book in French to the present one is, well,...

Nevertheless, with an eye on highlighting the most illustrative cases, we have tried to simplify certain proofs; we then give references for the stronger statements.

Finally, to the reader eager to learn more on the subject than is presented here, we could not do better than to recommend the beautiful article of Sarnak [112], entitled “Spectra of hyperbolic surfaces”, from which this book has borrowed its own title as well as a large part of its structural organization.

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