Preface

Moving interfaces – and in the stationary case, free boundaries – are ubiquitous in our environment and daily life. They are at the basis of many physical, chemical, and also biological processes.

Typically, a moving boundary problem consists of one or more partial differential equations which have to be solved in a domain that is a priori unknown and that has to be determined as part of the problem. Problems with moving boundaries are in general harder to solve, both analytically and numerically, than the underlying differential equations would be in a prescribed domain. They have an inherent nonlinear structure, as two separate solutions cannot be superposed. Einstein's words

In so far as the theorems of mathematics relate to reality, they are not certain, and in so as far as they are certain, they do not relate to reality

appear to be quite correct in the context of formulating and analyzing problems with moving interfaces or free boundaries. Many simple things, such as a whiskey glass with a melting ice cube or a pot of boiling water with potatoes, are already very difficult to be modeled in a physically accurate way.

But as a matter of fact, mathematicians never give up. If unable to model and analyze a complicated process, we concentrate on simpler ones which already exhibit the important difficulties and characteristics. Mathematicians have followed this route successfully ever since: let us solve model problems in a rigorous way in order to improve our tools and invent new ones, and to sharpen old and design new weapons to tackle real world problems.

The most famous model problem with a moving interface, perhaps, is the *Stefan problem* for the freezing of water, proposed by J. Stefan in the 19th century. This problem has attracted much mathematical research since then, resulting in hundreds of papers; see the biographical remarks at the end of this book. The second historically prominent problem which, likely, has been around for as long as the Stefan-problem, is the *two-phase Navier-Stokes problem*, which describes the motion of droplets of oil in water, for instance. This problem has caused as much mathematical interest as the Stefan problem.

In this book we extend and combine these two historical problems into classes of models for one-component two-phase flows with phase transitions. The proposed models are thermodynamically consistent in the sense that the total energy is preserved, while the total entropy is non-decreasing. The physical derivation of these models and their properties are explained in Chapter 1. A rigorous analysis of the resulting six model problems is presented in Chapters 9, 10, and 11.

Another source of problems with moving interfaces concerns geometric evolution laws which describe the dynamics of hypersurfaces. In these problems, the normal velocity of a surface is given by a law defined by its geometry. Steady states then are special "free boundaries," leading to certain classes of surfaces like *minimal surfaces* or *Willmore surfaces*. Important examples are the *mean curvature flow*, the *surface diffusion flow*, and the *Willmore flow*. On the other hand, some popular quasi-stationary problems like the *Mullins–Sekerka flow*, the *Stokes flow*, or the *Muskat flow*, are determined not only by the geometry of the interface, but also by diffusion in its environment. In this monograph, we extend this list to include what we call the *Stokes flow with phase transition* and the *Muskat flow with phase transition*.

In the last decades, it has become evident that the theory of maximal regularity provides an important tool to tackle problems with moving interfaces. By means of these methods, the quasilinear structure, which is inherent for most problems with moving boundaries that include mean curvature, can be exploited in a mathematically optimal way. Via certain linearizations we may resort to the contraction mapping principle and to the implicit function theorem, as no loss of regularity will occur. This refers to local well-posedness – sometimes called *short-time existence for arbitrary data* – but also to the regularity of solutions and their stability properties near equilibria – sometimes called *long-time existence for small data*. With our methods we can furthermore prove that the interfaces become instantaneously real analytic if the coefficients in the equations are so and the initial values are subject to only mild regularity assumptions. This encodes typical parabolic behaviour.

Techniques relying on variational inequalities and weak solutions have proven successful in analyzing a wide array of problems with moving interfaces that share a particular underlying structure, that is, which can be formulated in a weak sense or in terms of a variational inequality. This enables one to conclude without great efforts that a solution to the free boundary problem exists in some weak sense. One can then proceed to establish the regularity of the solution and then attempt to study the smoothness of the free boundary itself. The advantage of this method is that it provides the existence of a global solution. However, it is often difficult, if not impossible, to derive further information on the location and the qualitative properties of the free boundary. Moreover, problems that include surface tension on the free boundary do not have the luxury of a comparison principle, and this alters the mathematical structure of the equations in a fundamental way. Consequently, methods based on comparison principles, variational inequalities, and viscosity solutions do not seem well-adapted in the presence of surface tension.

The basis of our approach relies on the so-called *direct mapping method* which means that the problem is transformed to a problem with fixed interface. This can be achieved very easily by a Lagrange transform if the interface is advected with the underlying flow, as in the two-phase Navier-Stokes problem. If phase transitions are present, like in the Stefan problem, it is more convenient to employ a Hanzawa transform in which the moving interface is parameterized via a *height function* over a fixed reference hypersurface. This method seems to be better adapted than the Lagrange transform for two-phase flows, as it allows us to prove smoothing of the interface, even if no phase transitions take place but surface tension is present. For this method, some differential geometry of hypersurfaces in Euclidean space is needed as well as advanced knowledge of function spaces.

In this monograph, we employ the theory of maximal L_p-theory throughout. By now, in the L_p-framework, many classical as well as some very recent powerful results for vector-valued harmonic analysis are at our disposal.

To introduce the needed tools, we will explore numerous connections between maximal L_p-regularity, sectorial operators, $\mathcal{H}^\infty$-calculus, Fourier-multipliers, semi-groups, and function spaces. Chapters 3 and 4 are devoted to this general theory – a theory that can be used for many other problems besides those with moving boundaries, as is demonstrated in Chapters 5 and 12. Therefore, this book offers many things also to researchers who may not primarily be interested in moving boundaries, but want to learn about parabolic evolution equations.

The monograph is structured as follows. In the introductory Chapter 1, the necessary physical background is introduced and the main problems to be studied are formulated. It is shown that these problems are thermodynamically consistent; their equilibria are identified, and those equilibria which are local maxima of the total entropy are singled out. One major purpose of this book is to show that the latter are precisely the stable ones. We also give an outline of the strategies for their mathematical analysis.

Chapter 2 contains the basic differential geometry of hypersurfaces needed for the direct mapping principle. We investigate the notions of Weingarten tensor, principal curvatures, mean curvature, tubular neighbourhood, surface gradient, surface divergence, and Laplace-Beltrami operator. The main emphasis lies in deriving representations of these quantities for hypersurfaces that are given as parameterized surfaces in normal direction of a fixed reference surface by means of a height function. It is also important to study the mapping properties of these quantities in dependence on the height function, and to derive expressions for their variations. Among other things we study the first and second variations of the area and volume functional. Moreover, we show that C^2-hypersurfaces can be approximated in a suitable topology by smooth (e.g. analytic) hypersurfaces. We then show that the class of compact embedded hypersurfaces in $\mathbb{R}^n$ gives rise to a new manifold whose points are the compact embedded hypersurfaces. Finally, we consider moving hypersurfaces, and we state and prove various transport theorems. While most of the material is well-known, we nevertheless believe that our presentation contains new results and aspects that are also of interest to readers with more advanced knowledge in differential geometry and geometric analysis. This chapter can be read independently from the rest of the book.

In Chapter 3, some elementary results from operator and semigroup theory are recalled. Moreover, some interpolation theory and the concept of maximal L_p-regularity is introduced and discussed. More recent results on vector-valued harmonic analysis, in particular operator-valued Fourier multiplier theorems in L_p-spaces and their implication to maximal L_p-theory, are accounted for in Chapter 4. The results of these chapters form the functional analytic foundation for the maximal regularity results which are at the heart of this book.

To demonstrate the strength and flexibility of the maximal regularity approach, in Chapter 5 an L_p-theory for abstract quasilinear parabolic evolution equations is developed. This includes local well-posedness, regularity, compactness of the induced semiflow, as well as the generalized principle of linearized stability for the analysis of the semiflow near (a manifold of) equilibria. These results can be applied to a wide array of quasilinear parabolic systems, including geometric evolution equations, as shown in Chapter 12. Maximal regularity is also used in an essential way to show the existence of the stable and unstable foliations at normally hyperbolic equilibria. Chapter 5 only relies on the concept of maximal L_p-regularity and, hence, it is also useful for readers who are not primarily interested in problems with moving interfaces, but rather in quasilinear parabolic systems.

Also of independent interest are Chapters 6, 7 and 8 in which maximal L_p-regularity for large classes of linear elliptic and parabolic systems is proved. These classes include transmission problems for two-phase systems, problems with dynamics on the interface, and Stokes problems, needed later on. In these chapters the full strength of the multiplier results from Chapter 4 is used.

Chapters 9, 10, and 11 are devoted to the analysis of the six problems with moving interfaces introduced in Chapter 1. They form the core of this monograph. It turns out that these problems can, unfortunately, not be formulated as abstract evolution equations of the type considered in Chapter 5. This is caused by the presence of nonlinear stationary transmission conditions on the interface. Nevertheless, Chapter 5 is used as a guideline for the analysis of the more involved problems. Chapter 9 deals with their local well-posedness, which can be proved simultaneously for all six problems in question; only the chosen function spaces are specific for each one. The same is also valid for the regularity theory, and for the results on long-time behaviour discussed in Chapter 11. On the other hand, the analysis of the full linearizations at a given equilibrium in Chapter 10 depends on the specific problem.

Miscellaneous applications of the theory developed in Chapter 5 are presented in Chapter 12, which include sections on generalized Newtonian flows, nematic liquid crystal flows, Maxwell–Stefan diffusion, the Stefan problem with variable surface tension, and geometric evolution equations.

The monograph is supplemented with historical and bibliographical remarks, suggestions for extensions of the theory and for further studies and research, a list of symbols and a subject index, and with an extensive bibliography.

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