Time-Averaged Hydrodynamic Equations for Mobile-Bed Conditions

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Abstract The objective of this paper is to expand the framework of the conventional Reynolds-Averaged Navier-Stokes equations for the study of mobile-boundary flows. The temporal averaging concept is discussed first, including relevant definitions and theorems. Time-averaged continuity, momentum, mass-transport and stress balance equations are then derived. These new equations contain additional terms that represent the mobile-boundary effects. Potential applications of the proposed equations include flow-biota interactions and sediment dynamics, among others.

1 Introduction

Environmental flows, aquatic or atmospheric, involve fluid motion and its interaction with fixed or mobile boundaries. Examples include rivers with vegetated or gravel beds, and atmospheric flows over terrestrial canopies. Recently, double-averaging (in time and space) has been implemented to cope with the spatial heterogeneities induced by the presence of boundary roughness where the traditional Reynolds averaging approach is inappropriate (e.g., Finnigan 2000; Nikora et al. 2007; Pedras and de Lemos 2000). However, conventional Reynolds averaging is also unsuitable for flows where the boundary is mobile, due to the discontinuity of hydrodynamic variables within the averaging interval. In a recent paper (Nikora et al. 2013), the double-averaging methodology has been refined by adapting an approach used in the study of multiphase systems (Gray and Lee 1977) to arrive at an advanced set of double-averaged conservation equations applicable to mobile-bed

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flows. Following this direction, we first suggest a modification of the temporal averaging operation, including refined definitions and averaging theorems. Then, a new set of first- and second-order time-averaged conservation equations, suitable for the description of turbulent flows over mobile boundaries, is proposed. The new equations include terms responsible for bed mobility effects and, thus, can be used as a basis for physical analyses and numerical modelling of mobile-bed flows.

2 Theoretical Background

In the study of multiphase systems (e.g., Gray and Lee 1977), the use of a distribution function γ is introduced to cope with the spatial heterogeneity of the material within the averaging domain. In a similar way, a distribution function is used in our considerations and is set equal to unity when a point is in the fluid domain and zero otherwise. Two forms of the temporal average are defined: superficial and intrinsic. For an arbitrary variable $\theta(x_i, t)$, the superficial temporal average is defined over the interval T_o as:

$$\overline{\theta(x_i, t)}^s = \frac{1}{T_o} \int\limits_{T_o} \theta(x_i, t+\tau)\gamma(x_i, t+\tau)d\tau \tag{1}$$

where x_i are the spatial coordinates and τ is the local time coordinate; superficial temporal averaging is denoted with an overbar with an index s. The intrinsic temporal average is defined as:

$$\overline{\theta(x_i, t)} = \frac{1}{T_f} \int\limits_{T_o} \theta(x_i, t+\tau)\gamma(x_i, t+\tau)d\tau \tag{2}$$

where T_f is a part of T_o that contains the time instants at which the point x_i is occupied by fluid (Nikora et al. 2013); intrinsic temporal averaging is denoted with an overbar. At mobile boundary conditions, the averaging time interval T_o is set to be appropriately larger than the turbulent integral time scale and the time scale that characterises the variation of the moving boundary (e.g., Bendat and Piersol 2010; Monin and Yaglom 1971; Nikora et al. 2013). Superficial and intrinsic averages are linked through the relation:

$$\overline{\theta(x_i, t)}^s = \phi_T(x_i, t)\overline{\theta(x_i, t)} \tag{3}$$

where $\phi_T = T_f/T_o$ is the local time porosity (Nikora et al. 2013). Using the Reynolds decomposition, the local instantaneous variable θ can be separated into a mean and fluctuating part, i.e., $\theta(x_i, t) = \overline{\theta}(x_i) + \theta'(x_i, t)$. Due to the heterogeneities previously noted, the Reynolds conditions (e.g., Nikora et al. 2007) are not completely satisfied; those valid for this case are summarised below:

$$\overline{f+g} = \bar{f} + \bar{g}, \quad \overline{af} = a\bar{f}, \quad \bar{a} = a, \quad \overline{\bar{f}g} = \bar{f}\,\bar{g} \tag{4}$$

where f and g are arbitrary variables and a is a constant. Using Eqs. (1) and (2), theorems that link the average partial derivatives with the derivatives of averaged quantities can be obtained. The averaging theorem for the superficial temporal average of the time derivative is:

$$\begin{aligned}\overline{\frac{\partial \theta}{\partial t}}^{s} &= \frac{1}{T_o}\int\limits_{T_o} \frac{\partial \theta}{\partial t}\gamma dt = \frac{1}{T_o}\int\limits_{T_o} \frac{\partial \theta\gamma}{\partial t} dt - \frac{1}{T_o}\int\limits_{T_o} \theta \frac{\partial \gamma}{\partial t} dt \\ &= \frac{\partial}{\partial t}\left(\frac{1}{T_o}\int\limits_{T_o} \theta\gamma dt\right) + \frac{1}{T_o}\int\limits_{T_o} \theta w_i n_i \delta(x_i - x_{s_i}) dt \\ &= \frac{\partial \bar{\theta}^{s}}{\partial t} + \frac{1}{T_o}\int\limits_{T_o} \theta w_i n_i \delta(x_i - x_{s_i}) dt\end{aligned} \tag{5}$$

where the relations (e.g., Gray and Lee 1977):

$$\frac{\partial \gamma}{\partial t} = -w_i \frac{\partial \gamma}{\partial x_i}, \frac{\partial \gamma}{\partial x_i} = n_i \delta(x_i - x_{s_i}) \tag{6}$$

are used and where the dependence on time and space coordinates is not shown for brevity. In Eq. (6), the ith velocity component of the moving boundary is denoted by w_i, n_i is the unit vector normal to the interfacial surface and directed into the fluid, δ is the three dimensional analogue of the Dirac delta function, and x_{si} is the position coordinate of the boundary. Correspondingly, the averaging theorem for the spatial derivative is:

$$\overline{\frac{\partial \theta}{\partial x_i}}^{s} = \frac{\partial \bar{\theta}^{s}}{\partial x_i} - \frac{1}{T_o}\int\limits_{T_o} \theta n_i \delta(x_i - x_{s_i}) dt \tag{7}$$

Employing Eq. (3), the theorems for the intrinsically averaged derivatives can be obtained from Eqs. (5) and (7) as:

$$\begin{aligned}\overline{\frac{\partial \theta}{\partial t}} &= \frac{1}{\phi_T}\frac{\partial \phi_T \bar{\theta}}{\partial t} + \frac{1}{T_f}\int\limits_{T_o} \theta w_i n_i \delta(x_i - x_{s_i}) dt \\ \overline{\frac{\partial \theta}{\partial x_i}} &= \frac{1}{\phi_T}\frac{\partial \phi_T \bar{\theta}}{\partial x_i} - \frac{1}{T_f}\int\limits_{T_o} \theta n_i \delta(x_i - x_{s_i}) dt\end{aligned} \tag{8}$$

3 Time-Averaged Hydrodynamic Equations

The derivation starts with the equations of mass (continuity) and momentum conservation for instantaneous variables:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0, \ \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \rho g_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \tag{9}$$

where ρ is fluid density, u_i is a velocity component in the ith direction, g_i is the gravity acceleration component, p is pressure, and τ_{ij} is the viscous stress component (note that repeated indices imply Einstein's summation convention). For the Newtonian incompressible fluid, the viscous stress term is given by $\tau_{ij} = \rho v \partial u_i / \partial x_j$, where v is the coefficient of kinematic viscosity. The transport of a solute with a concentration c is described by the advection-diffusion equation as:

$$\frac{\partial c}{\partial t} + \frac{\partial u_j c}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\chi_m \frac{\partial c}{\partial x_j}\right) + r \tag{10}$$

where χ_m is the coefficient of molecular diffusion, and r represents the homogeneous reaction rate (i.e., sink or source of c).

3.1 Time-Averaged Continuity, Momentum, and Mass Transport Equations

Employing the superficial averaging (1), the averaging theorems (5) and (7), the Reynolds decomposition, and the Reynolds conditions (4), the time-averaged continuity and momentum equations can be derived from Eq. (9) as:

$$\frac{\partial \phi_T}{\partial t} + \frac{\partial \phi_T \bar{u}_j}{\partial x_j} = 0 \tag{11}$$

$$\begin{aligned} \underbrace{\frac{\partial \phi_T \bar{u}_i}{\partial t}}_{\substack{\text{local} \\ \text{acceleration}}} + \underbrace{\frac{\partial \phi_T \bar{u}_i \bar{u}_j}{\partial x_j}}_{\substack{\text{convective} \\ \text{acceleration}}} = \underbrace{\phi_T g_i}_{\text{gravity term}} - \underbrace{\frac{1}{\rho}\frac{\partial \phi_T \bar{p}}{\partial x_i}}_{\text{pressure gradient}} - \underbrace{\frac{\partial \phi_T \overline{u_i' u_j'}}{\partial x_j}}_{\substack{\text{turbulent stress} \\ \text{change}}} + \underbrace{\frac{1}{\rho}\frac{\partial \phi_T \bar{\tau}_{ij}}{\partial x_j}}_{\substack{\text{viscous stress} \\ \text{change}}} \\ + \underbrace{\frac{1}{\rho T_o} \int_{T_o} \left[p n_i \delta(x_i - x_{s_i}) - \tau_{ij} n_j \delta\left(x_j - x_{s_j}\right) \right] dt}_{\text{contribution from flow}-\text{boundary interactions}} \end{aligned} \tag{12}$$

In deriving Eqs. (11) and (12), the no-slip boundary condition is used and the fluid density is assumed to be constant. The interpretation of most terms involved in Eqs. (11) and (12) is analogous to that of the conventional Reynolds-averaged continuity and momentum equations. The seventh term of Eq. (12) represents a contribution to the fluid momentum balance from the flow-boundary interactions. Similarly, the temporal average of the advection-diffusion equation is:

$$\underbrace{\frac{\partial \phi_T \bar{c}}{\partial t}}_{\substack{\text{local}\\ \text{change of } c}} + \underbrace{\frac{\partial \phi_T \bar{u}_j \bar{c}}{\partial x_j}}_{\substack{\text{spatial}\\ \text{change of } c}} = \underbrace{\frac{\partial}{\partial x_j}\left(\phi_T \overline{\chi_m \frac{\partial c}{\partial x_j}}\right)}_{\text{molecular diffusion transport}} - \underbrace{\frac{\partial \phi_T \overline{u_j' c'}}{\partial x_j}}_{\text{turbulent transport}} + \underbrace{\phi_T \bar{r}}_{\text{reaction rate}}$$
$$- \underbrace{\frac{1}{T_o}\int_{T_o} \chi_m \frac{\partial c}{\partial x_j} n_j \delta\left(x_j - x_{s_j}\right) dt}_{\text{contribution from flow}-\text{boundary interactions}} \tag{13}$$

The last term of Eq. (13) represents a potential contribution to the transport of the solute mass due to the effect of the moving boundary.

3.2 Second-Order Balance Equations for Velocity Moments

To obtain the second-order hydrodynamic equations we employ an approach of Keller and Friedmann (Monin and Yaglom 1971). As a consequence of the Reynolds decomposition, the total average momentum flux (*I*) can be subdivided into mean field contribution (*II*) and turbulence field contribution (*III*), i.e.:

$$\underbrace{\phi_T \overline{u_i u_k}}_{I} = \underbrace{\phi_T \bar{u}_i \bar{u}_k}_{II} + \underbrace{\phi_T \overline{u_i' u_k'}}_{III}. \tag{14}$$

Assuming that the quantities *I* and *II* are given, the turbulent stress *III* can be determined by subtraction of *II* from *I*. The derivatives of the stress terms can be expanded following the product rule:

$$\begin{aligned} \frac{\partial u_i u_k}{\partial t} &= u_k \frac{\partial u_i}{\partial t} + u_i \frac{\partial u_k}{\partial t} \\ \frac{\partial \phi_T \bar{u}_i \bar{u}_k}{\partial t} &= \bar{u}_k \frac{\partial \phi_T \bar{u}_i}{\partial t} + \bar{u}_i \frac{\partial \phi_T \bar{u}_k}{\partial t} - \bar{u}_i \bar{u}_k \frac{\partial \phi_T}{\partial t} \end{aligned} \tag{15}$$

Substituting Eqs. (11) and (12) in the second relation of Eq. (15), the equation that describes the balance of $\phi_T \bar{u}_i \bar{u}_k$ can be produced:

$$
\begin{aligned}
\underbrace{\frac{\partial \phi_T \bar{u}_i \bar{u}_k}{\partial t}}_{\substack{\text{time rate}\\ \text{of change}}} + \underbrace{\frac{\partial \phi_T \bar{u}_i \bar{u}_k \bar{u}_j}{\partial x_j}}_{\text{mean convection}} = {} & \underbrace{\phi_T \left(\bar{u}_k g_i + \bar{u}_i g_k\right)}_{\substack{\text{work rate of gravity:}\\ \text{energy supply to the mean flow}}} \quad \underbrace{- \frac{1}{\rho} \frac{\partial \phi_T \left(\bar{u}_k \bar{p} \delta_{ij} + \bar{u}_i \bar{p} \delta_{kj}\right)}{\partial x_j}}_{\substack{\text{work rate of pressure:}\\ \text{pressure transport}}} \\
& - \underbrace{\frac{\partial \phi_T \left(\bar{u}_k \overline{u_i' u_j'} + \bar{u}_i \overline{u_k' u_j'}\right)}{\partial x_j}}_{\substack{\text{work rate of turbulent stress:}\\ \text{turbulent transport}}} + \underbrace{\frac{1}{\rho} \frac{\partial \phi_T \left(\bar{u}_k \bar{\tau}_{ij} + \bar{u}_i \bar{\tau}_{kj}\right)}{\partial x_j}}_{\substack{\text{work rate of viscous stress:}\\ \text{viscous transport}}} \\
& - \underbrace{\frac{\phi_T \bar{p}}{\rho} \left(\frac{\partial \bar{u}_k}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_k}\right)}_{\substack{\text{work rate of } p \text{ against}\\ \text{mean strain rate}}} + \underbrace{\frac{\phi_T}{\rho} \left(\bar{\tau}_{ij} \frac{\partial \bar{u}_k}{\partial x_j} + \bar{\tau}_{kj} \frac{\partial \bar{u}_i}{\partial x_j}\right)}_{\substack{\text{work rate of } \tau \text{ against mean strain rate:}\\ \text{dissipation of kinetic energy}}} \\
& + \underbrace{\phi_T \left(\overline{u_i' u_j'} \frac{\partial \bar{u}_k}{\partial x_j} + \overline{u_k' u_j'} \frac{\partial \bar{u}_i}{\partial x_j}\right)}_{\substack{\text{work rate of turbulent stress against mean}\\ \text{strain rate: turbulent energy production}}} \\
& + \underbrace{\frac{1}{\rho T_o} \int_{T_o} \left(\bar{u}_k p \delta_{ij} + \bar{u}_i p \delta_{kj} - \bar{u}_k \tau_{ij} - \bar{u}_i \tau_{kj}\right) n_j \delta\left(x_j - x_{s_j}\right) dt}_{\text{contribution from flow}-\text{boundary interactions}} .
\end{aligned}
\tag{16}
$$

The last term in Eq. (16) is the consequence of the last term of Eq. (12) and is considered as a contribution to the mean stress balance exerted by the boundary motion. Substituting Eq. (9) into the first relation of Eq. (15) yields the balance of $u_i u_k$. By its averaging and then subtracting Eq. (16), one obtains the equation for the Reynolds stress:

$$\underbrace{\frac{\partial \phi_T \overline{u_i' u_k'}}{\partial t}}_{\substack{\text{time rate}\\ \text{of change}}} + \underbrace{\frac{\partial \phi_T \bar{u}_j \overline{u_i' u_k'}}{\partial x_j}}_{\text{mean convection}} = \underbrace{- \phi_T \left(\overline{u_i' u_j'} \frac{\partial \bar{u}_k}{\partial x_j} + \overline{u_k' u_j'} \frac{\partial \bar{u}_i}{\partial x_j} \right)}_{\text{production}}$$
$$\underbrace{- \frac{1}{\rho} \frac{\partial \phi_T \left(\overline{u_k' p'} \delta_{ij} + \overline{u_i' p'} \delta_{kj} \right)}{\partial x_j}}_{\text{pressure transport}} \underbrace{- \frac{\partial \phi_T \overline{u_i' u_k' u_j'}}{\partial x_j}}_{\text{turbulent transport}}$$
$$\underbrace{+ \frac{1}{\rho} \frac{\partial \phi_T \left(\overline{u_k' \tau_{ij}'} + \overline{u_i' \tau_{kj}'} \right)}{\partial x_j}}_{\text{viscous transport}} \underbrace{+ \frac{\phi_T}{\rho} \overline{p' \left(\frac{\partial u_k'}{\partial x_i} + \frac{\partial u_i'}{\partial x_k} \right)}}_{\text{redistribution over space}}$$
$$\underbrace{- \frac{\phi_T}{\rho} \left(\overline{\tau_{ij}' \frac{\partial u_k'}{\partial x_j}} + \overline{\tau_{kj}' \frac{\partial u_i'}{\partial x_j}} \right)}_{\text{dissipation}}$$
$$\underbrace{+ \frac{1}{\rho T_o} \int\limits_{T_o} \left(u_k' p \delta_{ij} + u_i' p \delta_{kj} - u_k' \tau_{ij} - u_i' \tau_{kj} \right) n_j \delta\left(x_j - x_{s_j} \right) dt}_{\text{contribution due to flow}-\text{boundary interaction}} \tag{17}$$

Compared to the conventional Reynolds stress equation (e.g., Hanjalic and Launder 1972; Monin and Yaglom 1971), here the effect of the porosity function and the ninth term are included. In the reasoning followed in Eqs. (12) and (16), the ninth term of Eq. (17) is considered as an external contribution to the Reynolds stress balance, caused by the boundary motion.

4 Discussion

The suggested equations are developed for the study of flows over mobile boundaries, implying the existence of temporal heterogeneities. These heterogeneities result in the discontinuities of a hydrodynamic variable within the time averaging domain. This difficulty is surpassed with the use of the distribution function, which in turn leads to the appearance of the local time porosity and additional time integrals in the equations. These additional terms constitute the main difference between the proposed equations and the conventional Reynolds-Averaged hydrodynamic equations for flows over fixed boundaries. For the case in which the boundaries are fixed, we have $\phi_T \equiv 1$ and therefore operations of temporal averaging and differentiation commute (the distribution function is constant; i.e. $\partial \gamma / \partial x_i = 0$). Thus, for the fixed-bed conditions, Eqs. (11), (12),

(16) and (17) coincide with the conventional Reynolds-averaged continuity, momentum, mass transport and stress balance equations. Theorems for temporal averaging have also been proposed in Ishii and Hibiki (2006), in order to introduce interfacial terms in the conservation equations for a two-phase mixture. Material discontinuities and temporal heterogeneities are also reported there. However, the authors' definition of the temporal average follows the conventional one, while discontinuity is considered for the integration domain, when the relation between $\overline{\partial\theta/\partial s}$ and $\partial\bar{\theta}/\partial s$ (s is the domain of differentiation) is needed (Ishii and Hibiki 2006).

Applications of the proposed equations include the study of the interactions between aquatic flow and submerged vegetation or the study of sediment dynamics.

5 Conclusions

In the preceding sections, a method of temporal averaging was discussed, based on the concepts reported in Gray and Lee (1977) and Nikora et al. (2013). This method is followed for the development of equations for momentum, mass and stress transport for mobile-bed flows. The effect of a moving boundary is introduced through the local porosity function and interfacial terms. The suggested equations are intended to assist in studies involving the interaction of fluid flow with in-stream moving objects. The time-averaged transport equations alone may not be sufficient for the numerical investigation of such cases. However, complemented with spatial averaging as a second step, they may be used in the development of the theoretical framework that is needed for the experimental and numerical study of mobile-boundary flows.

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