2
Equations of Classical Hydrodynamics

The neglected borderline between two branches of knowledge is often that which best repays cultivation, or, to use a metaphor of Maxwell's, the greatest benefits may be derived from a cross-fertilization of the sciences.

– John William Strutt, 3rd Baron Rayleigh

In this chapter we give an overview of the equations of classical hydrodynamics. We provide their derivation, comment on the stress tensor, and thermodynamics, finally we present some elementary properties and also some exact solutions of the Navier–Stokes equations.

2.1 Derivation of the Equations of Motion

Fluid flow may be represented mathematically as a *continuous transformation* of three-dimensional Euclidean space into itself. The transformation is parametrized by a real parameter t representing time.

Let us introduce a fixed rectangular coordinate system (x_1, x_2, x_3). We refer to the coordinate triple (x_1, x_2, x_3) as the *position* and denote it by x. Now consider a particle P moving with the fluid, and suppose that at time $t = 0$ it occupies a position $X = (X_1, X_2, X_3)$ and that at some other time t, $-\infty < t < +\infty$, it has moved to a position $x = (x_1, x_2, x_3)$. Then x is determined as a function of X and t

$$x = x(X, t) \quad \text{or} \quad x_i = x_i(X, t)\,. \tag{2.1}$$

If X is fixed and t varies, Eq. (2.1) specifies the *path* of the particle initially at X. On the other hand, for fixed t, (2.1) determines a transformation of a region initially occupied by the fluid into its position at time t.

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We assume that the transformation (2.1) is *continuous* and *invertible*, that is, there exists its inverse

$$X = X(x,t), \quad (\text{or} \quad X_i = X_i(x,t))\,.$$

Also, to be able to differentiate, we assume that the functions x_i and X_i are sufficiently smooth.

From the condition that the transformation (2.1) possess a differentiable inverse it follows that its Jacobian

$$J = J(X,t) = \det\left(\frac{\partial x_i}{\partial X_j}\right)$$

satisfies

$$0 < J < \infty\,. \tag{2.2}$$

The initial coordinates X of the particle will be referred to as the *material coordinates* of the particle. The *spatial coordinates* x may be referred to as its *position*, or *place*. The representation of fluid motion as a *point transformation* violates the concept of the *kinetic theory* of fluids, as in this theory the particles are molecules, and they are in random motion. In the theory of *continuum mechanics* the state of motion at a given point x and at a given time t is described by a number of functions such as $\rho = \rho(x,t)$, $u = u(x,t)$, $\theta = \theta(x,t)$ representing density, velocity, temperature, and other *hydrodynamical variables*.

Due to the transformation (2.1), each such variable f can also be expressed in terms of material coordinates

$$f(x,t) = f(x(X,t),t) = F(X,t)\,. \tag{2.3}$$

The *velocity* u at time t of a particle initially at X is given, by definition, as

$$u(x,t) = U(X,t) = \frac{d}{dt}x(X,t)\,, \quad (x = x(X,t))\,. \tag{2.4}$$

Above, X is treated as a parameter representing a given fixed particle, and this is the reason that we use the ordinary derivative in (2.4).

Having the velocity field $u(x,t)$, we can (in principle) determine the transformation (2.1), solving the ordinary differential equation

$$\frac{d}{dt}x(X,t) = u(x(X,t),t),$$

with $x(X,0) = X$, where X is a parameter.

We shall always write

$$\frac{d}{dt}F(X,t) \quad \text{and} \quad \frac{\partial}{\partial t}f(x,t)\,,$$

where F and f are related by (2.3). We have thus

$$\frac{d}{dt}F(X,t) = \frac{d}{dt}f(x(X,t),t) = \frac{\partial f}{\partial x_i}(x(X,t),t)\frac{dx_i}{dt} + \frac{\partial f}{\partial t}(x(X,t),t)\,,$$

so that by (2.4) we obtain a general formula

$$\frac{d}{dt}F(X,t) = \frac{D}{Dt}f(x,t)\,, \tag{2.5}$$

where $\frac{D}{Dt}f(x,t) \equiv \frac{\partial f}{\partial t}(x,t) + u(x,t)\cdot\nabla f(x,t)$ is called the *material derivative* of f.

Transport Theorem Let $\Omega(t)$ denote an arbitrary volume that is moving with the fluid and let $f(x,t)$ be a scalar or vector function of position and time. The *transport theorem* states that

$$\frac{d}{dt}\int_{\Omega(t)} f(x,t)\,dx \tag{2.6}$$

$$= \int_{\Omega(t)} \left\{ \frac{\partial f}{\partial t}(x,t) + u(x,t)\cdot\nabla f(x,t) + f(x,t)\,\mathrm{div}\,u(x,t) \right\}\,dx\,.$$

For the proof consider the transformation

$$x : \Omega(0) \to \Omega(t), \quad x = x(X,t)\,,$$

as in (2.1). Then

$$\int_{\Omega(t)} f(x,t)\,dx$$

$$= \int_{\Omega(0)} f(x(X,t),t)J(X,t)\,dX = \int_{\Omega(0)} F(X,t)J(X,t)\,dX\,,$$

so that

$$\frac{d}{dt}\int_{\Omega(t)} f(x,t)dx = \frac{d}{dt}\int_{\Omega(0)} F(X,t)J(X,t)\,dX \tag{2.7}$$

$$= \int_{\Omega(0)} \left\{ \frac{d}{dt}F(X,t)J(X,t) + F(X,t)\frac{d}{dt}J(X,t) \right\}\,dX\,.$$

By (2.5) we have

$$\int_{\Omega(0)} \frac{d}{dt} F(X,t) J(X,t)\, dX$$

$$= \int_{\Omega(0)} \left\{ \frac{\partial f}{\partial t}(x(X,t),t) + u(x(X,t),t) \cdot \nabla f(x(X,t),t) \right\} J(X,t)\, dX$$

$$= \int_{\Omega(t)} \left\{ \frac{\partial f}{\partial t}(x,t) + u(x,t) \cdot \nabla f(x,t) \right\}\, dx\,.$$

To prove (2.6) it remains to prove the *Euler formula*

$$\frac{d}{dt} J(X,t) = \operatorname{div} u(x(X,t),t) J(X,t)\,, \tag{2.8}$$

the proof of which we leave to the reader as an exercise.

The fluid is called *incompressible* if for any domain $\Omega(0)$ and any t,

$$\text{volume } (\Omega(t)) = \text{volume } (\Omega(0))\,.$$

From (2.7) with $f(x,t) \equiv 1$ we have

$$\frac{d}{dt}\text{volume } (\Omega(t)) = \frac{d}{dt} \int_{\Omega(t)} dx = \int_{\Omega(0)} \frac{d}{dt} J(X,t) dX\,,$$

hence by (2.8), (2.2), and the arbitrariness of choice of the domain $\Omega(t)$ via $\Omega(0)$ a necessary and sufficient condition for the fluid to be incompressible is

$$\operatorname{div} u(x,t) = 0\,.$$

Exercise 2.1. Prove that the transport theorem can be written in the form

$$\frac{d}{dt} \int_{\Omega(t)} f(x,t)\, dx = \int_{\Omega(t)} \frac{\partial f}{\partial t}(x,t)\, dx + \int_{\partial\Omega(t)} f(x,t) u(x,t) \cdot n(x,t)\, dS\,,$$

where $n(x,t)$ is the outward unit normal to $\partial\Omega(t)$ at $x \in \partial\Omega(t)$.

Equation of Continuity Let $\rho = \rho(x,t)$ be the mass per unit volume of a fluid at point x and time t. Then the mass of any finite volume Ω is

$$m = \int_{\Omega} \rho(x,t)\, dx\,.$$

The *principle of conservation of mass* says that the mass of a fluid in a material volume Ω does not change as Ω moves with the fluid; that is,

$$\frac{d}{dt}\int_{\Omega(t)} \rho(x,t)\,dx = 0\,.$$

From the transport theorem (2.6) it follows that

$$\int_{\Omega(t)} \left\{ \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) \right\} dx = 0\,,$$

whence

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0\,. \tag{2.9}$$

Sometimes the principle of conservation of mass is expressed as follows. Let Ω be a fixed volume. Then

$$\frac{d}{dt}\int_{\Omega} \rho(x,t)\,dx = -\int_{\partial\Omega} \rho u \cdot n\,dS\,, \tag{2.10}$$

that is, the rate of change of mass in a fixed volume Ω is equal to the mass flux through its surface.

We notice also the general formula

$$\frac{d}{dt}\int_{\Omega(t)} \rho f\,dx = \int_{\Omega(t)} \rho \frac{D}{Dt} f\,dx\,. \tag{2.11}$$

Exercise 2.2. Derive (2.9) from (2.10).

Exercise 2.3 (Cf. [212]). Show that in material coordinates the equation of continuity is

$$\frac{d}{dt}\{\rho(X,t)J(X,t)\} = 0\,,$$

or

$$\rho(X,t)J(X,t) = \rho(X,0)\,.$$

Exercise 2.4 (Cf. [5]). Show that if $\rho_0(X)$ is the distribution of density of the fluid at time $t = 0$ and $\nabla(\operatorname{div} u) = 0$, then

$$\rho(x,t) = \rho_0(X(x,t)) \exp\left\{ -\int_0^t \operatorname{div} u(x,t)\,dt \right\}\,.$$

Exercise 2.5. Find $\rho(x,t)$ for the motion

$$u_i = \frac{x_i}{1 + a_i t} \quad (a_1 = 2\,, a_2 = 1\,, a_3 = 0),$$

if $\rho_0(X)$ is the distribution of density of the fluid at time $t = 0$.

Exercise 2.6. Prove (2.11).

Principle of Conservation of Linear Momentum We assume that the forces acting on an element of a continuous medium are of two kinds. *External*, or *body, forces*, such as gravitation or electromagnetic forces, can be regarded as reaching into the medium and acting throughout the volume. If f represents such a force *per unit mass*, then it acts on an element Ω as

$$\int_{\Omega} \rho f \, dx\,.$$

The *internal*, or *contact, forces* are to be regarded as acting on an element of volume Ω through its bounding surface. Let n be the unit outward normal at a point of the surface $\partial\Omega$, and t_n the force *per unit area* exerted there by the material volume outside $\partial\Omega$. Then the surface force exerted on the volume Ω can be expressed by the integral

$$\int_{\partial\Omega} t_n \, dS\,.$$

The *Cauchy principle* says that t_n depends at any given time only on the position and the orientation of the surface element dS; in other words,

$$t_n = t_n(x,t,n)\,.$$

The *principle of conservation of linear momentum* says that the rate of change of linear momentum of a material volume equals the resultant force on the volume

$$\frac{d}{dt}\int_{\Omega(t)} \rho u dx = \int_{\Omega(t)} \rho f \, dx + \int_{\partial\Omega(t)} t_n \, dS\,, \tag{2.12}$$

where f is assumed to be known.

By (2.11), (2.12) yields

$$\int_{\Omega(t)} \rho \frac{Du}{Dt} dx = \int_{\Omega(t)} \rho f \, dx + \int_{\partial\Omega(t)} t_n \, dS\,. \tag{2.13}$$

From this equation we derive a very important fact, namely, that the vector t_n (called *normal stress*) can be expressed as a linear function of n, in the form

$$t_n(x,t,n) = n(x,t)T(x,t)\,, \tag{2.14}$$

where $T = \{T_{ij}\}$ is a matrix called the *stress tensor*. This will allow us to pass from the integral form (2.13) of the equation of conservation of linear momentum to a differential one.

Let l^3 be the volume of $\Omega = \Omega(t)$. Dividing both sides of (2.13) by l^2 and letting the volume tend to zero we obtain

$$\lim_{|\Omega|\to 0} l^{-2} \int_{\partial\Omega} t_n \, dS = 0 \,, \tag{2.15}$$

that is, the stress forces are in local equilibrium.

Let Ω be a domain containing a fluid, and consider a regular tetrahedron with vertex at an arbitrary point $x \in \partial\Omega$, and with three of its faces parallel to the coordinate planes. Let the slanted face have normal $n = (n_1, n_2, n_3)$ and area Σ. The normals to the other faces are $-e_1$, $-e_2$, and $-e_3$, and their areas are $n_1\Sigma$, $n_2\Sigma$, and $n_3\Sigma$. Applying (2.15) to the family of tetrahedrons obtained by letting $\Sigma \to 0$, we obtain

$$t(n) + n_1 t(-e_1) + n_2 t(-e_2) + n_3 t(-e_3) = 0 \,, \tag{2.16}$$

where $t(n) = t_n = t_n(x, t, n)$, $t(-h) = t_{-h}$ for $h \in \{e_1, e_2, e_3\}$, and $n_i > 0$. By a continuity argument, (2.16) holds for all $n_i \geq 0$, and then we prove easily that $t(e_i) = -t(-e_i)$, $i = 1, 2, 3$, and that it holds for all n. This means that $t(n)$ may be expressed as a linear function of n; that is, we can write it in the form (2.14). Thus, by (2.13) and by the Green theorem we obtain

$$\int_{\Omega(t)} \rho \frac{Du}{Dt} \, dx = \int_{\Omega(t)} (\rho f + \operatorname{div} T) \, dx \,,$$

whence, by the arbitrariness of the domain of integration,

$$\rho \frac{Du}{Dt} = \rho f + \operatorname{div} T \,, \tag{2.17}$$

or

$$\rho \left(\frac{\partial}{\partial t} u_i + \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} u_i \right) = \rho f_i + T_{ji,j}, \quad i = 1, 2, 3 \,.$$

This is the general *Cauchy equation of motion* in differential form.

Exercise 2.7. Give a physical interpretation of the components of the stress tensor.

Notice that we have not specified T yet, that is, we have not made any assumptions concerning the nature of forces acting on surface elements. These forces depend on the kind of fluid, or, more generally, on the kind of medium under consideration.

In the simplest model the contact forces act perpendicularly to the surface elements. We have then

$$t(n) = -p(x)n\,,$$

and call p the *pressure*. The minus sign is chosen so that when $p > 0$, the contact forces on a closed surface tend to compress the fluid inside; p represents the pressure exerted from outside on a surface of the element of the fluid.

In particular, all fluids at rest exhibit this stress behavior, namely that an element of area always experiences a stress normal to itself, and this stress is independent of the orientation. Such stress is called *hydrostatic*.

We call this idealized model a *perfect fluid*. The equation of motion for perfect fluids is

$$\rho\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) = \rho f - \nabla p\,,$$

where

$$(u\cdot\nabla)u_i = \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} u_i\,, \quad i = 1, 2, 3\,.$$

All real fluids when in motion can exert tangential stresses across surface elements, in which case the tensor T is not diagonal.

The stress tensor may always be written in the form

$$T_{ij} = -p\delta_{ij} + P_{ij}\,.$$

In this case P_{ij} is called the *viscous stress tensor*.

In classical fluid dynamics it is assumed that the stress tensor is *symmetric*, that is,

$$T_{ij} = T_{ji}\,.$$

This assumption has very important consequences. It may be also considered as a theorem if we assume a specific form of the equation of conservation of angular momentum. We shall discuss this in Sect. 2.2.

Exercise 2.8 (Cf. [5]). Show that the Cauchy equation of motion can be written as

$$\frac{\partial}{\partial t}(\rho u_i) = \rho f_i + (T_{ji} - \rho u_j u_i)_{,j}\,,$$

and interpret it physically.

Exercise 2.9 (Cf. [5]). Show that if F is any function of position and time, then

$$\int_{\partial\Omega} FT_{ji}n_j\,dS = \int_{\Omega}\left[T_{ji}F_{,j} + \rho F\left(\frac{Du_i}{Dt} - f_i\right)\right]dx$$

(theorem of stress means).

Equation of Energy *The first law of thermodynamics* in classical hydrodynamics states that the increase of total energy (we shall consider here only kinetic and internal energies) in a material volume is the sum of the heat transferred and the work done on the volume. We denote by q the *heat flux* (then $-q\cdot n$ is the heat flux into the volume) and by E the *specific internal energy*. Then the balance expressed by the first law of thermodynamics is, cf. [5],

$$\frac{d}{dt}\int_{\Omega(t)}\rho\left(\frac{1}{2}|u|^2 + E\right)dx \tag{2.18}$$

$$= \int_{\Omega(t)}\rho f\cdot u\,dx + \int_{\partial\Omega(t)} t_n\cdot u\,dS - \int_{\partial\Omega(t)} q\cdot n\,dS\,.$$

The first integral on the right-hand side is the rate at which the body force does work, the second integral represents the work done by the stress, and the third integral is the total heat flux into the volume.

We shall write this equation in another form. From the theorem of stress means (Exercise 2.9) we have, with $F = u_i$,

$$\int_{\partial\Omega(t)} u_iT_{ji}n_j\,dS = \int_{\Omega(t)}\left(T_{ji}u_{i,j} + \rho u_i\frac{Du_i}{Dt} - \rho f_iu_i\right)dx\,.$$

Rearranging the terms and using the transport theorem, we obtain

$$\frac{d}{dt}\int_{\Omega(t)}\rho\frac{1}{2}|u|^2\,dx = \int_{\Omega(t)}\rho\frac{1}{2}\frac{D}{Dt}|u|^2\,dx \tag{2.19}$$

$$= \int_{\Omega(t)}\rho f_iu_i\,dx - \int_{\Omega(t)} T_{ji}u_{i,j}\,dx + \int_{\partial\Omega(t)} u_i(t_n)_i\,dS\,.$$

Thus the rate of change of kinetic energy of a material volume is the sum of three parts: the rate at which the body forces do work, the rate at which the internal stresses do work, and the rate at which the surface stresses do work.

From (2.18), (2.19), the transport theorem, and the Green theorem we obtain

$$\int_{\Omega(t)}\left(\rho\frac{DE}{Dt} + \nabla\cdot q - T:(\nabla u)\right)dx = 0\,,$$

where $T:(\nabla u)$ is the dyadic notation for $T_{ji}u_{i,j}$, the scalar product of T and ∇u.

Thus

$$\rho\frac{DE}{Dt} = -\nabla \cdot q + T : (\nabla u)\,.$$

Conservation Laws of Classical Hydrodynamics Above we obtained the following system of conservation laws of classical hydrodynamics

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot u\,, \tag{2.20}$$

$$\rho\frac{Du}{Dt} = \nabla \cdot T + \rho f\,, \tag{2.21}$$

$$\rho\frac{DE}{Dt} = -\nabla \cdot q + T : (\nabla u)\,. \tag{2.22}$$

They are laws of conservation of mass, momentum, and energy, respectively.

If we assume the *Fourier law* for the conduction of heat,

$$q = -k\nabla\theta \quad (k \geq 0)\,, \tag{2.23}$$

where k is the *thermal conductivity* of the fluid then the energy equation takes the form

$$\rho\frac{DE}{Dt} = \nabla \cdot (k\nabla\theta) + T : (\nabla u)\,.$$

2.2 The Stress Tensor

In the classical hydrodynamics the stress tensor T is defined by

$$T_{ij} = (-p + \lambda u_{k,k})\delta_{ij} + \mu(u_{i,j} + u_{j,i})\,. \tag{2.24}$$

If we define the *deformation tensor*

$$D_{i,j} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.25}$$

then the above formula takes the form

$$T_{ij} = (-p + \lambda u_{k,k})\delta_{ij} + 2\mu D_{ij}\,. \tag{2.26}$$

Remark 2.1. Formula (2.26) is a consequence of a number of postulates, coming originally from G. Stokes, about the fundamental properties of fluids. These postulates can be formulated as follows (cf. [5, 212]):

(a) The stress tensor T is a continuous function of the deformation tensor D and the local thermodynamic state, but independent of other kinematic quantities.

(b) The fluid is homogeneous; that is, T does not depend explicitly on x.
(c) The fluid is isotropic; that is, there is no preferred direction.
(d) When there is no deformation ($D = 0$), and the fluid is incompressible ($u_{k,k} = 0$), the stress is hydrostatic ($T = -pI$, I is the unit matrix).

Fluids that satisfy these postulates are called *Stokesian*. It can be proved (cf. [5, 212]) that the most general form of the stress tensor in this case is

$$T = (-p + \alpha)I + \beta D + \gamma D^2 ,$$

where p, α, β, γ are some functions that depend on the thermodynamic state, α, β, γ being dependent as well on the invariants of the tensor D.

Moreover, when the dependence of the components of T on the components of D is postulated to be *linear*, the stress tensor can be written as

$$T = (-p + \lambda \ \operatorname{div} u)I + 2\mu D ,$$

which coincides with (2.24). Such linear Stokesian fluids are called *Newtonian*. Fluids that are not Newtonian are called *non-Newtonian*. One important example of the latter are the micropolar fluids [92, 159].

The Stress Tensor and the Law of Conservation of Angular Momentum Looking at the form of the equation of conservation of linear momentum

$$\frac{d}{dt}\int_{\Omega(t)} \rho u \, dx = \int_{\Omega(t)} \rho f \, dx + \int_{\partial\Omega(t)} t_n \, dS ,$$

and recalling the definition of angular momentum in mechanics of mass points or rigid particles, it seems natural to *assume* the following form of the *law of conservation of angular momentum*:

$$\frac{d}{dt}\int_{\Omega(t)} \rho(x \times u) \, dx = \int_{\Omega(t)} \rho(x \times f) \, dx + \int_{\partial\Omega(t)} x \times t_n \, dS . \tag{2.27}$$

In fact, this form of the law of conservation of angular momentum holds if we assume that all torques arise from macroscopic forces. This is the case in most common fluids, but a fluid with a strongly polar character, e.g., a polyatomic fluid, is capable of transmitting stress torques and being subjected to body torques. We call such fluids *polar*.

Theorem 2.1. *For an arbitrary continuous medium satisfying the continuity equation (2.9) and the dynamical equation (2.17) the following statements are equivalent:*

(i) *the stress tensor is symmetric,*
(ii) *equation (2.27) holds.*

Remark 2.2. In classical hydrodynamics the stress tensor is symmetric, and the law of conservation of angular momentum is defined by Eq. (2.27). In consequence, in classical hydrodynamics the law of conservation of angular momentum can be derived from the law of conservation of mass and the law of conservation of linear momentum, and as such adds nothing to the description of the fluid.

Proof. Let us assume (ii), and we shall deduce (i). Applying formula (2.11), we have from (2.27)

$$\frac{d}{dt}\int_{\Omega(t)} \rho(x \times u)\, dx \tag{2.28}$$

$$= \int_{\Omega(t)} \rho \frac{D}{Dt}(x \times u)\, dx = \int_{\Omega(t)} \rho \left(x \times \frac{Du}{Dt} \right) dx$$

$$= \int_{\Omega(t)} \rho(x \times f)\, dx + \int_{\partial\Omega(t)} x \times t_n \, dS \, .$$

By the Green theorem,

$$\int_{\partial\Omega(t)} x \times t_n \, dS = \int_{\Omega(t)} (x \times (\nabla \cdot T) + T_x)\, dx \, , \tag{2.29}$$

where $\nabla \cdot T$ is another notation for $\operatorname{div} T$, and T_x is the vector $\epsilon_{ijk} T_{jk}$ (ϵ_{ijk} is the alternating tensor of Levi-Civita), so that by (2.28)

$$\int_{\Omega(t)} x \times \left(\rho \frac{Du}{Dt} - \rho f - \nabla \cdot T \right) dx = \int_{\Omega(t)} T_x \, dx \, .$$

The left-hand side vanishes identically by the Cauchy equation; hence the right-hand side vanishes for an arbitrary volume, and so $T_x = 0$. However, the components of T_x are $T_{23} - T_{32}$, $T_{31} - T_{13}$, $T_{12} - T_{21}$, and the vanishing of these implies $T_{ij} = T_{ji}$, so that T is symmetric.

We leave to the reader the proof that (i) implies (ii). □

2.3 Field Equations

Substituting the stress tensor (2.24) into the system (2.20)–(2.22) we obtain the system of *field equations* of classical hydrodynamics

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot u \, , \tag{2.30}$$

$$\rho \frac{Du}{Dt} = -\nabla p + (\lambda + \mu) \nabla \operatorname{div} u + \mu \Delta u + \rho f \, , \tag{2.31}$$

$$\rho \frac{DE}{Dt} = -p \operatorname{div} u + \rho \Phi - \nabla \cdot q\,, \tag{2.32}$$

where

$$\rho \Phi = \lambda (\operatorname{div} u)^2 + 2\mu D : D \tag{2.33}$$

is the *dissipation function* of mechanical energy per mass unit.

Let us assume that the fluid is *viscous* and *incompressible*, namely, that $\mu > 0$ and

$$\operatorname{div} u = 0\,, \tag{2.34}$$

that the specific internal energy of the fluid is proportional to its temperature,

$$E = c_r \theta\,, \quad \text{where} \quad c_r = \text{const} > 0\,, \tag{2.35}$$

and that Fourier's law (2.23) (with $k = const \geq 0$) holds. With (2.34), (2.35), (2.23), and (2.33), system (2.30)–(2.32) becomes

$$\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = 0\,, \qquad \operatorname{div} u = 0\,, \tag{2.36}$$

$$\rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = -\nabla p + \mu \Delta u + \rho f\,, \tag{2.37}$$

$$\rho c_r \left(\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta \right) = 2\mu D : D + k \Delta \theta. \tag{2.38}$$

2.4 Navier–Stokes Equations

Assuming that the density ρ of the fluid is uniform and denoting $\nu = \frac{\mu}{\rho}$, $\kappa = \frac{k}{\rho}$ (ν is called the *kinematic viscosity* coefficient), Eqs. (2.36)–(2.38) reduce to

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p + \nu \Delta u + f\,, \tag{2.39}$$

$$\operatorname{div} u = 0\,, \tag{2.40}$$

$$c_r \left(\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta \right) = 2\nu D : D + \kappa \Delta \theta. \tag{2.41}$$

When the body forces f do not depend on temperature, the first two equations of the above system,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p + \nu \Delta u + f\,, \tag{2.42}$$

$$\operatorname{div} u = 0 \tag{2.43}$$

constitute a closed system of equations with respect to variables u, p, and are called *Navier–Stokes equations* of viscous incompressible fluids with uniform density (we shall call them just the Navier–Stokes equations). The mechanical energy of the flow governed by (2.42) and (2.43) due to viscous dissipation is lost and appears as heat. This can be seen from Eq. (2.41) in which the term $2\nu D : D$ is positive, provided the flow is not uniform. In real fluids, however, density depends on temperature, so that our system (2.39)–(2.41) may be physically impossible. In fact, due to viscosity and high velocity gradients the temperature rises in view of (2.41), and this produces density fluctuations, contrary to our assumption that density is uniform in the flow domain. Thus, reduced problems often play the role of more or less justified approximations. For more considerations of this kind cf. [109, Chap. 1].

When the body forces depend on temperature, $f = f(\theta)$, we have to take into account the whole system (2.39)–(2.41). One of the considered in the literature system of equations of heat conducting viscous and incompressible fluid are the so-called *Boussinesq equations*,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho_0}\nabla p + \nu \Delta u + \frac{1}{\rho_0} g\alpha(\theta - \theta_0)\,, \tag{2.44}$$

$$\operatorname{div} u = 0\,, \tag{2.45}$$

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = \frac{\kappa}{c_r}\Delta\theta\,, \tag{2.46}$$

where g represents the vertical *gravity acceleration*, α is the *thermal expansion coefficient*, and $\frac{\kappa}{c_r}$ is the *thermal diffusion coefficient*. Moreover, ρ_0 and θ_0 are some reference density and temperature, respectively. In the velocity equation the vertical buoyancy force $\frac{1}{\rho_0} g\alpha(\theta - \theta_0)$ results from changes of density associated with temperature variations $\rho - \rho_0 = -\alpha(\theta - \theta_0)$. This is the only term in the system where changes of density were taken into account. We have also abandoned the viscous dissipation term in the temperature equation.

2.5 Vorticity Dynamics

Taking the *curl* of the equation of motion

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p + \nu \Delta u,$$

we obtain

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \Delta \omega, \tag{2.47}$$

where the vector field $\omega = \nabla \times u$ is called *vorticity* of the fluid. It has a simple physical interpretation. In the case of two-dimensional motion with

$$u = (u_1(x, y), u_2(x, y), 0),$$

the vorticity reduces to

$$\omega = (0, 0, \omega_3(x_1, x_2)) = \left(0, 0, \frac{\partial u_2(x_1, x_2)}{\partial x_1} - \frac{\partial u_1(x_1, x_2)}{\partial x_2}\right),$$

where the third component represents twice the angular velocity of a small (infinitesimal) fluid element at point (x_1, x_2). The vorticity field is, by definition, divergence free,

$$\operatorname{div} \omega = 0.$$

In the case of two-dimensional motions the Eq. (2.47) reduces to

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = \nu \Delta \omega,$$

and we can see that the vorticity in the fluid is transported by two agents: *convection* and *diffusion*, just as the temperature in the system (2.44)–(2.46).

For inviscid fluids ($\nu = 0$) the vorticity field has very important properties that allow us to imagine behavior of complicated turbulent flows [83]. In this case, vorticity is a *local variable* which means that we can isolate a patch of vorticity and observe how it is transported along the velocity field trajectories with a finite speed. For two-dimensional flows this is evident as then the vorticity equation reduces to

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = 0.$$

For more information, cf. [166].

Exercise 2.10. Vorticity has nothing in common with rotation of the fluid as a whole. Calculate the vorticity of the flows: (a) $u(x_1, x_2, x_3) = (u_1(x_2), 0, 0)$ and (b) $u(r, \phi, z) = (0, k/r, 0)$ for $r > 0$.

2.6 Thermodynamics

Equations of State From the point of view of thermodynamics the state of a homogeneous fluid can be described by some definite relations among a number of certain *state variables*, the most important being the volume V ($V = 1/\rho$), the entropy S, the internal energy E, the pressure p, and the absolute temperature θ, cf. [212].

In such a description one may start with a relation of the form (cf. [212])

$$E = E(S, V) \quad \text{(Gibbs relation)} \tag{2.48}$$

and define p and θ by

$$p = -\frac{\partial E}{\partial V}, \quad \theta = \frac{\partial E}{\partial S}, \tag{2.49}$$

with $p, \theta > 0$ by assumption. In this case, taking the total differential in (2.48) and using (2.49), we obtain

$$dE = \theta\, dS - p\, dV \quad \text{or} \quad dE = \theta\, dS - p\frac{1}{\rho^2}\, d\rho\,. \tag{2.50}$$

A simple phase system is said to undergo a *differentiable process* if its state variables are differentiable functions of time: $V = V(t)$, $S = S(t)$, etc. Assuming such a dependence one usually assumes, together with (2.50), that

$$\frac{DE}{Dt} = \theta\frac{DS}{Dt} - p\frac{DV}{Dt}$$

or

$$\frac{DE}{Dt} = \theta\frac{DS}{Dt} - p\frac{1}{\rho^2}\frac{D\rho}{Dt}\,. \tag{2.51}$$

Relation (2.51) makes it possible to write a definite form of the balance of entropy when we know the laws of conservation of mass and internal energy. We shall use this relation in the sequel.

Second Law of Thermodynamics and Constraints on Viscosity Coefficients Consider the law of conservation of energy (2.32)

$$\rho\frac{DE}{Dt} = \nabla\cdot(k\nabla\theta) - p\,\mathrm{div}\,u + \rho\Phi\,, \tag{2.52}$$

where Fourier's law is assumed, and $\rho\Phi$ is given by (2.33). We see that the internal energy increases with the influx of heat transfer, compression, and the viscous dissipation.

From the law of conservation of mass

$$\frac{D\rho}{Dt} + \rho \operatorname{div} u = 0,$$

we have

$$\operatorname{div} u = -\frac{1}{\rho}\frac{D\rho}{Dt}. \tag{2.53}$$

Substituting (2.53) into (2.52) we can write

$$\rho\frac{DE}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} = \nabla \cdot (k\nabla\theta) + \rho\Phi. \tag{2.54}$$

Now we shall transform the left-hand side of (2.54). From (2.51) we have

$$\rho\theta\frac{DS}{Dt} = \rho\frac{DE}{Dt} + \frac{p}{\rho}\frac{D\rho}{Dt}. \tag{2.55}$$

Thus

$$\rho\frac{DS}{Dt} = \frac{1}{\theta}\nabla \cdot (k\nabla\theta) + \frac{1}{\theta}\rho\Phi, \tag{2.56}$$

so that

$$\begin{aligned}\int_{\Omega(t)} \rho\frac{DS}{Dt}\,dx &= \frac{d}{dt}\int_{\Omega(t)} \rho S\,dx \qquad (2.57)\\ &= \int_{\Omega(t)} \left(\frac{1}{\theta}\nabla \cdot (k\nabla\theta) + \frac{1}{\theta}\rho\Phi\right) dx\\ &= \int_{\Omega(t)} \left(\nabla \cdot \left(\frac{k}{\theta}\nabla\theta\right) + \frac{k}{\theta^2}|\nabla\theta|^2 + \frac{1}{\theta}\rho\Phi\right) dx.\end{aligned}$$

In the end we obtain the following integral form of the *balance of entropy*

$$\int_{\Omega(t)} \rho\frac{DS}{Dt}dx = \int_{\partial\Omega(t)} \frac{k}{\theta}\frac{\partial\theta}{\partial n}dS + \int_{\Omega(t)} \left(\frac{k}{\theta^2}|\nabla\theta|^2 + \frac{1}{\theta}\rho\Phi\right) dx. \tag{2.58}$$

Equation (2.58) with Φ as in (2.33) and with $\mu \geq 0$, $3\lambda + 2\mu \geq 0$ (cf. [212]) is consistent with the *second law of thermodynamics*, which says that the rate of increase of entropy is not less than the heat transfer into the material volume. The first integral on the right-hand side of (2.58) is just the heat transferred into the material volume divided by the temperature

$$\int_{\partial\Omega(t)} \frac{k}{\theta}\frac{\partial\theta}{\partial n}\,dS = \int_{\partial\Omega(t)} \frac{-q\cdot n}{\theta}\,dS\,,$$

and the second integral on the right-hand side of (2.58) is nonnegative.

Remark 2.3. For a perfect fluid, with no heat conductivity and viscosity, Eq. (2.56) becomes

$$\frac{DS}{Dt} = 0\,,$$

so that the entropy is conserved.

Remark 2.4. If we assume that the Fourier law holds, the second law of thermodynamics demands

$$\frac{k}{\theta^2}|\nabla\theta|^2 + \frac{1}{\theta}\rho\Phi \geq 0\,, \tag{2.59}$$

where $\rho\Phi$ is given by (2.33). The inequality (2.59) gives us the restrictions on the coefficients in the constitutive equation (2.24). Without assuming the Fourier law, the second law of thermodynamics states that

$$-\frac{1}{\theta}q\cdot\nabla\theta + \rho\Phi \geq 0\,.$$

This law will be satisfied if $q\cdot\nabla\theta \leq 0$ (i.e., the heat flux is not directed against the temperature gradient) and $\Phi \geq 0$ (i.e., deformation always absorbs energy converting it to heat).

Exercise 2.11. Prove that inequality (2.59), with Φ given in (2.33), yields the following constraints on the coefficients

$$k \geq 0, \quad 3\lambda + 2\mu \geq 0, \quad \mu \geq 0\,.$$

Hint. The left-hand side of inequality (2.59) must be nonnegative for an arbitrary choice of $\theta_{,i}$ and D_{ij}. It is clear that this implies $k \geq 0$. The nonnegativity of the terms containing D_{ij} is equivalent to the condition

$$3\lambda + 2\mu \geq 0 \quad \text{and} \quad \mu \geq 0\,.$$

2.7 Similarity of Flows and Nondimensional Variables

Let us consider two initial boundary value problems

$$\rho\frac{Du}{Dt} = -\nabla p + \mu\Delta u, \qquad \operatorname{div} u = 0 \tag{2.60}$$

in $\Omega = [0, L]^n$, $u = u(x, t)$, $p = p(x, t)$, with initial condition $u(x, 0) = u_0(x)$, and

$$\rho' \frac{Du'}{Dt'} = -\nabla p' + \mu' \Delta u', \qquad \operatorname{div} u' = 0 \tag{2.61}$$

in $\Omega' = [0, L']^n$, $u' = u'(x', t')$, $p' = p'(x', t')$, with initial condition $u'(x', 0) = u'_0(x')$, respectively, and with periodic boundary conditions, $n = 2$ or 3.

Changing dependent and independent variables in (2.60) as follows:

$$u^\star(x^\star, t^\star) = \frac{u(x, t)}{U}, \quad p^\star(x^\star, t^\star) = \frac{p(x, t) - p_0}{\rho U^2} \tag{2.62}$$

and

$$x^\star = \frac{x}{L}, \quad t^\star = \frac{U}{L} t, \tag{2.63}$$

we obtain the system

$$\frac{Du^\star}{Dt^\star} = -\nabla p^\star + \frac{1}{Re} \Delta u^\star, \qquad \operatorname{div} u^\star = 0 \tag{2.64}$$

in $\Omega^\star = [0, 1]^n$, $u^\star = u^\star(x^\star, t^\star)$, $p^\star = p^\star(x^\star, t^\star)$, with initial condition $u^\star(x^\star, 0) = u_0(x^\star) = u_0(Lx^\star)/U$, with periodic boundary condition, and with

$$Re = \frac{\rho L U}{\mu}.$$

In the case of problem (2.60) the symbols ρ, L, U, μ denote *dimensional* characteristic quantities of the flow, describing the density of the fluid (ρ), characteristic linear dimension of the domain of the flow (L), some characteristic velocity (U), and viscosity (μ). For U we can take, for example, the square root of the average initial kinetic energy in Ω, see [88] for more details. According to (2.62) and (2.63), in (2.64) all dependent and independent variables are *nondimensional*, together with the *Reynolds number Re*. The characteristic quantities of the flow in (2.64) are equal to one, with the exception of viscosity which equals $1/Re$.

In the same way we can reduce system (2.61) to system (2.64), now with the Reynolds number

$$Re' = \frac{\rho' L' U'}{\mu'}.$$

Assume now that $Re = Re'$. Then both systems (2.60) and (2.61) (including the boundary conditions) are represented, in *nondimensional* variables, by the same system (2.64). We call systems (2.60) and (2.61) *dynamically similar*. Solving system (2.64) we generate solutions to an infinite three-parameter family of systems having the same Reynolds number.

The physical interpretation of the Reynolds number is that it describes the relation between the magnitudes of the inertial term $(u \cdot \nabla)u$ and the viscous term $\nu \Delta u$ $(\nu = \mu/\rho)$ in the equation of motion, according to

$$\frac{\textit{inertial term}}{\textit{viscous term}} = O\left(\frac{|(u \cdot \nabla)u|}{|\nu \Delta u|}\right) = O\left(\frac{U(U/L)}{\nu(U/L^2)}\right) = O(Re),$$

or, saying it differently, the Reynolds number can be interpreted as a ratio of the strength of inertial forces to viscous forces.

Exercise 2.12. Let $u^\star, p^\star$ solve system (2.64). Show that

$$u(x,t) = Uu^\star\left(\frac{x}{L}, \frac{U}{L}t\right), \quad p(x,t) = \rho U^2 p^\star\left(\frac{x}{L}, \frac{U}{L}t\right) + p_0$$

solve system (2.60).

Exercise 2.13. Let $\rho = \rho'$ and $\mu = \mu'$, and let u', p' solve system (2.61). Show that

$$\begin{aligned} u(x,t) &= \frac{U}{U'}u'\left(\frac{L'}{L}x, \frac{L'U}{LU'}t\right), \\ p(x,t) &= \frac{U^2}{U'^2}\left(p'\left(\frac{L'}{L}x, \frac{L'U}{LU'}t\right) - p'_0\right) + p_0 \end{aligned}$$

solve system (2.60). Here arbitrary constants p_0, p'_0 may represent typical values of pressures in the considered problems for systems (2.60) and (2.61), respectively, cf. (2.62).

Exercise 2.14. Let u', p' solve the system of equations

$$\rho' \frac{Du'}{Dt} = -\nabla p' + \mu \Delta u', \qquad \operatorname{div} u' = 0 \tag{2.65}$$

in the whole space. Show that

$$u(x,t) = lu'(lx, l^2 t), \quad p(x,t) = l^2(p'(lx, l^2 t) - p'_0) + p_0,$$

where $l > 0$, also solve this system of equations. Thus having one (nontrivial) solution of system (2.65) we can generate in principle an infinite number of different solutions of the same system.

Consider the transformations,

$$(x,t,u,p) \to \left(lx, l^2 t, \frac{1}{l}u, \frac{1}{l^2}p\right), \quad l > 0, \tag{2.66}$$

or

$$
\begin{aligned}
x' &= lx\,,\\
t' &= l^2 t\,,\\
u' &= \frac{1}{l}u\,,\\
p' &= \frac{1}{l^2}p.
\end{aligned}
$$

Show that transformations (2.66) form a group. This group is called a one parameter *symmetry group* of *scalings* of system (2.65).

A solution (u', p') is called *self-similar* if $u(x,t) = u'(x',t'),\ p(x,t) = p'(x',t')$, cf. [17].

Exercise 2.15. Let u', p' solve the system of equations

$$
\rho' \frac{Du'}{Dt} = -\nabla p' + \mu \Delta u', \qquad \operatorname{div} u' = 0
$$

in the whole space $\mathbb{R}^n$. Show that

$$
\begin{aligned}
u(x,t) &= u'(x-ct,t) + c,\\
p(x,t) &= p'(x-ct,t),
\end{aligned}
$$

where c is any vector in $\mathbb{R}^n$, also solve this system of equations (*Galilean invariance*), and

$$
\begin{aligned}
u(x,t) &= Q^t u'(Qx,t),\\
p(x,t) &= p'(Qx,t),
\end{aligned}
$$

where Q is any rotation matrix ($Q^t = Q^{-1}$) also solve this system of equations (*rotation symmetry*).

2.8 Examples of Simple Exact Solutions

The Flow Due to an Impulsively Moved Plain Boundary Let the domain of the flow be half-space

$$
\{(x,y,z) \in \mathbb{R}^3 \ :\ \ y > 0\},
$$

and assume that at times $t < 0$ the flow is at rest, and that at time $t = 0$ the plane $y = 0$ is suddenly jerked into motion in the x-direction, with a constant velocity U.

Due to the geometrical symmetry of the problem and under additional "intuitively apparent" preconceptions, we assume that the flow is of the form $u = (u(y,t), 0, 0)$. Substituting u of the above form into the Navier–Stokes equations (2.42) we obtain

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}, \tag{2.67}$$

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0. \tag{2.68}$$

As u depends only on y, from (2.67) together with (2.68) it follows that $\frac{\partial p}{\partial x}$ depends only on t. Assume additionally that $\frac{\partial p}{\partial x} = 0$ and that the flow is at rest for $y = \infty$. Then the above system together with the initial and boundary conditions reads

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{2.69}$$

with

$$u(y,0) = 0 \quad \text{for} \quad y > 0\,, \tag{2.70}$$

$$u(0,t) = U \quad \text{for} \quad t > 0\,, \tag{2.71}$$

$$u(\infty,t) = 0 \quad \text{for} \quad t > 0. \tag{2.72}$$

We say that a solution $u = u(y,t)$ of the initial and boundary value problem (2.69)–(2.72) is *invariant* or *self-similar* with respect to the one-parameter symmetry group of scalings

$$y' = \alpha y, \quad t' = \alpha^2 t, \quad u' = u, \quad \alpha > 0, \tag{2.73}$$

if the function $u' = u'(y',t') = u(y,t)$ is a solution of the problem

$$\frac{\partial u'}{\partial t'} = \nu \frac{\partial^2 u'}{\partial y'^2},$$

with

$$u'(y',0) = 0 \quad \text{for} \quad y' > 0\,,$$

$$u'(0,t') = U \quad \text{for} \quad t' > 0\,,$$

$$u(\infty,t') = 0 \quad \text{for} \quad t' > 0.$$

Observe that the transformation of the independent variables does not change the space-time domain.

The relation $u'(y', t') = u(y, t)$ satisfied by the invariant solution implies that the number of its independent variables can be reduced by one [17]. Together with the *dimensional homogeneity principle* [33] stating that physical phenomena must be described by laws that do not depend on the unit of measure applied to the dimensions of the variables that describe the phenomena, we arrive at the following form of the solution

$$u(y, t) = f(\eta), \quad \eta = \frac{y}{\sqrt{\nu t}}. \tag{2.74}$$

Notice that the new variable η, called *similarity variable* is nondimensional and is invariant with respect to the symmetry group of scalings $y' = \alpha y$, $t' = \alpha^2 t$.

Exercise 2.16. Assume that problem (2.69)–(2.72) is invariant with respect to a *dilation group*

$$y' = e^a y, \quad t' = e^b t, \quad u' = e^c u,$$

for some a, b, c, that is if $u(x, t)$ is a solution of the problem then $u'(x', t')$ is also a solution of the same problem. Then we get (2.73).

Observe, that if a problem is uniquely solvable and invariant with respect to a symmetry group of scalings $x \to x', t \to t', u \to u'$ then its unique solution is also invariant.

Substituting function of the form (2.74) to problem (2.69)–(2.72) we obtain

$$f'' + \frac{1}{2}\eta f' = 0,$$

with

$$f(0) = U, \quad f(\infty) = 0.$$

Integrating this equations we obtain

$$u(y, t) = f(\eta) = U\left(1 - \frac{1}{\sqrt{\pi}}\int_0^\eta e^{\frac{-s^2}{4}} ds\right), \tag{2.75}$$

or, in the nondimensional form

$$\frac{u(y, t)}{U} = 1 - \frac{1}{\sqrt{\pi}}\int_0^\eta e^{\frac{-s^2}{4}} ds.$$

The example of the velocity profile given by (2.75) is depicted in Fig. 2.1.

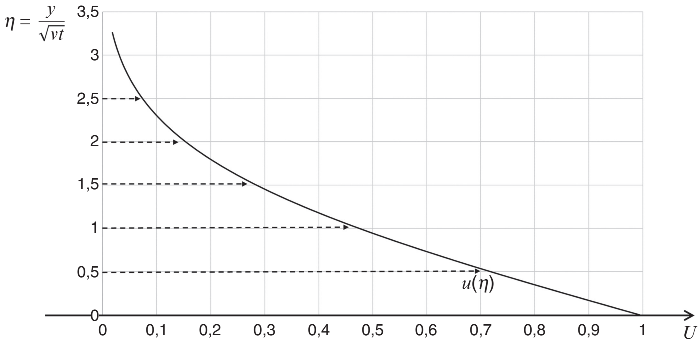

Fig. 2.1 Velocity profile for the flow in the half-space due to the impulsively moved boundary. In the plot we have $U = 1$. The variable on the horizontal axis is the velocity u, and on the vertical axis $\eta = \frac{y}{\sqrt{\nu t}}$

Exercise 2.17. Check that the pair $(u,p) = (u((y,t),0,0),C)$ where C is a constant and u is the function given in (2.75) is a solution of the Navier–Stokes equations (2.42) and (2.43) with initial and boundary conditions (2.70)–(2.72). Observe that we have found only one of, perhaps, many other solutions of the problem. We cannot claim that the solution is *unique* in u.

From (2.75) it follows that the velocity of the flow is a nonlocal variable, that is, it spreads information through the region of the flow with infinite speed. Although at negative times the flow was at rest, at any, however small, positive time t and any, however large, $y > 0$, we have $u(y,t) > 0$. The same observation concerns the vorticity vector (the third and only nonzero component of which is $\omega = -\frac{\partial u}{\partial y}$) as both functions u and ω satisfy the diffusion equation.

The Poiseuille Flow Consider a *stationary* flow $u = (u(y),0,0)$ in the domain

$$\{(x,y,z) \in \mathbb{R}^3 \ : \ 0 < y < h\},$$

and with homogeneous boundary conditions $u = 0$ at $y = 0$ and $y = h$. Substituting u of the above form to the Navier–Stokes equations we obtain the equations of motion

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{\partial p}{\partial x}\,,$$

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0\,,$$

from which it follows that

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{\partial p}{\partial x} = C \tag{2.76}$$

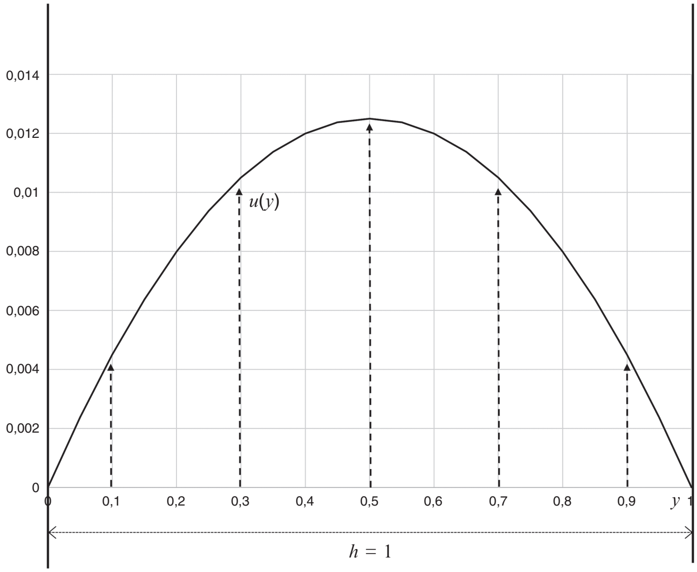

Fig. 2.2 Parabolic velocity profile for the Poiseuille flow. The constants were chosen as $h = 1$, $\nu = 10$, and $C = -1$. Velocity $u(y)$ is on the vertical axis, and the space variable y is on the horizontal axis

for some constant C. Taking into account the boundary conditions we obtain

$$u(y) = \frac{C}{2\nu} y(y-h)\,,$$

where

$$C = \frac{1}{\rho}\frac{\partial p}{\partial x}.$$

From the last equation we have $p(x) = C\rho x + B$, for an arbitrary constant B. Let us take $B = 0$. We can see that there is an *infinite number of solutions* (u, p) of the form

$$u(y) = \frac{C}{2\nu} y(y-h), \quad p(x) = C\rho x. \tag{2.77}$$

We call them the *Poiseuille flows*. For $C = 0$ the flow is just at rest, both velocity and pressure equal zero. For $C < 0$ the fluid flows towards the increasing x-direction, $u > 0$, driven by the force due to the pressure gradient (as $C < 0$ the pressure decreases with increasing x). The example of the velocity profile given by (2.77) for negative value of C is depicted in Fig. 2.2.

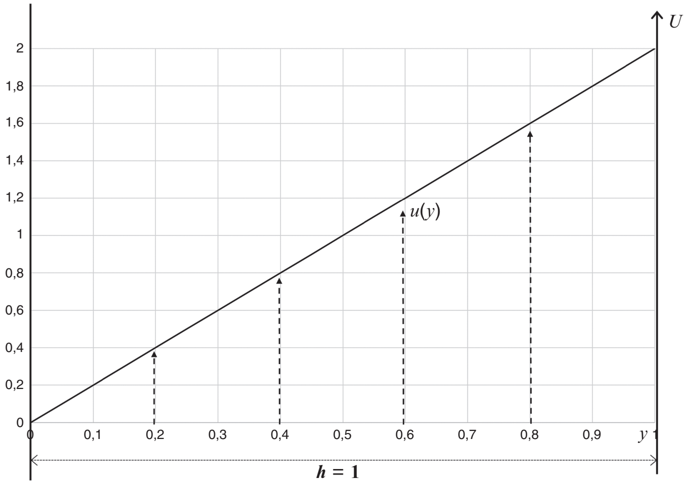

Fig. 2.3 Linear velocity profile for the Couette flow. The constants were chosen as $h = 1$ and $C = 2$. Velocity $u(y)$ is on the vertical axis, and the space variable y is on the horizontal axis

The Couette Flow Let us consider a flow $u = (u(y), 0, 0)$ in the same region as in the preceding example, this time, however, with different boundary conditions. We pose $u = 0$ at $y = 0$ and $u = U$ at $y = h$. As the driving force now comes from the motion of the upper boundary $y = h$, we pose $C = 0$ which corresponds to $p = const$ in the whole domain. Solving Eq. (2.76) in u with $C = 0$ and with the new boundary conditions we obtain the solution

$$u(y) = \frac{U}{h}y, \quad p = const. \tag{2.78}$$

of the problem, with a uniform pressure distribution. We call this flow the *Couette flow*. The velocity profile given by (2.78) for negative value of C is depicted in Fig. 2.3.

2.9 Comments and Bibliographical Notes

Remarks on Modeling Exact solutions serve, among other things, to test a given theory of hydrodynamics. By *theory* we mean here a set of governing equations (conservation laws) together with additional conditions (boundary or initial and boundary conditions), that is, a boundary or initial and boundary value problem. In the following chapters we shall prove several results about such theories of classical hydrodynamics.

A particular theory of hydrodynamics is *well-set*, that is, that it *possesses a unique solution* depending *continuously* upon the data (boundary data, external forces, etc.).

As was observed by Birkhoff in [16], "until it has been shown that the boundary value problem is well-set, one cannot conclude that its equations are erroneous." The latter means "false" according to the following definition, cf. [16]: A theory of rational hydrodynamics will be called *incomplete* if its conditions do not uniquely determine the flow, *overdetermined* if its conditions are mathematically incompatible, *false* if it is well-set but gives grossly incorrect predictions.

Since from a particular theory in its whole generality, it is in most cases quite impossible to predict qualitative properties of the flow (it is sometimes easier, however, to compare various flows among themselves), we are forced to "simplify," whatever that means. In this connection, there are very often exact solutions of special problems that serve to test whether a given theory gives correct predictions and consequently is not a false theory.

We enumerate a number of such simplifications (which we have already used in Sect. 2.8):

1. linearization of governing equations,
2. setting a special form of boundary conditions,
3. assumption of some symmetry in space (which reduces the number of spatial dimensions),
4. assumption that the problem is time-independent,
5. assumption that acting forces are potential or absent.

After suitable simplifications have been done, the original problem reduces to the one that can be (luckily) solved explicitly by solving, e.g., a linear system of ordinary differential equations with constant coefficients. Having the exact solutions, we try to compare them with experiments.

When simplifying, however, one has to be very careful in order not to oversimplify and obtain paradoxes, for example. It is well known (cf. [16]) that the various plausible intuitive assumptions we make *implicitly* when simplifying may be fallacious. Following [16], we state the assumptions that are especially suggestive:

1. intuition suffices for determining which physical variables require consideration,
2. small causes produce small effects, and infinitesimal causes produce infinitesimal effects,
3. symmetric causes produce effects with the same symmetry,
4. the flow topology can be guessed by intuition,
5. the processes of analysis can be used freely: the functions of rational hydrodynamics can be freely integrated, differentiated, expanded in series, integrals, etc.,
6. mathematical problems suggested by intuitive physical ideas are well-set.

In Chaps. 14–16 we consider problems whose solutions might not be unique.

For more information about classical hydrodynamics we refer the reader to [1, 14, 83, 149].

For a history of hydrodynamics and the derivation and development of the Navier–Stokes equations, in particular, see [81].

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