Chapter 2
Limits and Continuity of Functions

2.1 Continuity

As you have seen in beginning calculus, informally, a function is said to be continuous if its values change by small amounts corresponding to small changes in the value of its independent variable. We will give the precise definition. The discussion will be restricted to functions that are defined on intervals or unions of intervals since our discussion in later chapters will involve functions defined on such sets. We will introduce the important concept of the uniform continuity of a function on an interval. We will also discuss the continuity of some basic functions and their combinations.

2.1.1 The Definition of Continuity

Definition 1. Assume that f is a real-valued function that is defined in an open interval that contains the point x_0. We say that f is **continuous** at x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \text{ if } |x - x_0| < \delta.$$

You can think of the number δ as a positive number that is **sufficiently small** so that the magnitude of the difference between $f(x)$ and $f(x_0)$ is smaller than a given positive number ε that can be **arbitrarily small**, i.e., as **small as desired**, provided that the distance between x and x_0 is less than δ. You can also think of $\varepsilon > 0$ as an "error tolerance" in approximating $f(x_0)$ with the value of f at a nearby point x. Note that

$$|f(x) - f(x_0)| < \varepsilon \Leftrightarrow f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$$

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and

$$|x - x_0| < \delta \Leftrightarrow x_0 - \delta < x < x_0 + \delta.$$

Thus, f is continuous at x_0 if $f(x)$ is between $f(x_0) - \varepsilon$ and $f(x_0) + \varepsilon$ provided that x is between $x_0 - \delta$ and $x_0 + \delta$.

Remark 1. Note that the "$<$" sign in the definition of continuity can be replaced by the "$\leq$" sign: f is continuous at x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| \leq \varepsilon \text{ if } |x - x_0| \leq \delta.$$

Indeed, if the above statement is valid, given $\varepsilon > 0$ we can find $\delta > 0$ so that $|f(x) - f(x_0)| \leq \varepsilon/2$ if $x \in D$ and $|x - x_0| \leq \delta$. Then

$$|f(x) - f(x_0)| < \varepsilon \text{ if } x \in D \text{ and } |x - x_0| < \delta.$$

$\diamond$

Example 1. Let $f(x) = x^2$ for each $x \in \mathbb{R}$ and let x_0 be an arbitrary real number. Show that f is continuous at x_0.

Solution. We have

$$|f(x) - f(x_0)| = \left|x^2 - x_0^2\right| = |(x + x_0)(x - x_0)| = |x + x_0| |x - x_0|.$$

Since

$$|x + x_0| \leq |x| + |x_0|$$

by the triangle inequality, we have

$$|f(x) - f(x_0)| \leq (|x| + |x_0|) |x - x_0|.$$

Since we are entitled to have x as close to x_0 as necessary in order to have $f(x)$ as close to $f(x_0)$ as desired, we can restrict x so that $|x - x_0| < 1$. Then,

$$|x| = |(x - x_0) + x_0| \leq |x - x_0| + |x_0| < 1 + |x_0|.$$

Therefore,

$$\begin{aligned} |f(x) - f(x_0)| \leq (|x| + |x_0|) |x - x_0| &< (1 + |x_0| + |x_0|) |x - x_0| \\ &= (1 + 2|x_0|) |x - x_0|. \end{aligned}$$

Now we are ready to pick a δ for a given $\varepsilon > 0$ be given. Let us set

$$\delta = \min\left(1, \frac{\varepsilon}{1 + 2|x_0|}\right).$$

If $|x - x_0| < \delta$ then

$$|f(x) - f(x_0)| < (1 + 2|x_0|)|x - x_0| < (1 + 2|x_0|)\left(\frac{\varepsilon}{1 + 2|x_0|}\right) = \varepsilon.$$

Therefore, f is continuous at x_0, as claimed.

Note that the choice of δ for a given ε is not unique. For example, we could have restricted x so that $|x - x_0| < 0.1$. Then

$$|x| = |(x - x_0) + x_0| \leq |x - x_0| + |x_0| < 0.1 + |x_0|$$

so that

$$\begin{aligned} |f(x) - f(x_0)| \leq (|x| + |x_0|)|x - x_0| &< (0.1 + |x_0| + |x_0|)|x - x_0| \\ &= (0.1 + 2|x_0|)|x - x_0|. \end{aligned}$$

Then we are led to the choice

$$\delta = \min\left(0.1, \frac{\varepsilon}{0.1 + 2|x_0|}\right)$$

so that

$$|f(x) - f(x_0)| < \varepsilon \text{ if } |x - x_0| < \delta.$$

□

Remark 2. The definition of continuity can be rephrased as follows by setting $x = x_0 + h$:

A function f that is defined in an open interval containing x_0 is continuous at x_0 if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x_0 + h) - f(x_0)| < \varepsilon \text{ provided that } |h| < \delta.$$

In some cases this expression may be more convenient in confirming the continuity of a function. ◇

Example 2. Let $f(x) = x^3$ for each $x \in \mathbb{R}$. Show that f is continuous at any $x_0 \in \mathbb{R}$.

Solution. We have

$$\begin{aligned}|f(x_0+h)-f(x_0)| = \left|(x_0+h)^3 - x_0^3\right| &= \left|x_0^3 + 3x_0^2h + 3x_0h^2 + h^3 - x_0^3\right| \\ &= |h|\left|3x_0^2 + 3x_0h + h^2\right| \\ &\le |h|\left(3x_0^2 + 3|x_0||h| + h^2\right).\end{aligned}$$

Let us restrict h so that $|h| < 1$. Then,

$$|f(x_0+h)-f(x_0)| < |h|\left(3x_0^2 + 3|x_0| + 1\right)$$

Thus, given $\varepsilon > 0$ we can set

$$\delta = \min\left(1, \frac{\varepsilon}{3x_0^2 + 3|x_0| + 1}\right).$$

If $|h| < \delta$ then

$$\begin{aligned}|f(x_0+h)-f(x_0)| &< |h|\left(3x_0^2 + 3|x_0| + 1\right) < \delta\left(3x_0^2 + 3|x_0| + 1\right) \\ &< \left(\frac{\varepsilon}{3x_0^2 + 3|x_0| + 1}\right)\left(3x_0^2 + 3|x_0| + 1\right) = \varepsilon.\end{aligned}$$

Therefore, f is continuous at x_0. □

In the above example we made a choice for δ that yielded the inequality

$$|f(x_0+h)-f(x_0)| < \varepsilon.$$

We could have made another choice for δ as $\min(1, \varepsilon)$. Then $|h| < \delta$ implies that

$$|f(x_0+h)-f(x_0)| < |h|\left(3x_0^2 + 3|x_0| + 1\right) < \left(3x_0^2 + 3|x_0| + 1\right)\varepsilon.$$

Would that be sufficient to prove the continuity of f at x_0? Indeed it would: Once we have such an inequality, given $\varepsilon > 0$, we can easily amend the choice of δ by setting

$$\delta = \min\left(1, \frac{\varepsilon}{3x_0^2 + 3|x_0| + 1}\right)$$

in order to have $|f(x_0+h)-f(x_0)| < \varepsilon$ if $|h| < \delta$.

There are "one-sided" versions of continuity:

Definition 2. Assume that $f(x)$ is defined on an interval $[x_0, x_0 + \delta_0)$ for some $\delta_0 > 0$. The function f is **continuous at** x_0 **from the right** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x)-f(x_0)| < \varepsilon$ if $x_0 \le x < x_0 + \delta$. We say that f is

continuous at x_0 **from the left** If $f(x)$ is defined on an interval $(x_0 - \delta_0, x_0]$ for some $\delta_0 > 0$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ if $x_0 - \delta < x \leq x_0$. Assume that $f(x)$ is defined for each x in an interval J. We will say that f is **continuous on** J if f is continuous at any point in the interior of J (i.e., a point of J that is not an endpoint of J) and f is continuous from the right or from the left at an endpoint of J that belongs to J, depending on which concept is applicable.

Example 3. Let $f(x) = \sqrt{x}$ for each $x \geq 0$. Show that f is continuous on the interval $[0, +\infty)$.

Solution. Assume that $x > 0$ and that $|h| < x$. Then

$$f(x+h) - f(x) = \sqrt{x+h} - \sqrt{x} = \left(\sqrt{x+h} - \sqrt{x}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$
$$= \frac{(x+h) - x}{\sqrt{x+h} + \sqrt{x}} = \frac{h}{\sqrt{x+h} + \sqrt{x}}.$$

Since

$$\sqrt{x+h} + \sqrt{x} > \sqrt{x}$$

we have

$$|f(x+h) - f(x)| = \frac{|h|}{\sqrt{x+h} + \sqrt{x}} < \frac{|h|}{\sqrt{x}}$$

if $|h| < x$. Given $\varepsilon > 0$ in order to have $|f(x+h) - f(x)| < \varepsilon$ it is sufficient to have

$$\frac{|h|}{\sqrt{x}} < \varepsilon \Leftrightarrow |h| < \sqrt{x}\varepsilon.$$

Thus we will set

$$\delta = \min\left(x, \sqrt{x}\varepsilon\right).$$

If $|h| < \delta$ then

$$|f(x+h) - f(x)| < \frac{|h|}{\sqrt{x}} < \frac{\delta}{\sqrt{x}} = \frac{\sqrt{x}\varepsilon}{\sqrt{x}} = \varepsilon.$$

Therefore f is continuous at each x in the interior of the interval $[0, +\infty)$.

Now let us consider the continuity of f at the endpoint 0 of $[0, +\infty)$. The relevant concept is continuity at 0 from the right. We need to be able to choose $\delta > 0$ so that

$$|f(x) - f(0)| = \sqrt{x} < \varepsilon$$

if $0 \le x < \delta$. Given $\varepsilon > 0$ let us set $\delta = \varepsilon^2$. If

$$0 \le x < \varepsilon^2 \text{ then } 0 \le \sqrt{x} < \varepsilon.$$

Thus f is continuous at 0 from the right. Therefore we have shown that f is continuous on the interval $[0, +\infty)$. □

There is a connection between the concepts of continuity and limits of sequences:

Theorem 1 (The Sequential Characterization of Continuity). *Assume that f is defined in an open interval that contains x_0. The function f is continuous at x_0 if and only if*

$$\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0)$$

Proof. Assume that f is continuous at x_0. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $\lim_{n\to\infty} x_n = x_0$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ if $|x - x_0| < \delta$. Since $\lim_{n\to\infty} x_n = x_0$ there exists a positive integer N such that

$$|x_n - x_0| < \delta \text{ if } n \ge N.$$

In that case $|f(x_n) - f(x_0)| < \varepsilon$. Since we have shown that for a given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f(x_n) - f(x_0)| < \varepsilon$ we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Conversely, assume that $\lim_{n\to\infty} f(x_n) = f(x_0)$ for any sequence $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = x_0 \in D$. We would like to prove the continuity of f at x_0. We will prove the **contrapositive** of the implication. Thus, assume that f is **not continuous** at x_0. We will show that the statement about sequences that converge to x_0 is not true:

Since f is not continuous at x_0, there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$ there exists x where

$$|x - x_0| < \delta \text{ and } |f(x) - f(x_0)| \ge \varepsilon_0.$$

Thus, for any $n \in \mathbb{N}$ there exists x_n such that

$$|x_n - x_0| < \frac{1}{n} \text{ and } |f(x_n) - f(x_0)| \ge \varepsilon$$

Therefore, $\lim_{n\to\infty} x_n = x_0$ but it is not true that $\lim_{n\to\infty} f(x_n) = f(x_0)$. ■

Remark 3. In the statement of Theorem 1 it is assumed implicitly that $f(x_n)$ is defined for each n. This should be assumed in similar statements. The theorem can be rephrased as follows: A function f is continuous at x_0 if and only if for any sequence $\{x_n\}_{n=1}^{\infty}$ where $\lim_{n\to\infty} x_n = x_0$ we have

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

Note that "one-sided" versions of Theorem 1 are valid. For example, is $f(x)$ is defined for each x in an interval the form $[x_0, x_0 + \delta_0)$ then f is continuous at x_0 from the right if and only if for any sequence $\{x_n\}_{n=1}^{\infty}$ where $x_n \geq x_0$for each n and $\lim_{n\to\infty} x_n = x_0$ we have $\lim_{n\to\infty} f(x_n) = f(x_0)$. $\diamondsuit$

The sequential characterization of continuity is useful in ruling out continuity as in the following example:

Example 4. Let

$$f(x) = \begin{cases} -1 \text{ if } x < 0, \\ \ \ 1 \text{ if } x \geq 0. \end{cases}$$

Show that f is not continuous at 0.

Solution. Let

$$x_n = (-1)^n \frac{1}{n},\ n = 1, 2, 3, \ldots$$

Then $\lim_{n\to\infty} x_n = 0$, but it is not true that $\lim_{n\to\infty} f(x_n) = 0$. Indeed,

$$f(x_n) = f(-1) = -1 \text{ if } n = 1, 3, 5, \ldots,$$

and

$$f(x_n) = f(1) = 1 \text{ if } n = 2, 4, 6, \ldots$$

Thus, the sequence $\{f(x_n)\}_{n-1}^{\infty}$ has two subsequences that converge to different numbers. This rules out the existence of the limit of $\{f(x_n)\}_{n-1}^{\infty}$. Therefore f is not continuous at 0. $\square$

2.1.2 Uniform Continuity

In order to show that a function f is continuous at a point x_0 it is sufficient to be able to determine $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ if $|x - x_0| < \delta$. The choice of δ can depend on the particular point x_0. The uniform continuity of f on a set D requires that we should be able to choose $\delta > 0$ that works for all points in D:

Definition 3. A function f is **uniformly continuous on an interval** $J \subset \mathbb{R}$ if, given any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ if x_1 and x_2 are in J and $|x_1 - x_2| < \delta$.

Example 5. Let $f(x) = 1/x$. Show that f is uniformly continuous on $[1/2, +\infty)$.

Solution. Assume that $x_1 \geq 1/2$ and $x_2 \geq 1/2$. We have

$$|f(x_2) - f(x_1)| = \left|\frac{1}{x_2} - \frac{1}{x_1}\right| = \left|\frac{x_1 - x_2}{x_1 x_2}\right| = \frac{|x_2 - x_1|}{x_1 x_2}.$$

Since $x_1 \geq 1/2$ and $x_2 \geq 1/2$ we have $x_1 x_2 \geq 1/4$. Therefore

$$|f(x_2) - f(x_1)| = \frac{|x_2 - x_1|}{x_1 x_2} \leq \frac{|x_2 - x_1|}{1/4} = 4\,|x_2 - x_1|\,.$$

Given $\varepsilon > 0$ let us set $\delta = \varepsilon/4$. If $x_1 \geq 1/2$ and $x_2 \geq 1/2$ and $|x_1 - x_2| < \delta$ then

$$|f(x_2) - f(x_1)| \leq 4\,|x_2 - x_1| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon.$$

Thus f is uniformly continuous on $[1/2, \infty)$. □

Remark 4. As in the case of continuity at a point, we can rephrase uniform continuity as follows:
A function f is **uniformly continuous on the interval** J if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in J,\ x + h \in J \text{ and } |h| < \delta \Rightarrow |f(x+h) - f(x)| < \varepsilon.$$

◇

There is a sequential characterization of uniform continuity:

Theorem 2. *A function $f : J \to \mathbb{R}$ is uniformly continuous on the interval J if and only if the following condition is satisfied:*
If $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are sequences in J and $\lim_{n\to\infty}(u_n - v_n) = 0$ *then*

$$\lim_{n\to\infty}(f(u_n) - f(v_n)) = 0.$$

Proof. Assume that f is uniformly continuous and $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are sequences in J and $\lim_{n\to\infty}(u_n - v_n) = 0$. We will show that

$$\lim_{n\to\infty}(f(u_n) - f(v_n)) = 0.$$

Let $\varepsilon > 0$ be given. By the uniform continuity of f on J, there exists $\delta > 0$ such that

$$x_1 \in J,\ x_2 \in J \text{ and } |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon.$$

Since $\lim_{n\to\infty}(u_n - v_n) = 0$, there exists $N \in \mathbb{N}$ such that $|u_n - v_n| < \delta$ if $n \geq N$. Then $|f(u_n) - f(v_n)| < \varepsilon$ Therefore $\lim_{n\to\infty}(f(u_n) - f(v_n)) = 0$.

To prove the converse, we will assume that f is not uniformly continuous on J. Then there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists u_n and v_n in J with $|u_n - v_n| < 1/n$ and $|f(u_n) - f(v_n)| \geq \varepsilon$. Thus $\lim_{n\to\infty}(u_n - v_n) = 0$ but it is not true that $\lim_{n\to\infty}(f(u_n) - f(v_n)) = 0$. ■

The sequential characterization of uniform continuity is useful in ruling out uniform continuity, as in the following example:

Example 6. Let $f(x) = 1/x$ for each $x \neq 0$. Show that f is *not* uniformly continuous on $(0, 1]$.

Solution. Set

$$u_n = \frac{1}{n} \text{ and } v_n = \frac{1}{n+1}.$$

Then

$$\begin{aligned}\lim_{n\to\infty}(u_n - v_n) &= \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty}\left(\frac{n+1-n}{n(n+1)}\right)\\ &= \lim_{n\to\infty}\frac{1}{n(n+1)} = 0.\end{aligned}$$

On the other hand,

$$f(u_n) - f(v_n) = n - (n+1) = -1,$$

so that

$$\lim_{n\to\infty}(f(u_n) - f(v_n)) = \lim_{n\to\infty}(-1) = -1 \neq 0.$$

Therefore, f is not uniformly continuous on $(0, 1]$, even though f is continuous at each point in $(0, 1]$ (confirm). □

A continuous function on a closed and bounded interval is guaranteed to be uniformly continuous:

Theorem 3. *Assume that f is continuous on a closed and bounded interval $[a, b]$ (i.e., f is continuous at each point of J). Then f is uniformly continuous on $[a, b]$.*

Proof. Given the interval $[a, b]$, the contrapositive of the statement

$$f \text{ is continuous at each point of } [a, b] \Rightarrow f \text{ is uniformly continuous on } [a, b]$$

is

$$f \text{ is not uniformly continuous on } [a, b] \Rightarrow \text{ there exists a point of } [a, b]$$
$$\text{where } f \text{ is not continuous.}$$

We will prove the above statement. Thus, assume that f is not uniformly continuous on $[a, b]$. Then there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists points x_n and y_n in $[a, b]$ where $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. Since $[a, b]$ is sequentially compact (Theorem 5 of Sect. 1.5), there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ that converges to a point $x_0 \in [a, b]$. Again by the compactness of $[a, b]$ there exists a subsequence $\left\{y_{n_{k_j}}\right\}_{j=1}^{\infty}$ that converges to $y_0 \in J$. Since

$$\left|x_{n_{k_j}} - y_{n_{k_j}}\right| < \frac{1}{n_{k_j}}$$

we have

$$|x_0 - y_0| \leq \lim_{j \to \infty} \frac{1}{n_{k_j}} = 0.$$

Thus $x_0 = y_0$. The function f is not continuous at x_0. Indeed, if f were continuous at x_0, we would have

$$\lim_{j \to \infty} f\left(x_{n_{k_j}}\right) = f(x_0) \text{ and } \lim_{j \to \infty} f\left(y_{n_{k_j}}\right) = f(x_0)$$

since $\lim_{j \to \infty} x_{n_{k_j}} = \lim_{j \to \infty} y_{n_{k_j}} = x_0$. Thus

$$\lim_{j \to \infty} \left|f\left(x_{n_{k_j}}\right) - f\left(y_{n_{k_j}}\right)\right| = 0.$$

But

$$\left|f\left(x_{n_{k_j}}\right) - f\left(y_{n_{k_j}}\right)\right| \geq \varepsilon \text{ for each } j \in \mathbb{N}.$$

Thus f is not continuous at $x_0 \in D$. □

2.1.3 The Continuity of Basic Functions and Their Combinations

Many functions that are encountered frequently are continuous on their natural domains. Let us begin by noting that **a constant function is continuous on the entire number line**: Let $f(x) = c$ for each $x \in \mathbb{R}$. We have

$$|f(x + h) - f(x)| = |c - c| = 0.$$

Thus $|f(x+h) - f(x)| < \varepsilon$ for any positive number ε. Therefore given $\varepsilon > 0$ we have complete freedom in choosing a corresponding $\delta > 0$. For example, $\delta = 1$ will do.

Positive integer powers of x define continuous functions:

Proposition 1. *Assume that n is a positive integer and $f_n(x) = x^n$ for each $x \in \mathbb{R}$. Then f_n is continuous on the entire number line.*

Proof. We have $f_1(x) = x$ for each $x \in \mathbb{R}$. Since $|f_1(x+h) - f_1(x)| = |h|$ it is sufficient to set $\delta = \varepsilon$ for a given $\varepsilon > 0$.

Let x be an arbitrary real number. If $n \geq 2$ we have

$$\begin{aligned} f_n(x+h) - f_n(x) &= (x+h)^n - x^n \\ &= \left(x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n\right) - x^n \end{aligned}$$

by the Binomial Theorem. Therefore

$$\begin{aligned} |f_n(x+h) - f_n(x)| &= |h| \left| nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1} \right| \\ &\leq |h| \left(n|x|^{n-1} + \frac{n(n-1)}{2}|x|^{n-2}|h| + \cdots + |h|^{n-2} \right), \end{aligned}$$

by the triangle inequality. Let us restrict h so that $|h| < 1$. Then

$$|f_n(x+h) - f_n(x)| < |h| \left(n|x|^{n-1} + \frac{n(n-1)}{2}|x|^{n-2} + \cdots + 1 \right).$$

If we set

$$C(x) = n|x|^{n-1} + \frac{n(n-1)}{2}|x|^{n-2} + \cdots + 1$$

we have

$$|f_n(x+h) - f_n(x)| < C(x)|h|.$$

Thus, given $\varepsilon > 0$, it is sufficient to choose δ so that

$$\delta = \min\left(1, \frac{\varepsilon}{C(x)}\right).$$

Since $|f_n(x+h) - f_n(x)| < \varepsilon$ if $|h| < \delta$ the function f_n is continuous at x. ■

The following theorem states that arithmetic operations on continuous functions lead to continuous functions:

Theorem 4. *Assume that the functions f and g are continuous at x_0. Then*

1. *$f+g$ is continuous at x_0,*
2. *fg is continuous at x_0,*
3. *f/g is continuous at x_0 if $g(x_0) \neq 0$.*

The assertions of Theorem 4 follow easily from the corresponding facts for sequences (exercise). It is also instructive to prove each statement by referring directly to ε-δ definition of continuity.

Corollary 1. *A polynomial is continuous at each $x \in \mathbb{R}$.*

Proof. A polynomial is defined by an expression of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where the coefficients $a_0, a_0, \ldots, a_n$ are given numbers. We have shown that constants and functions defined by positive integer powers of x are continuous functions on $\mathbb{R}$. Since sums and products of continuous functions are continuous a polynomial defines a continuous function on $\mathbb{R}$ (we may simply say that a polynomial is continuous on $\mathbb{R}$). ■

Proposition 2. *A rational function is continuous on its natural domain.*

Proof. A rational function f is the quotient of polynomials: If $P(x)$ and $Q(x)$ are polynomials

$$f(x) = \frac{P(x)}{Q(x)} \text{ for each } x \in \mathbb{R} \text{ such that } Q(x) \neq 0.$$

Since $P(x)$ and $Q(x)$ are continuous at each $x \in \mathbb{R}$ the continuity of f on its natural domain rule follows from Theorem 4. ■

Remark 5. Thanks to Theorem 3 a polynomial is uniformly continuous on any closed and bounded interval. A rational function g is uniformly continuous on any closed and bounded interval that does not contain a point where the denominator in the expression for $g(x)$ vanishes. ◇

Remark 6. **The trigonometric functions sine and cosine are continuous at any** $x \in \mathbb{R}$.

We are not in a position to provide a rigorous proof for this statement since we have not even provided precise definitions for sine and cosine. In Chap. 5 we will be able to discuss these functions rigorously via power series (Sect. 5.6). In any case, it may be worth mentioning that

$$|\sin(x+h) - \sin(x)| \leq |h| \text{ and } |\cos(x+h) - \cos(x)| \leq |h|$$

for each x and h in $\mathbb{R}$. These inequalities lead to the uniform continuity of sine and cosine on the entire number line (we can set $\delta = \varepsilon$ for any $\varepsilon > 0$.

The trigonometric functions tangent and secant are continuous at any $x \in \mathbb{R}$ that is not an odd integer multiple of $\pi/2$. Indeed,

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \text{ and } \sec(x) = \frac{1}{\cos(x)},$$

and a quotient of continuous functions is continuous at any point where the denominator does not vanish ($\cos(x) = 0$ iff x is an odd integer multiple of $\pi/2$). ◇

Remark 7. **We will also take it for granted that exponential functions are continous on the entire number line and logarithms are continuous on** $(0, +\infty)$. We will provide the justification for these facts in Chaps. 4 and 5. ◇

Composition of continuous functions leads to continuous functions:

Theorem 5. *Assume that f is continuous at x_0 and g is continuous at $f(x_0)$. Then the composite function $g \circ f$ is continuous at x_0.*

Proof. Let $\varepsilon > 0$ be given. Since g is continuous at $f(x_0)$, we can choose $\delta_1 > 0$ so that

$$|u - f(x_0)| < \delta_1 \Rightarrow |g(u) - g(f(x_0))| < \varepsilon.$$

Since f is continuous at x_0, we can choose $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \delta_1.$$

Thus,

$$|x - x_0| < \delta \Rightarrow |g(f(x)) - g(f(x_0))| < \varepsilon.$$

This shows that $g \circ f$ is continuous at x_0. ■

Example 7. Let $F(x) = \sin\left(x^2\right)$ for each $x \in \mathbb{R}$. Then f is continuous on the entire number line. Indeed, if we set $f(x) = x^2$ and $g(u) = \sin(u)$ then $F = g \circ f$ is continuous on $\mathbb{R}$ since both f and g have that property. □

2.1.4 Problems

In problems 1 and 2 prove that f is continuous at x_0 in accordance with the $\varepsilon - \delta$ definition of continuity (Suggestion: It may be easier to work with the definition that involves $f(x_0 + h)$).

1.

$$f(x) = x^4;\ x_0 = 3.$$

2.

$$f(x) = \frac{x}{x^2+1};\ x_0 = 2.$$

In problems 3 and 4, prove that f is continuous for each $x_0 \in D$ in accordance with the $\varepsilon - \delta$ definition of continuity.

3.

$$f(x) = \frac{1}{x-4},\ D = \{x \in \mathbb{R} : x > 4\}$$

Hint: Restrict h so that $|h| < (x-4)/2$.

4.

$$f(x) = x^{1/3},\ D = [0, +\infty).$$

Hint: Consider the cases $x = 0$ and $x > 0$ separately. If $x > 0$ multiply and divide $f(x+h) - f(x)$ by

$$(x+h)^{2/3} + (x+h)^{1/3} x^{1/3} + x^{2/3}$$

and restrict h so that $|h| < x/2$.

In problems 5 and 6, show that f is uniformly continuous on D. You may find it convenient to make use of the following version of the $\varepsilon - \delta$ definition of uniform continuity:

A function $f : D \to \mathbb{R}$ is uniformly continuous on D if, given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x \in D, x+h \in D \text{ and } |h| < \delta \Rightarrow |f(x+h) - f(x)| < \delta.$$

5.

$$f(x) = x^2 + x - 2, D = [0, 3].$$

6.

$$f(x) = \frac{1}{x^2},\ D = [1, +\infty)$$

In problems 7 and 8 show that f is **not** uniformly continuous on D by appealing to the sequential characterization of uniform continuity.

7.

$$f(x) = \frac{1}{x^2 - 16},\ D = (4, 8].$$

Hint: Consider sequences that converge to 4.

8.

$$f(x) = \frac{1}{x^4}, \ D = (0, 2].$$

Hint: Consider sequences that converge to 0.

2.2 The Limit of a Function at a Point

In the previous section we discussed the continuity of a function at a point. Informally f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a. In some cases $f(x)$ may approach a definite value as x approaches a point a even though f may not be defined at a. Even if f is defined at a that value may not be the same as $f(a)$. The relevant concept is the limit of a function at a point. That is the topic that we will discuss in this section.

2.2.1 The Definition of the Limit of a Function

Definition 1. Assume that $f(x)$ is defined for each x in an open interval that contains the point x_0, with the possible exception of x_0. **The limit of** f **at** x_0 is L if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ if $x \in D, \ x \neq x_0$ and $|x - x_0| < \delta$.

If the limit of f at x_0 is L we write

$$\lim_{x \to x_0} f(x) = L$$

(read "**the limit of** $f(x)$ **as** x**approaches** x_0 **is** L").

Example 1. Let

$$f(x) = \frac{4\left(x^2 - 9\right)}{x - 3} \text{ if } x \neq 3.$$

The function is not defined at 3 but we are entitled to investigate the limit of f at 3 since it is defined if $x \neq 3$. We have

$$f(x) = \frac{4(x+3)(x-3)}{x-3} = 4(x+3)$$

if $x \neq 3$. This expression indicates that $f(x)$ approaches 24 as x approaches 3. Let us prove that in accordance with Definition 1. We have

$$|f(x) - 24| = |4(x+3) - 24| = |4x - 12| = |4(x-3)| = 4|x-3|.$$

Given $\varepsilon > 0$ we need to choose $\delta > 0$ so that $4|x-3| < \varepsilon$ if $|x-3| < \delta$. It is sufficient to set $\delta = \varepsilon/4$. If $x \neq 3$ and $|x-3| < \delta$ then

$$|f(x) - 24| = 4|x-3| < 4\delta < \varepsilon.$$

Therefore $\lim_{x \to 3} f(x) - 24$, as claimed. □

Remark 1. Note that if $f(x)$ is defined in an open interval that contains x_0 then f is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$. ◇

Remark 2. If we set $x = x_0 + h$ we can rephrase the definition of the limit as follows:

The limit of f at x_0 is L if, given any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x_0 + h) - L| < \varepsilon$ if $x_0 + h \in D$, $h \neq 0$ and $|h| < \delta$. ◇

Example 2. Let

$$f(x) = \frac{x^2 - 4}{3(x-2)} \text{ if } x \neq 2.$$

Show that $\lim_{x \to 2} f(x) = 4/3$.

Solution. If $x \neq 2$

$$f(x) = \frac{(x+2)(x-2)}{3(x-2)} = \frac{x+2}{3}.$$

This expression for $f(x)$ indicates that $\lim_{x \to 2} f(x) = 4/3$. Let us justify this in accordance with the alternative definition of the limit as in Remark 2:

If $x = 2 + h$ where $h \neq 0$ then

$$f(2+h) = \frac{(2+h)+2}{3} = \frac{4+h}{3}.$$

Thus

$$\left| f(2+h) - \frac{4}{3} \right| = \left| \frac{4+h}{3} - \frac{4}{3} \right| = \left| \frac{h}{3} \right| = \frac{1}{3}|h|.$$

Therefore, for a given $\varepsilon > 0$ it is sufficient to have $h \neq 0$ and

$$\frac{1}{3}|h| < \varepsilon.$$

Thus, we can set $\delta = 3\varepsilon$. If $h \neq 0$ and $|h| < \delta$ then

$$\left| f(2+h) - \frac{4}{3} \right| = \frac{1}{3}|h| < \frac{1}{3}(3\varepsilon) = \varepsilon.$$

Therefore $\lim_{x \to 2} f(x) = 4/3$, as claimed. The choice of δ corresponding to a given ε is not unique, of course. For example, the choice $\delta = \varepsilon$ is also sufficient. □

2.2.2 *Basic Facts about Limits*

As in Example 1 and Example 2, in many cases we can show that a given function has a certain limit at a point by identifying a continuous function with values that coincide with the values of the given function near that point:

Proposition 1. *Assume that $f(x)$ is defined in an open interval J that contains x_0 with the possible exception of x_0. If the function g is continuous at x_0 and $g(x) = f(x)$ for each $x \in J$ other than x_0 then*

$$\lim_{x \to x_0} f(x) = g(x_0).$$

Proof. Let $\varepsilon > 0$ be given. Since g is continuous at x_0,

$$\lim_{x \to x_0} g(x) = g(x_0).$$

Since $f(x) = g(x)$ if $x \neq x_0$ and $x \in J$, we have

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = g(x_0).$$

■

Example 3. Let

$$f(x) = \frac{\sqrt{x} - 2}{x - 4} \text{ if } x > 0 \text{ and } x \neq 4.$$

a) Determine $\lim_{x \to 4} f(x)$ by finding a function g that is continuous at 4 such that $g(x) = f(x)$ if $x \neq 4$ and x is in some open interval containing 4.
b) Justify your assertion that g is continuous at 4 in accordance with the $\varepsilon - \delta$ definition of continuity.

Solution. a) We have

$$f(x) = \frac{\sqrt{x} - 2}{x - 4} = \left(\frac{\sqrt{x} - 2}{x - 4} \right) \left(\frac{\sqrt{x} + 2}{\sqrt{x} + 2} \right) = \frac{x - 4}{(x - 4)\left(\sqrt{x} + 2\right)} = \frac{1}{\sqrt{x} + 2}$$

if $x > 0$ and $x \neq 4$. Set

$$g(x) = \frac{1}{\sqrt{x} + 2}.$$

Then g is continuous at 4 since $\sqrt{x}$ defines a function that is continuous at 4 (Example 3 of Sect. 2.1). We have

$$g(4) = \frac{1}{\sqrt{4} + 1} = \frac{1}{2 + 2} = \frac{1}{4}.$$

Since $f(x) = g(x)$ if $x > 0$ and $x \neq 4$,

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} g(x) = g(4) = \frac{1}{4}.$$

b) If $|h| < 4$ and $h \neq 0$,

$$\begin{aligned} g(4+h) - \frac{1}{4} = \frac{1}{\sqrt{4+h} + 2} - \frac{1}{4} &= \frac{4 - \sqrt{4+h} - 2}{4\left(\sqrt{4+h} + 2\right)} \\ &= \frac{2 - \sqrt{4+h}}{4\left(\sqrt{4+h} + 2\right)} \\ &= \left(\frac{2 - \sqrt{4+h}}{4\left(\sqrt{4+h} + 2\right)}\right)\left(\frac{2 + \sqrt{4+h}}{2 + \sqrt{4+h}}\right) \\ &= \frac{4 - (4+h)}{4\left(2 + \sqrt{4+h}\right)^2} = -\frac{h}{4\left(2 + \sqrt{4+h}\right)^2}. \end{aligned}$$

Therefore,

$$\left| g(4+h) - \frac{1}{4} \right| = \frac{|h|}{4\left(2 + \sqrt{4+h}\right)^2} < \frac{|h|}{4\left(2^2\right)} = \frac{1}{16}|h|.$$

Let $\varepsilon > 0$ be given. Let

$$\delta = \min(16\varepsilon, 4).$$

If $h \neq 0$ and $|h| < \delta$ then

$$\left| g(4+h) - \frac{1}{4} \right| < \frac{1}{16}|h| < \frac{1}{16}(16\varepsilon) = \varepsilon.$$

□

We can define "one-sided limits," just as we can refer to one-sided continuity:

Definition 2. Assume that there exists $\delta_0 > 0$ such that $f(x)$ is defined in the open interval $(x_0, x_0 + \delta_0)$. **The limit of $f(x)$ as x approaches x_0 from the right is L_+** if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L_+| < \varepsilon$ provided that $x_0 < x < x_0 + \delta$. In this case we write

$$\lim_{x \to x_0+} f(x) = L_+$$

(read "the limit of $f(x)$ as x approaches x_0 from the right is L_+).
Similarly, the **limit of $f(x)$ as x approaches x_0 from the left is L_-** if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L_-| < \varepsilon$ if $x_0 - \delta < x < x_0$. In this case we write

$$\lim_{x \to x_0-} f(x) = L_-$$

(read "the limit of $f(x)$ as ax approaches x_0 from the left is L_-).

Remark 3. Clearly, the limit of $f(x)$ as x approaches x_0 exists if and only if the limits of $f(x)$ as x approaches x_0 from the right and from the left exist and have the same value. Also note that f is continuous at x_0 from the right if and only if

$$\lim_{x \to x_0+} f(x) = f(x_0).$$

Similarly, f is continuous at x_0 from the left if and only if

$$\lim_{x \to x_0-} f(x) = f(x_0).$$

◇

Just as there is a sequential characterization of continuity (Theorem 1 of Sect. 2.1) there is a sequential characterization of a limit:

Theorem 1. *Assume that $f(x)$ is defined for each x in an open interval that contains the point x_0, with the possible exception of x_0. Then $\lim_{x \to x_0} f(x) = L$ if and only if for any sequence $\{x_n\}_{n=1}^{\infty}$ such that each $x_n \neq x_0$, $f(x_n)$ is defined at each x_n and $\lim_{n \to \infty} x_n = x_0$ we have $\lim_{n \to \infty} f(x_n) = L$.*

The proof is similar to the sequential characterization of continuity and is left as an exercise.

The rules for the limits of sums, products, and quotients of functions are similar to the corresponding rules for sequences.

The following theorem is relevant to the composition of functions:

Theorem 2. *Assume that $f(x)$ is defined for each x in an open interval J containing x_0 with the possible exception of x_0 and that $\lim_{x \to x_0} f(x) = y_0$. Assume that g is continuous at y_0. Then the limit of $g \circ f$ at x_0 exists and*

$$\lim_{x\to x_0} g(f(x)) = g(y_0),$$

i.e.,

$$\lim_{x\to x_0} g(f(x)) = g\left(\lim_{x\to x_0} f(x)\right).$$

Proof. The proof of Theorem 2 is similar to the proof of the corresponding theorem for the composition of continuous functions (Theorem 5 of Sect. 2.1):

Let $\varepsilon > 0$ be given. Since g is continuous at y_0, we can choose $\delta_1 > 0$ so that

$$|u - y_0| < \delta_1 \Rightarrow |g(u) - g(y_0)| < \varepsilon.$$

Since $\lim_{x\to x_0} f(x) = y_0$, we can choose $\delta > 0$ such that

$$x \neq x_0 \text{ and } |x - x_0| < \delta \Rightarrow |f(x) - y_0| < \delta_1.$$

Thus,

$$x \neq x_0 \text{ and } |x - x_0| < \delta \Rightarrow |g(f(x)) - g(y_0)| < \varepsilon.$$

This shows that $\lim_{x\to x_0} g(f(x)) = g(y_0)$. ■

Example 4. Evaluate

$$\lim_{x\to 1} \cos\left(\frac{\pi\left(x^2-1\right)}{6(x-1)}\right)$$

Assume that cosine is continuous at any real number.

Solution. We have

$$\lim_{x\to 1} \frac{\pi\left(x^2-1\right)}{6(x-1)} = \lim_{x\to 1} \frac{\pi(x-1)(x+1)}{6(x-1)} = \lim_{x\to 1} \frac{\pi(x+1)}{6} = \frac{2\pi}{6} = \frac{\pi}{3}.$$

Since cosine is continuous at $\pi/3$, we can apply Theorem 2:

$$\lim_{x\to 1} \cos\left(\frac{\pi\left(x^2-1\right)}{6(x-1)}\right) = \cos\left(\lim_{x\to 1} \frac{\pi\left(x^2-1\right)}{6(x-1)}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

□

Now we will state and prove two theorems that will be useful when we discuss integrals in Chap. 4, especially in our discussion of improper integrals. The first such theorem is about monotone functions. Such functions may have discontinuities but one-sided limits exist:

Theorem 3. *Assume that $f : [a,b] \to \mathbb{R}$ is an increasing or decreasing function. Then*

$$\lim_{x\to c-} f(x) \text{ and } \lim_{x\to c+} f(x)$$

exist at each $c \in (a,b)$. In case f is increasing we have

$$\lim_{x\to c-} f(x) \le f(c) \le \lim_{x\to c+} f(x).$$

In case f is decreasing

$$\lim_{x\to c-} f(x) \ge f(c) \ge \lim_{x\to c+} f(x)$$

The limits $\lim_{x\to a+} f(x)$ *and* $\lim_{x\to b-} f(x)$ *exist as well.*

Proof. We will consider the case of an increasing function and $c \in (a,b)$. Let

$$S = \{f(x) : c < x \le b\}.$$

Since f is an increasing function $f(c)$ is a lower bound for S_+. Therefore

$$L = \inf S$$

exists. We claim that

$$\lim_{x\to c+} f(x) = L.$$

Indeed, let $\varepsilon > 0$ be given. By the definition of the greatest lower bound of a set there exists $\delta > 0$ such that $c + \delta \le b$ and

$$f(c) \le f(c+\delta) < L + \varepsilon$$

Since f is increasing we have

$$f(c) \le f(x) \le f(c+\delta) < L + \varepsilon$$

if $c < x < c + \delta$. Since L is a lower bound of the values $f(x)$ in the interval $(c,b]$ we have

$$L \le f(x) \le f(c+\delta) < L + \varepsilon$$

if $c < x < c + \delta$. Therefore $\lim_{x\to c+} f(x)$ exists and

$$\lim_{x\to c+} f(x) = L.$$

Since

$$f(c) < L + \varepsilon \text{ for each } \varepsilon > 0$$

we have

$$f(c) \leq L = \lim_{x \to c+} f(x).$$

The proof of the existence of $\lim_{x \to c-} f(x)$ and the fact that

$$\lim_{x \to c-} f(x) \leq f(c)$$

is along similar lines. We need to consider

$$\sup(\{f(x) : a \leq x < c\})$$

(confirm as an exercise). ■

Example 5. Let

$$f(x) = \begin{cases} \frac{1}{n} \text{ if } \frac{1}{n+1} < x \leq \frac{1}{n}, \; n = 1, 2, 3, \ldots \\ 0 \text{ if } \qquad\qquad x = 0 \end{cases}.$$

Then f is monotone increasing on $[0, 1]$. Note that f has infinitely many discontinuities,

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \cup \{0\}.$$

At each point of discontinuity $1/n$ we have

$$\lim_{x \to 1/n-} f(x) = \frac{1}{n} \text{ and } \lim_{x \to 1/n+} f(x) = \frac{1}{n-1}, n = 2, 3, 4,$$

We also have

$$\lim_{x \to 0+} f(x) = 0 \text{ and } \lim_{x \to 1} f(x) = 1.$$

□

Another criterion for the existence of the limit of a function is a Cauchy condition that is applicable to functions that are not necessarily monotone increasing.

Theorem 4 (Cauchy Condition for the Limit of a Function). *Assume that* $f(x)$ *is defined for each* $x \in (c, c+\delta_0)$ *for some* $\delta_0 > 0$. *Also assume that given any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $\delta \leq \delta_0$ *and*

$$|f(u) - f(v)| < \varepsilon \text{ if } c < u < c + \delta \text{ and } c < v < c + \delta.$$

Then $\lim_{x \to c+} f(x)$ *exists. The obvious counterpart of the statement is valid for the existence of* $\lim_{x \to c-} f(x)$.

Proof. By the given condition, if $\varepsilon = 1$ there exists positive $\delta_1 < 1$ such that

$$|f(u) - f(v)| < 1 \text{ if } c < u < c + \delta_1 \text{ and } c < v < c + \delta_1.$$

Select such a point x_1. Thus $c < x_1 < c + \delta_1 < c + 1$ and

$$|f(u) - f(x_1)| < 1 \text{ if } c < u < c + \delta_1$$

Again by the given condition, if $\varepsilon = 1/2$ there exists a positive $\delta_2 < \min(1/2, \delta_1)$ such that

$$|f(u) - f(v)| < \frac{1}{2} \text{ if } c < u < c + \delta_2 \text{ and } c < v < c + \delta_2.$$

Select a point $x_2 < x_1$ such that $c < x_2 < c + \delta_2 < c + 1/2$ and

$$|f(u) - f(x_2)| < \frac{1}{2} \text{ if } c < u < c + \delta_2.$$

Note that

$$|f(x_1) - f(x_2)| < 1.$$

Having selected $x_1 > x_2 > \ldots > x_n$ and positive numbers $\delta_1 > \delta_2 > \ldots > \delta_n$ such that

$$c < x_k < c + \delta_k < c + \frac{1}{k}$$

and

$$|f(u) - f(x_k)| < \frac{1}{k} \text{ if } c < u < c + \delta_k, \; k = 1, 2, \ldots, n$$

we select $\delta_{n+1} < \min(1/(n+1), \delta_n)$ and $x_{n+1} < x_n$ such that $c < x_{n+1} < c + \delta_{n+1}$ and

$$|f(u) - f(x_{n+1})| < \frac{1}{n+1} \text{ if } c < u < c + \delta_{n+1}.$$

Thus, $\lim_{n\to\infty} x_n = c$ and the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence. Indeed, given $\varepsilon > 0$ we can select $N \in \mathbb{N}$ such that $1/N < \varepsilon$. If $n \geq N$ and $k = 1, 2, 3, \ldots$ then

$$|f(x_{n+k}) - f(x_n)| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

By the Cauchy convergence principle $L = \lim_{n\to\infty} f(x_n)$ exists. We claim that $\lim_{x\to c+} f(x) = L$. Let $\varepsilon > 0$ be given. Pick $N \in \mathbb{N}$ such that $N > 2/\varepsilon$. Set $\delta = \delta_N$. If $c < x < c + \delta$ then

$$|f(x) - L| \leq |f(x) - f(x_n)| + |f(x_n) - L|$$

for any n. If $n \geq N$ then

$$\begin{aligned}|f(x) - L| &\leq |f(x) - f(x_n)| + |f(x_n) - L| \\ &< \frac{1}{N} + |f(x_n) - L| < \frac{\varepsilon}{2} + |f(x_n) - L|\end{aligned}$$

Since $\lim_{n\to\infty} f(x_n) = L$ we can select $n \geq N$ large enough so that

$$|f(x_n) - L| < \frac{\varepsilon}{2}.$$

Thus

$$|f(x) - L| < \varepsilon \text{ if } c < x < c + \delta.$$

This shows that $\lim_{x\to c+} f(x) = L$. ■

2.2.3 Problems

1. Prove that

$$\lim_{x\to 4} \frac{\left(2x^2 - 32\right)}{x - 4} = 16$$

in accordance with the ε-δ definition of the limit.

2. Prove that

$$\lim_{x\to 3} \frac{x^3 - 2x^2 - 2x - 3}{(x - 3)} = 13$$

in accordance with the ε-δ definition of the limit.

Hint: Divide and then set $x = 3 + h$.

3. Show that

$$\lim_{x\to 8} \frac{x^{1/3}-2}{x-8} = \frac{1}{12}$$

by finding a function g that is continuous at 8 such that

$$g(x) = \frac{x^{1/3}-2}{x-8} \text{ if } x \neq 8$$

You need not give an ε-δ proof for the continuity of g.
Hint:

$$(a-b)\left(a^2+ab+b^2\right) = a^3-b^3$$

4. Prove "**the squeeze theorem**":
If $f(x)$, $g(x)$, and $h(x)$ are defined for each x in an open interval that contains x_0, with the possible exception of x_0,

$$g(x) \leq f(x) \leq h(x)$$

for each such x, and

$$\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x),$$

then $\lim_{x\to x_0} f(x)$ exists and we have

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x),$$

2.3 Infinite Limits and Limits at Infinity

In beginning calculus you have studied the vertical asymptotes for the graphs of functions. The relevant concept is that of an "infinite limit." We will provide the precise definitions and justify some techniques that are useful in the determination of such limits.

2.3.1 Infinite Limits

Definition 1. The limit of f at a is $+\infty$ if, given any $\boldsymbol{M > 0}$, there exists $\boldsymbol{\delta > 0}$ such that $\boldsymbol{f(x) > M}$ provided that $\boldsymbol{0 < |x-a| < \delta}$. We write

$$\lim_{\boldsymbol{x}\to\boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = +\infty$$

The limit of $\boldsymbol{f}$ at $\boldsymbol{a}$ is $-\infty$ if, given any $\boldsymbol{M > 0}$, there exists $\boldsymbol{\delta > 0}$ such that $\boldsymbol{f(x) < -M}$ provided that $\boldsymbol{0 < |x - a| < \delta}$. We write

$$\lim_{\boldsymbol{x}\to\boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = -\infty$$

The definition of a one-sided infinite limit such as $\lim_{x\to a+} f(x) = +\infty$ is a modification of the definition of $\lim_{x\to a+} f(x) = +\infty$ by restricting x so that $x > a$. Note that

$$\lim_{x\to a} f(x) = -\infty \text{ if and only if } \lim_{x\to a} (-f(x)) = +\infty.$$

Remark 1 (Caution). **In any of the cases covered by Definition 1, the relevant limit of f does not exist as a real number.** Indeed, if $\lim_{x\to a} f(x) = L$, there exists $\delta > 0$ and $M > 0$ such that $|f(x)| < M$ if $0 < |x - a| < \delta$. Here, we have an example of "mathematical doublespeak": We are using the same word "limit", and the same symbol "lim" in connection with "finite limits" and "infinite limits" . The doublespeak is traditional and convenient, and we will use it. The particular context should clarify which usage of the word "limit" we have in mind. Nevertheless, if there is any possibility of confusion, we may stress that we are talking about a **"finite limit"**, or an **"infinite limit"** in the sense of Definition . $\diamond$

Example 1. Let

$$f(x) = \frac{1}{(x+3)(x-2)}.$$

Prove that $\lim_{x\to 2+} f(x) = +\infty$. and $\lim_{x\to 2-} f(x) = -\infty$.

Solution. In order to prove that $\lim_{x\to 2+} f(x) = +\infty$, we will restrict x so that $2 < x < 3$. Thus $5 < x + 3 < 6$ so that

$$\frac{1}{x+3} > \frac{1}{6}.$$

Therefore,

$$f(x) = \frac{1}{(x+3)(x-2)} > \frac{1}{6(x-2)}$$

Thus, given $M > 0$, in order to ensure that

$$\frac{1}{(x+3)(x-2)} > M,$$

it is sufficient to have

$$\frac{1}{6(x-2)} > M.$$

This is the case if

$$0 < x - 2 < \frac{1}{6M}, \text{ i.e., } 2 < x < 2 + \frac{1}{6M},$$

with the restriction that $x < 3$. Thus we can set

$$\delta = \min\left(1, \frac{1}{6M}\right).$$

If $2 < x < 2 + \delta$ then

$$f(x) > \frac{1}{6(x-2)} > \frac{1}{6\left(\frac{1}{6M}\right)} = M.$$

Therefore $\lim_{x \to 2+} f(x) = +\infty$.

Now let us prove that $\lim_{x \to 2-} f(x) = -\infty$. It may be more convenient to prove the equivalent statement that $\lim_{x \to 2-} (-f(x)) = +\infty$: We will restrict x so that $0 < x < 2$. Thus $0 < x + 3 < 5$ so that

$$\frac{1}{x+3} > \frac{1}{5}.$$

Therefore

$$-f(x) = \frac{1}{(x+3)(2-x)} > \frac{1}{5(2-x)}$$

Thus, given $M > 0$, in order to ensure that $-f(x) > M$ it is sufficient to have

$$\frac{1}{5(2-x)} > M.$$

This is the case if

$$\frac{1}{5M} > 2 - x$$

i.e.,

$$x > 2 - \frac{1}{5M}.$$

We can set

$$\delta = \min\left(2, \frac{1}{5M}\right)$$

to ensure that $x > 0$. If $2 - \delta < x < 2$ then

$$-f(x) > \frac{1}{5(2-x)} > M.$$

Therefore $\lim_{x\to 2-}(-f(x)) = +\infty$. □

The following proposition is helpful in the determination of infinite limits:

Proposition 1. *Assume that* $f(x) > 0$ *if* x *is in an open interval that contains* a, $x \neq a$ *and* $\lim_{x\to a} f(x) = 0$. *Then*

$$\lim_{x\to a} \frac{1}{f(x)} = +\infty.$$

Similarly, if $f(x) < 0$ *when* x *is in an open interval that contains* a, $x \neq a$, *and* $\lim_{x\to a} f(x) = 0$ *then*

$$\lim_{x\to a} \frac{1}{f(x)} = -\infty.$$

Proof. The proof is similar to the proof of Proposition 2 of Sect. 1.6. Let us establish the statement about f that has positive values near a. Let $M > 0$ be given. Since $f(x) > 0$ if x is in an open interval that contains a, $x \neq a$ and $\lim_{x\to a} f(x) = 0$ there exists $\delta > 0$ such that

$$0 < f(x) < \frac{1}{M} \text{ if } x \neq a \text{ and } |x - a| < \delta.$$

Therefore

$$\frac{1}{f(x)} > M \text{ if } x \neq a \text{ and } |x - a| < \delta.$$

Thus

$$\lim_{x\to a} \frac{1}{f(x)} = +\infty.$$

The statement about f that has negative values near follows by considering $-f$: Then

$$\lim_{x\to\infty} \left(-\frac{1}{f(x)}\right) = +\infty$$

so that

$$\lim_{x \to a} \frac{1}{f(x)} = -\infty,$$

as we noted before. ■

One-sided versions of Proposition 1are valid.

Example 2. Determine $\lim_{x \to \pi/2\pm} \sec(x)$.

Solution. We have

$$\sec(x) = \frac{1}{\cos(x)}.$$

If $0 < x < \pi/2$ then $\cos(x) > 0$, and

$$\lim_{x \to \pi/2-} \cos(x) = \lim_{x \to \pi/2} \cos(x) = \cos\left(\frac{\pi}{2}\right) = 0.$$

Therefore,

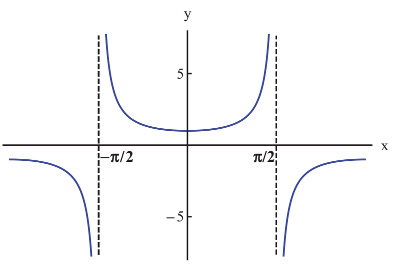
$$\lim_{x \to \pi/2-} \sec(x) = \lim_{x \to \pi/2-} \frac{1}{\cos(x)} = +\infty.$$

If $\pi/2 < x < 3\pi/2$ then $\cos(x) < 0$, and $\lim_{x \to \pi/2} \cos(x) = 0$. Therefore,

$$\lim_{x \to \pi/2+} \sec(x) = \lim_{x \to \pi/2+} \frac{1}{\cos(x)} = -\infty$$

Figure 2.1 shows the graph of secant on the interval $[\pi, \pi]$.

Fig. 2.1



The picture is consistent with the infinite limits that we calculated. The line $x = \pi/2$ is a vertical asymptote for the graph of secant. Since secant is an even function, the graph is symmetric with respect to the vertical axis, and the line $x = -\pi/2$ is also a vertical asymptote. □

Proposition 2. *Assume that* $\lim_{x\to a} f(x) = L > 0$ *or* $\lim_{x\to a} f(x) = +\infty$ *and* $\lim_{x\to a} g(x) = +\infty$. *Then*

$$\lim_{x\to a} f(x) g(x) = +\infty.$$

Proof. Assume that $\lim_{x\to a} f(x) = L > 0$. Then there exists $\delta_1 > 0$ such that

$$|x-a| < \delta_1 \text{ and } x \neq a \Rightarrow |f(x) - L| < \frac{L}{2}.$$

Then

$$f(x) - L > -\frac{L}{2} \Rightarrow f(x) > L - \frac{L}{2} = \frac{L}{2}.$$

Let $M > 0$ be given. Since $\lim_{x\to\infty} g(x) = +\infty$ there exists $\delta_2 > 0$ such that

$$g(x) > \frac{2M}{L} \text{ if } |x-a| < \delta_2 \text{ and } x \neq a.$$

Let us set $\delta = \min(\delta_1, \delta_2)$. If $|x-a| < \delta$ and $x \neq a$ then

$$f(x)\, g(x) > \left(\frac{L}{2}\right)\left(\frac{2M}{L}\right) = M.$$

Therefore $\lim_{x\to a} f(x)\, g(x) = +\infty$.

The case where $\lim_{x\to\infty} f(x) = +\infty$ is handled similarly. Since $\lim_{x\to a} f(x) = +\infty$ there exists $\delta_3 > 0$ such that

$$|x-a| < \delta_3 \text{ and } x \neq a \Rightarrow f(x) > 1.$$

Given $M > 0$ there exists $\delta_4 > 0$ such that

$$g(x) > M \text{ if } |x-a| < \delta_4 \text{ and } x \neq a$$

since $\lim_{x\to\infty} g(x) = +\infty$. If we set $\delta = \min(\delta_3, \delta_4)$ we have

$$f(x)\, g(x) > M \text{ if } |x-a| < \delta \text{ and } x \neq a.$$

Therefore $\lim_{x\to a} f(x)\, g(x) = +\infty$. ■

One-sided versions of Proposition 2 are valid.

Example 3. Determine

$$\lim_{x\to\pi/2-} \tan(x) \text{ and } \lim_{x\to\pi/2+} \tan(x).$$

Proof. We have

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

As in Example 2,

$$\lim_{x \to \pi/2-} \frac{1}{\cos(x)} = +\infty.$$

We also have

$$\lim_{x \to \pi/2} \sin(x) = \sin\left(\frac{\pi}{2}\right) = 1 > 0.$$

Therefore

$$\lim_{x \to \pi/2-} \tan(x) = \lim_{x \to \pi/2-} \sin(x) \left(\frac{1}{\cos(x)}\right) = +\infty,$$

by the one-sided version of Proposition 2.

As in Example 2

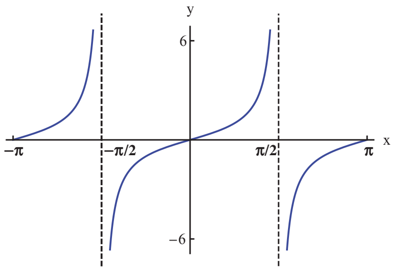
$$\lim_{x \to \pi/2+} \frac{1}{\cos(x)} = -\infty.$$

Therefore

$$\lim_{x \to \pi/2+} \tan(x) = \lim_{x \to \pi/2+} \sin(x) \left(\frac{1}{\cos(x)}\right) = -\infty,$$

by Proposition 2. Figure 2.2 is consistent with our assertions. □

Fig. 2.2



Remark 2. A word of caution: Even though

$$\lim_{x\to a} f(x) > 0 \text{ and } \lim_{x\to a} g(x) = +\infty$$

implies that $\lim_{x\to a} f(x)\, g(x) = +\infty$, we cannot make a general statement about $\lim_{x\to a+} f(x)\, g(x)$ if $\lim_{x\to a+} f(x) = 0$ and $\lim_{x\to a+} g(x) = +\infty$. The expression $0 \cdot \infty$ is indeterminate.

For example, we have

$$\lim_{x\to 1} \left(x^2 - 1\right) = 0, \ \lim_{x\to 1+} \frac{1}{x-1} = +\infty$$

and

$$\lim_{x\to 1+} \left(x^2 - 1\right) \left(\frac{1}{x-1}\right) = \lim_{x\to 1+} (x+1) = 2.$$

In this case the indeterminate expression $\infty \cdot 0$ seems to hide the number 2.
Similarly,

$$\lim_{x\to 1} (x-1) = 0, \ \lim_{x\to 1+} \frac{1}{x-1} = +\infty,$$

and

$$\lim_{x\to 1+} (x-1) \left(\frac{1}{x-1}\right) = \lim_{x\to 1+} 1 = 1.$$

In this case, $\infty \cdot 0$ seems to hide the number 1. ◇

Proposition 3. *Assume that* $\lim_{x\to a} f(x) = L$ *or* $\lim_{x\to a} f(x) = +\infty$ *and* $\lim_{x\to a} g(x) = +\infty$. *Then*

$$\lim_{x\to a} (f(x) + g(x)) = +\infty.$$

The proof is similar to the proof of Proposition 4 of Sect. 1.6 (exercise).

2.3.2 Limits at Infinity

The behavior of a function for large positive or negative values of the independent variable are usually of interest. The relevant concepts are limits at $+\infty$ or $-\infty$. We will provide the precise definitions. The evaluation of such limits is similar to the evaluation of the limits of sequences. Thus our discussion will be brief.

Definition 2. **The limit of** f **at** $+\infty$ **is** L if, given any $\boldsymbol{\varepsilon > 0}$ there exists $\boldsymbol{A > 0}$ such that $|\boldsymbol{f}(x) - \boldsymbol{L}| < \boldsymbol{\varepsilon}$ for each $\boldsymbol{x > A}$. **The limit of** f **at** $-\infty$ **is** L if, given any $\boldsymbol{\varepsilon > 0}$ there exists $\boldsymbol{A > 0}$ such that $|\boldsymbol{f}(x) - \boldsymbol{L}| < \boldsymbol{\varepsilon}$ for each $x < -A$.

Example 4. Let

$$f(x) = \frac{2x}{x+3}.$$

Show that $\lim_{x\to+\infty} f(x) = 2$.

Solution. We have

$$|f(x) - 2| = \left|\frac{2x}{x+3} - 2\right| = \left|\frac{2x - 2(x+3)}{x+3}\right| = \frac{6}{|x+3|}.$$

Therefore, if $x > -3$,

$$|f(x) - 2| = \frac{6}{x+3}.$$

Let $\varepsilon > 0$ be given, and assume that $x > -3$. Then,

$$\frac{6}{x+3} < \varepsilon \Leftrightarrow \frac{x+3}{6} > \frac{1}{\varepsilon} \Leftrightarrow x + 3 > \frac{6}{\varepsilon} \Leftrightarrow x > \frac{6}{\varepsilon} - 3.$$

It is certainly sufficient to have $x > 6/\varepsilon$. With reference to Definition 2, we can set $A = 6/\varepsilon$. By the above calculations, if $x > A$ we have

$$|f(x) - 2| < \varepsilon.$$

Therefore, $\lim_{x\to+\infty} f(x) = 2$. □

We may also speak of infinite limits at infinity:

Definition 3. **The limit of** f **at** $+\infty$ **is** $+\infty$ if, given any $\boldsymbol{M > 0}$, there exists $\boldsymbol{A > 0}$ such that $\boldsymbol{f(x) > M}$ for each $\boldsymbol{x > A}$. **The limit of** f **at** $+\infty$ **is** $-\infty$ if, given any $\boldsymbol{M > 0}$, there exists $\boldsymbol{A > 0}$ such that $\boldsymbol{f}(x) < \boldsymbol{-M}$ for each $\boldsymbol{x > A}$.

Example 5. Let $f(x) = x^2 - x$. Show that $\lim_{x\to+\infty} f(x) = +\infty$.

Solution. We have

$$f(x) = x^2 - x = x^2\left(1 - \frac{1}{x}\right).$$

If $x > 2$, then

$$\frac{1}{x} < \frac{1}{2} \Rightarrow -\frac{1}{x} > -\frac{1}{2}.$$

Therefore,

$$f(x) = x^2\left(1 - \frac{1}{x}\right) > x^2\left(1 - \frac{1}{2}\right) = \frac{x^2}{2}.$$

Let $M > 0$ be given. By the above inequality, in order to have $f(x) > M$ it is sufficient to have $x > 2$ and

$$\frac{x^2}{2} > M.$$

This is the case if $x > 2$ and $x > \sqrt{2M}$. With reference to Definition 3, we can set A to be the maximum of 2 and $\sqrt{2M}$. Thus, $f(x) > M$ if $x > A$. Therefore, $\lim_{x\to+\infty} f(x) = +\infty$. □

There is a counterpart of Theorem 3 of Sect. 2.2 for limits at infinity:

Theorem 1. *Assume that f is monotone increasing on $[A, +\infty)$. If there exists $M > 0$ such that $f(x) < M$ for each $x \geq A$ then $\lim_{x\to+\infty} f(x)$ exists (as a finite limit). If there is no upper bound on the values of f on $[A, +\infty)$ then $\lim_{x\to+\infty} f(x) = +\infty$.*

Proof. Assume that there exists a number M such that $f(x) \leq M$ for each $x \geq a$. Then

$$L = \sup\{f(x) : x \geq a\}$$

is finite. We claim that $\lim_{x\to+\infty} f(x) = L$. Indeed, by the definition of the least upper bound, given any $\varepsilon > 0$ there exists x^* such that

$$L - \varepsilon < f(x^*) \leq L.$$

Since f is monotone increasing, if $x \geq x^*$

$$L - \varepsilon < f(x^*) \leq f(x) \leq L.$$

Therefore, $\lim_{x\to+\infty} f(x) = L$, as claimed.

On the other hand, if the set $\{f(x) : x \geq a\}$is not bounded above, given any M there exists $x^* \geq a$ such that $f(x^*) > M$. Since f is increasing, we have

$$f(x) \geq f\left(x^*\right) > M$$

for any $x \geq x^*$. Therefore, $\lim_{x\to+\infty} f(x) = +\infty$. ■

There is also a counterpart of Theorem 4 of Sect. 2.2 for limits at infinity:

Theorem 2 (Cauchy Condition for a Limit at Infinity). *Assume that $f(x)$ is defined if $x \geq a$. Then $\lim_{x\to+\infty} f(x)$ exists (as a finite number) if and only if given $\varepsilon > 0$ there exists A such that*

$$c > b \geq A \Rightarrow |f(c) - f(b)| < \varepsilon.$$

Proof. Assume that $\lim_{x\to\infty} f(x) = L$. Let $\varepsilon > 0$ be given. Pick A so that

$$b \geq A \Rightarrow |f(b) - L| < \frac{\varepsilon}{2}.$$

If $c > b \geq A$ then

$$\begin{aligned}|f(c) - f(b)| &= |(f(c) - L) + (L - f(b))| \\ &\leq |f(c) - L| + |f(b) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Conversely, assume that the given "Cauchy condition" is valid. There exists $n_1 \geq a$ such that

$$c > b \geq n_1 \Rightarrow |f(c) - f(b)| < 1.$$

There exists $n_2 > n_1$ such that

$$c > b \geq n_2 \Rightarrow |f(c) - f(b)| < \frac{1}{2}.$$

There exists $n_3 > n_2$ such that

$$c > b \geq n_3 \Rightarrow |f(c) - f(b)| < \frac{1}{3}.$$

Having chosen positive integers $n_k > n_{k-1} > \cdots > n_1$ such that

$$c > b \geq n_j \Rightarrow |f(c) - f(b)| < \frac{1}{j},\ j = 1, 2, \cdots, k,$$

we choose $n_{k+1} > n$ such that

$$c > b \geq n_{k+1} \Rightarrow |f(c) - f(b)| < \frac{1}{k+1}.$$

Thus we construct a strictly increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that

$$|f(n_m) - f(n_k)| < \frac{1}{k} \text{ if } m > k.$$

Therefore the sequence $\{f(n_k)\}_{k=1}^{\infty}$ is a Cauchy sequence so that $L = \lim_{k\to\infty} f(n_k)$ exists. We will show that $\lim_{x\to\infty} f(x) = L$ as well. Let $\varepsilon > 0$ be given. Pick A such that

$$c > b \geq A \Rightarrow |f(c) - f(b)| < \frac{\varepsilon}{2}.$$

Let $x \geq A$. Since $L = \lim_{k\to\infty} f(n_k)$ we can pick the integer K so that $n_K > A$ and

$$|f(n_K) - L| < \frac{\varepsilon}{2}.$$

Thus

$$|f(x) - L| \leq |f(x) - f_{n_K}| + |f_{n_K} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $\lim_{x\to\infty} f(x) = L$. ■

We will make use of Theorems 1 and 2 when we study improper integrals in Chap. 4.

2.3.3 Problems

In problems 1 and 2 justify the statement in accordance with the precise definition of an infinite limit:

1.

$$\lim_{x\to 2+} \frac{x+1}{(x-1)(x-2)} = +\infty$$

2.

$$\lim_{x\to 4+} \frac{(x-1)(x-4)}{(x-4)^2} = +\infty$$

In problems 3 and 4 justify the statement in accordance with the relevant precise definition:

3.

$$\lim_{x\to+\infty}\frac{x-4}{4x-1}=\frac{1}{4}$$

4.

$$\lim_{x\to+\infty}\frac{x^2-4}{x-9}=+\infty$$

5. Assume that $\lim_{x\to a+}(-f(x))=+\infty$. Prove that $\lim_{x\to a+}f(x)=-\infty$.

2.4 The Intermediate Value Theorem

In beginning calculus it is assumed that a continuous function attains its maximum and minimum values on a closed and bounded interval. Now we will justify that assumption. We will also clarify the graphical implication of the continuity of a function as the "continuity" of its graph by showing that the image of an interval under a continuous function is also an interval. We will also discuss the existence and continuity of inverse functions.

2.4.1 The Extreme Value Theorem

Theorem 1. *Assume that f is continuous on a closed and bounded interval $[a,b]$. Then f attains its (absolute) maximum and minimum values on $[a,b]$.*

Proof. We will establish the existence of the absolute maximum. The statement about the minimum follows by considering $-f$ (provide the details as an exercise).

To begin with, let us establish that f is bounded above on $[a,b]$. If we assume that f is not bounded above on $[a,b]$, then for any $n\in\mathbb{N}$ there exists $x_n\in[a,b]$ such that

$$f(x_n)>n.$$

Since the closed and bounded interval $[a,b]$ is sequentially compact**,** there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and $x_0\in[a,b]$ such that

$$\lim_{k\to\infty}x_{n_k}=x_0.$$

Since f is continuous at x_0, we have

$$\lim_{k\to\infty}f(x_{n_k})=f(x_0).$$

Since a convergent sequence is bounded, the sequence $\{f(x_{n_k})\}_{k=1}^{\infty}$ must be bounded. But this contradicts the statement,

$$f(x_{n_k}) > n_k$$

for each $k \in \mathbb{N}$. Thus, f is bounded above on $[a, b]$.

By the least upper bound principle, the set

$$\{f(x) : a \leq x \leq b\}$$

has a least upper bound. Set $M = \sup_{a \leq x \leq b} f(x)$. We will show that there exists $x_0 \in [a, b]$ such that $f(x_0) = M$.

By the definition of the least upper bound, for each $n \in \mathbb{N}$ there exists $x_n \in [a, b]$ such that

$$M - \frac{1}{n} < f(x_n) \leq M.$$

Since the closed and bounded interval $[a, b]$ is sequentially compact, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and $x_0 \in [a, b]$ such that

$$\lim_{k \to \infty} x_{n_k} = x_0.$$

By the continuity of f at x_0, we have

$$\lim_{k \to \infty} f(x_{n_k}) = f(x_0).$$

On the other hand, the inequalities

$$M - \frac{1}{n_k} < f(x_{n_k}) \leq M$$

imply that

$$\lim_{k \to \infty} f(x_{n_k}) = M.$$

By the uniqueness of the limit, we must have $f(x_0) = M$. ■

2.4.2 The Intermediate Value Theorem

Theorem 2. *Assume that f is continuous on $[a, b]$ and $f(a) \neq f(b)$. If c is in the open interval with endpoints $f(a)$ and $f(b)$ then there exists $x_0 \in (a, b)$ such that $f(x_0) = c$.*

Proof. We will assume that

$$f(a) < c < f(b).$$

Set $g(x) = f(x) - c$ so that

$$g(a) < 0 \text{ and } g(b) > 0.$$

We need to show the existence of $x_0 \in (a, b)$ such that $g(x_0) = 0$. We will make use of the bisection method.

Let $a_1 = a$ and $b_1 = b$. Set

$$m_1 = \frac{a_1 + b_1}{2} = \frac{a + b}{2},$$

so that m_1 is the midpoint of $[a, b]$. If $g(m_1) = 0$, we are done: We can set $x_0 = m$. Otherwise,

- If $g(m_1) < 0$ we set $a_2 = m_1$ and $b_2 = b_1 = b$.
- If $g(m_1) > 0$ we set $a_2 = a_1 = a$ and $b_2 = m_1$.

Thus, $[a_2, b_2] \subset [a_1, b_1] = [a, b]$ and

$$g(a_2) < 0, g(b_2) > 0$$

Having determined $[a_2, b_2]$, we set

$$m_2 = \frac{a_2 + b_2}{2},$$

so that m_2 is the midpoint of $[a_1, b_1]$. If $g(m_2) = 0$, we set $x_0 = m_2$. Otherwise,

- If $g(m_2) < 0$ we set $a_3 = m_2$ and $b_3 = b_2$.
- If $g(m_2) > 0$ we set $a_3 = a_2$ and $b_3 = m_2$.

Note that $[a_3, b_3] \subset [a_2, b_2]$ and

$$g(a_3) < 0, \ g(b_3) > 0$$

This procedure is terminated at step n if

$$m_n = \frac{a_n + b_n}{2},$$

and $g(m_n) = 0$. In this case, we set $x_0 = m_n$. Otherwise,

- If $g(m_n) < 0$ we set $a_{n+1} = m_n$ and $b_{n+1} = b_n$.
- If $g(m_n) > 0$ we set $a_{n+1} = a_n$ and $b_{n+1} = m_n$.

Thus, $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ and

$$g(a_{n+1}) < 0, \ g(b_{n+1}) > 0.$$

If the procedure is never terminated, we obtain the nested sequence of intervals

$$[a_n, b_n], \ n = 1, 2, 3, \ldots,$$

such that

$$g(a_n) < 0, \ g(b_n) > 0,$$

and

$$b_n - a_n = \frac{b-a}{2^{n-1}}.$$

By the nested interval property, there exists a unique x_0 such that

$$a_n \leq x_0 \leq b_n$$

for all n, since

$$\lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{b-a}{2^{n-1}} = 0.$$

We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x_0.$$

By the continuity of g,

$$\lim_{n\to\infty} g(a_n) = \lim_{n\to\infty} g(b_n) = g(x_0).$$

Since $g(a_n) < 0$ for each n, we have $g(x_0) \leq 0$. Since $g(b_n) > 0$ for each n, we have $g(x_0) \geq 0$. Therefore, $g(x_0) = 0$. ■

2.4.3 The Existence and Continuity of Inverse Functions

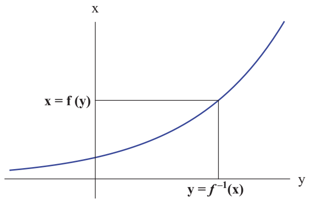
Let us begin by recalling the definition of the inverse of a function.

Definition 1. Assume that for each x in the range of f there is a **unique** y in the domain of f such that $f(y) = x$. **The inverse f^{-1} of f** is defined by the following relationship:

$$y = f^{-1}(x) \Leftrightarrow x = f(y).$$

Thus, the value of f^{-1} at x is the solution of the equation $x = f(y)$, provided that the solution exists and is unique. Figure 2.3 illustrates the relationship between f and the inverse f^{-1} graphically in the yx-plane (the y-axis is horizontal and the x-axis is vertical).

Fig. 2.3



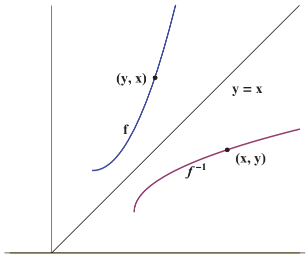
By the relationship between a function f and its inverse f^{-1}, the domain of f^{-1} is the same as the range of f, and the range of f^{-1} is the same as the domain of f. We must emphasize that **the notation f^{-1} in the present context should not be confused with the reciprocal $1/f$ of the function f**. The meaning of the notation should be clear within a particular context.

Assume that the function f has an inverse. Then

$$\left(f^{-1} \circ f\right)(y) = f^{-1}(f(y)) = y \text{ for each } y \text{ in the domain of } f,$$
$$\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = x \text{ for each } x \text{ in the domain of } f^{-1}.$$

If the scale on the vertical axis is the same as the scale on the horizontal axis, the graph of f^{-1} appears as the reflection of the graph of f with respect to the diagonal $y = x$, as illustrated in Fig. 2.4:

Fig. 2.4



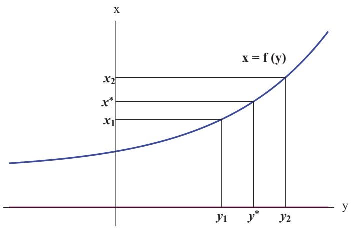
There is a general fact about the existence and continuity of the inverse of a function:

Theorem 3. *Assume that f is strictly increasing or decreasing and continuous on the interval J. The range I of f is also an interval. The inverse of f exists and f^{-1} is continuous on I. The function f^{-1} is increasing if f is increasing, and decreasing if f is decreasing.*

Proof. We will assume that f is increasing on the interval J (the case of a decreasing function is similar).

Let us first show that the range of f is an interval. Thus, assume that $x_1 = f(y_1)$ and $x_2 = f(y_2)$ are points in the range of f, and that $x_1 < x_2$. Let $x^* \in (x_1, x_2) = (f(y_1), f(y_2))$. By the Intermediate Value Theorem there exists $y^* \in J$ such that $f(y^*) = x^*$. Therefore the range of f is an interval. Let us label it as I.

Fig. 2.5



Let $x^* \in I$. We have shown that there exists $y^* \in J$ such that $f(y^*) = x^*$. Since f is strictly increasing if $y > y^*$ then $f(y) > f(y^*)$, if $y < y^*$ then $f(y) < f(y^*)$. Therefore y^* is the only point in J at which f attains the value x^*. Therefore f has an inverse $f^{-1} : I \to J$.

The inverse of f is also (strictly) increasing: Assume that x_1 and x_2 are in I and $x_1 < x_2$. Let $y_1 = f^{-1}(x_1)$ and $y_2 = f^{-1}(x_2)$. Thus $x_1 = f(y_1)$ and $x_2 = f(y_2)$. if $y_1 > y_2$ then $f(y_1) > f(y_2)$ so that $x_1 > x_2$. Since that is not the case we must have $y_1 < y_2$.

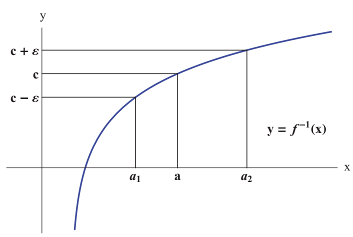
Now let us show that f^{-1} is continuous. We will consider the case of a point a in the interior of I (the appropriate one-sided continuity is discussed in a similar manner at an endpoint of I which belongs to I).

Let $c = f^{-1}(a)$, so that $a = f(c)$. Let $\varepsilon > 0$ be a given. With reference to Fig. 2.6, let $c - \varepsilon = f^{-1}(a_1)$ and $c + \varepsilon = f^{-1}(a_2)$. Since f is increasing, so is f^{-1}. Therefore, if $a_1 < x < a_2$, then $c - \varepsilon < f^{-1}(x) < c + \varepsilon$. Set δ to be the minimum of $|a - a_1|$ and $|a - a_2|$. Then,

$$|x - a| < \delta \Rightarrow x \in (a_1, a_2) \Rightarrow c - \varepsilon < f^{-1}(x) < c + \varepsilon$$
$$\Rightarrow \left|f^{-1}(x) - c\right| = \left|f^{-1}(x) - f^{-1}(a)\right| < \varepsilon.$$

This establishes the continuity of f^{-1}. ■

Fig. 2.6

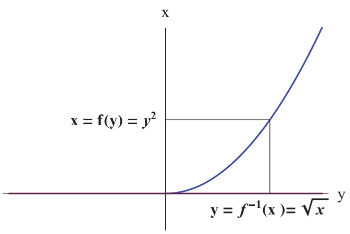


Example 1. Let $f(y) = y^2$. We restrict y so that $y \geq 0$. The function f is continuous and increasing on $[0, +\infty)$. The range of f is also $[0, +\infty)$. We have

$$y = \sqrt{x} \text{ where } x \geq 0 \Leftrightarrow x = f(y) = y^2 \text{ and } y \geq 0.$$

Therefore, $f^{-1}(x) = \sqrt{x}$. Figure 2.7 illustrates the definition of $\sqrt{x}$ graphically. The square-root function f^{-1} is continuous on $[0, \infty)$. □

Fig. 2.7



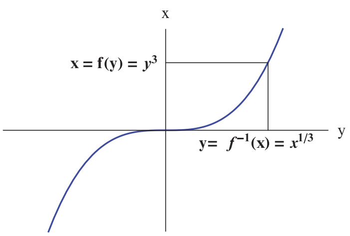
The square-root function illustrates a function defined by $x^{1/n}$, where n is an even positive integer. We have

$$y = x^{1/n} \Leftrightarrow x = y^n$$

where $x \geq 0$ and $y \geq 0$. Therefore, if we set $f(y) = y^n$, where $y \geq 0$, then $f^{-1}(x) = x^{1/n}$, $x \geq 0$.

Example 2. Let $f(y) = y^3$, where y is an arbitrary real number. The function is continuous and increasing on $\mathbb{R}$. The range of f is $\mathbb{R}$. We have $f^{-1}(x) = x^{1/3}$ for each $x \in \mathbb{R}$. Figure 2.8 illustrates the relationship between $x = f(y) = y^3$ and $y = f^{-1}(x) = x^{1/3}$. □

Fig. 2.8



Example 2 illustrates $x^{1/n}$ where n is an odd positive integer. If n is an odd positive integer and $f(y) = y^n$ for each $y \in \mathbb{R}$ then $f^{-1}(x) = x^{1/n}$ for each $x \in \mathbb{R}$.

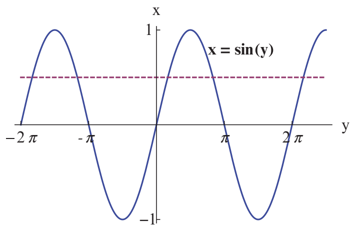
Appropriate restrictions of sine, cosine, and tangent have inverses. Let us recall the definitions of these important special functions of mathematics (Fig. 2.9).

Let us begin with the **sine** function. The equation $x = \sin(y)$ has infinitely many solutions for a given $x \in [-1, 1]$. Indeed, if y is a solution of the equation $x = \sin(y)$, then $y + 2n\pi$, $n = \pm 1, \pm 2, \ldots$ are also solutions, since sine is periodic with period 2π:

$$\sin(y + 2n\pi) = \sin(y) = x.$$

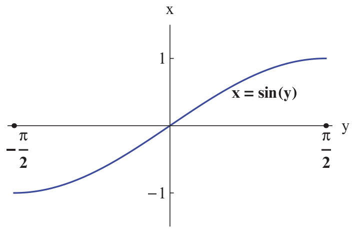
Thus, the sine function does not have an inverse.

Fig. 2.9



Let us restrict sine to the interval $[-\pi/2, \pi/2]$ and call the resulting function f. The function f is continuous and increasing on $[-\pi/2, \pi/2]$, and the range of f is the interval $[-1, 1]$. Figure 2.10 shows the graph of f.

Fig. 2.10 $x = f(y) = \sin(y)$ on $[-\pi/2, \pi/2]$



By Theorem 3, the inverse of f exists and f^{-1} is continuous on $[-1, 1]$. We have

$$y = f^{-1}(x) \Leftrightarrow x = \sin(y),$$

where $-1 \leq x \leq 1$ and $-\pi/2 \leq y \leq \pi/2$. We will refer to f^{-1} as **arcsine**, and abbreviate it as arcsin. Thus,

$$\boldsymbol{y = \arcsin(x) \Leftrightarrow x = \sin(y)}$$

where $\boldsymbol{-1 \leq x \leq 1}$and $\boldsymbol{-\pi/2 \leq y \leq \pi/2}$. Figure 2.11 illustrates the definition of arcsine.

Fig. 2.11

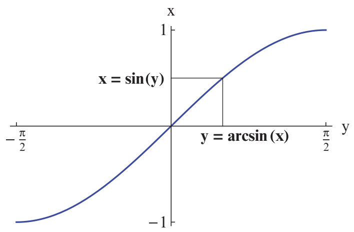
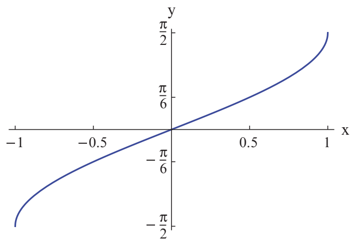


Figure 2.12 shows the graph of $y = \arcsin(x)$.

Fig. 2.12 $y = \arcsin(x)$



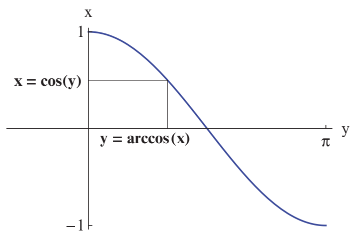
Another notation for $\arcsin(x)$ is $\sin^{-1}(x)$. We will favor the notation $\arcsin(x)$ in order to avoid any confusion with the reciprocal of sine.

Just as in the case of sine, we cannot define the inverse of the periodic function **cosine**. On the other hand, cosine is continuous and decreasing on $[0, \pi]$, so that the restriction of cosine to the interval $[0, \pi]$ has an inverse. We will refer to that function as **arccosine**, and use the abbreviation arccos:

$$y = \arccos(x) \Leftrightarrow x = \cos(y),$$

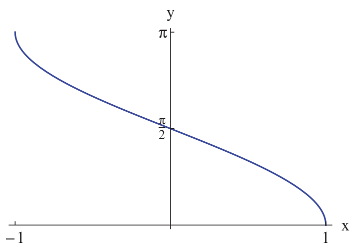
where $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$. Thus, the value of arccosine at $x \in [-1, 1]$ is the unique solution y of the equation $\cos(y) = x$ that is in the interval $[0, \pi]$. Figure 2.13 illustrates the definition of arccosine.

Fig. 2.13



By Theorem 3, arccosine is continuous on $[-1, 1]$. Another notation for $\arccos(x)$ is $\cos^{-1}(x)$. We will favor the notation $\arccos(x)$. Figure 2.14 shows the graph of arccosine.

Fig. 2.14



Remark 1. The functions arcsine and arccosine are related to each other in a simple manner. It can be shown that

$$\arccos(x) + \arcsin(x) = \frac{\pi}{2}, \ -1 \leq x \leq 1.$$

$\diamond$

The function **tangent** is periodic with period π, and its range is the entire number line. Therefore, the equation $\tan(y) = x$ has infinitely many solutions for any real number x, so that the inverse of tangent does not exist. On the other hand, **the restriction of tangent to the open interval** $(-\pi/2, \pi/2)$ is continuous, increasing, and has range equal to $\mathbb{R}$, so that it has an inverse that is continuous on the entire number line. We will refer to that function as **arctangent**, and use the abbreviation **arctan**. Thus,

$$y = \arctan(x) \Leftrightarrow \ x = \tan(y),$$

where x is an arbitrary real number and $-\pi/2 < y < \pi/2$. You may think of y as the unique angle between $-\pi/2$ and $\pi/2$ such that $\tan(y) = x$. Figure 2.15 illustrates the definition of arctangent. Another notation for $\arctan(x)$ is $\tan^{-1}(x)$.

Fig. 2.15

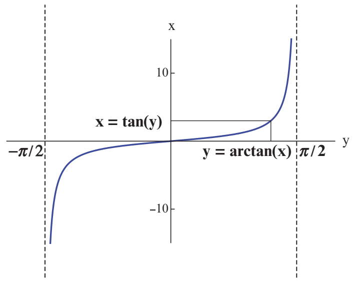
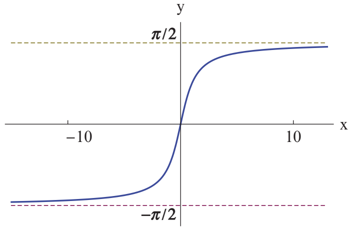


Figure 2.16 shows the graph of arctangent.

Fig. 2.16 $y = \arctan(x)$



We have

$$\lim_{x \to +\infty} \arctan(x) = \frac{\pi}{2} \text{ and } \lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}.$$

These facts are parallel to the facts,

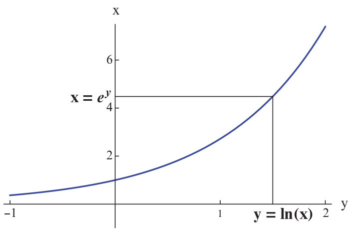
$$\lim_{y \to \frac{\pi}{2}-} \tan(y) = +\infty \text{ and } \lim_{y \to -\frac{\pi}{2}+} \tan(y) = -\infty.$$

The natural exponential and **the natural logarithm** are inverses of each other: We have

$$y = \ln(x) \Leftrightarrow x = e^y$$

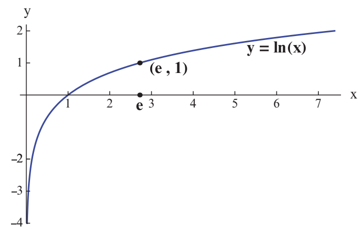
where $x > 0$ and $y \in \mathbb{R}$ (Fig. 2.17).

Fig. 2.17



As we will discuss in Chap. 4, both functions are continuous on their respective domains (Fig. 2.18).

Fig. 2.18



2.4.4 Problems

1. Let $f(x) = x^3 + 2x - 7$. Show that there exists $x_0 \in (0, 2)$ such that $f(x_0) = 0$.
2. Let

$$f(x) = 2\sin(3x) + 3\cos(2x)$$

Show that there exists $c \in (0, 2)$ such that $f(c) = 2$.

3. Let $f(x) = x^3 + 2x - 7$. Show that the inverse function f^{-1} exists (you may use the derivative test for monotonicity from beginning calculus). What is the domain of f^{-1}?

4. Let $f(x) = \sin^3(x)$ where $x \in [0, \pi/2]$.

a) Show that the inverse f^{-1} exists. What is the domain of f^{-1}? What is the range of f^{-1}?
b) Determine $f^{-1}(x)$ explicitly in terms of $\arcsin(x)$.

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