Chapter 2

Nonlinear and Multivalued Maps

2.1 Introduction

2.1.1 Compact, Completely Continuous, and Proper Maps

Definition 2.1
Let X and Y be two Banach spaces, let $D \subseteq X$ be a set, and let $f \colon D \longrightarrow Y$ be a map.

(a) *We say that f is **compact**, if it is continuous and maps bounded subsets of D to relatively compact sets (i.e., if $B \subseteq D$ is bounded, then $\overline{f(B)} \subseteq Y$ is compact). We denote the set of all compact maps $f \colon D \longrightarrow Y$ by $K(D;Y)$. If $D = X$, then we set $\mathcal{L}_c(X;Y) = \mathcal{L}(X;Y) \cap K(X;Y)$.*

(b) *We say that f is **completely continuous**, if for every sequence $\{u_n\}_{n \geqslant 1} \subseteq D$ such that $u_n \xrightarrow{w} u \in D$ in X, we have $f(u_n) \longrightarrow f(u)$ in Y (i.e., f maps weakly convergent sequences in D to strongly convergent sequences in Y).*

(c) *We say that f is a **finite rank map**, if $f(D)$ lies in a finite dimensional subspace of Y. We denote the set of all finite rank maps $f \colon D \longrightarrow Y$ by $K_f(D;Y)$. If $D = X$, then we set $\mathcal{L}_f(X;Y) = \mathcal{L}(X;Y) \cap K_f(X;Y)$.*

L. Gasiński, N.S. Papageorgiou, *Exercises in Analysis: Part 2*,
Problem Books in Mathematics, DOI 10.1007/978-3-319-27817-9_2

(d) *We say that f is a **weakly compact map**, if it maps bounded sets in D to relatively weakly compact subsets of Y (that is, if $B \subseteq D$ is bounded, then $\overline{f(B)}^w \subseteq Y$ is weakly compact). We denote the set of all weakly compact maps $f\colon D \longrightarrow Y$ by $K_{wc}(D;Y)$. If $D = X$, then we set $\mathcal{L}_{wc}(X;Y) = \mathcal{L}(X;Y) \cap K_{wc}(X;Y)$.*

Remark 2.2

Evidently we have $K_f(X;Y) \subseteq K(X;Y)$. A completely continuous linear operator $A\colon X \longrightarrow Y$ is sometimes called ***Dunford–Pettis operator***. Clearly the completely continuous maps are continuous.

Proposition 2.3

If X and Y are two Banach spaces and $K \in \mathcal{L}_c(X;Y)$,
then K is completely continuous.

The converse of this result is not in general true.

Proposition 2.4

If X is a reflexive Banach space, $D \subseteq X$ is nonempty closed and $f\colon D \longrightarrow Y$ is completely continuous,
then $f \in K(D;Y)$.

Combining Propositions 2.3 and 2.4, we infer the following.

Corollary 2.5

If X is a reflexive Banach space, Y is a Banach space and $K \in \mathcal{L}(X;Y)$,
then $K \in \mathcal{L}_c(X;Y)$ if and only if K is completely continuous.

The next theorem presents a useful approximation property that compact maps have and explains why those maps are the right class to extend the finite dimensional theory.

Theorem 2.6

If X and Y are two Banach spaces, $D \subseteq X$ is a bounded closed set and $f\colon D \longrightarrow Y$ is a map,
then $f \in K(D;Y)$ if and only if f is the uniform limit of finite rank maps

Remark 2.7
In fact given $\varepsilon > 0$, we can find a finite rank map $f_\varepsilon \colon D \longrightarrow Y$ such that

$$\|f(u) - f_\varepsilon(u)\|_Y \ < \ \varepsilon \qquad \forall\, u \in D$$

and

$$\overline{f_\varepsilon(D)} \ \subseteq \ \overline{\operatorname{conv}}\, f(D).$$

Compact maps exhibit a nice extension property, which is a consequence of the following theorem, which generalizes the well-known Tietze extension theorem (see Theorem I.2.138).

Theorem 2.8 *(Dugundji Extension Theorem)*
<u>If</u> X is a metric space, Y is a locally convex space, $D \subseteq X$ is a nonempty closed set, and $f \colon D \longrightarrow Y$ is a continuous map,
<u>then</u> there exists a continuous map $\widehat{f} \colon X \longrightarrow Y$ such that

$$\widehat{f}|_D \ = \ f \quad \text{and} \quad \widehat{f}(X) \ \subseteq \ \operatorname{conv} f(D).$$

Using the above theorem together with the Mazur theorem on the convex hull of compact sets (according to which the convex hull of a compact set in X is relatively compact; see Theorem I.5.86), we have the following result.

Theorem 2.9
<u>If</u> X and Y are two Banach spaces, $D \subseteq X$ is a nonempty closed and bounded set and $f \in K(D;Y)$,
<u>then</u> there exists $\widehat{f} \in K(X;Y)$ such that

$$\widehat{f}|_D \ = \ f \quad \text{and} \quad \widehat{f}(X) \ \subseteq \ \operatorname{conv} f(D).$$

The space $\mathcal{L}_c(X;Y)$ is closed and so

$$\overline{\mathcal{L}_f(X;Y)}^{\|\cdot\|_{\mathcal{L}}} \ \subseteq \ \mathcal{L}_c(X;Y).$$

In general this inclusion can be strict.

Definition 2.10
A Banach space Y is said to have the ***approximation property*** *(the* ***AP*** *for short), if for every Banach space X we have*

$$\overline{\mathcal{L}_f(X;Y)}^{\|\cdot\|_{\mathcal{L}}} \ = \ \mathcal{L}_c(X;Y).$$

Remark 2.11
The Hilbert spaces and the Banach spaces $c_0 = \big\{\widehat{u} = \{u_k\}_{k\geqslant 1} : u_k \longrightarrow 0\big\}$ and l^p (with $p \in [1, +\infty)$) have the AP.

Theorem 2.12 *(Schauder Theorem)*
If X and Y are two Banach spaces and $K \in \mathcal{L}(X;Y)$, then
(a) *$K \in \mathcal{L}_c(X;Y)$ if and only if $K^* \in \mathcal{L}_c(Y^*;X^*)$;*
(b) *$K \in \mathcal{L}_f(X;Y)$ if and only if $K^* \in \mathcal{L}_f(Y^*;X^*)$.*

The next notion is important because it restricts the size of the solution set of an operator equation of the form $f(u) = y$.

Definition 2.13
*Let X and Y be two Hausdorff topological spaces and let $f \in C(X;Y)$. We say that f is **proper**, if for every compact set $C \subseteq Y$, the set $f^{-1}(C) \subseteq X$ is compact.*

This notion is related to the coercivity of f which is important in variational analysis.

Proposition 2.14
If X and Y are two Banach spaces and $f \in C(X;Y)$,
then the following conditions are equivalent:
(a) *f is proper.*
(b) *f is a closed map (i.e., maps closed sets in X to closed sets in Y). Moreover, if X and Y are finite dimensional, then the above conditions are also equivalent to the following one:*
(c) *f is coercive (i.e., $\|f(u)\|_Y \longrightarrow +\infty$ as $\|u\|_X \longrightarrow +\infty$).*

In some special cases for maps between infinite dimensional Banach spaces, coercivity implies properness.

Proposition 2.15
If X and Y are two Banach spaces, $f \in C(X;Y)$ is coercive (i.e., $\|f(u)\|_Y \longrightarrow +\infty$ as $\|u\|_X \longrightarrow +\infty$) and at least one of the following conditions holds:
(i) *$f = g + h$ where g is proper and h is compact; or*
(ii) *X is reflexive and f is completely continuous (see Definition 2.1(b)),*
then f is proper (see Definition 2.13).

The notions of compactness and properness are related as follows.

Proposition 2.16
If X *is a Banach space,* $D \subseteq X$ *is a bounded closed set and* $f \in K(D;X)$*,*
then $I_X - f$ *is proper (*I_X *being the identity on* X*).*

Now we turn our attention to the space $\mathcal{L}_c(X;Y)$ and in particular to the spectral properties of such operators.

Proposition 2.17
If X *and* Y *are two Banach spaces,*
then $\mathcal{L}_c(X;Y)$ *furnished with the operator norm is a Banach space*

Definition 2.18
Let X *be a complex Banach space and let* $A \in \mathcal{L}(X)$*. The* ***spectrum*** $\sigma(A)$ *of* A *is defined by*

$$\sigma(A) \;=\; \big\{\lambda \in \mathbb{C} : \; \lambda I_X - A \textit{ is not invertible}\big\}.$$

The set $\mathbb{C} \setminus \sigma(A)$ *is called the* ***resolvent set*** *of* A *and it is denoted by* $\varrho(A)$*. Also, the operator* $R(\lambda) = (\lambda I_X - A)^{-1}$ *is called the* ***resolvent*** *of* A*.*

Remark 2.19
If we insist on real Banach spaces, then some important results of the spectral theory fail. For this reason we consider complex Banach spaces. For example, the next result is no longer true, if X is a real Banach space.

Proposition 2.20
If X *is a complex Banach space and* $A \in \mathcal{L}(X)$*,*
then $\sigma(A) \neq \emptyset$ *and in particular* $\sigma(A)$ *is a nonempty compact subset of* $\big\{\lambda \in \mathbb{C} : \; |\lambda| \leqslant \|A\|_{\mathcal{L}}\big\}$*.*

Definition 2.21
Let X *be a complex Banach space and let* $A \in \mathcal{L}(X)$*.*

(a) *We say that* $\lambda \in \mathbb{C}$ *is an* ***eigenvalue*** *of* A*, if there is* $u \in X$*,* $u \neq 0$ *such that* $(\lambda I_X - A)(u) = 0$*. Then* u *is an* ***eigenvector*** *corresponding to the eigenvalue* λ*. The subspace* $(\lambda I_X - A)^{-1}(0)$ *is the* ***eigenspace*** *corresponding to the eigenvalue* λ*. The set of all eigenvalues of* A *is called the* ***point spectrum*** *of* A *and is denoted by* $\sigma_p(A)$*.*

(b) *If* $\lambda \in \sigma(A) \setminus \sigma_p(A)$ *and* $R(\lambda I_X - A)$ *is not dense in* X*, then we say that* λ *is in the* ***residual spectrum*** *of* A*.*

Remark 2.22
The reason that we single out the residual spectrum is that it does not occur for a large class of operators, for example, for self-adjoint operators on a Hilbert space.

Theorem 2.23
If X *is an infinite dimensional complex Banach space and* $A \in \mathcal{L}_c(X)$*, then*
(a) $\sigma(A)$ *is a countably compact set (see Definition I.2.88) with* 0 *as the only possible limit point;*
(b) $\sigma(A) = \sigma_p(A) \cup \{0\}$*;*
(c) *if* $\lambda \in \sigma_p(A) \setminus \{0\}$*, then the eigenspace corresponding to* λ *is finite dimensional and*

$$\dim \ker (\lambda I_X - A) \ = \ \operatorname{codim} R(\lambda I_X - A)$$

(recall that $\operatorname{codim} R(\lambda I_X - A) = \dim \big(X/_{R(\lambda I_X - A)}\big)$*).*

We can expand **(c)** in the above theorem.

Proposition 2.24
If X *is a Banach space,* $A \in \mathcal{L}_c(X)$ *and* λ *is a nonzero scalar, then*
(a) $\ker (\lambda I_X - A)$ *is finite dimensional;*
(b) $R(\lambda I_X - A)$ *is closed and* $R(\lambda I_X - A) = \ker (\lambda I_X^* - A^*)^\perp$*;*
(c) $\ker (\lambda I_X - A) = \{0\}$ *if and only if* $R(\lambda I_X - A) = X$*;*
(d) $\dim \ker (\lambda I_X - A) = \dim \ker (\lambda I_X^* - A^*)$*.*

Remark 2.25
Statement **(c)** says that $\lambda I_X - A$ is injective if and only if it is surjective. This is a well-known fact for operators between finite dimensional spaces. So, according to this statement, the equation

$$(\lambda I_X - A)(u) \ = \ y$$

has a solution for every $y \in X$ if and only if the equation

$$(\lambda I_X - A)(u) \ = \ 0$$

admits only the trivial solution. In this form, the result is known as the ***Fredholm alternative theorem***.

For compact self-adjoint operators defined on a separable Hilbert space, we have the following result.

Theorem 2.26 *(Spectral Theorem)*
If H is an infinite dimensional separable Hilbert space and $A \in \mathcal{L}_c(H)$, then there exists an orthonormal basis $\{e_n\}_{n\geqslant 1}$ of H consisting of eigenvectors of A such that for every $u \in H$ we have

$$A(u) \;=\; \sum_{k\geqslant 1} \lambda_k\, (u, e_k)_H\, e_k,$$

with λ_k being the eigenvalue corresponding to e_k and $(\cdot,\cdot)_H$ denoting the inner product of H.

Definition 2.27
Let H be a Hilbert space and let $A \in \mathcal{L}(H)$.
(a) *We say that A is **normal**, if $AA^* = A^*A$;*
(b) *We say that A is **unitary**, if $A^* = A^{-1}$ (i.e., $AA^* = A^*A = I_H$);*
(c) *We say that A is **projection**, if $A^2 = A$ and it is an **orthogonal projection**, if it is a projection and $R(A) = (\ker A)^\perp$.*

Proposition 2.28
If H is a complex separable Hilbert space and $A \in \mathcal{L}_c(H)$ is normal, then there exists an orthonormal basis $\{e_n\}_{n\geqslant 1}$ of H consisting of eigenvectors of A such that for every $u \in H$, we have

$$A(u) \;=\; \sum_{k\geqslant 1} \lambda_k\, (u, e_k)_H\, e_k,$$

with λ_k being the eigenvalue corresponding to e_k and $(\cdot,\cdot)_H$ denoting the inner product of H.

Definition 2.29
*Let X and Y be two Banach spaces and let $L \in \mathcal{L}(X;Y)$. We say that L is a **Fredholm operator**, if $\ker L$ is finite dimensional and $R(L) = L(X)$ is finite codimensional (that is, $\dim(Y/_{R(L)}) < +\infty$). The number $i(L) = \dim\ker L - \dim(Y/_{R(L)})$ is called the **index** of L. The set of Fredholm operators $L\colon X \longrightarrow Y$ is denoted by $\mathrm{Fred}\,(X;Y)$.*

Remark 2.30
If $L \in \mathrm{Fred}\,(X;Y)$, then $R(L) \subseteq Y$ is closed. Moreover, we have

$$X \;=\; \ker L \oplus V$$

and $L|_V$ is an isomorphism of V onto $L(X)$. $\mathrm{Fred}\,(X;Y)$ is an open subset of $\mathcal{L}(X;Y)$ and the map $L \longmapsto i(L)$ is continuous (hence, it is constant on each connected component of $\mathrm{Fred}\,(X;Y)$). Finally every $L \in \mathrm{Fred}\,(X;Y)$ is invertible modulo finite rank operators, that is, there exists $T \in \mathcal{L}(Y;X)$ such that both $L \circ T - I_Y$ and $T \circ L - I_X$ are finite rank operators.

Finally we introduce a notion that allows us to go beyond compact maps.

Definition 2.31
Let X be a Banach space and let $\mathcal{B}$ be the family of bounded subsets of X.

(a) *The **Kuratowski measure of noncompactness** $\alpha\colon \mathcal{B} \longrightarrow \mathbb{R}_+$ is defined by*

$$\alpha(C) = \inf\big\{d > 0 : \ C \text{ can be covered by a finite number of sets of diameter} \leqslant d\big\}.$$

(b) *The **Hausdorff measure of noncompactness** $\beta\colon \mathcal{B} \longrightarrow \mathbb{R}_+$ is defined by*

$$\beta(C) = \inf\big\{r > 0 : \ C \text{ can be covered by a finite number of balls of radius } r\big\}.$$

Let $D \subseteq X$, let Y be another Banach space and let $f\colon D \longrightarrow Y$ be a continuous and bounded (i.e., maps bounded sets to bounded sets) map. In what follows by γ_X (respectively γ_Y) we denote either α or β on X (respectively on Y).

(c) *We say that f is a **k-set-Lipschitz map**, if for every bounded set $C \subseteq D$ we have*

$$\gamma_Y(f(C)) \leqslant k\gamma_X(C).$$

(d) *We say that f is a **k-set contraction**, if it is k-set Lipschitz with $k < 1$. We denote by $SC_k(D;Y)$ the family of all such maps.*

(e) *We say that f is a **condensing**, if for every bounded set $C \subseteq D$ with $\gamma_X(C) > 0$, we have $\gamma_Y(f(C)) < \gamma_X(C)$. We denote by $S(D;Y)$ the family of all such maps.*

Proposition 2.32
If X *is Banach space and* $\gamma\colon \mathcal{B} \longrightarrow \mathbb{R}_+$ *is either* α *or* β, *then*
(a) $\gamma(C) = 0$ *if and only if* $\overline{C}$ *is compact (regularity);*
(b) $\gamma(\lambda C) = |\lambda|\gamma(C)$ *for all* $\lambda \in \mathbb{R}$ *and* $\gamma(C_1 + C_2) \leqslant \gamma(C_1) + \gamma(C_2)$ *(seminorm);*
(c) *if* $C_1 \subseteq C_2$, *then* $\gamma(C_1) \leqslant \gamma(C_2)$ *(monotonicity);*
(d) $\gamma(C_1 \cup C_2) = \max\{\gamma(C_1), \gamma(C_2)\}$ *(semiadditivity);*
(e) $\gamma(C) = \gamma(\operatorname{conv} C)$ *and* $\gamma(C) = \gamma(\overline{C})$.

Another measure of noncompactness that we will use is related to the weak topology on X.

Definition 2.33
Let X *be a Banach space and let* $\mathcal{B}$ *be the family of bounded subsets of* X. *The weak measure of noncompactness* $\xi\colon \mathcal{B} \longrightarrow [0, +\infty)$ *is defined by*

$$\begin{aligned} \xi(C) \quad = \quad \inf\big\{\varepsilon > 0 :\ & \textit{there exists a } w\textit{-compact set } D \subseteq X \\ & \textit{such that } C \subseteq D + \varepsilon\overline{B}_1\big\}, \end{aligned}$$

where $\overline{B}_1 = \{u \in X :\ \|u\|_X \leqslant 1\}$.

The properties of ξ are similar to those of α and β.

Proposition 2.34
If X *is a Banach space and* $\xi\colon \mathcal{B} \longrightarrow \mathbb{R}_+$ *is the weak measure of noncompactness, then*
(a) $\xi(C) = 0$ *if and only if* $\overline{C}^w$ *is w-compact;*
(b) $\xi(\lambda C) = |\lambda|\xi(C)$ *for all* $\lambda \in \mathbb{R}$ *and* $\xi(C_1 + C_2) \leqslant \xi(C_1) + \xi(C_2)$;
(c) *if* $C_1 \subseteq C_2$, *then* $\xi(C_1) \leqslant \xi(C_2)$;
(d) $\xi(C_1 \cup C_2) = \max\{\xi(C_1), \xi(C_2)\}$;
(e) $\xi(C) = \xi(\operatorname{conv} C)$ *and* $\xi(C) = \xi(\overline{C}^w)$.

2.1.2 Multifunctions

Throughout this section, we will use the following notation for certain hyperspaces. So, let X be a Hausdorff topological space. We define

$$\begin{aligned} P_f(X) &= \big\{A \subseteq X :\ A \text{ is nonempty and closed}\big\} \\ P_k(X) &= \big\{A \subseteq X :\ A \text{ is nonempty and compact}\big\} \\ \widehat{P}_f(X) &= P_f(X) \cup \{\emptyset\}. \end{aligned}$$

Moreover, if X is a normed space, then

$$\begin{aligned}
P_{fc}(X) &= \{A \in P_f(X) : A \text{ is convex}\}\\
P_{kc}(X) &= \{A \in P_k(X) : A \text{ is convex}\}\\
P_{wkc}(X) &= \{A \subseteq X : A \text{ is nonempty, } w\text{-compact and convex}\}\\
P_{bf(c)}(X) &= \{A \subseteq X : A \text{ is nonempty, bounded, closed (and convex)}\}.
\end{aligned}$$

Definition 2.35
Let X and Y be two sets, let $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ be a multifunction and let $D \subseteq Y$. We define
(a) *The **weak inverse image** of D under F is the set*

$$F^-(D) \;=\; \{u \in X : \; F(u) \cap D \neq \emptyset\}.$$

(b) *The **strong inverse image** of D under F is the set*

$$F^+(D) \;=\; \{u \in X : \; F(u) \subseteq D\}.$$

Using these notions, we can define some continuity concepts for multifunctions between two Hausdorff topological spaces.

Definition 2.36
Let X and Y be two Hausdorff topological spaces and let $F\colon X \longrightarrow 2^Y$ be a multifunction.

(a) *We say that F is **upper semicontinuous** at $u_0 \in X$ (**usc** at x_0 for short), if for every open set $V \subseteq Y$ such that $F(u_0) \subseteq V$, we can find $U \in \mathcal{N}(u_0)$ (the filter of neighborhoods of u_0), such that $F(u) \subseteq V$ for all $u \in U$. We say that F is **upper semicontinuous** (**usc** for short), if it is upper semicontinuous at any $u \in X$.*

(b) *We say that F is **lower semicontinuous** at $u_0 \in X$ (**lsc** at x_0 for short), if for every open set $V \subseteq Y$ such that $F(u_0) \cap V \neq \emptyset$, we can find $U \in \mathcal{N}(u_0)$, such that $F(u) \cap V \neq \emptyset$ for all $u \in U$. We say that F is **lower semicontinuous** (**lsc** for short), if it is lower semicontinuous at any $u \in X$.*

(c) *We say that F is **continuous** (or **Vietoris continuous**) at $u_0 \in X$, if it is both upper and lower semicontinuous at u_0. We say that F is **continuous** (or **Vietoris continuous**) if it is continuous at any $u \in X$.*

These definitions lead to the following alternative descriptions of the above continuity notions for multifunctions.

Proposition 2.37
If X *and* Y *be two Hausdorff topological spaces and* $F\colon X \longrightarrow 2^Y$ *is a multifunction,*
then the following statements are equivalent:
(a) *F is upper semicontinuous.*
(b) *For every closed set $C \subseteq Y$, the set $F^-(C)$ is closed in X.*
(c) *If $x \in X$, $\{x_\alpha\}_{\alpha \in J}$ is a net in X, $x_\alpha \longrightarrow x$, $V \subseteq Y$ is an open set such that $F(x) \subseteq V$, then there exists $\alpha_0 \in J$, such that for all $\alpha \in J$, $\alpha \geqslant \alpha_0$ we have $F(x_\alpha) \subseteq V$.*
(d) *For every open set $U \subseteq Y$, the set $F^+(U)$ is open in X.*

Proposition 2.38
If X *and* Y *be two Hausdorff topological spaces and* $F\colon X \longrightarrow 2^Y$ *is a multifunction,*
then the following statements are equivalent:
(a) *F is lower semicontinuous.*
(b) *For every closed set $C \subseteq Y$, the set $F^+(C)$ is closed in X.*
(c) *If $x \in X$, $\{x_\alpha\}_{\alpha \in J}$ is a net in X, $x_\alpha \longrightarrow x$, $V \subseteq Y$ is an open set such that $F(x) \cap V \neq \emptyset$, then there exists $\alpha_0 \in J$, such that for all $\alpha \in J$, $\alpha \geqslant \alpha_0$ we have $F(x_\alpha) \cap V \neq \emptyset$.*
(d) *If $x \in X$, $\{x_\alpha\}_{\alpha \in J}$ is a net in X, $x_\alpha \longrightarrow x$ and $y \in F(x)$, then for every $\alpha \in J$ we can find $y_\alpha \in F(x_\alpha)$, such that $y_\alpha \longrightarrow y$ in Y.*
(e) *For every open set $U \subseteq Y$, the set $F^-(U)$ is open in X.*

Proposition 2.39
If X *and* Y *are two Hausdorff topological spaces and* $F\colon X \longrightarrow 2^Y$ *is a multifunction,*
then the following statements are equivalent:
(a) *F is continuous.*
(b) *For every closed set $C \subseteq Y$, the sets $F^+(C)$ and $F^-(C)$ are both closed in X.*
(c) *If $x \in X$, $\{x_\alpha\}_{\alpha \in J}$ is a net in X, $x_\alpha \longrightarrow x$, $V, W \subseteq Y$ are open sets such that $F(x) \subseteq V$ and $F(x) \cap W \neq \emptyset$, then there exists $\alpha_0 \in J$, such that for all $\alpha \in J$, $\alpha \geqslant \alpha_0$ we have $F(x_\alpha) \subseteq V$ and $F(x_\alpha) \cap W \neq \emptyset$.*

Definition 2.40
Let X and Y be two Hausdorff topological spaces and let $F\colon X \longrightarrow 2^Y$ be a multifunction. The ***graph*** *of F is the set*

$$\operatorname{Gr} F = \{(x,y) \in X \times Y : \ y \in F(x)\}.$$

We say that F is ***closed****, at $u_0 \in X$, if for every net $\{(x_\alpha, y_\alpha)\}_{\alpha \in J} \subseteq \operatorname{Gr} F$ such that $(x_\alpha, y_\alpha) \longrightarrow (x_0, y_0)$ in $X \times Y$, we have $y_0 \in F(x_0)$. We say that F is* ***closed****, if F is closed at every $x \in X$ (that is, $\operatorname{Gr} F \subseteq X \times Y$ is closed).*

Proposition 2.41

(a) *If X and Y are two Hausdorff topological spaces with Y being regular and $F\colon X \longrightarrow P_f(Y)$ is an upper semicontinuous multifunction,*
then F is closed.

(b) *If X is a complete metric space, Y is a Hausdorff topological space with topology τ, $F\colon X \longrightarrow P_k(Y)$ is an upper semicontinuous multifunction and there is a metrizable topology τ_1 of Y such that $\tau_1 \subseteq \tau$,*
then F is continuous on a residual set (that is, on the complement of a set of first Baire category).

For multifunctions with compact values, we can drop the regularity condition on Y. This is a consequence of the following proposition.

Proposition 2.42
If X and Y are two Hausdorff topological spaces and $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is a multifunction,
then F is $P_k(Y)$-valued and upper semicontinuous if and only if for every net $\{(x_\alpha, y_\alpha)\}_{\alpha \in J} \subseteq \operatorname{Gr} F$ such that $x_\alpha \longrightarrow x$ in X, $\{y_\alpha\}_{\alpha \in J}$ has a limit point in $F(x)$.

With this result, we can have a version of Proposition 2.41(a) in which the regularity of the space Y is no longer required.

Proposition 2.43
If X and Y are two Hausdorff topological spaces and $F\colon X \longrightarrow P_k(Y)$ is an upper semicontinuous multifunction,
then F is closed.

Now we introduce a metric structure on the range space Y and for every $y \in Y$, we examine the function $x \longmapsto \operatorname{dist}_Y(y, F(x))$.

Proposition 2.44
If X is a Hausdorff topological space, (Y, d_Y) is a metric space and $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is a multifunction,
then F is lower semicontinuous if and only if for every $y \in Y$, the function $x \longmapsto \operatorname{dist}_Y(y, F(x))$ is upper semicontinuous (as an $\mathbb{R}_+$-valued function).

There is no corresponding result for upper semicontinuous multifunctions. We only have the following result.

Proposition 2.45
If X is a Hausdorff topological space, (Y, d_Y) is a metric space and $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is a locally compact multifunction (i.e., for every $x \in X$, we can find $U \in \mathcal{N}(x)$ such that $\overline{F(U)} \in P_k(Y)$) and for every $y \in Y$, the function $x \longmapsto \operatorname{dist}_Y(y, F(x))$ is lower semicontinuous,
then F is upper semicontinuous.

If on Y we introduce linear structure, then there is another important functional that we can associate with a multifunction $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$, namely the support function.

Definition 2.46
Let Y be a normed space, Y^* its topological dual and $D \subseteq Y$ a nonempty set. The ***support function*** *of D, $\sigma_D\colon Y^* \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is defined by*

$$\sigma_D(y^*) = \sup_{y \in D} \langle y^*, y \rangle ,$$

where by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (Y^*, Y).

Proposition 2.47
If X is a Hausdorff topological space, Y is a normed space furnished with its weak topology and $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is an upper semicontinuous multifunction,
then for every $y^* \in Y^*$, the function $x \longmapsto \sigma_{F(x)}(y^*)$ is upper semicontinuous.

To have the converse of the above proposition, we need to strengthen the conditions on the multifunction F.

Proposition 2.48
If X is a Hausdorff topological space, Y is a normed space and $F\colon X \longrightarrow P_{wkc}(Y)$ is a multifunction such that for all $y^* \in Y^*$, the function $x \longmapsto \sigma_{F(x)}(y^*)$ is upper semicontinuous,
then F is upper semicontinuous from X into Y equipped with the weak topology.

In the presence of a metric structure on Y, we can have a pseudometric structure on the hyperspace 2^Y.

Definition 2.49
Let (Y, d_Y) be a metric space and let $C, D \in 2^Y$ be two sets. We define:
(a) $h^*(C,D) = \sup\limits_{c\in C} \operatorname{dist}_Y(c,D)$ *(the **excess** of C over D).*
(b) $h(C,D) = \max\big\{h^*(C,D), h^*(D,C)\big\}$ *(the **Hausdorff distance** of C from D).*

Remark 2.50
Evidently h is a generalized pseudometric on 2^Y (that is, h is a pseudometric that can also take the value $+\infty$). Moreover, $h(C,D) = 0$ if and only if $\overline{C} = \overline{D}$ and so $P_f(Y) \cup \{\emptyset\}$ is a generalized metric space, with $\emptyset$ an isolated point. Then we call h the ***Hausdorff metric***. The Hausdorff metric h is not a topological construction, that is, the Hausdorff metric topology on $P_f(Y)$ is not determined by the topology of Y. It is easy to check that

$$h(\lambda C, \lambda D) \;=\; |\lambda| h(C,D) \qquad \forall\, \lambda \in \mathbb{R}$$

and if Y is a normed space, then

$$h(C_1 + C_2, D_1 + D_2) \;\leqslant\; h(C_1, D_1) + h(C_2, D_2)$$

and

$$h(\lambda C_1 + (1-\lambda) C_2, \lambda D_1 + (1-\lambda) D_2) \;\leqslant\; \lambda h(C_1, D_1) + (1-\lambda) h(C_2, D_2),$$

for all $\lambda \in [0,1]$. Therefore, we have

$$h(\overline{\operatorname{conv}}\, C, \overline{\operatorname{conv}}\, D) \;\leqslant\; h(C,D).$$

The same inequalities are also valid for h^*. Also, note that

$$h^*(C,D) \;=\; \sup_{y\in Y}\left(\operatorname{dist}(y,C)-\operatorname{dist}(y,D)\right)$$

and is Y is a normed space and $C,D \in P_{bfc}(Y)$, then

$$h^*(C,D) \;=\; \sup\left\{\sigma_{C}(y^*)-\sigma_{D}(y^*):\ \|y^*\|_* \leqslant 1\right\}$$

An immediate consequence of Definition 2.49 are the following two useful expressions for the Hausdorff metric h.

Proposition 2.51

(a) *If* (Y,d_{Y}) *is a metric space and* $C,D \in 2^Y \setminus \{\emptyset\}$,
then $h(C,D) = \sup\limits_{y\in Y}\left|\operatorname{dist}_Y(y,C)-\operatorname{dist}_Y(y,D)\right|$;

(b) *If* Y *is a normed space and* $C,D \in P_{bfc}(Y)$,
then $h(C,D) \;=\; \sup\limits_{\|y^*\|\leqslant 1}\left|\sigma_{C}(y^*) - \sigma_{D}(y^*)\right|$ *(the* ***Hörmander formula****)*.

(c) *If* Y *is a normed space and* $C \subseteq Y$ *is a nonempty, convex set then for every* $v \in Y$, *we have*

$$\operatorname{dist}_Y(v,C) \;=\; \sup\Big\{\langle v^*,v\rangle - \sigma_{C}(v^*):\ v^*\in Y^*,\ \|v^*\|_* \leqslant 1\Big\}$$

(see Problem 3.88).

(d) *If* Y *is a Banach space and* $C \subseteq Y$,
then $C \in P_{wkc}(Y)$ *if and only if the map* $v^* \longmapsto \sigma_{C}(v^*)$ *is* $m(Y^*,Y)$*-continuous with* $m(Y^*,Y)$ *being the Mackey topology.*

Assuming extra structure on Y, we can say more about the metric space $(P_f(Y),h)$.

Proposition 2.52

(a) *If* (Y,d_{Y}) *is a complete metric space,*
then $(P_f(Y),h)$ *is a complete metric space too.*

(b) $P_{bf}(Y)$ *is a closed subset of* $(P_f(Y),h)$ *and if* (Y,d_{Y}) *is a separable metric space, then so is* $(P_f(Y),h)$.

(c) *If* Y *is a normed space,*
then $P_{kc}(Y) \subseteq P_{bfc}(Y)$ *and* $P_k(Y) \subseteq P_{bf}(Y)$ *are all closed subsets of the metric space* $(P_f(Y),h)$.

Using these metric notions, we can introduce new continuity concepts for a multifunction F, in general distinct from those of Definition 2.36.

Definition 2.53
Let X be a Hausdorff topological space and let (Y, d_Y) be a metric space.

(a) *A multifunction $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is said to be h-**upper semicontinuous** at $x_0 \in X$ (h-**usc** at x_0 for short), if the function $x \longmapsto h^*\big(F(x), F(x_0)\big)$ is continuous at x_0. We say that F is h-**upper semicontinuous** (h-**usc** for short), if it is h-upper semicontinuous at every $x \in X$.*

(b) *A multifunction $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is said to be h-**lower semicontinuous** at $x_0 \in X$ (h-**lsc** at x_0 for short), if the function $x \longmapsto h^*\big(F(x_0), F(x)\big)$ is continuous at x_0. We say that F is h-**lower semicontinuous** (h-**lsc** for short), if it is h-lower semicontinuous at every $x \in X$.*

(c) *A multifunction $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is said to be h-**continuous** at $x_0 \in X$, if it is both h-lower and h-upper semicontinuous at $x_0 \in X$ (that is, the multifunction function $F\colon X \longrightarrow (2^Y \setminus \{\emptyset\}, h)$ is continuous at x_0). We say that F is h-**continuous**, if it is h-continuous at every $x \in X$.*

How are these continuity notions related to those introduced in Definition 2.36?

Proposition 2.54
<u>If</u> X is a Hausdorff topological space, (Y, d_Y) is a metric space, and $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is upper semicontinuous,
<u>then</u> F is h-upper semicontinuous.

Proposition 2.55
<u>If</u> X is a Hausdorff topological space, (Y, d_Y) is a metric space, and $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is h-lower semicontinuous,
<u>then</u> F is lower semicontinuous.

In general the converse implications are not true (see Problem 2.67). For compact valued multifunctions, the situation is better.

Proposition 2.56
If X *is a Hausdorff topological space,* (Y, d_Y) *is a metric space, and* $F\colon X \longrightarrow P_k(Y)$ *is a multifunction, then*

(a) F *is upper semicontinuous if and only if* F *is h-upper semicontinuous.*
(b) F *is lower semicontinuous if and only if* F *is h-lower semicontinuous.*
(c) F *is continuous if and only if* F *is h-continuous.*

Now we turn our attention to the fundamental question of the existence of a continuous selection.

Definition 2.57
Let X *and* Y *be two Hausdorff topological spaces and let* $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ *be a multifunction. A* ***continuous selection*** *(or* ***selector****) of* F *is a continuous function* $f\colon X \longrightarrow Y$ *such that* $f(x) \in F(x)$ *for all* $x \in X$*.*

We have the following existence result.

Theorem 2.58 *(Michael Selection Theorem)*
If X *is a paracompact space,* Y *is a Banach space and* $F\colon X \longrightarrow P_{fc}(Y)$ *is a lower semicontinuous multifunction,*
then F *admits a continuous selection.*

When X is a metric space (thus also a paracompact space) and Y is a separable Banach space, we can produce a whole sequence of continuous selectors of F which is dense in $F(x)$ for all $x \in X$.

Proposition 2.59
If X *is a metric space,* Y *is a separable Banach space and* $F\colon X \longrightarrow P_{fc}(Y)$ *is a lower semicontinuous multifunction,*
then there exists a sequence $\{f_n\}_{n \geqslant 1}$ *of continuous selectors of* F *such that*

$$F(x) \;=\; \overline{\{f_n(x)\}}_{n \geqslant 1} \qquad \forall\, x \in X.$$

In general Theorem 2.58 is optimal in the sense that the hypotheses of the theorem cannot be relaxed. However, if Y is a separable Banach space, then we can enlarge the range space of F.

Proposition 2.60
If X *is a paracompact, perfectly normal topological space (see Definitions I.2.142 and I.2.137),* Y *is a separable Banach space, and* $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ *is a lower semicontinuous multifunction with convex values such that for all* $x \in X$ *either* $\operatorname{int} F(x) \neq \emptyset$ *or* $F(x)$ *is finite dimensional,*
then F *admits a continuous selection.*

So far, we have focused on the topological properties of multifunctions. Next we turn our attention to the measure theoretic ones.

Definition 2.61
Let (Ω, Σ) *be a measurable space,* X *a metric space and* $F\colon \Omega \longrightarrow 2^X$ *a multifunction.*
(a) *We say that* F *is* ***measurable****, if for every open set* $U \subseteq X$*, we have* $F^-(U) \in \Sigma$ *(see Definition 2.35(a)).*
(b) *We say that* F *is* ***graph measurable****, if* $\operatorname{Gr} F \in \Sigma \times \mathcal{B}(X)$ *(*$\mathcal{B}(X)$ *being the Borel* σ*-field of* X*).*

Remark 2.62
The ***domain*** of F is defined to be the set

$$\operatorname{dom} F \;=\; \big\{\omega \in \Omega :\; F(\omega) \neq \emptyset\big\}.$$

Evidently, if F is measurable, then $\operatorname{dom} F \in \Sigma$. So, when dealing with measurable multifunctions F, we can always assume that $\operatorname{dom} F = \Omega$.

The next theorem summarizes what is known about the measurability of multifunctions with closed values.

Theorem 2.63
Let (Ω, Σ) *be a measurable space,* (X, d_X) *a separable metric space and* $F\colon \Omega \longrightarrow P_f(X)$ *is a multifunction. We consider the following properties of* F*:*
(1) $F^-(D) \in \Sigma$ *for every* $D \in \mathcal{B}(X)$ *(*$\mathcal{B}(X)$ *being the Borel* σ*-field of* X*).*
(2) $F^-(C) \in \Sigma$ *for every closed set* $C \subseteq X$*.*
(3) F *is measurable (see Definition 2.61(a)).*
(4) For every $x \in X$*, the function* $\omega \longrightarrow \operatorname{dist}(x, F(x))$ *is* Σ*-measurable.*
(5) F *is graph measurable.*

We have the following relations among the above properties:
(a) $(1) \implies (2) \implies (3) \iff (4) \implies (5)$.
(b) *If X is σ-compact, then* $(2) \iff (3)$.
(c) *If Σ is complete (that is, $\Sigma = \widehat{\Sigma}$ ($\widehat{\Sigma}$ being the universal σ-field; see Definition I.4.45)) and X is complete, then* $(1) \iff (2) \iff (3)$.

Now we pass to the problem of existence of selections. In this case we look for a measurable selections.

Theorem 2.64 *(Kuratowski–Ryll Nardzewski Selection Theorem)*
If (Ω, Σ) is a measurable space, X is a Polish space and $F\colon \Omega \longrightarrow P_f(X)$ is a measurable multifunction,
then F admits a Σ-measurable selection.

Remark 2.65
In fact we can be slightly more general and instead assume that X is a separable metrizable space (that is, we drop the completeness hypothesis on X) and that F has complete values.

We can improve the above selection theorem (compare with Proposition 2.59).

Theorem 2.66
If (Ω, Σ) is a measurable space, X is a Polish space and $F\colon \Omega \longrightarrow P_f(X)$ is a multifunction,
then the following statements are equivalent:
(a) *F is measurable.*
(b) *There exists a sequence of Σ-measurable selectors $\{f_n\colon \Omega \longrightarrow X\}_{n \geqslant 1}$ of the multifunction F such that*

$$F(\omega) \;=\; \overline{\{f_n(\omega)\}}_{n \geqslant 1} \qquad \forall\, \omega \in \Omega.$$

This result leads to the following improved version of Theorem 2.63.

Theorem 2.67
Let (Ω, Σ) be a measurable space, (X, d_X) a separable metric space and $F\colon \Omega \longrightarrow P_f(X)$ a multifunction. We consider the following statements:
(1) $F^-(D) \in \Sigma$ for every $D \in \mathcal{B}(X)$ (with $\mathcal{B}(X)$ being the Borel σ-field of X).

(2) $F^{-}(C) \in \Sigma$ *for every closed set* $C \subseteq X$.
(3) F *is measurable (see Definition 2.61(a)).*
(4) *For every* $x \in X$, *the function* $\omega \longmapsto \operatorname{dist}(x, F(\omega))$ *is* Σ*-measurable.*
(5) *There exists a sequence* $\{f_n : \Omega \longrightarrow X\}_{n \geqslant 1}$ *of* Σ*-selectors of* F *such that*

$$F(\omega) = \overline{\{f_n(\omega)\}_{n \geqslant 1}} \qquad \forall\, \omega \in \Omega.$$

(6) F *is graph measurable.*

We have the following relations among the above properties:

(a) $(1) \implies (2) \implies (3) \implies (4) \implies (6)$.
(b) *If* X *is complete (that is,* X *is a Polish space), then* $(3) \iff (4) \iff (5)$.
(c) *If* X *is* σ*-compact, then* $(2) \iff (3)$.
(d) *If* Σ *is complete (that is,* $\Sigma = \widehat{\Sigma}$ *(*$\widehat{\Sigma}$ *being the universal* σ*-field)) and* X *is a Polish space, then all properties (1) to (6) are equivalent.*

In Theorem 2.64 (see also Theorem 2.66), we require that F has closed values. We have a second measurable selection theorem for multifunctions which need not to be closed valued.

Theorem 2.68 *(Yankov–von Neumann–Aumann Selection Theorem)*
If (Ω, Σ) *is a complete measurable space (that is,* $\Sigma = \widehat{\Sigma}$ *(*$\widehat{\Sigma}$ *being the universal* σ*-field)),* X *is a Souslin space and* $F : \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ *is a graph measurable multifunction (see Definition 2.61),*
then F *admits a* Σ*-measurable selection.*

Remark 2.69
Recall that a Hausdorff topological space X is said to be ***Souslin***, if there is a Polish space Y and a continuous surjection from Y onto X. For locally compact spaces, the notions of Souslin space and Polish space coincide. A separable Banach space with the weak topology is a Souslin nonmetrizable space. Similarly, if X^* is the topological dual of a separable Banach space and we furnish X^* with the w^*-topology (denoted by $X^*_{w^*}$), then $X^*_{w^*}$ is a Souslin nonmetrizable space. In fact the above theorem can be strengthened by saying that there is a whole sequence $\{f_n\}_{n \geqslant 1}$ of Σ-measurable selections of F such that $F(\omega) \subseteq \overline{\{f_n(\omega)\}_{n \geqslant 1}}$ for all $\omega \in \Omega$.

The notion of decomposability which we will introduce next formally resembles that of convexity and is important in the theory of measurable multifunctions.

Definition 2.70
Let (Ω, Σ, μ) be a σ-finite measure space, X a separable Banach space, $L^0(\Omega; X) = \{u \colon \Omega \longrightarrow X : \ u \text{ is } \Sigma\text{-measurable}\}$ and $D \subseteq L^0(\Omega; X)$. We say that the set D is ***decomposable****, if for every triple $(A, x_1, x_2) \in \Sigma \times D \times D$, we have*

$$\chi_A x_1 + \chi_{A^c} x_2 \ \in \ D.$$

Remark 2.71
Since $\chi_{A^c} = 1 - \chi_A$, we see that the notion of decomposability reminds us that of convexity. Only now the coefficients instead of scalars are functions.

Decomposable sets are closely related to measurable multifunctions.

Theorem 2.72
If (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space and the set $D \subseteq L^p(\Omega; X)$ (with $1 \leqslant p < +\infty$) is closed,
then the set D is decomposable if and only if there exists a unique measurable multifunction $F \colon \Omega \longrightarrow P_f(X)$ such that

$$D \ = \ S_F^p \ = \ \{u \in L^p(\Omega; X) : \ u(\omega) \in F(\omega) \ \mu\text{-a.e.}\}.$$

Theorem 2.73
If (Ω, Σ, μ) is a finite nonatomic measure space (see Definition I.3.40), X is a Banach space and $D \subseteq L^p(\Omega; X)$ (with $1 \leqslant p < +\infty$) is a nonempty, decomposable and w-closed set,
then D is convex.

The next theorem is another remarkable consequence of decomposability and is a basic tool in variational analysis.

Theorem 2.74
If (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space, $h \colon \Omega \times X \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is measurable, $F \colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ is graph measurable, $\xi(u) = \int\limits_\Omega h(\omega, u(\omega))\, d\mu$ for all $u \in L^p(\Omega; X)$ (with

$1 \leqslant p \leqslant +\infty$) and it is defined (maybe $+\infty$) for all $u \in S_F^p$ and there exists $u_0 \in S_F^p$ such that $\xi(u_0) > -\infty$, <u>then</u>

$$\sup_{u \in S_F^p} \xi(u) \;=\; \int\limits_{\Omega} \sup_{x \in F(\omega)} h(\omega, x)\, d\mu.$$

Definition 2.75

*Let (Ω, Σ, μ) be a σ-finite measure space, X a separable Banach space, and $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ a multifunction. We say that F is L^p-**integrably bounded** if $1 < p \leqslant +\infty$ or simply **integrably bounded** if $p = 1$, when there exists $h \in L^p(\Omega)$ such that*

$$|F(\omega)| \;=\; \sup_{x \in F(\omega)} \|x\|_X \;\leqslant\; h(\omega) \quad \textit{for } \mu\textit{-a.a. } \omega \in \Omega.$$

Theorem 2.76

<u>If</u> (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space and $F\colon \Omega \longrightarrow P_f(X)$ is a graph measurable and integrably bounded multifunction,
<u>then</u> the set $S_F^1 = \big\{u \in L^1(\Omega; X) :\ u(\omega) \in F(\omega) \text{ for } \mu\text{-a.a. } \omega \in \Omega\big\}$ is nonempty, w-compact, and convex if and only if

$$F(\omega) \;\in\; P_{wkc}(X) \quad \textit{for } \mu\textit{-a.a. } \omega \in \Omega.$$

Decomposability leads to a continuous selection theorem for nonconvex valued multifunctions (compare with Theorem 2.58).

Theorem 2.77

<u>If</u> (Ω, Σ, μ) is a σ-finite measure space, V is a separable metric space, X is a separable Banach space, and $F\colon V \longrightarrow P_f\big(L^1(\Omega; X)\big)$ is lower semicontinuous and has decomposable values,
<u>then</u> F admits a continuous selection.

Let $T = [0, b]$ (a bounded closed interval in $\mathbb{R}$) and let X be a separable Banach space. We introduce the following weaker norm on the Lebesgue–Bochner space $L^1(T; X)$.

Definition 2.78

The weak norm $\|\cdot\|_w$ on $L^1(T; X)$ is defined by

$$\|u\|_w \;=\; \sup_{0 \leqslant t \leqslant t' \leqslant b} \Big\| \int\limits_t^{t'} u(s)\, ds \Big\|_X \qquad \forall\, u \in L^1(T; X).$$

Remark 2.79
An equivalent way to define the weak norm is to set

$$\|u\|_w = \sup_{0 \leqslant t \leqslant b} \Big\| \int_0^t u(s)\, ds \Big\|_X \qquad \forall\, u \in L^1(T;X)$$

We want to compare the weak topology on $L^1(T;X)$ with the weak norm topology. To do this, we introduce the following notion.

Definition 2.80
We say that the set $E \subseteq L^1(T;X)$ has ***property U****, if*
(a) *E is uniformly integrable;*
(b) *for every $\varepsilon > 0$, we can find $K_\varepsilon \in P_k(X)$ such that for every $u \in E$ there exists a measurable set $T_{u,\varepsilon} \subseteq T$ with $\lambda(T \setminus T_{u,\varepsilon}) < \varepsilon$ (λ being the Lebesgue measure on $\mathbb{R}$) such that $u(t) \in K_\varepsilon$ for all $t \in T_{u,\varepsilon}$.*

Remark 2.81
If $E \subseteq L^1(T;X)$ has property U, then E is relatively w-compact in $L^1(T;X)$.

Proposition 2.82
If $E \subseteq L^1(T;X)$ has property U,
then the w-topology and $\|\cdot\|_w$-topology coincide on E.

Definition 2.83
*Let Y be a Polish space and let $F\colon T \times Y \longrightarrow P_{wkc}(X)$ be a multifunction. We say that F is an h-**Carathéodory multifunction**, if*
(a) *for every $y \in Y$, the multifunction $t \longmapsto F(t,y)$ is measurable;*
(b) *for almost all $t \in T$, the multifunction $y \longmapsto F(t,y)$ is h-continuous;*
(c) *for every $C \in P_k(Y)$, we can find $\varphi_C \in L^1(T)$ such that*

$$|F(t,y)| = \sup_{x \in F(t,y)} \|x\|_X \leqslant \varphi_C(t) \quad \textit{for a.a. } t \in T, \textit{ all } y \in C.$$

Given a nonempty compact set $K \subseteq C(T;Y)$, let $\Gamma\colon K \longrightarrow P_{wkc}\big(L^1(T;X)\big)$ be defined by

$$\Gamma(y) = S^1_{F(\cdot,y(\cdot))} \qquad \forall\, y \in K.$$

Let CS^w_Γ (respectively $CS^w_{\operatorname{ext}\Gamma}$) denote the set of selectors of Γ (respectively of $\operatorname{ext}\Gamma$) which are continuous from K into $(L^1(T;X), \|\cdot\|_w)$ (usually denoted by $L^1_w(T;X)$).

Theorem 2.84
If $F\colon T\times Y \longrightarrow P_{wkc}(X)$ is an h-continuous Carathéodory multifunction and $K \subseteq C(T;Y)$ is a compact set, then

$$CS^w_\Gamma \;=\; \overline{CS^w_{\operatorname{ext}\Gamma}}^{\|\cdot\|_w}.$$

Definition 2.85
*Let (Ω,Σ,μ) be a σ-finite measure space, X a separable Banach space and $F\colon \Omega \longrightarrow 2^X\setminus\{\emptyset\}$ a multifunction with $S^1_F \neq \emptyset$. The **set-valued integral** of F (or **Aumann integral** of F) is defined by*

$$\int_\Omega F(\omega)\,d\mu \;=\; \Big\{\int_\Omega u\,d\mu:\; u\in S^1_F\Big\}.$$

Next we introduce some convergence notions for sets and multifunctions, which arise often in variational analysis.

Definition 2.86
*Let (X,d_X) be a metric space, $A\in P_f(X)$ and $\{A_n\}_{n\geqslant 1}\subseteq P_f(X)$ a sequence of sets. We say that the sequence $\{A_n\}_{n\geqslant 1}$ converges to A in the **Hausdorff sense**, denoted by $A_n \xrightarrow{h} A$ or $h\text{-}\lim\limits_{n\to+\infty} A_n = A$ if $h(A_n,A)\longrightarrow 0$, with h being the Hausdorff metric on $P_f(X)$ (see Definition 2.49(c)).*

Remark 2.87
This mode of set-convergence is actually convergence in the metric space $(P_f(X),h)$.

Definition 2.88
Let X be a Hausdorff topological space with τ denoting the topology of X. Let $\{A_n\}_{n\geqslant 1}\subseteq 2^X\setminus\{\emptyset\}$. We define

$$\tau\text{-}\liminf_{n\to+\infty} A_n \;=\; \big\{x\in X:\; x=\tau\text{-}\lim_{n\to+\infty} x_n,\; x_n\in A_n,\; n\geqslant 1\big\},$$

$$\tau\text{-}\limsup_{n\to+\infty} A_n \;=\; \big\{x\in X:\; x=\tau\text{-}\lim_{k\to+\infty} x_{n_k},\; x_{n_k}\in A_{n_k},\; n_1<n_2<\ldots\big\}.$$

We call $\tau\text{–}\liminf\limits_{n\to+\infty} A_n$ *the* τ***-Kuratowski limit inferior*** *of the sequence* $\{A_n\}_{n\geqslant 1}$ *and* $\tau\text{–}\limsup\limits_{n\to+\infty} A_n$ *the* τ***-Kuratowski limit superior*** *of the sequence* $\{A_n\}_{n\geqslant 1}$*. If* $A = \tau\text{–}\liminf\limits_{n\to+\infty} A_n = \tau\text{–}\limsup\limits_{n\to+\infty} A_n$*, then we say that* A *is the* τ***-Kuratowski limit*** *of the sequence* $\{A_n\}_{n\geqslant 1}$ *and we write* $A = \tau\text{–}\lim\limits_{n\to+\infty} A_n$ *or* $A_n \xrightarrow{K_\tau} A$*.*

Remark 2.89

Evidently $\tau\text{–}\liminf\limits_{n\to+\infty} A_n \subseteq \tau\text{–}\limsup\limits_{n\to+\infty} A_n$ and the inclusion can be strict. If X is a metric space with metric d_X, then we drop the letter τ and have

$$\begin{aligned}\liminf_{n\to+\infty} A_n &= \Big\{x \in X : \lim_{n\to+\infty} \operatorname{dist}(x, A_n) = 0\Big\},\\ \limsup_{n\to+\infty} A_n &= \Big\{x \in X : \liminf_{n\to+\infty} \operatorname{dist}(x, A_n) = 0\Big\}.\end{aligned}$$

Moreover, if X is a first countable space (see Definition I.2.24), then

$$\tau\text{–}\limsup_{n\to+\infty} A_n = \bigcap_{k\geqslant 1} \overline{\bigcup_{n\geqslant k} A_n}.$$

Note that in general if the topology τ is clear from the context, for notational simplicity we drop the letter τ. This mode of convergence is useful in the context of locally compact spaces, but exhibits serious problems if X is an infinite dimensional Banach space. For this reason in the next definition, we mix the topologies.

Definition 2.90

Let X *be a Banach space. By* w *(respectively* s*) we denote the weak (respectively strong) topology on* X*. Let* $\{A_n\}_{n\geqslant 1} \subseteq 2^X \setminus \{\emptyset\}$ *be a sequence. We say that the sequence* $\{A_n\}_{n\geqslant 1}$ *converges to* A *in the* ***Mosco sense****, denoted by* $A_n \xrightarrow{M} A$*, if we have*

$$A = s\text{–}\liminf_{n\to+\infty} A_n = w\text{–}\limsup_{n\to+\infty} A_n.$$

Remark 2.91

Evidently we have that $A_n \xrightarrow{M} A$ if and only if $A_n \xrightarrow{K_w} A$ and $A_n \xrightarrow{K_s} A$.

This mode of set convergence is more effective in the context of reflexive Banach spaces.

Definition 2.92
*Let (X, d_X) be a metric space, $A \in P_f(X)$ and let $\{A_n\}_{n\geqslant 1} \subseteq P_f(X)$ be a sequence. We say that the sequence $\{A_n\}_{n\geqslant 1}$ converges to A in the **Wijsman sense**, denoted by $A_n \xrightarrow{W} A$, if we have*

$$\operatorname{dist}(x, A_n) \longrightarrow \operatorname{dist}(x, A) \qquad \forall\, x \in X.$$

Remark 2.93
From Proposition 2.51(a), it is clear that, if $A_n \xrightarrow{h} A$, then $A_n \xrightarrow{W} A$.

Definition 2.94
*Let X be a Banach space, $A \in P_{fc}(X)$ and let $\{A_n\}_{n\geqslant 1} \subseteq P_{fc}(X)$ be a sequence. We say that the sequence $\{A_n\}_{n\geqslant 1}$ **converges weakly** (or **scalarly**) to A, denoted by $A_n \xrightarrow{w} A$, if for all $x^* \in X^*$, we have*

$$\sigma_{A_n}(x^*) \longrightarrow \sigma_A(x^*)$$

(see Definition 2.46).

Remark 2.95
From the Hörmander formula (see Proposition 2.51(b)), we see that for a sequence $\{A_n\}_{n\geqslant 1} \subseteq P_{fc}(X)$, if $A_n \xrightarrow{h} A$, then $A_n \xrightarrow{w} A$.

Recall that a sequence in $L^1(\Omega)$, which converges weakly but not strongly, oscillates wildly around its weak limit (see Problem 1.1). For vector-valued functions, we have an extremality condition which prevents such a behavior.

Proposition 2.96
<u>If</u> (Ω, Σ, μ) is a finite measure space and $\{u_n\}_{n\geqslant 1} \subseteq L^1(\Omega; \mathbb{R}^N)$ is a sequence such that $u_n \xrightarrow{w} u$ in $L^1(\Omega; \mathbb{R}^N)$ and $u(\omega) \in \operatorname{ext}\left\{\overline{\operatorname{conv}}\ \limsup\limits_{n\to+\infty} \{u_n(\omega)\}\right\}$ for almost all $\omega \in \Omega$,
<u>then</u> $u_n \longrightarrow u$ in $L^1(\Omega; \mathbb{R}^N)$.

The next proposition elaborates further the above result.

Proposition 2.97
<u>If</u> (Ω, Σ, μ) is a finite measure space, $u \in L^1(\Omega; \mathbb{R}^N)$, $\{u_n\}_{n \geqslant 1} \subseteq L^1(\Omega; \mathbb{R}^N)$ is a sequence such that $u_n \xrightarrow{w} u$ in $L^1(\Omega; \mathbb{R}^N)$ and there exists a measurable multifunction $F\colon \Omega \longrightarrow P_{fc}(\mathbb{R}^N)$ such that:
(i) $\operatorname{dist}(u_n(\omega), F(\omega)) \longrightarrow 0$ *for μ-almost all $\omega \in \Omega$;*
(ii) $u(\omega) \in \operatorname{ext} F(\omega)$ *for μ-almost all $\omega \in \Omega$,*
<u>then</u> $u_n \longrightarrow u$ in $L^1(\Omega; \mathbb{R}^N)$.

2.1.3 Maximal Monotone Maps and Generalizations

A function $u\colon \mathbb{R} \longrightarrow \mathbb{R}$ is said to be nondecreasing if

$$x \leqslant y \implies u(x) \leqslant u(y).$$

Unfortunately this definition involves the order of $\mathbb{R}$ and so it cannot be extended to maps between Banach spaces. However, we can define alternatively the monotonicity of u, by saying that

$$(x-y)\cdot(u(x)-u(x)) \geqslant 0 \qquad \forall\, x, y \in \mathbb{R}.$$

This definition avoids the order structure of $\mathbb{R}$ and can be easily extended to maps from a Banach space X into its dual X^* by replacing the product with the duality brackets for the pair (X^*, X).

So, let X be a Banach space, X^* its topological dual and let $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair (X^*, X). Consider the map $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$. Here

$$D(A) = \big\{x \in X : A(x) \neq \emptyset\big\} \quad \text{(the } \textbf{\textit{domain}} \text{ of } A).$$

Also,

$$\operatorname{Gr} A = \big\{(x, x^*) \in X \times X^* : x^* \in A(x)\big\} \quad \text{(the } \textbf{\textit{graph}} \text{ of multifunction } A).$$

Based on the initial remark, we make the following definitions.

Definition 2.98
Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ be a multivalued map.
(a) *We say that A is **monotone**, if*

$$\langle u^* - v^*, u - v \rangle \geqslant 0 \qquad \forall\, (u, u^*), (v, v^*) \in \operatorname{Gr} A.$$

(b) *We say that A is **strictly monotone**, if*

$$\langle u^* - v^*, u - v\rangle \;>\; 0 \qquad \forall\ (u,u^*),(v,v^*) \in \operatorname{Gr} A,\ u \neq v.$$

(c) *We say that A is **strongly monotone**, if*

$$\langle u^* - v^*, u - v\rangle \;\geqslant\; c\|u-v\|_X^2 \qquad \forall\ (u,u^*),(v,v^*) \in \operatorname{Gr} A,$$

with some $c > 0$.

(d) *We say that A is **uniformly monotone**, if there exists a continuous function $\vartheta\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which is strictly increasing, $\vartheta(0) = 0$, $\vartheta(t) \longrightarrow +\infty$ as $t \to +\infty$ and*

$$\langle u^*-v^*, u-v\rangle \;\geqslant\; \vartheta(\|u-v\|_X)\|u-v\|_X \quad \forall\ (u,u^*),(v,v^*) \in \operatorname{Gr} A.$$

(e) *We say that A is **coercive**, if $D(A)$ is bounded or $D(A)$ is unbounded and*

$$\inf_{u^* \in A(u)} \|u^*\|_* \;\longrightarrow\; +\infty \quad \text{as } \|u\|_X \to +\infty,\ u \in D(A).$$

(f) *We say that A is **strongly coercive**, if $D(A)$ is bounded or $D(A)$ is unbounded and*

$$\frac{\inf\limits_{(u,u^*) \in \operatorname{Gr} A} \langle u^*, u\rangle}{\|u\|_X} \;\longrightarrow\; +\infty \quad \text{as } \|u\|_X \to +\infty,\ u \in D(A).$$

Remark 2.99

From the above definition it is clear that we have

$$\begin{gathered}\text{strong monotonicity}\\ \Downarrow\\ \text{uniform monotonicity}\\ \Downarrow\\ \text{strict monotonicity}\\ \Downarrow\\ \text{monotonicity}\end{gathered}$$

and also

$$\begin{gathered}\text{uniform monotonicity}\\ \Downarrow\\ \text{strong coercivity}\\ \Downarrow\\ \text{coercivity.}\end{gathered}$$

The next definition identifies a subclass of monotone maps which is central in the theory and in applications.

Definition 2.100
Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ be a monotone map. We say that A is ***maximal monotone****, if*

$$\big[\,\langle u^* - v^*, u - v\rangle \geqslant 0 \qquad \forall\ (u, u^*) \in \operatorname{Gr} A\big] \implies (v, v^*) \in \operatorname{Gr} A.$$

Remark 2.101
The above definition says that A is maximal monotone if and only if $\operatorname{Gr} A$ is maximal with respect to inclusion among the graphs of monotone maps.

Definition 2.102
Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ be a map. We say that the map A is ***locally bounded*** *at $u \in D(A)$, if there exist $\varrho > 0$ and $M > 0$ such that*

$$\|v^*\|_* \ \leqslant\ M \qquad \forall\ v^* \in A(v),\ v \in D(A) \cap \overline{B}_\varrho(u).$$

We say that A is ***bounded****, if it maps bounded sets in $X \cap D(A)$ to bounded sets in X^*. Also, if $C \subseteq X$ and $u \in C$, then we say that u is an* ***absorbing point*** *of C, if*

$$X \ =\ \bigcup_{\lambda>0} \lambda(C - u).$$

Remark 2.103
If $\operatorname{int} C \neq \emptyset$, then every interior point is absorbing. However, the set C can have absorbing points even if $\operatorname{int} C = \emptyset$.

Proposition 2.104
<u>If</u> X is a Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a monotone map, and $u \in D(A)$ is an absorbing point of $D(A)$,
<u>then</u> A is locally bounded at u.

Proposition 2.105
<u>If</u> X is a Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a maximal monotone map and $\operatorname{int} D(A) \neq \emptyset$,
<u>then</u> $A|_{\operatorname{int} D(A)}$ is upper semicontinuous from X with the norm topology into X^* with the w^*-topology.

Proposition 2.106
If X is a Banach space and $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a maximal monotone map,
then for every $u \in D(A)$, the set $A(u) \subseteq X^*$ is nonempty, convex, and w^*-closed.

Combining Propositions 2.104 and 2.106, we obtain the following result.

Corollary 2.107
If X is a reflexive Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a maximal monotone map, and $u \in D(A)$ is absorbing,
then $A(u)$ is nonempty, convex, and w-compact.
In particular if $D(A) = X$, then A has values in $P_{wkc}(X^*)$.

The next result is a partial converse of Proposition 2.105.

Proposition 2.108
If X is a Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a monotone map with $D(A) = X$, has w^*-closed and convex values and for every $u, h \in X$, the multifunction $t \longmapsto A(u + th)$ is upper semicontinuous from $[0, 1]$ into X^* with the w^*-topology,
then A is maximal monotone.

Remark 2.109
If A is upper semicontinuous from X with the norm topology into X^* with the w^*-topology, then the previous proposition holds.

Definition 2.110
Let $A\colon X \longrightarrow X^*$ be a single valued map with $D(A) = X$.
- ***(a)*** *We say that A is **demicontinuous,** if for any sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ with $x_n \longrightarrow x$, we have $A(x_n) \xrightarrow{w^*} A(x)$.*
- ***(b)*** *We say that A is **hemicontinuous,** if for any $x, h \in X$, the map $t \longmapsto A(x + th)$ is continuous from $[0, 1]$ into X^* with the w^*-topology.*

As a consequence of Proposition 2.108, we have

Corollary 2.111
If X is a Banach space and $A\colon X \longrightarrow X^*$ is a monotone and hemicontinuous map,
then A is maximal monotone.

Definition 2.112
Let X be a Banach space. The map $\mathcal{F}\colon X \longrightarrow 2^{X^*}$ defined by

$$\mathcal{F}(x) \;=\; \left\{x^* \in X^* : \; \langle x^*, x\rangle = \|x\|_X^2 = \|x^*\|_*^2\right\}$$

*is called the **(normalized) duality map** on X.*

Remark 2.113
As a consequence of the Hahn–Banach theorem (see Corollary I.5.26), we have that $D(\mathcal{F}) = X$. Moreover, $\mathcal{F}$ has closed and convex values.

The duality map is essentially dependent on the norm of X. In fact, two equivalent norms on X need not generate the same duality map. The next proposition shows that the geometry of X and X^* is crucial concerning the properties of $\mathcal{F}$.

Proposition 2.114

(a) *If X is a reflexive Banach space and X^* is strictly convex (see Definition I.5.168 and Remark I.5.169),*
then $\mathcal{F}$ is single valued, bounded, odd, demicontinuous, and maximal monotone.

(b) *If X is a reflexive Banach space and both X and X^* are strictly convex,*
then $\mathcal{F}$ is strictly monotone.

(c) *If X is a reflexive Banach space and X^* is locally uniformly convex,*
then $\mathcal{F}$ is continuous.

(d) *If X^* is uniformly convex,*
then $\mathcal{F}$ is uniformly continuous on bounded sets.

Remark 2.115
The Troyanski renorming theorem is a basic tool in the theory of maximal monotone maps (see Theorem I.5.192). We recall that this theorem says that every reflexive Banach space X be can equivalently renormed so that both X and X^* are locally uniformly convex and have Fréchet differentiable norms. Note that equivalently renorming X and X^* does not affect important properties of the map A, such as maximal monotonicity and coercivity (simple or strong).

Using the duality map, we can have a criterion for the maximality of a map $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$.

Theorem 2.116
If X *is a reflexive Banach space with* X *and* X^* *both strictly convex (see Definition I.5.168 and Remark I.5.169) and* $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ *is monotone,*
then A *is maximal monotone if and only if* $R(A + \mathcal{F}) = X^*$.

Maximal monotone operators are very useful because they exhibit remarkable surjectivity properties.

Definition 2.117
Let X *be a Banach space and let* $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ *be a multivalued map. We define* $A^{-1}\colon X^* \supseteq D^* \longrightarrow 2^X$ *by setting*

$$\operatorname{Gr} A^{-1} = \big\{(x^*, x) \in X^* \times X : \ (x, x^*) \in \operatorname{Gr} A\big\}.$$

Using this notion, we have the following criterion for the surjectivity of a maximal monotone map.

Theorem 2.118
If X *is a reflexive map and* $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ *is maximal monotone,*
then A *is surjective if and only if* A^{-1} *is locally bounded.*

Evidently, if A is coercive (see Definition 2.98(e)), then A^{-1} is locally bounded. So, from Theorem 2.118 we deduce the following fundamental surjectivity result for maximal monotone maps.

Theorem 2.119
If X *is a reflexive Banach space and* $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ *is maximal monotone and coercive,*
then A *is surjective.*

Corollary 2.120
If X *is a reflexive Banach space and* $A\colon X \supseteq D(A) \longrightarrow X^*$ *is monotone, hemicontinuous and coercive,*
then A *is surjective.*

In finite dimensional spaces, we can drop the monotonicity requirement.

Proposition 2.121
If X *is a finite dimensional Banach space and* $A\colon X \supseteq D(A) \longrightarrow P_{kc}(X^*)$ *is upper semicontinuous and strongly coercive (see Definition 2.98(f)),*
then A *is surjective.*

The surjectivity results lead to some useful single-valued approximations of a maximal monotone map. These approximations are more useful in the context of pivot Hilbert spaces (a Hilbert space is ***pivot***, if it is identified with its dual).

Definition 2.122
Let H *be a pivot Hilbert space (i.e.,* $H^* = H$*) and let* $A\colon H \supseteq D(A) \longrightarrow 2^H$ *be a maximal monotone map. For every* $\lambda > 0$*, we define*

$$\begin{aligned} J_\lambda^A &= (I_H + \lambda A)^{-1} \quad \textit{(the } \textbf{\textit{resolvent}} \textit{ of } A\textit{)} \\ A_\lambda &= \tfrac{1}{\lambda}(I_H - J_\lambda^A)^{-1} \quad \textit{(the } \textbf{\textit{Yosida approximation}} \textit{ of } A\textit{).} \end{aligned}$$

Proposition 2.123
If H *is a pivot Hilbert space,* $A\colon H \supseteq D(A) \longrightarrow 2^H$ *is a maximal monotone map and* $\lambda > 0$, *then*
(a) $D(J_\lambda^A) = D(A_\lambda) = H$.
(b) J_λ^A *is nonexpansive (i.e., Lipschitz continuous with Lipschitz constant* 1*).*
(c) $A_\lambda(x) \in A(J_\lambda^A(x))$ *for all* $x \in H$.
(d) A_λ *is monotone and Lipschitz continuous with Lipschitz constant* $\frac{1}{\lambda}$.
(e) $\|A_\lambda(x)\|_H \leqslant \|A^0(x)\|_H$ *for all* $x \in D(A)$*, where* $A^0(x) \in A(x)$ *is the unique element of* $A(x)$ *with smallest norm.*
(f) $\lim\limits_{\lambda \searrow 0} A_\lambda(x) = A^0(x)$ *for all* $x \in D(A)$.
(g) $\overline{D(A)}$ *is convex and* $\lim\limits_{\lambda \searrow 0} J_\lambda^A(x) = \operatorname{proj}_{\overline{D(A)}}(x)$ *for all* $x \in H$ *(here* $\operatorname{proj}_{\overline{D(A)}}$ *denotes the metric projection on the closed convex set* $\overline{D(A)}$*).*

Remark 2.124
If $x \notin D(A)$, then $\|A_\lambda(x)\|_H \longrightarrow +\infty$ and $\lambda \searrow 0$.

Perturbation results for maximal monotone operators are important in applications. We start with two results in Hilbert spaces.

Theorem 2.125
If H *is a pivot Hilbert space (i.e.,* $H^* = H$*) and* $A\colon H \supseteq D(A) \longrightarrow 2^H$, $B\colon H \supseteq D(B) \longrightarrow 2^H$ *are two maximal monotone maps such that* $D(A) \cap D(B) \neq \emptyset$ *and*

$$0 \leqslant (y, B_\lambda(x))_H \qquad \forall\ (x,y) \in \mathrm{Gr}\, A,\ \lambda > 0,$$

then $A + B$ *is maximal monotone.*

Theorem 2.126
If H *is a pivot Hilbert space (i.e.,* $H^* = H$*) and* $A\colon H \supseteq D(A) \longrightarrow 2^H$, $B\colon H \supseteq D(B) \longrightarrow 2^H$ *are two maximal monotone maps such that*

$$0 \in \operatorname{int}\big(D(A) - D(B)\big),$$

then $A + B$ *is maximal monotone.*

For maps defined on a reflexive Banach space with values in its dual, we have the following result.

Theorem 2.127
If X *is a reflexive Banach space and* $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$, $B\colon X \supseteq D(B) \longrightarrow 2^{X^*}$ *are two maximal monotone maps such that*

$$\operatorname{int} D(A) \cap D(B) \neq \emptyset,$$

then $A + B$ *is maximal monotone.*

Remark 2.128
Since $\operatorname{int} D(A) - D(B) \subseteq \operatorname{int}\big(D(A) - D(B)\big)$, we see that the hypothesis of Theorem 2.126 is weaker than that of Theorem 2.127. Note that $0 \in \operatorname{int}(D(A) - D(B))$ can happen even if $\operatorname{int} D(A) = \operatorname{int} D(B) = \emptyset$.

Next we pass to generalizations of the notion of maximal monotonicity.

Definition 2.129
Let X *be a reflexive Banach space and let* $A\colon X \longrightarrow 2^{X^*}$ *be a multifunction. We say that* A *is* ***pseudomonotone****, if*

(a) $D(A) = X$ *and for every* $x \in X$, $A(x) \in P_{wkc}(X)$*;*

(b) *A is upper semicontinuous from every finite dimensional subspace of X into X^* with the weak topology;*

(c) *if* $x_n \xrightarrow{w} x$ *in* X, $x_n^* \in A(x_n)$ *for all* $n \geqslant 1$ *and* $\limsup\limits_{n\to+\infty} \langle x_n^*, x_n - x\rangle \leqslant 0$, *then for each* $v \in X$, *we can find* $y^*(v) \in A(x)$ *such that*

$$\langle y^*(v), x - v\rangle \leqslant \liminf_{n\to+\infty} \langle x_n^*, x_n - v\rangle .$$

Remark 2.130

A completely continuous map $A\colon X \longrightarrow X^*$ is pseudomonotone. Also, if X is finite dimensional, then any continuous operator $A\colon X \longrightarrow X^*$ is pseudomonotone.

Definition 2.131

Let X *be a reflexive Banach space and let* $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ *be a multifunction. We say that* A *is* ***generalized pseudomonotone***, *if for every sequences* $\{x_n\}_{n\geqslant 1} \subseteq X$, $\{x_n^*\}_{n\geqslant 1} \subseteq X^*$ *such that* $x_n \xrightarrow{w} x$ *in* X *and* $x_n^* \xrightarrow{w} x^*$ *in* X^* *with* $x_n^* \in A(x_n)$ *for all* $n \geqslant 1$ *and* $\limsup\limits_{n\to+\infty} \langle x_n^*, x_n - x\rangle \leqslant 0$, *we have*

$$x^* \in A(x) \quad \text{and} \quad \langle x_n^*, x_n\rangle \longrightarrow \langle x^*, x\rangle .$$

Proposition 2.132

If X *is a reflexive Banach space and* $A\colon X \longrightarrow 2^{X^*}$ *is a pseudomonotone map,*
then A *is generalized pseudomonotone.*

There is a version of the converse to this proposition.

Proposition 2.133

If X *is a reflexive Banach space and* $A\colon X \longrightarrow P_{fc}(X^*)$ *is a bounded and generalized pseudomonotone map,*
then A *is pseudomonotone.*

Proposition 2.134

If X *is a reflexive Banach space and* $A\colon X \supseteq D(A) \longrightarrow P_{fc}(X^*)$ *is a maximal monotone map with* $D(A) = X$,
then A *is pseudomonotone.*

The property of pseudomonotonicity is preserved by addition.

Proposition 2.135
If X *is a reflexive Banach space and* $A_1, A_2\colon X \longrightarrow 2^{X^*}$ *are two pseudomonotone maps,*
then $A_1 + A_2$ *is pseudomonotone.*

Pseudomonotone maps exhibit remarkable surjectivity properties and for this reason are important in applications.

Theorem 2.136
If X *is a reflexive Banach space and* $A\colon X \longrightarrow 2^{X^*}$ *is a pseudomonotone and strongly coercive map (see Definition 2.98(f)),*
then A *is surjective.*

A notion closely related to pseudomonotonicity and to generalized pseudomonotonicity is that of an $(S)_+$-operator, which is particularly useful in the application of variational methods. In particular it is the right property to check the Palais–Smale condition for a functional $\varphi \in C^1(X)$ (see Definition 5.45(a) in Chap. 5).

Definition 2.137
Let X *be a reflexive Banach space,* $C \subseteq X$ *a nonempty subset and let* $A\colon C \longrightarrow X^*$ *be a map. We say that* A *is an* $(S)_+$***-map****, if for every sequence* $\{u_n\}_{n\geqslant 1} \subseteq C$ *such that* $x_n \xrightarrow{w} x$ *in* X *and* $\limsup\limits_{n\to+\infty} \langle A(x_n), x_n - x\rangle \leqslant 0$*, we have* $x_n \longrightarrow x$ *in* X*.*

The next proposition summarizes some basic properties of $(S)_+$-maps.

Proposition 2.138
Let X *be a reflexive Banach space and let* $C \subseteq X$ *be a nonempty subset.*

(a) *If* $A\colon C \longrightarrow X^*$ *is an* $(S)_+$*-map and* $\lambda > 0$*,*
then λA *is an* $(S)_+$*-map too.*

(b) *If* $A, B\colon C \longrightarrow X^*$ *are two* $(S)_+$*-maps and one of them is also demicontinuous,*
then $A + B$ *is an* $(S)_+$*-map too.*

(c) *If* $A\colon C \longrightarrow X^*$ *is an* $(S)_+$*-map and* $B\colon X \longrightarrow X^*$ *is a monotone map,*
then $A + B\colon C \longrightarrow X^*$ *is an* $(S)_+$*-map.*

(d) *If* $A\colon C \longrightarrow X^*$ *is an* $(S)_+$*-map and* $B\colon X \longrightarrow X^*$ *is a completely continuous map,*
then $A + B\colon C \longrightarrow X^*$ *is an* $(S)_+$*-map.*

2.1.4 Accretive Maps

So far we have considered maps from a Banach space X into its dual X^*. Now, we focus on maps from X into itself and introduce the notion of accretive maps. Such operators are closely related to the generation theory of semigroups of operators (linear and nonlinear alike).

Definition 2.139
Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^X$ be a multivalued map.

(a) *We say that A is **accretive**, if for every $(x,u),(y,v) \in \operatorname{Gr} A$ we can find $x^* \in \mathcal{F}(x-y)$ such that $\langle x^*, u-v\rangle \geqslant 0$ (where $\mathcal{F}\colon X \longrightarrow 2^{X^*}$ is the normalized duality map; see Definition 2.112).*

(b) *We say that A is **maximal accretive**, if its graph is maximal with respect to inclusion among the graphs of all accretive maps.*

(c) *We say that A is m-**accretive**, if $R(I_X + A) = X$.*

(d) *We say that A is **dissipative** (respectively m-**dissipative**), if $-A$ is accretive (respectively m-accretive).*

Remark 2.140
If H is a pivot Hilbert space (i.e., $H^* = H$), then $\mathcal{F} = I_H$ and A is accretive if and only if A is monotone. Moreover, the notions of maximal accretivity and m-accretivity coincide and correspond to maximal monotonicity (see Definition 2.100 and Theorem 2.116).

Proposition 2.141
<u>If</u> X is a Banach space and $A\colon X \supseteq D(A) \longrightarrow 2^X$ is a multivalued map,
<u>then</u> A is an accretive map if and only if

$$\|x-y+\lambda(u-v)\|_X \;\geqslant\; \|x-y\|_X \qquad \forall\, \lambda>0,\ (x,u),(y,v) \in \operatorname{Gr} A.$$

In analogy to Definition 2.112, we introduce the following single-valued approximation of the identity and of the map itself.

Definition 2.142
Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^X$ be an accretive map. For $\lambda > 0$ we define

$$\begin{aligned} J_\lambda(x) &= (I_X + \lambda A)^{-1}(x), \\ A_\lambda(x) &= \tfrac{1}{\lambda}(I_X - J_\lambda)(x), \\ D(A_\lambda) &= D_\lambda \;=\; R(I_X + \lambda A), \\ |A(x)| &= \inf_{u \in A(x)} \|u\|_X. \end{aligned}$$

As an easy consequence of Proposition 2.141, we have

Proposition 2.143
If X is a Banach space and $A\colon X \supseteq D(A) \longrightarrow 2^X$ is a multivalued map,
then A is accretive if and only if for every $\lambda > 0$, J_λ is single-valued and nonexpansive, i.e.,

$$\|J_\lambda(x) - J_\lambda(y)\|_X \leqslant \|x - y\|_X \qquad \forall\, x, y \in D_\lambda.$$

Moreover, for any accretive map $A\colon X \supseteq D(A) \longrightarrow 2^X$, we have
(a) *$A_\lambda\colon R(I_X + \lambda A) \longrightarrow X$ is single-valued, accretive and Lipschitz continuous;*
(b) *$A_\lambda(u) \in A(J_\lambda(u))$ for all $u \in R(I_X + \lambda A)$;*
(c) *$\|A_\lambda(u)\|_X \leqslant \inf\limits_{v \in A(y)} \|v\|_X$ for all $v \in D(A) \cap R(I_X + \lambda A)$;*
(d) *$\lim\limits_{\lambda \searrow 0} J_\lambda(u) = u$ for all $u \in D(A) \cap \big(\bigcap\limits_{\lambda > 0} R(I_X + \lambda A)\big)$.*

Proposition 2.144
If X is a Banach space and $A\colon X \supseteq D(A) \longrightarrow 2^X$ is an m-accretive map,
then A is maximal accretive and $R(I_X + \lambda A) = X$ for all $\lambda > 0$.

Definition 2.145
*Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^X$ be a multivalued map. The **minimal section** of A is the map $A^0\colon X \supseteq D(A^0) \longrightarrow 2^X$ defined by*

$$A^0(x) \;=\; \big\{u \in A(x) : \; \|u\|_X = |A(x)|\big\}.$$

Proposition 2.146
If X is a reflexive and strictly convex Banach space, the dual X^* is strictly convex too (see Definition I.5.168 and Remark I.5.169) and $A\colon X \supseteq D(A) \longrightarrow 2^X$ is maximal accretive,
then $D(A^0) = D(A)$ and A^0 is single-valued.

The importance of A^0 is illustrated by the next theorem.

Theorem 2.147
If X is a Banach space with uniformly convex dual X^* and $A, B\colon X \supseteq D \longrightarrow 2^X$ are two m-accretive maps such that

$$A^0(x) \cap B^0(x) \neq \emptyset \qquad \forall\, x \in D,$$

then $A = B$. Hence, if $A^0 = B^0$, then $A = B$.

The next theorem is analogous to Theorem 2.125.

Theorem 2.148
If X is a Banach space with uniformly convex dual X^* and $A\colon X \supseteq D(A) \longrightarrow 2^X$, $B\colon X \supseteq D(B) \longrightarrow 2^X$ are two m-accretive maps such that:
(i) *$D(A) \cap D(B) \neq \emptyset$,*
(ii) *$\langle \mathcal{F}(B_\lambda(x)), y\rangle \geqslant 0$ for all $(x, y) \in \operatorname{Gr} A$ and $\lambda > 0$,*
then $A + B$ is m-accretive.

As we already mentioned m-accretive maps are related to the generation of contraction semigroups of operators.

Definition 2.149
*Let X be a Banach space and let $\{S(t)\}_{t\geqslant 0} \subseteq \mathcal{L}(X)$. We say that the family $\{S(t)\}_{t\geqslant 0}$ is a C_0**-semigroup**, if the following is true:*
(a) *$S(0) = I_X$,*
(b) *$S(t + \tau) = S(t) \circ S(\tau)$ for all $t, \tau \geqslant 0$ (semigroup property),*
(c) *for every $x \in X$, $S(t)x \longrightarrow x$ as $t \searrow 0$.*
*If $\|S(t)\|_{\mathcal{L}} \leqslant 1$ for all $t \geqslant 0$, then $\{S(t)\}_{t\geqslant 0}$ is called a **contraction semigroup**.*

Definition 2.150
Let X be a Banach space and let $\{S(t)\}_{t\geqslant 0}$ be a C_0-semigroup. The ***infinitesimal generator*** *of the semigroup is a linear operator A defined by*

$$A(x) = \lim_{t\searrow 0} \frac{S(t)x-x}{t} \qquad \forall\, x \in D(A),$$

with $D(A) = \Big\{x \in X : \lim\limits_{t\searrow 0} \frac{S(t)x-x}{t} \text{ exists}\Big\}$.

Proposition 2.151
If X is a Banach space and $\{S(t)\}_{t\geqslant 0}$ is a C_0-semigroup with infinitesimal operator A,
then A is closed and densely defined.

Remark 2.152
If $A \in \mathcal{L}(X)$, then A is the infinitesimal generator of

$$S(t) = e^{At} = \sum_{k\geqslant 0} \frac{(At)^k}{k!}.$$

For an unbounded linear operator A to be the infinitesimal generator of a C_0-semigroup, the precise conditions are provided by the following theorem.

Theorem 2.153 *(Hille–Yosida Theorem)*
Necessary and sufficient conditions for a linear unbounded operator A on a Banach space X to be the infinitesimal generator of a C_0-semigroup satisfying that

$$\|S(t)\|_{\mathcal{L}} \leqslant Me^{\omega t} \qquad \forall\, t \geqslant 0,$$

for some $M \geqslant 1$ and $\omega \geqslant 0$, are
(a) *A is closed and densely defined;*
(b) *$(\lambda I_X - A)^{-1}$ exists for $\lambda > \omega$;*
(c) *$\|(\lambda I_X - A)^{-m}\|_{\mathcal{L}} \leqslant \frac{M}{(\lambda-\omega)^m}$ for $\lambda > \omega$, $m \geqslant 1$.*

Remark 2.154
The family of the operators $R(\lambda) = (\lambda I_X - A)^{-1}$ for $\lambda > \omega$ is called the ***resolvent*** of A.

Proposition 2.155
If X *is a Banach space and* $\{S(t)\}_{t\geqslant 0} \subseteq \mathcal{L}(X)$ *is a contraction semigroup with infinitesimal generator* A,
then for any $x \in X$,

$$S(t)x = \lim_{n\to+\infty} \left(I_X - \tfrac{t}{n}A\right)^{-n}x = \lim_{n\to+\infty} \left(\tfrac{n}{t}R(\tfrac{n}{t})\right)^n x.$$

We can have also nonlinear semigroups.

Definition 2.156
Let X *be a Banach space and let* $C \subseteq X$ *be a nonempty subset. A family of nonlinear maps* $\{S(t)\colon C \longrightarrow C\}_{t\geqslant 0}$ *is said to be a* ***semigroup of nonexpansive maps*** *on* C, *if*
(a) $\|S(t)x - S(t)y\|_X \leqslant \|x-y\|_X$ *for all* $t \geqslant 0$, $x, y \in C$;
(b) $S(t+\tau) = S(t)\circ S(\tau)$ *for all* $t, \tau \geqslant 0$ *(semigroup property),*
(c) $S(0) = I_C$,
(d) $\lim\limits_{t\searrow 0} S(t)x = x$ *for every* $x \in C$.

Remark 2.157
The map $(t,x) \longmapsto S(t)x$ is jointly continuous from $\mathbb{R}_+ \times C$ into C.

The main generation theorem for nonlinear semigroups reads as follows.

Theorem 2.158
If $A\colon X \supseteq D(A) \longrightarrow 2^X$ *is an m-accretive operator,*
then for every $x \in \overline{D(A)}$,

$$S(t)x = \lim_{n\to+\infty} \left(I_X + \tfrac{t}{n}A\right)^{-n}x$$

exists uniformly in t *on compact sets in* $\mathbb{R}_+$, $\{S(t)\}_{t\geqslant 0}$ *is a semigroup of nonexpansive maps and for each* $x \in D(A)$ *and* $t > 0$, *we have*

$$\|S(t)x - x\|_X \leqslant t|A(x)|$$

(see Definition 2.142). Moreover, if X *and* X^* *are both uniformly convex and* $u_0 \in D(A)$, *then*
(a) *the map* $t \longmapsto S(t)u_0$ *is right differentiable on* $\mathbb{R}_+ = [0,+\infty)$ *and we have*

$$\tfrac{d^+}{dt^+}S(t)u_0 = AS(t)u_0 \qquad \forall\ t \geqslant 0;$$

(b) *the map* $t \longmapsto \|AS(t)u_0\|_X$ *is nonincreasing on* $\mathbb{R}_+$ *and the map* $t \longmapsto AS(t)u_0$ *is right continuous at every* $t \geqslant 0$*;*
(c) $\frac{d}{dt}S(t)u_0 = AS(t)u_0$ *exists and is continuous on* $\mathbb{R}_+$ *except for a countable set of point.*

Definition 2.159
Let X *be a Banach space.*
(a) *A nonlinear semigroup* $\{S(t): C \longrightarrow C\}_{t \geqslant 0}$ *of nonexpansive maps on* C *is said to be* ***compact****, if for every* $t > 0$*,* $S(t)$ *is compact.*
(b) *A nonlinear semigroup* $\{S(t): C \longrightarrow C\}_{t \geqslant 0}$ *of nonexpansive maps on* C *is said to be* ***equicontinuous****, if for every bounded set* $B \subseteq X$*, the family* $\big\{S(\cdot)x : \ x \in B\big\}$ *is equicontinuous at every* $t > 0$*.*

Remark 2.160
Recall that $S(0) = I_C$. So, if in Definition 2.159(a), $S(t)$ is compact for all $t \geqslant 0$, then the set C imbeds compactly in X. In particular, if $C = X$, then X is finite dimensional.

Proposition 2.161
<u>If</u> X *is a Banach space and* $A: X \supseteq D(A) \longrightarrow 2^X$ *is accretive,*
<u>then</u> for all $x \in X$ *and all* $\lambda, \mu > 0$*, we have*

$$J_\lambda(x) \;=\; J_\mu\big(\tfrac{\mu}{\lambda}x + \tfrac{\lambda-\mu}{\lambda}J_\lambda(x)\big)$$

(resolvent identity).

Remark 2.162
Using the resolvent identity we see that J_λ is compact for all $\lambda > 0$ if and only if it is compact for some $\lambda > 0$.

2.1.5 Miscellaneous Results

Theorem 2.163
<u>If</u> X *is a Banach space and* $C \subseteq X$*,*
<u>then</u> C *is relatively compact if and only if there is a sequence* $\{u_n\}_{n \geqslant 1} \subseteq X$ *converging to the origin such that* $C \subseteq \overline{\operatorname{conv}}\,\{u_n\}_{n \geqslant 1}$*.*

Theorem 2.164
A set $K \subseteq c_0$ *is relatively compact if and only if it is bounded and the limit* $\lim\limits_{n \to +\infty} u_n$ *exists uniformly for* $\{u_n\}_{n \geqslant 1} \in K$*.*

Theorem 2.165 *(Lyapunov Convexity Theorem)*

(a) *If* (Ω, Σ) *is a measurable space,* X *is a finite dimensional Banach space and* $m\colon \Sigma \longrightarrow X$ *is a vector measure which is nonatomic,* *then the set* $m(\Sigma) \subseteq X$ *is compact and convex.*

(b) *If* (Ω, Σ) *is a measurable space,* X *is a Banach space with the RNP and* $m\colon \Sigma \longrightarrow X$ *is a vector measure which is nonatomic and of bounded variation,* *then the set* $\overline{m(\Sigma)} \subseteq X$ *is compact and convex.*

The Lusin theorem (see Theorem I.3.77) says that an $\mathbb{R}$-valued Borel function is actually continuous outside a "small" set. Below we provide an abstract version of this result for maps with values in a separable metric space.

Theorem 2.166 *(Lusin Theorem II)*

If X *is a Polish space,* Y *is a separable metric space,* $f\colon X \longrightarrow Y$ *is a Borel measurable function and* μ *is a finite Borel measure on* X*,* *then for every* $\varepsilon > 0$*, we can find a compact set* $K_\varepsilon \subseteq X$ *such that* $\mu(X \setminus K_\varepsilon) < \varepsilon$ *and* $f|_{K_\varepsilon}$ *is continuous.*

Remark 2.167

Recall that every finite Borel measure on a Polish space is Radon (see Theorem I.4.12).

Proposition 2.168

If (Ω, Σ, μ) *is a finite measure space and* $\mathcal{D}$ *is a family of* Σ*-measurable* $\mathbb{R}_+$*-valued functions,*
then there exists a unique (modulo μ*-a.e. equality)* Σ*-measurable function* $h\colon \Omega \longrightarrow \mathbb{R}_+$ *such that:*

(a) $f(\omega) \leqslant h(\omega)$ *for* μ*-almost all* $\omega \in \Omega$ *and all* $f \in \mathcal{D}$*;*

(b) *if* $g\colon \Omega \longrightarrow \mathbb{R}_+$ *is* Σ*-measurable and* $f(\omega) \leqslant g(\omega)$ *for* μ*-almost all* $\omega \in \Omega$ *and* $f \in \mathcal{D}$*, then* $h(\omega) \leqslant g(\omega)$ *for* μ*-almost all* $\omega \in \Omega$*.*

Moreover, there exists a sequence $\{f_n\}_{n \geqslant 1} \subseteq \mathcal{D}$ *such that*

$$h(\omega) \;=\; \sup_{n \geqslant 1} f_n(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Finally, if $\mathcal{D}$ *is upward directed (that is, if* $f_1, f_2 \in \mathcal{D}$*, then there exists* $\widehat{f} \in \mathcal{D}$ *such that* $f_1 \leqslant \widehat{f}$ *and* $f_2 \leqslant \widehat{f}$*), then the sequence* $\{f_n\}_{n \geqslant 1}$ *can be chosen to be increasing.*

2.2 Problems

Problem 2.1*
Let X and Y be two Banach spaces and let $A\colon X \longrightarrow Y$ be a linear map such that for each bounded sequence $\{u_n\}_{n\geqslant 1} \subseteq X$, the sequence $\{A(u_n)\}_{n\geqslant 1} \subseteq Y$ admits a strongly convergent subsequence. Show that $A \in \mathcal{L}_c(X;Y)$.

Problem 2.2*
Suppose that X and Y are two Banach spaces, $D \subseteq X$ is a nonempty subset and $\{f_\alpha\colon D \longrightarrow Y\}_{\alpha\in J}$ is a net of compact maps (i.e., $\{f_\alpha\}_{\alpha\in J} \subseteq K(D;Y)$) such that $f_\alpha \longrightarrow f$ uniformly on bounded subsets of D. Show that $f \in K(D;Y)$.

Problem 2.3*
Suppose that X is a Banach space, $D \subseteq X$ is a nonempty bounded and closed subset, and $f\colon D \longrightarrow X$ is a compact map. Suppose that given $\varepsilon > 0$, we can find $u_\varepsilon \in D$ such that $\|u_\varepsilon - f(u_\varepsilon)\|_X < \varepsilon$. Show that f has a fixed point in D, i.e., there exists $v \in D$ such that $v = f(v)$.

Problem 2.4**
Let X and Y be two Banach spaces and let $A \in \mathcal{L}_c(X;Y)$. Show that:
(a) the subspace $A(X) \subseteq Y$ is separable;
(b) if $A(X)$ is of second category in itself, then $A \in \mathcal{L}_f(X;Y)$.

Problem 2.5**
Let X and Y be two metric spaces and let $f \in C(X;Y)$. Show that the following two conditions are equivalent:
(a) Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $f(u_n) \longrightarrow y \in Y$ admits a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ such that $u_{n_k} \longrightarrow u \in X$.
(b) f is a closed map and for every $y \in Y$, the set $f^{-1}(y) \subseteq Y$ is compact.

Problem 2.6**
Let X and Y be two metric spaces and suppose that $f \in C(X;Y)$ satisfies one of the equivalent properties **(a)** or **(b)** of Problem 2.5. Show that f is proper.

Problem 2.7**
Let X and Y be two metric spaces with X being compact. Show that every function $f \in C(X;Y)$ is closed and proper.

Problem 2.8**
Let $\{\vartheta_n\}_{n\geqslant 1} \subseteq \mathbb{R}$ be a sequence such that $\vartheta_n \longrightarrow 0$ and let $K\colon l^2 \longrightarrow l^2$ be the linear operator defined by

$$K(u) \;=\; \{\vartheta_n x_n\}_{n\geqslant 1} \qquad \forall\, u = \{x_n\}_{n\geqslant 1} \in l^2.$$

Show that $K \in \mathcal{L}_c(l^2)$.

Problem 2.9*
Suppose that X is a Banach space, $A \in \mathcal{L}_c(X)$ and $A^2 = A$ (that is, A is a projection; see Definition 2.27(c)). Show that $A \in \mathcal{L}_f(X)$.

Problem 2.10*
Suppose that X is a Banach space, $D \subseteq X$ is a nonempty closed and bounded subset, and $f\colon D \longrightarrow X$ is a condensing map (that is, $f \in S(D)$). Show that $I_X - f$ is proper.

Problem 2.11***
Let X be an infinite dimensional Banach space and let $B_1 = \{u \in X :\ \|u\|_X = 1\}$. Show that $\alpha(B_1) = 2$ and $\beta(B_1) = 1$.

Problem 2.12**
Suppose that X is a Banach space, $\mathcal{B}$ is the family of bounded subsets of X, $\{C_n\}_{n\geqslant 1} \subseteq \mathcal{B}$ is a decreasing sequence of sets (i.e., $C_{n+1} \subseteq C_n$ for all $n \geqslant 1$), and $\lim\limits_{n\to+\infty} \gamma(C_n) = 0$, where $\gamma = \alpha$ or $\gamma = \beta$. Show that the set $\bigcap\limits_{n\geqslant 1} \overline{C}_n$ is nonempty and compact.

Problem 2.13**
Suppose that X is a reflexive Banach space, Y is a Banach space, $A \in \mathcal{L}_c(X;Y)$, and $C \subseteq X$ is a nonempty, bounded, closed, and convex set. Show that the set $A(C) \subseteq Y$ is compact.

Problem 2.14*
Suppose that X and Y are two Banach spaces, $A \in \mathcal{L}_c(X;Y)$ and $L \in \mathcal{L}(Y;X)$. Show that $AL \in \mathcal{L}_c(Y)$ and $LA \in \mathcal{L}_c(X)$.

Problem 2.15**
Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X;Y)$. Suppose that there exists $c > 0$ such that

$$\|A(u)\|_Y \;\geqslant\; c\|u\|_X \qquad \forall\, u \in X.$$

Show that A is compact if and only if X is finite dimensional.

Problem 2.16**
Let X be a Banach space and let $A \in \mathcal{L}_c(X) \setminus \mathcal{L}_f(X)$. Show that $0 \in \overline{A(\partial B_1)}$ (recall that $\partial B_1 = \{x \in X : \|x\|_X = 1\}$).

Problem 2.17*
Let $S\colon l^2 \longrightarrow l^2$ be the forward shift operator, i.e.,

$$S(u_1, u_2, \ldots) = (0, u_1, u_2, \ldots) \qquad \forall\ \{u_n\}_{n\geqslant 1} \in l^2.$$

Evidently $S \in \mathcal{L}(l^2)$. Is S compact? Justify your answer.

Problem 2.18**
Suppose that X and Y are two Banach spaces, $A \in \mathcal{L}(X;Y)$ and A is also continuous from X with the weak topology into Y with the strong topology. Show that $A \in \mathcal{L}_f(X;Y)$.

Problem 2.19*
Let X be a reflexive Banach space and let $A \in \mathcal{L}_c(X)$. Show that there exists $u_0 \in X$ with $\|u_0\|_X \leqslant 1$ such that $\|A\|_{\mathcal{L}} = \|A(u_0)\|_X$.

Problem 2.20**
Find a Banach space X and $A \in \mathcal{L}(X)$ such that A is not compact but A^2 is.

Problem 2.21***
Let H be a Hilbert space and let $A \in \mathcal{L}(H)$. Show that the following properties are equivalent:

(a) $A \in \mathcal{L}_c(H)$.

(b) For every orthonormal basis $\{e_\alpha\}_{\alpha\in J}$ of H and $\varepsilon > 0$, the set

$$\big\{\alpha \in J : \ |\,(A(e_\alpha), e_\alpha)_H\,| \geqslant \varepsilon\big\}$$

is finite.

(c) There is a sequence $\{A_n\}_{n\geqslant 1} \subseteq \mathcal{L}_f(H)$ such that $\|A - A_n\|_{\mathcal{L}} \longrightarrow 0$.

Problem 2.22**
Let (S, Σ_1, μ) and (T, Σ_2, ξ) be two finite separable measure spaces and let $K \in L^2(T \times S)$. Consider the integral operator $L\colon L^2(S) \longrightarrow L^2(T)$ defined by

$$L(u)(t) = \int_S K(t,s)u(s)\,d\mu \qquad \forall\ t \in T.$$

Show that L is compact.

Problem 2.23**
Let $h \in C([0,1])$, $h \neq 0$ and let $A\colon C([0,1]) \longrightarrow C([0,1])$ be defined by

$$A(u)(t) = h(t)u(t) \qquad \forall\, u \in C([0,1]),\ t \in [0,1].$$

Show that A is well defined, linear, continuous but not compact.

Problem 2.24**
Show that l^1 is embedded continuously in l^2 and examine if the embedding is compact.

Problem 2.25**
Let X be a reflexive Banach space. Show that every $A \in \mathcal{L}(X;l^1)$ is compact.

Problem 2.26***
Show that the space $\mathcal{L}_c(l^2)$ is not reflexive.

Problem 2.27**
Let X and Y be two Banach spaces with X of infinite dimension and let $A \in \mathcal{L}_c(X;Y)$. Show that there exists a sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\|u_n\|_X = 1$ for all $n \geqslant 1$ and $\|A(u_n)\|_Y \longrightarrow 0$.

Problem 2.28**
Let X be a reflexive Banach space and let $A \in \mathcal{L}(c_0;X)$. Show that A is compact (recall that

$$c_0 = \big\{\{u_n\}_{n\geqslant 1} : \{u_n\}_{n\geqslant 1} \text{ is a real sequence such that } u_n \longrightarrow 0\big\}$$

with the norm $\|\{u_n\}_{n\geqslant 1}\|_{c_0} = \sup\limits_{n\geqslant 1} |u_n|$).

Problem 2.29**
Suppose that X and Y are two Banach spaces, $A \in \mathcal{L}_c(X;Y)$ and $R(A) \subseteq Y$ is closed. Show that $A \in \mathcal{L}_f(X;Y)$. In addition, if $\dim\ker A < +\infty$, then X is finite dimensional.

Problem 2.30**
Let X and Y be two Banach spaces and let $A \in \mathcal{L}_{wc}(X;Y)$ (see Definition 2.1(d)). Show that $R(A)$ is closed if and only if $R(A)$ is reflexive.

Problem 2.31***

Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X;Y)$. Show that the following four properties are equivalent:

(a) A is weakly compact (see Definition 2.1(d)).

(b) $A^{**}(X^{**}) \subseteq Y$.

(c) $A^*\colon Y^*_{w^*} \longrightarrow X^*_w$ is continuous (here by $Y^*_{w^*}$ we denote the space Y^* furnished with the w^*-topology and by X^*_w, the space X^* furnished with its w-topology).

(d) $A^*\colon Y^* \longrightarrow X^*$ is weakly compact.

Problem 2.32***

Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X;Y)$. Show that $A \in \mathcal{L}_c(X;Y)$ if and only if there exists a sequence $\{u_n^*\}_{n\geqslant 1} \subseteq X^*$ such that $\|u_n^*\|_{X^*} \longrightarrow 0$ and

$$\|A(u)\|_Y \ \leqslant\ \sup_{n\geqslant 1} |\,\langle x_n, u\rangle\,| \qquad \forall\ u \in X$$

Problem 2.33**

Let X be a reflexive Banach space and let $f\colon X \longrightarrow X^*$ be a coercive map satisfying the condition: "for any sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $u_n \xrightarrow{w} u$ in X and $\lim\limits_{n\to+\infty} \langle f(u_n) - f(u), u_n - u\rangle = 0$, we have that $u_n \longrightarrow u$ in X." Show that f is proper.

Problem 2.34**

Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X;Y)$. Show that $A \in \mathcal{L}_c(X;Y)$ if and only if A factors with compact factors through a closed subspace of c_0.

Problem 2.35**

Let X be a Banach space and let $A \in \mathcal{L}(c_0, X)$. Show that $A \in \mathcal{L}_{wc}(c_0, X)$ if and only if $A \in \mathcal{L}_c(c_0; X)$.

Problem 2.36**

Suppose that X is a reflexive Banach space, Y is a Banach space, $A \in \mathcal{L}_c(X;Y)$, $C \subseteq X$ is a nonempty, bounded, closed, and convex set and $y \in Y$. Show that there exists $u_0 \in C$ such that

$$\|A(u_0) - y\|_Y \ =\ \inf_{u\in C} \|A(u) - y\|_Y.$$

Problem 2.37**
Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X;Y)$. Show that $A \in \mathcal{L}_f(X;Y)$ if and only if $A^* \in \mathcal{L}_f(Y^*;X^*)$. Moreover $\dim R(A) = \dim R(A^*)$.

Problem 2.38**
Let Y be a Banach space and let $A \in \mathcal{L}(l^1;X)$. Show that A is weakly compact if and only if the sequence $\{A(e_n)\}_{n\geqslant 1} \subseteq Y$ is relatively w-compact (here $\{e_n\}_{n\geqslant 1}$ denotes the standard Schauder basis of l^1).

Problem 2.39**
Suppose that X is a Banach space, $\{u_n\}_{n\geqslant 1} \subseteq X$ is a relatively w-compact sequence and $A\colon l^1 \longrightarrow X$ is defined by

$$A(x) \;=\; \sum_{n\geqslant 1} x_n u_n \qquad \forall\, x = \{x_n\}_{n\geqslant 1} \in l^1.$$

Show that A is weakly compact.

Problem 2.40**
Let X be an infinite dimensional Banach space and let $A \in \mathcal{L}_c(X)$. Show that $0 \in \sigma(A)$.

Problem 2.41**
Let $A \in \mathcal{L}(L^2(0,1))$ be defined by

$$A(u)(t) \;=\; tu(t) \qquad \forall\, t \in [0,1].$$

Show that $\sigma_p = \emptyset$ (that is, A has no eigenvalues) and $[0,1] \subseteq \sigma(A)$.

Problem 2.42**
Let H be a Hilbert space and let $A \in \mathcal{L}(H)$ be self-adjoint. Show that:
(a) $\|A\|_{\mathcal{L}} = \sup\limits_{\|u\|_H \leqslant 1} (A(u),u)_H$;
(b) all eigenvalues of A are real and eigenvectors corresponding to distinct eigenvalues are orthogonal.

Problem 2.43**
Suppose that H is a Hilbert space, $A \in \mathcal{L}(H)$ is a self-adjoint operator and $\lambda \in \mathbb{C}$. Show that $\lambda \in \sigma(A)$ if and only if

$$\inf_{\|u\|_H = 1} \|(\lambda I_H - A)(u)\|_H = 0.$$

Problem 2.44**
Suppose that H is a Hilbert space, $A \in \mathcal{L}(H)$ is a self-adjoint operator and

$$m = \inf_{\|u\|_H=1} (A(u), u)_H \quad \text{and} \quad M = \sup_{\|u\|_H=1} (A(u), u)_H .$$

Show that $\sigma(A) \subseteq [m, M]$ and $m, M \in \sigma(A)$.

Problem 2.45*
Suppose that X is a Banach space and $A \in \mathcal{L}(X)$ is such that $A \neq 0$, $A \neq I_X$ and A is a projection (see Definition 2.27(c)). Show that $\sigma_p(A) = \sigma(A) = \{0, 1\}$.

Problem 2.46**
Suppose that X is a Banach space, $A \in \mathcal{L}(X)$ and $\xi \in \mathbb{C} \setminus \{0\}$. Assume that the series $\sum\limits_{n \geqslant 0} \xi^{-(n+1)} A^n(u)$ converges for every $u \in X$. Show that $\xi \in \varrho(A)$.

Problem 2.47**
Let $X = L^p(0,1)$ with $1 \leqslant p < +\infty$ and let $A\colon X \longrightarrow X$ be defined by

$$A(u)(t) = \int_0^t u(s)\, ds \qquad \forall\, t \in (0,1).$$

Show that $A \in \mathcal{L}_c(X)$ and find $\sigma(A)$ and $\sigma_p(A)$.

Problem 2.48**
Let H be a Hilbert space and let $A \in \mathcal{L}(H)$ be a self-adjoint isomorphism of H which is positive (i.e., $(A(u), u)_H \geqslant 0$ for all $u \in H$). Show that

$$\langle u, h \rangle = (A(u), h)_H \qquad \forall\, u, h \in H$$

defines a new inner product on H and $|u| = \langle u, u \rangle^{\frac{1}{2}}$ is an equivalent norm on H.

Problem 2.49*
Let X and Y be two Banach spaces and let $L \in \mathcal{L}(X; Y)$. Suppose that $T, S \in \mathcal{L}(Y; X)$ are such that $L \circ T = I_Y + K$ and $S \circ L = I_X + \widehat{K}$ with K and $\widehat{K}$ compact operators. Show that $L \in \mathrm{Fred}\,(X; Y)$.

Problem 2.50*
Let X and Y be two Banach spaces, $L \in \operatorname{Fred}(X;Y)$ and $K \in \mathcal{L}_c(X;Y)$. Show that $L+K \in \operatorname{Fred}(X;Y)$ and $i(K) = i(L+K)$.

Problem 2.51**
Let X be a Hausdorff topological space and let $\varphi\colon X \longrightarrow \mathbb{R}$ be a function. Show that:

(a) φ is a lower semicontinuous function if and only if the multifunction $u \longmapsto E_\varphi(u)$ is upper semicontinuous (where $E_\varphi(u) = \{\lambda \in \mathbb{R} : \varphi(u) \leqslant \lambda\}$);

(b) φ is an upper semicontinuous function if and only if the multifunction $u \longmapsto E_\varphi(u)$ is lower semicontinuous.

Problem 2.52**
Let X and Y be two Hausdorff topological spaces and let $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ be a multifunction with connected values which is upper semicontinuous or lower semicontinuous. Show that F maps connected sets to connected sets.

Problem 2.53*
Let X and Y be two Hausdorff topological spaces and let $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ be a multifunction. We introduce the multifunction $\overline{F}\colon X \longrightarrow P_f(X)$ defined by

$$\overline{F}(u) = \overline{F(u)} \qquad \forall\, u \in X.$$

Show that F is lower semicontinuous if and only if $\overline{F}$ is lower semicontinuous.
Is the result true for upper semicontinuous functions? Justify your answer.

Problem 2.54**
Suppose that X is a Hausdorff topological space, Y is a normed space and $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is a lower semicontinuous multifunction. Show that the multifunctions $u \longmapsto \operatorname{conv} F(u)$ and $u \longmapsto \overline{\operatorname{conv}}\, F(u)$ are both lower semicontinuous.

Problem 2.55*
Suppose that X is a Hausdorff topological space, Y is a Banach space and $F\colon X \longrightarrow P_k(Y)$ is an upper semicontinuous multifunction. Show that the multifunction $u \longmapsto \overline{\operatorname{conv}}\, F(u)$ is upper semicontinuous too.

Problem 2.56**

Let X and Y be two Hausdorff topological spaces and let $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ be a multifunction. Show that:

(a) if F is $P_k(Y)$-valued and upper semicontinuous and $K \in P_k(X)$, then $F(K) \in P_k(Y)$;

(b) if $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is closed, $G\colon X \longrightarrow P_k(Y)$ is upper semicontinuous and $F(u) \cap G(u) \neq \emptyset$ for all $u \in X$, then $X \ni u \longmapsto H(u) = F(u) \cap G(u) \in 2^Y \setminus \{\emptyset\}$ is upper semicontinuous.

(c) Is the intersection of two lower semicontinuous multifunctions necessary a lower semicontinuous multifunction? Justify your answer.

Problem 2.57**

Let (X, d_X) be a metric space and suppose that $\varphi \in C(X)$ is in the closure of $\{\mathrm{dist}(\cdot, E) : E \in P_f(X)\}$ for the topology of pointwise convergence. Assume that $D = \{x \in X : \varphi(x) = 0\} \neq \emptyset$ and for each $u \in X$ we have $\mathrm{dist}(u, D) \leqslant \varphi(u)$. Show that $\varphi(\cdot) = \mathrm{dist}(\cdot, D)$

Problem 2.58**

Let (X, d_X) be a metric space and let $\{C_n\}_{n\geqslant 1} \subseteq P_f(X)$ be a sequence such that $C_n \xrightarrow{h} C \in P_k(X)$. Show that

$$C = \bigcap_{m\geqslant 1} \overline{\bigcup_{n\geqslant m} C_n} = \bigcap_{\varepsilon>0} \bigcup_{m\geqslant 1} \bigcap_{n\geqslant m} (C_n)_\varepsilon,$$

where $(C_n)_\varepsilon = \{u \in X : \mathrm{dist}(u, C_n) < \varepsilon\}$.

Problem 2.59**

Let (X, d_X) be a metric space and let $K \subseteq (P_f(X), h)$ be compact. Show that the set $\bigcup\limits_{C\in K} C \subseteq X$ is closed.

Problem 2.60**

Let (X, d_X) be a metric space and let $\mathcal{F}$ be the family of finite subsets of X. Show that $\mathcal{F}$ is dense in $(P_f(X), h)$ if and only if X is totally bounded.

Problem 2.61*
Suppose that X, Y, and V are three Hausdorff topological spaces, $\varphi\colon X \times Y \longrightarrow V$ is a continuous map, and $M\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is a lower semicontinuous multifunction. We set

$$F(u) \;=\; \varphi(u, M(u)) \qquad \forall\, u \in X.$$

Show that $u \longmapsto F(u)$ is lower semicontinuous form X into $2^V \setminus \{\emptyset\}$.

Problem 2.62*
Suppose that X, Y, and V are three metric spaces, $\varphi\colon X \times Y \longrightarrow V$ is a continuous map, and $M\colon X \longrightarrow P_k(Y)$ is an upper semicontinuous multifunction. We set

$$F(u) \;=\; \varphi(u, M(u)) \qquad \forall\, u \in X.$$

Show that $u \longmapsto F(u)$ is upper semicontinuous form X into $P_k(V)$.

Problem 2.63*
Let (X, d_X) be a metric space and suppose that $(P_f(X), h)$ is compact. Show that X is compact.

Problem 2.64**
Suppose that X and Y are two Hausdorff topological spaces, $\varphi\colon X \times Y \longrightarrow \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ is a lower semicontinuous function and $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is a lower semicontinuous multifunction. Let

$$v(u) \;=\; \sup_{y \in F(u)} \varphi(u, y)$$

(the ***value function***). Show that $v\colon X \longrightarrow \mathbb{R}^*$ is lower semicontinuous.

Problem 2.65**
Let X and Y be two Hausdorff topological spaces and let $F\colon X \longrightarrow P_f(Y)$ be a multifunction which is closed (see Definition 2.40) and locally compact (see Proposition 2.45). Show that F is upper semicontinuous.

Problem 2.66**

Suppose that X and Y are two Hausdorff topological spaces, $\varphi\colon X \times Y \longrightarrow \mathbb{R}$ is a continuous function, and $F\colon X \longrightarrow P_k(X)$ is a continuous multifunction. We define

$$\begin{aligned} v(u) &= \sup_{y \in F(u)} \varphi(u,y) \qquad \forall\, u \in X, \\ S(u) &= \{y \in F(x):\ v(u) = \varphi(u,y)\} \qquad \forall\, u \in X. \end{aligned}$$

Show that $v\colon X \longrightarrow \mathbb{R}$ is continuous and $S\colon X \longrightarrow P_k(Y)$ is upper semicontinuous.

Problem 2.67*

Show that in general a lower semicontinuous multifunction need not be h-lower semicontinuous and an h-upper semicontinuous multifunction need not be upper semicontinuous (cf. Propositions 2.54 and 2.55).

Problem 2.68*

Suppose that X is a Hausdorff topological space, (Y, d_Y) is a metric space, and $F\colon X \longrightarrow P_f(Y)$ is an h-upper semicontinuous multifunction. Show that F is closed.

Problem 2.69*

Suppose that X is a Hausdorff topological space, (Y, d_Y) is a metric space and $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is an h-upper semicontinuous multifunction. Show that for every $y \in X$ the function $u \longmapsto \varphi_y(u) = \operatorname{dist}_Y(y, F(u))$ is lower semicontinuous.

Problem 2.70**

Let (X, d_X) be a complete metric space and let $F\colon X \longrightarrow P_k(X)$ be a multifunction such that

$$h(F(u), F(v)) \ \leqslant\ k d_X(u,v) \qquad \forall\, u, v \in X,$$

with $k > 0$. For any compact set $K \subseteq X$, we set $R_F(K) = \bigcup\limits_{u \in K} F(u)$.

Show that:

(a) $R_F(K) \in P_k(X)$;

(b) $R_F\colon P_k(X) \longrightarrow P_k(X)$ is Lipschitz continuous with Lipschitz constant k (on $P_k(X)$ we consider the Hausdorff metric corresponding to d_X).

Problem 2.71**
Suppose that X and Y are two Hausdorff topological spaces, $F\colon X \longrightarrow 2^Y\setminus\{\emptyset\}$ is a lower semicontinuous multifunction, and $G\colon X \longrightarrow 2^Y\setminus\{\emptyset\}$ is a multifunction with open graph. Suppose that

$$F(u)\cap G(u) \neq \emptyset \qquad \forall\, u \in X.$$

Show that the multifunction $u \longmapsto F(u) \cap G(u)$ is lower semicontinuous.

Problem 2.72**
Let X be a paracompact, perfectly normal topological space, let Y be a Banach space, and let $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ be a lower semicontinuous multifunction with convex values. Show that for every $r > 0$, there exists a continuous map $h\colon X \longrightarrow Y$ such that

$$\mathrm{dist}_Y(h(u), F(u)) \ < \ r \qquad \forall\, u \in X.$$

Problem 2.73**
Suppose that X is a compact Hausdorff topological space, Y is a locally convex vector space, and $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is a multifunction with convex values. Assume that for every $y \in Y$, the set $F^-(\{y\}) = \{u \in X :\ y \in F(u)\}$ is open. Show that there exists a continuous map $f\colon X \longrightarrow Y$ such that $f(u) \in F(u)$ for all $u \in X$.

Problem 2.74**
Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X;Y)$ be surjective. Show that there exists a continuous map $f\colon Y \longrightarrow X$ such that $A(f(y)) = y$ for all $y \in Y$.

Problem 2.75*
Suppose that X is a paracompact space, Y is a Banach space, $C \in P_f(X)$ and $F\colon X \longrightarrow P_{fc}(Y)$ is a lower semicontinuous multifunction. Show that any continuous selection of $F|_C$ can be extended to a continuous selection of F.

Problem 2.76*
Suppose that X is metric space, $\varphi\colon X \longrightarrow \mathbb{R}$ is an upper semicontinuous function and $\psi\colon X \longrightarrow \mathbb{R}$ is a lower semicontinuous function such that $\varphi(u) < \psi(u)$ for all $u \in X$. Show that there exists a continuous function $f\colon X \longrightarrow \mathbb{R}$ such that

$$\varphi(u) \ < \ f(u) \ < \ \psi(u) \qquad \forall\, u \in X.$$

Problem 2.77**
Suppose that X is a Hausdorff topological space, Y is a topological vector space, $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is a lower semicontinuous multifunction, $\varphi \in C(X;Y)$ and U is an open neighborhood of the origin in Y. Assume that

$$H(u) \;=\; F(u) \cap \big(\varphi(u) + U\big) \;\neq\; \emptyset \qquad \forall\, u \in X.$$

Show that the multifunction $u \longmapsto H(u)$ is lower semicontinuous from X into Y.

Problem 2.78**
Suppose that X is a Hausdorff topological space, Y is a topological vector space, and $F, G\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ are two lower semicontinuous multifunctions. Show that the multifunction $X \ni u \longmapsto H(u) = F(u) + G(u) \in 2^Y \setminus \{\emptyset\}$ is lower semicontinuous.

Problem 2.79**
Let X be a paracompact space and let Y be a normed space. We say that a multifunction $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is ***almost lower semicontinuous*** (a-***lsc*** for short), if for every $\varepsilon > 0$ and every $u \in X$, there exists $U \in \mathcal{N}(u)$ (where $\mathcal{N}(u)$ is the filter of neighborhoods of u) such that

$$\bigcap_{u' \in U} F(u')_\varepsilon \;\neq\; \emptyset$$

(recall that $F(u')_\varepsilon = \{y \in Y : \ \mathrm{dist}_Y(y, F(u')) < \varepsilon)\}$). Show that, if F is an almost lower semicontinuous multifunction and has convex values, then for every $\varepsilon > 0$ there exists $f_\varepsilon \in C(X;Y)$ such that $\mathrm{dist}_Y(f_\varepsilon(u), F(u)) < \varepsilon$ for all $u \in X$.

Problem 2.80*
Suppose that X is a paracompact space, Y is a Banach space, $F\colon X \longrightarrow P_{fc}(Y)$ is a lower semicontinuous multifunction and $(\widehat{u}, \widehat{y}) \in \operatorname{Gr} F$. Show that there exists a continuous selector $\widehat{f}$ of F such that $\widehat{f}(\widehat{u}) = \widehat{y}$.

Problem 2.81*
Suppose that X and Y are two Hausdorff topological spaces, $D \subseteq X$ is a set and $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is a multifunction which is lower semicontinuous on $\overline{D} \setminus D$. Show that $F(\overline{D}) \subseteq \overline{F(D)}$.

Problem 2.82*
Show that the Cartesian product of two lower semicontinuous multifunctions is lower semicontinuous. Is this also true for upper semicontinuous multifunctions? Justify your answer.

Problem 2.83***
Suppose that (X, d_X) is a metric space, Y is a Banach space and $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ is an upper semicontinuous multifunction with convex values. Show that for every $\varepsilon > 0$, we can find a locally Lipschitz function $f_\varepsilon\colon X \longrightarrow Y$ such that

$$f_\varepsilon(X) \subseteq \operatorname{conv} F(X) \quad \text{and} \quad h^*(\operatorname{Gr} f_\varepsilon, \operatorname{Gr} F) < \varepsilon.$$

Problem 2.84***
Suppose that X is a metric space, Y is a Banach space, $W \subseteq X$ is an open set, $K \subseteq W$ is a compact set and $F\colon \overline{W} \longrightarrow 2^Y \setminus \{\emptyset\}$ is an upper semicontinuous multifunction with convex values. Show that for every $\varepsilon > 0$, there exists an open neighborhood V_ε of K and a locally Lipschitz function $f_\varepsilon\colon V_\varepsilon \longrightarrow \operatorname{conv} F(K)$ with finite dimensional range such that

$$f_\varepsilon(u) \in F(K \cap B_\varepsilon(u)) + \varepsilon B_1 \qquad \forall\, u \in V_\varepsilon.$$

Problem 2.85**
Suppose that X is a compact topological space and $F\colon X \longrightarrow P_f(X)$ is an upper semicontinuous multifunction. Show that there exists a nonempty closed set $C \subseteq X$ such that $F(C) = C$.

Problem 2.86**
Find a compact valued multifunction which is closed but not upper semicontinuous.

Problem 2.87**
Let X be a Banach space and let $K \subseteq X$ be a nonempty set which is boundedly compact (respectively boundedly w-compact), i.e., for every $r > 0$, the set $K \cap \overline{B}_r$ is compact (respectively w-compact). Let

$$\operatorname{proj}_K(u) = \{h \in K : \ \|u - h\|_X = \operatorname{dist}(u, K)\}.$$

Show that the multifunction $u \longmapsto \operatorname{proj}_K(u)$ (known as the ***metric projection*** on K) is upper semicontinuous from X into X (respectively from X into X_w, where X_w denotes the Banach space X furnished with the weak topology).

Problem 2.88**
Suppose that X is a Banach space, X^* is its topological dual, $\mathcal{F}\colon X \longrightarrow 2^{X^*} \setminus \{\emptyset\}$ is the duality map. Show that $\mathcal{F}$ is upper semicontinuous from X with norm topology into X^*_{w*} (X^*_{w*} denoting X^* with the w^*-topology).

Problem 2.89**
Let X be a metric space and let $F\colon X \longrightarrow P_k(X)$ be a lower semicontinuous and closed multifunction with connected values. Show that F is continuous.

Problem 2.90*
Produce an upper semicontinuous multifunction with closed and convex values which does not have a continuous selector.

Problem 2.91**
Let T and X be two separable metric spaces which are Borel sets in their respective completions (that is, T and X are Borel spaces) and let $F\colon T \longrightarrow P_{fc}(X)$ be a measurable multifunction with σ-compact values. Show that for every closed set $C \subseteq X$, we have $F^-(C) \in \mathcal{B}(X)$.

Problem 2.92*
Suppose that (Ω, Σ) is a measurable space, X is a Polish space and $\varphi\colon X \longrightarrow \Omega$ is a map such that:
(i) for every $\omega \in \Omega$, $\varphi^{-1}(\{\omega\}) \in P_f(X)$; and
(ii) for every open set $V \subseteq X$, $\varphi(V) \in \Sigma$.
Show that there exists a Σ-measurable map $f\colon \Omega \longrightarrow X$ such that $\varphi(f(\omega)) = \omega$ for all $\omega \in \Omega$.

Problem 2.93**
Suppose that (Ω, Σ) is a measurable space, X is a separable metric space and $F\colon \Omega \longrightarrow P_k(X)$ is a multifunction. Show that F is measurable if and only if $F^-(C) \in \Sigma$ for every closed set $C \subseteq X$.

Problem 2.94**
Suppose that (Ω, Σ) is a measurable space, X is a separable metric space, and $F\colon \Omega \longrightarrow P_f(X)$ is a measurable multifunction. Show that for every compact set $K \subseteq X$, we have $F^-(K) \in \Sigma$.

Problem 2.95**
Suppose that (Ω, Σ) is a measurable space, X is a separable metric space, Y is a metric space, $\varphi\colon \Omega \times X \longrightarrow Y$ is a Carathéodory map (i.e., for all $u \in X$, the function $\omega \longmapsto \varphi(\omega, u)$ is Σ-measurable, while for every $\omega \in \Omega$, the function $u \longmapsto \varphi(\omega, u)$ is continuous) and $U \subseteq Y$ is a nonempty open set. Show that the multifunction $F\colon \Omega \longrightarrow 2^X$ defined by

$$F(\omega) \;=\; \{u \in X:\; \varphi(\omega, u) \in U\}\,.$$

is measurable.

Problem 2.96**
Let X and Y be two σ-compact metric spaces and let $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ be a closed multifunction. Show that F is measurable.

Problem 2.97***
Suppose that (Ω, Σ) is a measurable space, X is a separable metric space and $\{F_n\colon \Omega \longrightarrow P_f(X)\}_{n \geqslant 1}$ is a sequence of measurable multifunctions. Suppose that at least one of them is compact valued. Show that the map $\omega \longmapsto H(\omega) = \bigcap\limits_{n \geqslant 1} F_n(\omega)$ is measurable.

Problem 2.98**
Suppose that (Ω, Σ) is a measurable space, X is a Polish space, Y is a metric space, $\varphi\colon \Omega \times X \longrightarrow Y$ is a Carathéodory map, $U\colon \Omega \longrightarrow P_f(X)$ is a measurable multifunction, and

$$G(\omega) \;=\; \varphi(\omega, U(\omega)) \qquad \forall\, \omega \in \Omega.$$

Show that $G\colon \Omega \longrightarrow Y$ is measurable.

Problem 2.99**
Suppose that (Ω, Σ) is a measurable space, X is a Polish space, $F\colon \Omega \longrightarrow P_k(X)$ is a measurable multifunction, and $g\colon \Omega \longrightarrow X$ is a Σ-measurable function. Show that there exists a Σ-measurable selector f of F such that

$$\operatorname{dist}(g(\omega), F(\omega)) \;=\; d_X(g(\omega), f(\omega)) \qquad \forall\, \omega \in \Omega.$$

Problem 2.100*
Suppose that (Ω, Σ) is a measurable space, X is a σ-compact metric space, and $F\colon \Omega \longrightarrow P_f(X)$ is a multifunction such that $F^-(K) \in \Sigma$ for every $K \in P_k(X)$. Show that F is measurable.

Problem 2.101***
Suppose that (Ω, Σ) is a complete measurable space, X is a Polish space, Y is a σ-compact Polish space, and $F\colon \Omega \longrightarrow P_f(X \times Y)$ is a multifunction. We define the multifunction $G\colon \Omega \times X \longrightarrow P_f(Y)$ by setting

$$G(\omega, u) \;=\; \{y \in Y : \; (u, y) \in F(\omega)\} \qquad \forall\, (\omega, u) \in \Omega \times X.$$

Show that F is measurable if and only if G is measurable.

Problem 2.102*
Suppose that (Ω, Σ) is a complete measurable space, X is a separable Banach space and $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ is a graph measurable multifunction (see Definition 2.61(b)). Show that for every $u^* \in X^*$, the map

$$\omega \;\longmapsto\; \sigma_{F(\omega)}(u^*) \;=\; \sup_{u \in F(\omega)} \langle u^*, u\rangle$$

is Σ-measurable.

Problem 2.103***
Suppose that (Ω, Σ) is a measurable space, X is a separable Banach space, and $F\colon \Omega \longrightarrow P_{wkc}(X)$ is a multifunction. Show that F is measurable if and only if for every $u^* \in X^*$, the map $\omega \longmapsto \sigma_{F(\omega)}(u^*)$ in Σ-measurable.

Problem 2.104**
Suppose that (Ω, Σ) is a complete measurable space, X is a Souslin space $\varphi\colon \Omega \times X \longrightarrow \mathbb{R}^*$ is a jointly measurable function and $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ is a graph measurable multifunction. We define

$$m(\omega) \;=\; \inf_{u \in F(\omega)} \varphi(\omega, u) \qquad \forall\, \omega \in \Omega.$$

Show that $m\colon \Omega \longrightarrow \mathbb{R}^*$ is Σ-measurable.

Problem 2.105**

(a) Suppose that (Ω, Σ) is a measurable space, X is a separable Banach space and $F\colon \Omega \longrightarrow P_f(X)$ is a measurable multifunction. Show that multifunctions $\omega \longmapsto \operatorname{conv} F(\omega)$ and $\omega \longmapsto \overline{\operatorname{conv}}\, F(\omega)$ are both measurable.

(b) Suppose that (Ω, Σ) is a complete measurable space, X is a separable Banach space and $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ is a graph measurable multifunction. Show that the multifunction $\omega \longmapsto \overline{\operatorname{conv}} F(\omega)$ is measurable.

Problem 2.106*
Suppose that (Ω, Σ) is a measurable space, X is a separable Banach space, and $u\colon \Omega \longrightarrow X$ and $\varrho\colon \Omega \longrightarrow (0, +\infty)$ are Σ-measurable functions. Show that the multifunction

$$\Omega \ni \omega \;\longmapsto\; B_{\varrho(\omega)}(u(\omega)) = \{y \in X : \; \|y - u(\omega)\|_X < \varrho(\omega)\}$$

is measurable.

Problem 2.107**
Suppose that (Ω, Σ) is a measurable space, X is a compact metric space, and $\varphi\colon \Omega \times X \longrightarrow \mathbb{R}$ is a Carathéodory function (see Problem 2.95). Show that the multifunction $F\colon \Omega \longrightarrow 2^X$ defined by

$$F(\omega) \;=\; \{u \in X : \; \varphi(\omega, u) = 0\}$$

is measurable.

Problem 2.108**
We suppose that (Ω, Σ) and $(T, \mathcal{T})$ are two measurable spaces with Σ being complete (i.e., $\Sigma = \widehat{\Sigma}$; see Definition I.4.45), X is a Souslin space, $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$, and $G\colon \Omega \longrightarrow 2^T \setminus \{\emptyset\}$ are two graph measurable multifunctions, $h\colon \Omega \times X \longrightarrow T$ is a $(\Sigma \times \mathcal{B}, \mathcal{T})$-measurable map and

$$h(\omega, F(\omega)) \cap G(\omega) \;\neq\; \emptyset \qquad \forall\, \omega \in \Omega.$$

Show that there exists a Σ-measurable selector $f\colon \Omega \longrightarrow X$ of F such that

$$h(\omega, f(\omega)) \;\in\; G(\omega) \qquad \forall\, \omega \in \Omega.$$

Problem 2.109*
Suppose that (Ω, Σ) is a complete measurable space, X is a Polish space and $\{F_n\colon \Omega \longrightarrow P_f(X)\}_{n \geqslant 1}$ is a sequence of measurable multifunctions. Show that the multifunctions

$$\omega \;\longmapsto\; G(\omega) \;=\; \overline{\bigcup_{n \geqslant 1} F_n(\omega)} \quad \text{and} \quad \omega \;\longmapsto\; H(\omega) \;=\; \bigcap_{n \geqslant 1} F_n(\omega)$$

are both measurable.

Problem 2.110**
Suppose that (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space and $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ is a multifunction with $\operatorname{Gr} F \in \Sigma_\mu \times \mathcal{B}(X)$ (Σ_μ being the μ-completion of Σ; see Definition I.4.45). Show that for $1 \leqslant p \leqslant +\infty$, we have $S_F^p \neq \emptyset$ (see Theorem 2.72) if and only if

$$\inf_{u \in F(\omega)} \|u\|_X \ \leqslant\ \vartheta(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

with $\vartheta \in L^p(\Omega)_+$.

Problem 2.111**
Suppose that (Ω, Σ, μ) is a complete σ-finite measure space, X is a Polish space and $\{F_n\colon \Omega \longrightarrow P_f(X)\}_{n\geqslant 1}$ is a sequence of measurable multifunctions. Show that the multifunctions

$$\omega \ \longmapsto\ \liminf_{n\to+\infty} F_n(\omega) \quad \text{and} \quad \omega \ \longmapsto\ \limsup_{n\to+\infty} F_n(\omega)$$

(see Definition 2.88) are both measurable.

Problem 2.112**
Suppose that (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space and $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ a graph measurable multifunction such that $S_F^p \neq \emptyset$ with $1 \leqslant p < \infty$. Show that we can find a sequence $\{f_n\}_{n\geqslant 1} \subseteq S_F^p$ such that

$$F(\omega) \ \subseteq\ \overline{\{f_n(\omega)\}}_{n\geqslant 1} \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Problem 2.113**
Suppose that (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space, $F\colon \Omega \longrightarrow P_f(X)$ a graph measurable multifunction such that $S_F^p \neq \emptyset$ with $1 \leqslant p < +\infty$, $\{f_n\}_{n\geqslant 1} \subseteq S_F^p$ is a sequence such that

$$F(\omega) \ =\ \overline{\{f_n(\omega)\}}_{n\geqslant 1} \quad \text{for } \mu\text{-a.a. } \omega \in \Omega$$

(see Problem 2.112), $f \in S_F^p$ and $\varepsilon > 0$. Show that there exists a finite Σ-partition $\{A_1, \ldots, A_l\}$ of Ω such that

$$\Big\| f - \sum_{k=1}^{l} \chi_{A_k} f_k \Big\|_X \ <\ \varepsilon.$$

Problem 2.114**

Suppose that (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space and $F, G\colon \Omega \longrightarrow P_f(X)$ are two measurable multifunctions such that $S_F^p, S_G^p \neq \emptyset$ with some $1 \leqslant p \leqslant +\infty$. Set $H(\omega) = \overline{F(\omega) + G(\omega)}$. Show that $S_H^p = \overline{S_F^p + S_G^p}$.

Problem 2.115**

Suppose that (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space, $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ a graph measurable multifunction such that $S_F^p \neq \emptyset$ with $1 \leqslant p < +\infty$. Show that

$$\overline{\operatorname{conv}}\, S_F^p \;=\; S^p_{\overline{\operatorname{conv}}\, F}.$$

Problem 2.116**

Suppose that (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space, $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ a graph measurable multifunction such that $S_F^p \neq \emptyset$ with $1 \leqslant p < +\infty$. Show that $\overline{S_F^p} = S^p_{\overline{F}}$.

Problem 2.117***

Suppose that (Ω, Σ, μ) is a σ-finite nonatomic measure space, X, Y are two Banach spaces with Y having the RNP (see Definition 1.45), $D \subseteq L^p(\Omega; X)$ (with $1 \leqslant p < +\infty$) is a decomposable set (see Definition 2.70) and $T \in \mathcal{L}(L^p(\Omega; X); Y)$. Show that the set $\overline{T(D)} \subseteq Y$ is convex.

Problem 2.118***

Suppose that (Ω, Σ, μ) is a σ-finite nonatomic measure space, X is a Banach space, $D \subseteq L^p(\Omega; X)$ (with $1 \leqslant p < +\infty$) is a decomposable set which is w-closed. Show that the set D is convex.

Problem 2.119**

Suppose that (Ω, Σ, μ) is a σ-finite nonatomic measure space, X is a separable Banach space, and $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ is a graph measurable multifunction, such that $S_F^p \neq \emptyset$ (with $1 \leqslant p < +\infty$). Show that $\overline{S_F^p}^{w} = S^p_{\overline{\operatorname{conv}}\, F}$ (here by w we denote the weak topology on the Lebesgue–Bochner space $L^p(\Omega; X)$).

Problem 2.120**

Suppose that (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space and $F, G\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ are two graph measurable multifunction, such that $S_F^p = S_G^p$ for some $1 \leqslant p \leqslant +\infty$. Show that $F(\omega) = G(\omega)$ for μ-almost all $\omega \in \Omega$.

Problem 2.121**
Suppose that (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space, and $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ is a graph measurable multifunction.

(a) Suppose that S_F^p (with $1 \leqslant p < +\infty$) is nonempty and closed. Show that $F(\omega) \in P_f(X)$ for μ-almost all $\omega \in \Omega$.

(b) Suppose that μ is nonatomic and S_F^p (with $1 \leqslant p < +\infty$) is nonempty, closed, and convex. Show that $F(\omega) \in P_{fc}(X)$ for μ-almost all $\omega \in \Omega$.

Problem 2.122**
Suppose that (Ω, Σ, μ) is a nonatomic, σ-finite measure space, X is a separable Banach space and $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ is a graph measurable multifunction with $S_F^1 \neq \emptyset$. Show that the set $\operatorname{cl} \int\limits_\Omega F\, d\mu \subseteq X$ is convex (see Definition 2.85).

Problem 2.123*
Suppose that (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space and $F\colon \Omega \longrightarrow P_{wkc}(X)$ is a graph measurable and integrably bounded multifunction. Show that $\int\limits_\Omega F(\omega)\, d\mu \in P_{wkc}(X)$.

Problem 2.124*
Suppose that (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space and $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ is a graph measurable multifunction with $S_F^1 \neq \emptyset$. Show that

$$\sigma_{S_F^1}(h^*) \;=\; \int\limits_\Omega \sigma_{F(\omega)}(h^*(\omega))\, d\mu \qquad \forall\, h^* \in L^\infty(\Omega; X^*_{w^*}).$$

Problem 2.125**
Suppose that V is a Polish space, X is a separable metric space, $\mu\colon \mathcal{B}(V) \longrightarrow \mathbb{R}$ is a finite Borel measure on V and $F\colon V \longrightarrow P_f(X)$ is a measurable multifunction. Show that for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq V$ with $\mu(V \setminus K_\varepsilon) < \varepsilon$ such that $F|_{K_\varepsilon}$ is closed.

Problem 2.126**
Suppose that V is a Polish space, X is a separable metric space, $\mu\colon \mathcal{B}(V) \longrightarrow \mathbb{R}$ is a finite Borel measure on V and $F\colon V \longrightarrow P_f(X)$ is a measurable multifunction. Show that for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq V$ with $\mu(V \setminus K_\varepsilon) < \varepsilon$ such that $F|_{K_\varepsilon}$ is lower semicontinuous.

Problem 2.127*

Suppose that (Ω, Σ) and $(T, \mathcal{T})$ are two measurable spaces, X is a separable metric space and $F\colon \Omega \times T \longrightarrow P_f(X)$ is a $\Sigma \times \mathcal{T}$-measurable multifunction. Fix $t_0 \in T$ and let u_0 be a Σ-measurable selector of $F(\cdot, t_0)$. Show that given $\varepsilon > 0$, we can find a $\Sigma \times \mathcal{T}$-measurable selector $\widehat{f}$ of F such that

$$d_X(\widehat{f}(\omega, t_0), u_0(\omega)) \leqslant \varepsilon \qquad \forall\, \omega \in \Omega.$$

Problem 2.128*

Suppose that (Ω, Σ) is a complete measurable space, X is a Souslin space and $F\colon \Omega \longrightarrow 2^X$ is a graph measurable multifunction. Show that for every $D \in \mathcal{B}(X)$, we have $F^-(D) \in \Sigma$.

Problem 2.129**

Suppose that (Ω, Σ) is a complete measurable space, X is a Polish space and $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ is a graph measurable multifunction. Show that the multifunction $\omega \longmapsto \partial F(\omega)$ is measurable and $\omega \longmapsto \operatorname{int} F(\omega)$ is graph measurable (where $\partial F(\omega)$ denotes the boundary of $F(\omega)$).

Problem 2.130**

Suppose that (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space and $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ is a graph measurable multifunction with open values such that $S_F^1 \neq \emptyset$. Show that for every set $A \in \Sigma$, the set $\int_A F\, d\mu \subseteq X$ is open.

Problem 2.131**

Suppose that (Ω, Σ, μ) is a finite measure space, X is a separable Banach space and $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ is a graph measurable multifunction such that $S_F^1 \neq \emptyset$, $F(\omega)$ is convex for all $\omega \in \Omega$ and

$$\operatorname{int} F(\omega) \neq \emptyset \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Show that

$$\operatorname{int} \int_A F\, d\mu = \int_A \operatorname{int} F\, d\mu \qquad \forall\, A \in \Sigma.$$

Problem 2.132**

Suppose that (Ω, Σ, μ) is a finite measure space, X is a Banach space and $D \subseteq L^1(\Omega; X)$ is a bounded and decomposable set. Show that D is uniformly integrable.

Problem 2.133**
Let $T = [0,b]$ and let $F\colon T \longrightarrow P_{kc}(\mathbb{R}^N)$ be a graph measurable and integrably bounded multifunction. For every $t \in T$ we set

$$V(t) \;=\; \int_0^t F(s)\,ds.$$

Show that $V\colon T \longrightarrow P_{kc}(\mathbb{R}^N)$ is h-continuous and continuous and the set CS_V of continuous selectors of V is nonempty and compact in $C(T;\mathbb{R}^N)$.

Problem 2.134*
Let X be a Banach space and let $\{A_n\}_{n\geqslant 1} \subseteq 2^X \setminus \{\emptyset\}$ be a sequence of sets. Show that for every $u \in X$, we have

$$\limsup_{n\to+\infty} \operatorname{dist}(u, A_n) \;\leqslant\; \operatorname{dist}(u, s\text{–}\liminf_{n\to+\infty} A_n).$$

Problem 2.135*
Suppose that X is a Banach space, $C \subseteq X$ is a nonempty, w-closed set such that for every $r > 0$, we have $C \cap \overline{B}_r \in P_{wk}(X)$ (where $\overline{B}_r = \{u \in X :\ \|u\|_X \leqslant r\}$) and $\{A_n\}_{n\geqslant 1} \subseteq 2^X \setminus \{\emptyset\}$ is a sequence of sets such that $A_n \subseteq C$ for all $n \geqslant 1$. Show that for every $u \in X$, we have

$$\operatorname{dist}(u, w\text{–}\limsup_{n\to+\infty} A_n) \;\leqslant\; \liminf_{n\to+\infty} \operatorname{dist}(u, A_n).$$

Problem 2.136*
Suppose that X is a reflexive Banach space, $A \in P_{fc}(X)$, $\{A_n\}_{n\geqslant 1} \subseteq P_{fc}(X)$ is a sequence such that $A_n \xrightarrow{M} A$ and $\{u_n\}_{n\geqslant 1} \subseteq X$ is a sequence such that $u_n \longrightarrow u$ in X. Show that $\operatorname{dist}(u_n, A_n) \longrightarrow \operatorname{dist}(u, A)$.

Problem 2.137**
Suppose that (Ω, Σ) is a measurable space, X is a σ-compact metric space and $\{F_n\colon \Omega \longrightarrow P_f(X)\}_{n\geqslant 1}$ is a sequence of measurable multifunctions with

$$\limsup_{n\to+\infty} F_n(\omega) \;\neq\; \emptyset \qquad \forall\, \omega \in \Omega.$$

Show that the multifunction $\omega \longmapsto H(\omega) = \limsup\limits_{n\to+\infty} F_n(\omega)$ is measurable.

Problem 2.138*
Suppose that X is a Banach space, $A \in 2^X \setminus \{\emptyset\}$, $\{A_n\}_{n \geqslant 1} \subseteq 2^X \setminus \{\emptyset\}$ is a sequence such that

$$\limsup_{n \to +\infty} \operatorname{dist}(u, A_n) \ \leqslant\ \operatorname{dist}(u, A) \qquad \forall\, u \in X.$$

Show that $A \subseteq s\text{--}\liminf\limits_{n \to +\infty} A_n$.

Problem 2.139**
Assume that X is a Banach space, $C \in P_{wk}(X)$ and $\{A_n\}_{n \geqslant 1} \subseteq 2^X \setminus \{\emptyset\}$ is a sequence such that $A_n \subseteq C$ for all $n \geqslant 1$. Show that $w\text{--}\limsup\limits_{n \to +\infty} A_n \neq \emptyset$ and

$$\sigma_{\substack{w\text{--}\limsup A_n \\ n \to +\infty}}(u^*) \ =\ \limsup_{n \to +\infty} \sigma_{A_n}(u^*) \qquad \forall\, u^* \in X^*.$$

Problem 2.140*
Suppose that X is a Banach space, $A \in 2^X \setminus \{\emptyset\}$ and $\{A_n\}_{n \geqslant 1} \subseteq 2^X \setminus \{\emptyset\}$ is a sequence such that

$$\limsup_{n \to +\infty} \sigma_{A_n}(u^*) \ \leqslant\ \sigma_A(u^*) \qquad \forall\, u^* \in X^*.$$

Show that $w\text{--}\limsup\limits_{n \to +\infty} A_n \subseteq \overline{\operatorname{conv}}\, A$.

Problem 2.141***
Suppose that (Ω, Σ, μ) is a σ-finite measure space, X is a Banach space, $h \in L^p(\Omega; X)$, $\{h_n\}_{n \geqslant 1} \subseteq L^p(\Omega; X)$ (with $1 \leqslant p < +\infty$) is a sequence such that $h_n \xrightarrow{w} h$ in $L^p(\Omega; X)$ and

$$h_n(\omega) \ \in\ G(\omega) \ \in\ P_{wk}(X) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Show that

$$h(\omega) \ \in\ \overline{\operatorname{conv}}\ w\text{--}\limsup_{n \to +\infty} \{h_n(\omega)\} \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Problem 2.142***
Suppose that (Ω, Σ, μ) is a σ-finite measure space, X and Y are two separable Banach spaces and $F\colon \Omega \times X \longrightarrow P_{fc}(Y)$ is a multifunction such that:

(i) for all $u \in X$, the multifunction $\omega \longmapsto F(\omega,u)$ is graph measurable;
(ii) for μ-almost all $\omega \in \Omega$, the multifunction $u \longmapsto F(\omega,u)$ has a graph which is closed in $X \times Y_w$ (where Y_w denotes the Banach space Y with the w-topology);
(iii) $F(\omega,u) \subseteq G(\omega)$ for μ-almost all $\omega \in \Omega$, all $u \in X$ with a graph measurable and integrably bounded multifunction $G\colon \Omega \longrightarrow P_{wkc}(Y)$.

Show that $S^1_{F(\cdot,v(\cdot))} \neq \emptyset$ for any $v \in L^1(\Omega;X)$.

Problem 2.143*
Suppose that (Ω,Σ,μ) is a σ-finite measure space, X is a separable Banach spaces, and $\{h_n\}_{n\geqslant 1} \subseteq L^1(\Omega;X)$ is a sequence which is uniformly integrable and such that

$$\overline{\{h_n(\omega)\}_{n\geqslant 1}}^{w} \in P_{wk}(X) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Show that the sequence $\{h_n\}_{n\geqslant 1} \subseteq L^1(\Omega;X)$ is relatively w-compact.

Problem 2.144*
Suppose that (Ω,Σ,μ) is a finite measure space, $h \in L^1(\Omega)$, $\{h_n\}_{n\geqslant 1} \subseteq L^1(\Omega)$ is a sequence such that $h_n \xrightarrow{w} h$ in $L^1(\Omega)$ and assume that at least one of the following conditions holds:
(i) $h(\omega) \leqslant \liminf\limits_{n\to+\infty} h_n(\omega)$ for μ-almost all $\omega \in \Omega$; or
(ii) $\limsup\limits_{n\to+\infty} h_n(\omega) \leqslant h(\omega)$ for μ-almost all $\omega \in \Omega$.

Show that $h_n \longrightarrow h$ in $L^1(\Omega)$.

Problem 2.145**
Show that in general for a sequence $\{A_n\}_{n\geqslant 1} \subseteq P_f(X)$ in a Banach space X, the set $w\text{--}\limsup\limits_{n\to+\infty} A_n$ need not be closed or weakly closed.

Problem 2.146*
Let X be a Banach space and let $\{A_n\}_{n\geqslant 1} \subseteq P_{fc}(X)$ be a sequence such that

$$A_{n+1} \subseteq A_n \quad (\text{respectively } A_{n+1} \supseteq A_n) \qquad \forall\, n \geqslant 1.$$

Show that $A_n \xrightarrow{M} \bigcap\limits_{n\geqslant 1} A_n$ (respectively $A_n \xrightarrow{M} \overline{\bigcup\limits_{n\geqslant 1} A_n}$).

Problem 2.147**
Suppose that (Ω, Σ, μ) is a σ-finite measure space, X and Y are two separable
Banach spaces and $F\colon \Omega \times X \longrightarrow 2^Y \setminus \{\emptyset\}$ is a multifunction such that:

(i) F is measurable on $\Omega \times X$;
(ii) for μ-almost all $\omega \in \Omega$, the multifunction $u \longmapsto F(\omega, u)$ is lower semicontinuous;
(iii) there exists $\xi \in L^1(\Omega)$ such that

$$|F(\omega, u)| = \sup_{y \in F(\omega,u)} \|y\|_Y \leqslant \xi(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega, \text{ all } u \in X.$$

Show that the multifunction $h \longmapsto S^1_{F(\cdot,h(\cdot))}$ is lower semicontinuous from $L^1(\Omega; X)$ into $L^1(\Omega; Y)$.

Problem 2.148*
Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ be a monotone map with $D(A) \neq \emptyset$. Show that there exists a maximal monotone extension $\widehat{A}\colon X \supseteq D(\widehat{A}) \longrightarrow 2^{X^*}$ of A (that is, a maximal monotone map $\widehat{A}\colon X \supseteq D(\widehat{A}) \longrightarrow 2^{X^*}$ such that $\operatorname{Gr} A \subseteq \operatorname{Gr} \widehat{A}$).

Problem 2.149**
Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ be a maximal monotone map. Show that $\operatorname{Gr} A$ is closed in $X_w \times X^*$ and in $X \times X^*_{w^*}$ (here X_w (respectively $X^*_{w^*}$) denotes the Banach space X (respectively X^*) furnished with the weak (respectively weak*) topology). Can we say that $\operatorname{Gr} A$ is closed in $X_w \times X^*_{w^*}$? Justify your answer.

Problem 2.150*
Suppose that $T = (0, b)$, H is a Hilbert space, and $u_0 \in H$. We define $A\colon L^2(T; H) \supseteq D(A) \longrightarrow L^2(T; H)$ by

$$A(u) = \tfrac{d}{dt} u \qquad \forall\, u \in D(A) = \big\{u \in W^{1,2}(T; H) :\ u(0) = u_0\big\}$$

(see Problem 1.104). Show that A is maximal monotone.

Problem 2.151**
Let X be a finite dimensional Banach space and let $A\colon X \longrightarrow X^*$ be a monotone map with $D(A) = X$ which is also hemicontinuous (see Definition 2.110(b)). Show that A is continuous.

Problem 2.152**
Let X be a Banach space and let $A\colon X \longrightarrow X^*$ be a monotone map with $D(A) = X$. Show that A is demicontinuous if and only if it is hemicontinuous.

Problem 2.153*
Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with a Lipschitz boundary $\partial\Omega$, $1 < p < +\infty$ and let $a\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous and monotone map satisfying the following growth condition

$$|a(y)| \;\leqslant\; c(1 + |y|^{p-1}) \qquad \forall\, y \in \mathbb{R}^N,$$

for some $c > 0$. For every $u \in W_0^{1,p}(\Omega)$, let $A(u) = -\operatorname{div} a(Du)$. Show that $A\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ (where $\frac{1}{p} + \frac{1}{p'} = 1$) is bounded (that is, maps bounded sets to bounded ones), continuous, and maximal monotone.

Problem 2.154**
Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with a Lipschitz boundary $\partial\Omega$, $1 < p < +\infty$ and let $a\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous and monotone map satisfying the following conditions

$$|a(y)| \;\leqslant\; c(1 + |y|^{p-1}) \qquad \forall\, y \in \mathbb{R}^N,$$

for some $c > 0$ and

$$(a(y), y)_{\mathbb{R}^N} \;\geqslant\; \widehat{c}|y|^p \qquad \forall\, y \in \mathbb{R}^N,$$

for some $\widehat{c} > 0$. Let $g \in W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$. Show that there exists $u \in W_0^{1,p}(\Omega)$ such that

$$-\operatorname{div} a(Du) \;=\; g \quad \text{in } W^{-1,p'}(\Omega).$$

Problem 2.155*
Let X be a reflexive Banach space and let $A\colon X \supseteq D(A) \longrightarrow X^*$ be a linear monotone map. Show that the map A is maximal monotone if and only if $\operatorname{Gr} A$ is maximal among all linear monotone subsets (graphs) of $X \times X^*$.

Problem 2.156**
Let X be a reflexive Banach space and let $A\colon X \longrightarrow X^*$ be a linear demicontinuous monotone map. Show that $A \in \mathcal{L}(X; X^*)$.

Problem 2.157**
Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ be a monotone map which is lower semicontinuous at $\widehat{u} \in D(A) \subseteq X$ into $X^*_{w^*}$ (where $X^*_{w^*}$ denotes the space X^* furnished with the w^*-topology). Show that $A(\widehat{u})$ is a singleton.

Problem 2.158***
Let X be a separable Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ be a maximal monotone map with $\operatorname{int} D(A) \neq \emptyset$. Show that the set

$$\Gamma \; = \; \{u \in D(A) : \; A(u) \text{ is not a singleton}\}$$

is of first category.

Problem 2.159**
Show that the duality map of a Banach space X is linear if and only if X is a Hilbert space.

Problem 2.160***
Suppose that X is a reflexive Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a monotone map, and C is a closed convex set such that $D(A) \subseteq C$. Show that there exists a maximal monotone map $\widehat{A}\colon X \supseteq D(\widehat{A}) \longrightarrow 2^{X^*}$ such that $\operatorname{Gr} A \subseteq \operatorname{Gr} \widehat{A}$ and $D(\widehat{A}) \subseteq C$.

Problem 2.161**
Let X be a reflexive Banach space with X and X^* both locally uniformly convex (which is always possible thanks to the Troyanski renorming theorem; see Theorem I.5.192 or Remark 2.115) and let $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ be a maximal monotone map. Show that for every $u \in X$ and for every $\lambda > 0$, there exists unique $u_\lambda \in D(A)$ such that $0 \in \lambda A(u_\lambda) + \mathcal{F}(u_\lambda - u)$.

Remark. According to this problem, for every $\lambda > 0$ we can define the map $J_\lambda\colon X \longrightarrow D(A)$ by setting

$$J_\lambda(u) \; = \; u_\lambda \; \in \; D(A)$$

and then the map $A_\lambda\colon X \longrightarrow X^*$ by setting

$$A_\lambda(u) \; = \; -\tfrac{1}{\lambda}\mathcal{F}(u_\lambda - u) \; = \; \tfrac{1}{\lambda}\mathcal{F}(u - u_\lambda).$$

The map J_λ is called the ***resolvent of*** A and A_λ is called the ***Yosida approximation of*** A. They are extensions to Banach spaces of the items introduced in Definition 2.122.

Problem 2.162**
Suppose that X is a reflexive Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a maximal monotone map, $\{(u_n, u_n^*)\}_{n\geqslant 1} \subseteq \operatorname{Gr} A$ is a sequence such that $u_n \xrightarrow{w} u$ in X and $u_n^* \xrightarrow{w} u^*$ in X^* and assume that

$$\limsup_{n,k\to+\infty} \langle u_n^* - u_k^*, u_n - u_k\rangle \ \leqslant\ 0 \quad \text{or} \quad \limsup_{n\to+\infty} \langle u_n^* - u^*, u_n - u\rangle \ \leqslant\ 0.$$

Show that $(u, u^*) \in \operatorname{Gr} A$ and $\langle u_n^*, u_n\rangle \longrightarrow \langle u^*, u\rangle$.

Problem 2.163***
Let X be a reflexive Banach space with X and X^* both locally uniformly convex. Show that for every $\lambda > 0$, the Yosida approximation $A_\lambda\colon X \longrightarrow X^*$ (see the Remark after Problem 2.161) is everywhere defined, single valued, monotone, bounded, demicontinuous and satisfies

$$A_\lambda(u) \ \in\ A(J_\lambda(u)) \qquad \forall\, u \in X$$

and $J_\lambda\colon X \longrightarrow D(A)$ is continuous.

Problem 2.164**
Let X be a reflexive Banach space with X and X^* both locally uniformly convex and let $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ be a maximal monotone map. Show that

$$A_\lambda(u) \ \longrightarrow\ A^0(u) \quad \text{in } X^* \quad \text{as } \lambda \searrow 0 \qquad \forall\, u \in D(A),$$

with $A^0(u) \in A(u)$ such that

$$\|A^0(u)\|_{X^*} \ =\ \inf_{u^*\in A(u)} \|u^*\|_{X^*}$$

(the ***minimal section of*** A) and

$$J_\lambda(u) \ \longrightarrow\ u \quad \text{in } X \quad \text{as } \lambda \searrow 0 \qquad \forall\, u \in \overline{\operatorname{conv}}\, D(A).$$

Problem 2.165**
Let X be a reflexive Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ be a maximal monotone map. Show that $\overline{D(A)}$ and $\overline{R(A)}$ are both convex.

Problem 2.166**
Let $A\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a surjective monotone map. Show that $\lim\limits_{|u|\to+\infty} |A(u)| = +\infty$.

Problem 2.167*
Let $A\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a monotone map on $B_{r+\varepsilon} = \{u \in \mathbb{R}^N : |u| < r+\varepsilon\}$ (where $r, \varepsilon > 0$). Show that the set $A(\overline{B}_r) \subseteq \mathbb{R}^N$ is bounded.

Problem 2.168**
Suppose that $T = (a,b)$, H is a Hilbert space, $u_0 \in H$ and $A\colon L^2(T;H) \supseteq D(A) \longrightarrow L^2(T;H)$ is defined by

$$A(u) = \tfrac{d}{dt}u \qquad \forall\, u \in D(A),$$

with $D(A) = \{u \in W^{1,2}(T;H) : u(0) = u_0\}$ (see Problem 1.104). Find the resolvent J_1^A (see Definition 2.122).

Problem 2.169**
Let X be a reflexive Banach space and let $A\colon X \longrightarrow X^*$ be an everywhere defined uniformly monotone and hemicontinuous map. Show that A is surjective.

Problem 2.170**
Suppose that X is a reflexive Banach space and $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a bounded map (that is, A maps bounded sets to bounded sets) which satisfies the following condition: if $\{(u_n, u_n^*)\}_{n\geqslant 1} \subseteq \operatorname{Gr} A$ is a sequence such that $u_n \xrightarrow{w} u$ in X, $u_n^* \xrightarrow{w^*} u^*$ in X^* and $\limsup\limits_{n\to+\infty} \langle u_n^*, u_n - u\rangle \leqslant 0$, then $(u, u^*) \in \operatorname{Gr} A$. Show that A is upper semicontinuous from every finite dimensional subspace of X into X_w^* (where X_w^* denotes the space X^* furnished with the w-topology).

Problem 2.171**
Suppose that X is a reflexive Banach space and $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ and $C\colon X \supseteq D(C) \longrightarrow X^*$ are two monotone maps with $D(C) = X$. Assume that the map $u \longmapsto (A+C)(u)$ is maximal monotone. Show that A is maximal monotone.

Problem 2.172**
Suppose that (Ω, Σ, μ) is a σ-finite measure space, H is a Hilbert space, $A\colon H \supseteq D(A) \longrightarrow 2^H$ is a maximal monotone map with $(0,0) \in \operatorname{Gr} A$ and $\widehat{A}\colon L^2(\Omega;H) \supseteq D(\widehat{A}) \longrightarrow L^2(\Omega;H)$ is the realization of A on the Hilbert space $L^2(\Omega;H)$, that is,

$$\widehat{A}(u) = \{h \in L^2(\Omega;H) : h(\omega) \in A(u(\omega)) \text{ for } \mu\text{-a.a. } \omega \in \Omega\}$$

for all $u \in D(\widehat{A}) = \{v \in L^2(\Omega; H) : S^2_{A(v(\cdot))} \neq \emptyset\}$. Show that $\widehat{A}$ is maximal monotone and find its Yosida approximation $\widehat{A}_\lambda$, for $\lambda > 0$. If $\mu(\Omega) < +\infty$, then we can drop the requirement that $(0,0) \in \operatorname{Gr} A$.

Problem 2.173*
Given two maximal monotone maps A and C which are defined everywhere, is it true that $R(A + C) = R(A) + R(C)$? Justify your answer.

Problem 2.174**
Suppose that (Ω, Σ) is a complete measurable space, X is a separable reflexive Banach space and for every $\omega \in \Omega$, the map $A(\omega)\colon X \supseteq D(A(\omega)) \longrightarrow 2^{X^*}$ is maximal monotone. Show that the following two properties are equivalent:

(a) The multifunction $\omega \longmapsto \operatorname{Gr} A(\omega)$ is measurable from Ω into $P_f(X \times X^*)$.

(b) For every $u^* \in X^*$, the map $\omega \longmapsto (A(\omega) + \mathcal{F})^{-1}(u^*)$ is Σ-measurable from Ω into X.

Problem 2.175**
Suppose that (Ω, Σ) is a complete measurable space, X is a separable reflexive Banach space and

$$\{A(\omega)\colon X \supseteq D(A(\omega)) = D(\omega) \longrightarrow 2^{X^*}\}_{\omega \in \Omega}$$

is a family of maximal monotone maps satisfying property **(a)** (or equivalently property **(b)**) in Problem 2.174. Show that:

(1) for every $E \in \mathcal{B}(X)$, we have $D^-(E) = \{\omega \in \Omega : D(\omega) \cap E \neq \emptyset\} \in \Sigma$;

(2) for every $u \in X$, the multifunction $\omega \longmapsto A(\omega)(u)$ is measurable from Ω into X^*;

(3) for every $u \in X$, we have $\Omega(u) = \{\omega \in \Omega : u \in D(\omega)\} \in \Sigma$ and the minimal section map $\omega \longmapsto A^0(\omega)(u)$ (see Problem 2.164) is Σ-measurable from $\Omega(u)$ into X^*.

Problem 2.176**
Suppose that X is a reflexive Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a maximal monotone surjective map with $D(A) = X$ and $C \subseteq X$ is a nonempty, closed, and convex set. Show that the set $A(C) \subseteq X^*$ is closed.

Problem 2.177**
Suppose that X is a strictly convex (see Definition I.5.168 and Remark I.5.169), reflexive Banach space with a strictly convex dual X^*, $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ and $C\colon X \supseteq D(C) \longrightarrow 2^{X^*}$ are two maximal monotone maps such that $D(A) \cap D(C) \neq \emptyset$. Show that for every $h^* \in X^*$, the set of solutions $\{u_\lambda\}_{\lambda>0}$ of the equations $h^* \in (A + C_\lambda + \mathcal{F})(u_\lambda)$ remains bounded in X as $\lambda \searrow 0$.

Problem 2.178**
Let X be a Banach space and let $u, y \in X$. Show that the following two properties are equivalent:
(a) $\|u\|_X \leqslant \|u + \lambda y\|_X$ for all $\lambda > 0$.
(b) There exists $u^* \in \mathcal{F}(u)$ such that $\langle u^*, y\rangle \geqslant 0$.

Problem 2.179**
Suppose that X is a reflexive Banach space with a strictly convex dual X^* (see Definition I.5.168 and Remark I.5.169), $\mathcal{F}\colon X \longrightarrow X^*$ is the duality map, $C \subseteq X$ is a nonempty and convex set and $u_0 \in C$. Show that $\|u_0\|_X = \inf\limits_{u\in C} \|u\|_X$ if and only if

$$\langle \mathcal{F}(u_0), u_0\rangle \;\leqslant\; \langle \mathcal{F}(u_0), u\rangle \qquad \forall\, u \in C.$$

Problem 2.180*
Suppose that X is a reflexive Banach space with both X and X^* being strictly convex, $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a maximal monotone multifunction, $\{\lambda_n\}_{n\geqslant 1} \subseteq \mathbb{R}$ and $\{u_n\}_{n\geqslant 1} \subseteq X$ are two sequences such that

$$\lambda_n \longrightarrow 0, \quad u_n \xrightarrow{w} u \quad \text{in } X \quad \text{and} \quad A_{\lambda_n}(u_n) \xrightarrow{w} y^* \quad \text{in } X^*$$

and

$$\limsup_{n,m\to+\infty} \langle A_{\lambda_n}(u_n) - A_{\lambda_m}(u_m), u_n - u_m\rangle \;\leqslant\; 0.$$

Show that $(u, y^*) \in \operatorname{Gr} A$ and

$$\limsup_{n,m\to+\infty} \langle A_{\lambda_n}(u_n) - A_{\lambda_m}(u_m), u_n - u_m\rangle \;=\; 0.$$

Problem 2.181***
Suppose that X is a strictly convex reflexive Banach space with a strictly convex dual X^* (see Definition I.5.168 and Remark I.5.169),

$A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ and $C\colon X \supseteq D(C) \longrightarrow 2^{X^*}$ are two maximal monotone maps such that $D(A) \cap D(C) \neq \emptyset$. Show that if for every $\lambda > 0$, $u_\lambda \in X$ is the unique solution of the operator inclusion

$$h^* \;\in\; (A + C_\lambda + \mathcal{F})(u_\lambda)$$

and $\{C_\lambda(u_\lambda)\}_{\lambda>0} \subseteq X^*$ is bounded as $\lambda \searrow 0$, then $h^* \in R(A+C+\mathcal{F})$.

Problem 2.182*
Let X be a reflexive Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ be a maximal monotone map. We set

$$m(u) \;=\; \inf_{u^* \in A(u)} \|u^*\|_*$$

(with $\inf \emptyset = +\infty$). Show that the function $m\colon X \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.

Problem 2.183**
Suppose that X is a reflexive Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ and $C\colon X \supseteq D(C) \longrightarrow X^*$ are two monotone maps such that $D(C) = X$, $R(A+C) = X^*$ and $(A+C)^{-1}$ is continuous. Show that A is maximal monotone.

Problem 2.184**
Let X be a Banach space and let $\mathcal{F}$ be its duality map. Show that X is reflexive if and only if $R(\mathcal{F}) = X^*$.

Problem 2.185*
Let X be a locally uniformly convex Banach space with a strictly convex dual X^* (see Definition I.5.168 and Remark I.5.169). Show that the duality map $\mathcal{F}\colon X \longrightarrow X^*$ is of type $(S)_+$ (see Definition 2.137).

Problem 2.186**
Suppose that X is a reflexive Banach space, $A\colon X \longrightarrow X^*$ is a demicontinuous map of type $(S)_+$, and $K\colon X \longrightarrow X^*$ is a compact map. Show that the map $T\colon X \longrightarrow X^*$ defined by

$$T(u) \;=\; (A+K)(u) \qquad \forall\, u \in X$$

is generalized pseudomonotone (see Definition 2.131).

Problem 2.187**
Suppose that X is a reflexive Banach space, $A\colon X \longrightarrow X^*$ is a generalized pseudomonotone map with $D(A) = X$ and $K\colon X \longrightarrow X^*$ is a map which is sequentially continuous from X_w (where X_w denotes the Banach space X furnished with the weak topology) into X_w^* (where X_w^* denotes the Banach space X^* furnished with the weak topology) and such that the function $u \longmapsto \varphi(u) = \langle K(u), u\rangle$ is sequentially weakly lower semicontinuous. Let $\{u_n\}_{n\geqslant 1} \subseteq X$ be a sequence such that $u_n \xrightarrow{w} u$ in X, $(A+K)(u_n) \xrightarrow{w} u^*$ in X^* and $\limsup\limits_{n\to+\infty} \langle (A+K)(u_n), u_n - u\rangle \leqslant 0$. Show that $u^* \in (A+K)(u)$.

Problem 2.188*
Suppose that X is a reflexive Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a generalized pseudomonotone map and $C \subseteq X$ is a nonempty, bounded, and weakly closed set. Show that the set

$$A(C) \;=\; \{u^* \in X^* : \; u^* \in A(u) \text{ for some } u \in C\}$$

is strongly closed in X^*.

Problem 2.189**
Let H be a Hilbert space and let $K\colon H \longrightarrow H$ be a compact map. Show that the map $u \longmapsto u + K(u)$ is generalized pseudomonotone.

Problem 2.190**
Let X be a reflexive Banach space and let $K\colon X \longrightarrow X^*$ be a compact map. Is K necessarily generalized pseudomonotone? Justify your answer.

Problem 2.191*
Let X be a reflexive Banach space and let $A\colon X \longrightarrow X^*$ be a monotone map which is sequentially weakly continuous. Show that the function $u \longmapsto \varphi(u) = \langle A(u), u\rangle$ is sequentially weakly lower semicontinuous.

Problem 2.192***
Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with a Lipschitz boundary and let $G\colon \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$ satisfies:

(i) for all $y \in \mathbb{R}^N$, the function $z \longmapsto G(z,y)$ is measurable;
(ii) for almost all $z \in \Omega$, the function $y \longmapsto G(z,y)$ is C^1, strictly convex and $G(z,0) = 0$;

(iii) there exist $\widehat{a} \in L^\infty(\Omega)$ and $1 < p < +\infty$ such that

$$|\nabla_y G(z,y)| \leqslant \widehat{a}(z)(1 + |y|^{p-1})$$

for almost all $z \in \Omega$ and all $y \in \mathbb{R}^N$;

(iv) we have

$$(\nabla_y G(z,y), y)_{\mathbb{R}^N} \leqslant pG(z,y)$$

for almost all $z \in \Omega$ and all $y \in \mathbb{R}^N$;

(v) there exists $c_0 > 0$ such that

$$c_0|y|^p \leqslant pG(z,y)$$

for almost all $z \in \Omega$ and all $y \in \mathbb{R}^N$.

Let $a(z,y) = \nabla_y G(z,y)$ and let $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int\limits_\Omega (a(z, Du(z)), Dh(z))_{\mathbb{R}^N}\, dz \qquad \forall\, u, h \in W^{1,p}(\Omega).$$

Show that A is maximal monotone and of type $(S)_+$.

Remark. If $G(z,y) = G(y) = \frac{1}{p}|y|^p$ for all $y \in \mathbb{R}^N$ with $1 < p < +\infty$, then $a(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^N$ and so A corresponds to the p-Laplacian differential operator with Neumann boundary condition.

Problem 2.193**
Suppose that H is a Hilbert space, $A\colon H \supseteq D(A) \longrightarrow 2^H$ is a maximal monotone map, and $C\colon H \supseteq D(C) \longrightarrow H$ is a monotone map with $D(C) \subseteq H$ closed and satisfies

$$\|C(u) - C(y)\|_H \leqslant k\|u - y\|_H \qquad \forall\, u, y \in D(C),$$

for some $k \in (0,1)$. Show that the map $u \longmapsto (A + C)(u)$ is maximal monotone.

Problem 2.194*
Let X be a reflexive Banach space and let $A\colon X \longrightarrow X^*$ be a demicontinuous map of type $(S)_+$. Show that A is pseudomonotone.

Problem 2.195**
Let X be a reflexive Banach space and let $A\colon X \longrightarrow X^*$ be a demicontinuous, strongly coercive (see Definition 2.98(f)) and bounded map which is of type $(S)_+$. Show that A is surjective.

Problem 2.196**
Let $\Omega \subseteq \mathbb{R}^N$ (with $N \leqslant 3$) be a bounded open set and let $A\colon H_0^1(\Omega) \longrightarrow H^{-1}(\Omega) = H_0^1(\Omega)^*$ be defined by

$$\langle A(u), h\rangle \;=\; \int_\Omega \sum_{k=1}^N (\sin u(z))(D_k u(z))h(z)\,dz \qquad \forall\, u,h \in H_0^1(\Omega).$$

Show that A is completely continuous.

Problem 2.197***
Let H be a Hilbert space and let $A\colon H \supseteq D(A) \longrightarrow H$ be a linear operator (not necessarily bounded) which is monotone (positive) and symmetric. Show that A is self-adjoint if and only if A is maximal monotone.

Problem 2.198**
Suppose that H is a Hilbert space, $A\colon H \supseteq D(A) \longrightarrow H$ and $T\colon H \supseteq D(T) \longrightarrow H$ are two linear monotone operators such that A is self-adjoint and $A \subseteq T$ (i.e., $\operatorname{Gr} A \subseteq \operatorname{Gr} T$). Show that $A = T$.

Problem 2.199*
Let X be a Banach space with a strictly convex dual X^* and let $A\colon X \supseteq D(A) \longrightarrow 2^X$ be a maximal accretive map. Show that for every $u \in D(A)$, we have $A(u) \in P_{fc}(X)$.

Problem 2.200**
- **(a)** Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^X$ be a maximal accretive map. Show that $\operatorname{Gr} A \subseteq X \times X$ is closed.
- **(b)** Let X be a Banach space with locally uniformly convex dual X^* and let $A\colon X \supseteq D(A) \longrightarrow 2^X$ be a maximal accretive map. Show that $\operatorname{Gr} A$ is closed in $X \times X_w$ (where X_w denotes the Banach space X furnished with the weak topology).

Problem 2.201**
Let X be a uniformly convex Banach space with a uniformly convex dual X^* and let $A\colon X \supseteq D(A) \longrightarrow 2^X$ be an m-accretive operator. Show that for all $u \in D(A)$ we have

$$\lim_{\lambda\searrow 0} A_\lambda(u) \;=\; A^0(u) \quad\text{and}\quad \lim_{\lambda\searrow 0} A^0(J_\lambda(u)) \;=\; A^0(u).$$

Problem 2.202**
Let X be a reflexive Banach space with uniformly convex dual X^* and let $A\colon X \supseteq D(A) \longrightarrow 2^X$ be a maximal accretive map, which is locally bounded. Show that A is upper semicontinuous from X into X_w (where X_w denotes the Banach space X furnished with the weak topology).

Problem 2.203**
Suppose that X is a Banach space, $u\colon \mathbb{R} \longrightarrow X$ is a map which is almost everywhere weakly differentiable and the map $t \longmapsto \|u(t)\|_X$ is almost everywhere differentiable. Show that for almost all $t \in \mathbb{R}$, we have

$$\|u(t)\|_X \tfrac{d}{dt}\|u(t)\|_X \;=\; \langle u^*, u'(t)\rangle \qquad \forall\, u^* \in \mathcal{F}(u(t)).$$

Problem 2.204*
Let X be a Banach space and let $A\colon X \longrightarrow X$ be a continuous accretive map such that $D(A) = X$. Show that A is m-accretive.

Problem 2.205**
Let X be a reflexive Banach space with a uniformly convex dual X^* and let $A\colon X \longrightarrow P_{fc}(X)$ be an accretive map which is upper semicontinuous from X into X_w (where X_w denotes the Banach space X furnished with the weak topology). Show that A is maximal accretive.

Problem 2.206*
Suppose that X is a uniformly convex Banach space with uniformly convex dual X^*, $C \subseteq X$ is a nonempty and closed set and $\{S(t)\colon C \longrightarrow C\}_{t\geqslant 0}$ is a semigroup of nonexpansive maps (see Definition 2.156) with infinitesimal generator A and which has the following property: "if $\{u_n\}_{n\geqslant 1} \subseteq D(A)$ is a sequence such that $\|u_n\|_X \longrightarrow +\infty$, then $\|A(u_n)\|_X \longrightarrow +\infty$." Show that for every $u \in D(A)$, we have $\sup\limits_{t\geqslant 0} \|S(t)u_0\|_X < +\infty$.

Problem 2.207*
Suppose that X is a Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^X$ is an m-accretive map and $\{S(t)\colon \overline{D(A)} \longrightarrow \overline{D(A)}\}_{t\geqslant 0}$ is the semigroup of nonexpansive maps generated by A (see Theorem 2.158). Show that for every $u \in \overline{D(A)}$ and $t > 0$, we have

$$\|S(t)u - u\|_X \leqslant \frac{2}{t} \int_0^t \|S(\tau)u - u\|_X \, d\tau.$$

Problem 2.208*
Suppose that X is a Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^X$ is an m-accretive map and $\{S(t)\colon \overline{D(A)} \longrightarrow \overline{D(A)}\}_{t \geqslant 0}$ is the nonlinear semigroup of nonexpansive maps generated by A (see Theorem 2.158). Show that for every $u \in \overline{D(A)}$, every $t > 0$ and every $\lambda > 0$, we have

$$\|S(t)u - u\|_X \leqslant \left(2 + \tfrac{t}{\lambda}\right) \|u - J_\lambda(u)\|_X$$

and

$$\|u - J_\lambda(u)\|_X \leqslant \tfrac{2}{t}\left(1 + \tfrac{\lambda}{t}\right) \int_0^t \|S(\tau)u - u\|_X \, d\tau.$$

Problem 2.209***
Suppose that X is a Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^X$ is an m-accretive map and $\{S(t)\colon \overline{D(A)} \longrightarrow \overline{D(A)}\}_{t \geqslant 0}$ is the nonlinear semigroup of nonexpansive maps generated by A (see Theorem 2.158). Show that the following two properties are equivalent:

(a) The nonlinear semigroup $\{S(t)\}_{t \geqslant 0}$ is compact (see Definition 2.159(a)).

(b) For every $\lambda > 0$, the map J_λ is compact and $\{S(t)\}_{t \geqslant 0}$ is equicontinuous (see Definition 2.159(b)).

Problem 2.210**
Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^X$ be an m-accretive map. Suppose that $\{u_n\}_{n \geqslant 1} \subseteq D(A)$ is a sequence such that $u_n \longrightarrow u$ in X and $A_{\lambda_n}(u_n) \longrightarrow h$ with $\lambda_n \searrow 0$. Show that $(u, h) \in \operatorname{Gr} A$. Also show that if X is reflexive with locally uniformly convex dual, then we may assume that $A_{\lambda_n}(u_n) \xrightarrow{w} h$ (instead of $A_{\lambda_n}(u_n) \longrightarrow h$).

Problem 2.211**
Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^X$ be an m-accretive map. Show that the following two properties are equivalent:

(a) For every $\lambda > 0$, J_λ is compact.

(b) For every $\eta > 0$, the sublevel set $L_\eta = \{u \in D(A) : \|u\|_X + |A(u)| \leqslant \eta\}$ is relatively compact in X.

Problem 2.212*

Let $L^2_{2\pi}(\mathbb{R})$ be the space of $L^2(\mathbb{R})$-functions which are 2π-period. We equip $L^2_{2\pi}(\mathbb{R})$ with the norm $\|u\| = \|u|_{[0,2\pi]}\|_2$. It becomes a Hilbert space. Let $A\colon L^2_{2\pi}(\mathbb{R}) \supseteq D(A) \longrightarrow L^2_{2\pi}(\mathbb{R})$ be defined by $A(u) = u'$ for every $u \in D(A)$, where $D(A) = \big\{u \in L^2_{2\pi}(\mathbb{R}) : u' \in L^2_{2\pi}(\mathbb{R})\big\}$. Show that A is maximal monotone and it generates a C_0-semigroup $\{S(t)\}_{t\geqslant 1}$ which is not compact.

Problem 2.213**

Suppose that X is a Banach space, (E, d_E) is a compact metric space, and $F\colon E \longrightarrow P_{w^*kc}(X^*)$ is an upper semicontinuous multifunction from E into $X^*_{w^*}$ (with $X^*_{w^*}$ being the Banach space X^* furnished with the w^*-topology) and

$$\gamma(u) \;=\; \inf_{v^* \in F(u)} \|v^*\|_* \qquad \forall\, u \in E \quad \text{and} \quad \eta \;=\; \inf_{u \in E} \gamma(u).$$

Show that for every $\varepsilon > 0$, we can find a continuous map $\xi\colon E \longrightarrow X$ such that for every $u \in E$ and every $v^* \in F(u)$ we have

$$\|\xi(u)\|_X \;\leqslant\; 1 \quad \text{and} \quad \langle v^*, \xi(u)\rangle \;\geqslant\; \eta - \varepsilon.$$

Problem 2.214**

Suppose that X is a reflexive Banach space, Y, Z are two Banach spaces with X being continuously embedded in Z and $L \in \mathcal{L}_c(X;Y)$. Show that for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$\|L(u)\|_Y \;\leqslant\; \varepsilon\|u\|_X + c_\varepsilon\|u\|_Z \qquad \forall\, u \in X.$$

Problem 2.215**

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. Suppose that $p \in (1, N)$ and $q \in (1, \frac{Np-p}{N-p})$. Show that for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$\|u\|_{L^p(\partial\Omega)} \;\leqslant\; \varepsilon\|Du\|_p + c_\varepsilon\|u\|_{L^p(\Omega)} \qquad \forall\, u \in W^{1,p}(\Omega).$$

Problem 2.216**

Suppose that X is a Banach space, X^* is its topological dual, $U \subseteq X$ is a nonempty open set and $A\colon U \longrightarrow 2^{X^*} \setminus \{\emptyset\}$ is a monotone map with convex and w^*-closed values which is upper semicontinuous from X into $X^*_{w^*}$ (by $X^*_{w^*}$ we denote the Banach space X^* furnished with the w^*-topology). Show that A is maximal monotone.

Problem 2.217**

Suppose that X is a Banach space, X^* is its topological dual, $A\colon X \longrightarrow 2^{X^*} \setminus \{\emptyset\}$ is a monotone, locally bounded map with convex values and $\operatorname{Gr} A$ is closed in $X \times X^*_{w^*}$ (by $X^*_{w^*}$ we denote the Banach space X^* furnished with the w^*-topology). Show that A is maximal monotone.

Problem 2.218*

Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ be a strongly coercive and surjective map. Show that A^{-1} is locally bounded.

2.3 Solutions

Solution of Problem 2.1

Evidently for each bounded set $B \subseteq X$, the set $\overline{A(B)} \subseteq Y$ is compact. We need to show that A is also continuous (see Definition 2.1(a)). Arguing by contradiction, suppose that A is not continuous. Then for each $n \geqslant 1$, we can find $u_n \in X$ such that

$$\|A(u_n)\|_Y \;>\; n\|u_n\|_X.$$

Then the sequence $\left\{\frac{u_n}{\|u_n\|_X}\right\}_{n\geqslant 1}$ is bounded (it is on the unit sphere of X). But the sequence $\left\{A(\frac{u_n}{\|u_n\|_X})\right\}_{n\geqslant 1}$ is unbounded, a contradiction to the hypothesis.

Solution of Problem 2.2

Clearly f is continuous. Let $B \subseteq D$ be a nonempty bounded set. Given $\varepsilon > 0$, we can find $\alpha_0 = \alpha_0(\varepsilon, B) \in J$ such that

$$\|f_\alpha(u) - f(u)\|_Y \;<\; \tfrac{\varepsilon}{2} \qquad \forall\, \alpha \geqslant \alpha_0,\; u \in B. \tag{2.1}$$

We fix $\alpha \in J$, $\alpha \geqslant \alpha_0$. We have that $f_\alpha(B) \subseteq Y$ is relatively compact, in particular then totally bounded (see Definition I.1.70 and Theorem I.1.71). So, we can find $N = N(\varepsilon, B) \geqslant 1$ and $y_1, \ldots, y_N \in Y$ such that

$$f_\alpha(B) \;\subseteq\; \bigcup_{n=1}^{N} B_{\frac{\varepsilon}{2}}(y_n), \tag{2.2}$$

where $B_{\frac{\varepsilon}{2}}(y_n) = \left\{y \in Y :\; \|y - y_n\|_Y < \frac{\varepsilon}{2}\right\}$. Let $u \in B$. We can find $n_0 \in \{1, \ldots, N\}$ such that

$$\|f_\alpha(u) - y_{n_0}\|_Y \;<\; \tfrac{\varepsilon}{2}$$

(see (2.2)), so

$$\|f(u) - y_{n_0}\|_Y \;\leqslant\; \|f(u) - f_\alpha(u)\|_Y + \|f_\alpha(u) - y_{n_0}\|_Y \;<\; \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \;=\; \varepsilon$$

(see (2.1) and (2.2)), thus

$$f(B) \;\subseteq\; \bigcup_{n=1}^{N} B_\varepsilon(y_n)$$

(since $u \in B$ is arbitrary). This means that $f(B)$ is totally bounded, hence relatively compact. We conclude that $f \in K(D;Y)$ (see Definition 2.1(a)).

Solution of Problem 2.3

By hypothesis, for every $n \geqslant 1$, we can find $u_n \in D$ such that

$$\|u_n - f(u_n)\|_X < \frac{1}{n}. \tag{2.3}$$

Since f is compact (see Definition 2.1(a)), passing to a subsequence if necessary, we may assume that $f(u_n) \longrightarrow v \in \overline{F(D)}$. It follows that $u_n \longrightarrow v$ (see (2.3)) and since D is closed, we have $v \in D$. The continuity of f implies that $f(u_n) \longrightarrow f(v)$. Hence $v = f(v)$ with $v \in D$.

Solution of Problem 2.4

(a) Let $\overline{B}_1$ be the closed unit ball in X. We have

$$A(X) = \bigcup_{n \geqslant 1} nA(\overline{B}_1). \tag{2.4}$$

But since $A \in \mathcal{L}_c(X;Y)$ (see Definition 2.1(a)), we have that the set $\overline{A(\overline{B}_1)}$ is compact, hence separable. It follows that $\bigcup_{n\geqslant 1} n\overline{A(\overline{B}_1)}$ is separable and so we conclude that $A(X)$ is separable (see (2.4)).

(b) Since $A(X)$ is of second category in itself (see Definition I.1.25), from the open mapping theorem (see Theorem I.5.47), we have that $\overline{A(\overline{B}_1)}$ is a neighborhood of the origin in $(A(X), \|\cdot\|_Y)$. The compactness of A (see Definition 2.1(a)) implies that $\overline{A(\overline{B}_1)}$ is compact. So, $(A(X), \|\cdot\|_Y)$ is locally compact, hence finite dimensional (see Proposition I.5.9(a)). Therefore $A \in \mathcal{L}_f(X;Y)$ (see Definition 2.1(c)).

Solution of Problem 2.5
"**(a)** $\Longrightarrow$ **(b)**": Let $\{u_n\}_{n\geqslant 1} \subseteq f^{-1}(y)$ be a sequence. Then $f(u_n) = y$ for all $n \geqslant 1$ and so from property **(a)**, we can find a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ such that $u_{n_k} \longrightarrow u \in X$. The continuity of f implies that $f(u_{n_k}) \longrightarrow f(u)$ in Y. Hence $y = f(u)$. Therefore $u \in f^{-1}(y)$ and this shows that $f^{-1}(y)$ is compact. Let $C \subseteq X$ be closed and let $y \in \overline{f(C)}$. We can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq C$ such that $f(u_n) \longrightarrow y$ in Y. According to property **(a)**, we can find a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ such that $u_{n_k} \longrightarrow u \in C$ (recall that the set C is closed). The continuity of f implies that $f(u_{n_k}) \longrightarrow f(u)$ in Y, hence $y = f(u) \in f(C)$. This proves that $f(C)$ is closed and so f is a closed map (see Proposition 2.14).

"**(b)** $\Longrightarrow$ **(a)**": Let $\{u_n\}_{n\geqslant 1} \subseteq X$ be a sequence such that $f(u_n) \longrightarrow y \in Y$. Let $C_n = \{u_k : k \geqslant n\}$ for $n \geqslant 1$. Since f is closed, we have $f(\overline{C}_n) = \overline{f(C_n)}$. Since $f(u_n) \longrightarrow y$, we have

$$\{y\} = \bigcap_{n\geqslant 1} \overline{f(C_n)} = \bigcap_{n\geqslant 1} f(\overline{C}_n).$$

So, if $x \in f^{-1}(y)$, then

$$f(x) \in \bigcap_{n\geqslant 1} f(\overline{C}_n).$$

Then the closed sets $D_n = f^{-1}(y) \cap \overline{C}_n$ for $n \geqslant 1$, are nonempty and have the finite intersection property (see Definition I.2.80). The compactness of $f^{-1}(y)$ implies that

$$f^{-1}(y) \cap \Big(\bigcap_{n\geqslant 1} \overline{C}_n\Big) \neq \emptyset$$

(see and Theorem I.2.81). So, there is a limit point of the sequence $\{u_n\}_{n\geqslant 1}$ which belongs in $f^{-1}(y)$.

Solution of Problem 2.6
Let $C \subseteq Y$ be a compact set and let $\{u_n\}_{n\geqslant 1} \subseteq f^{-1}(C)$ be a sequence. Since $\{f(u_n)\}_{n\geqslant 1} \subseteq C$, we can find a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ such that

$f(u_{n_k}) \longrightarrow y \in C$. From property **(a)** of Problem 2.5, we know that we can find a subsequence $\{u_{n_{k_m}}\}_{m\geqslant 1}$ of $\{u_{n_k}\}_{k\geqslant 1}$ such that $u_{n_{k_m}} \longrightarrow v \in X$. The continuity of f implies that $f(u_{n_{k_m}}) \longrightarrow f(v) \in C$. Then $y = f(v) \in C$ and so $v \in f^{-1}(C)$ which establishes the compactness of $f^{-1}(C)$. Therefore f is proper (see Definition 2.13).

Solution of Problem 2.7

Let $C \subseteq X$ be a closed subset. The compactness of X implies that C is compact too (see Proposition I.1.69(b)). Then from the continuity of f, we have that the set $f(C) \subseteq Y$ is compact (see Proposition I.1.74), hence closed too (see Proposition 2.14(b)). Thus we have proved that f is a closed map (see Proposition 2.14). Also, for every $y \in Y$, the set $f^{-1}(y) \subseteq X$ is closed (due to the continuity of f), hence compact in X (since X is compact; see Proposition I.1.69(b)). So, we have verified property **(b)** of Problem 2.5. Invoking Problem 2.6, we conclude that f is proper (see Definition 2.13).

Solution of Problem 2.8

Evidently $K \in \mathcal{L}(l^2)$. For every $n \geqslant 1$, we consider $K_n \in \mathcal{L}_f(l^2)$ defined by

$$K_n(u) = \big(\vartheta_1 x_1, \vartheta_2 x_2, \ldots, \vartheta_n x_n, 0, \ldots\big) \qquad \forall\, u = \{x_n\}_{n\geqslant 1} \in l^2.$$

Since by hypothesis $\vartheta_n \longrightarrow 0$, given $\varepsilon > 0$, we can find $n_0 \geqslant 1$ such that

$$|\vartheta_n| \leqslant \varepsilon \qquad \forall\, n \geqslant n_0.$$

We have

$$\|(K-K_n)(u)\|_{l^2} = \Big(\sum_{k\geqslant n+1} \vartheta_k^2 x_k^2\Big)^{\frac{1}{2}} \leqslant \varepsilon \|u\|_{l^2} \qquad \forall\, n \geqslant n_0,\ u \in l^2,$$

so

$$\|K - K_n\|_{\mathcal{L}} \leqslant \varepsilon \qquad \forall\, n \geqslant n_0,$$

thus $K_n \longrightarrow K$ and hence $K \in \mathcal{L}_c(l^2)$ (see Problem 2.2).

Solution of Problem 2.9
Let $V = A(X) \subseteq X$ and consider $A|_V : V \longrightarrow V$. If $v \in V$, then we can find $x \in X$ such that $v = A(x)$. Hence $A(v) = A^2(x) = A(x) = v$. If follows that $A|_V = I_V$, which implies that V is finite dimensional (see Theorem I.5.22). Therefore $A \in \mathcal{L}_f(X)$ (see Definition 2.1(c)).

Solution of Problem 2.10
Let $C \subseteq D$ be a compact subset and let $E = (I_X - f)^{-1}(C)$. The continuity of $I_X - f$ implies that E is closed in X. Also, note that $E \subseteq C + f(E)$ and so from Proposition 2.32, we have

$$\gamma(E) \leqslant \gamma(C + f(E)) \leqslant \gamma(C) + \gamma(f(E)) = \gamma(f(E)). \tag{2.5}$$

If E is not compact, then $\gamma(E) > 0$ (see Proposition 2.32) and since γ is condensing (see Definition 2.31(e)), we have

$$\gamma(f(E)) < \gamma(E). \tag{2.6}$$

Comparing (2.5) and (2.6) we reach a contradiction and so we conclude that E is compact. This means that $I_X - f$ is proper (see Definition 2.13).

Solution of Problem 2.11
First let us show that $\alpha(B_1) = 2$ (see Definition 2.31). Note that by virtue of Proposition 2.32(e), we may replace B_1 by $\partial B_1 = \{x \in X : \|x\|_X = 1\}$. From Definition 2.31, it is clear that $\alpha(\partial B_1) \leqslant 2$. Suppose that $\alpha(\partial B_1) < 2$. This means that

$$\partial B_1 = \bigcup_{n=1}^{N} C_n,$$

with $\operatorname{diam} C_n < 2$ for all $n \in \{1, \ldots, N\}$. Since $\overline{\bigcup_{n=1}^{N} C_n} = \bigcup_{n=1}^{N} \overline{C}_n$, we may assume that the sets C_n are closed. Let V be an N-dimensional subspace of X. Then

$$\partial B_1 \cap V = \bigcup_{n=1}^{N} (C_n \cap V).$$

Therefore $\{C_1 \cap V, C_2 \cap V, \ldots, C_N \cap V\}$ is a closed cover of $\partial B_1 \cap V$ (the unit sphere of V) and so at least for one $n_0 \in \{1, \ldots, N\}$, the set $C_{n_0} \cap V$ must contain a pair of antipodal points. Thus we have

$$2 \;=\; \operatorname{diam}(C_{n_0} \cap V) \;\leqslant\; \operatorname{diam} C_{n_0},$$

a contradiction. This proves that $\alpha(B_1) = 2$.

Next let us show that $\beta(B_1) = 1$. Again from Definition 2.31, we have $\beta(B_1) \leqslant 1$. Suppose that $\beta(B_1) = \lambda < 1$. Let $\varepsilon > 0$ be such that $\lambda + \varepsilon < 1$. From Definition 2.31, we know that we can find $u_1, \ldots, u_N \in X$ such that

$$B_1 \;\subseteq\; \bigcup_{n=1}^{N} B_{\lambda+\varepsilon}(u_n) \;=\; \bigcup_{n=1}^{N} (u_n + (\lambda + \varepsilon) B_1).$$

Then using Proposition 2.32, we have

$$\lambda \;=\; \gamma(B_1) \;\leqslant\; (\lambda+\varepsilon)\gamma(B_1) \;=\; (\lambda+\varepsilon)\lambda \;<\; \lambda,$$

a contradiction. This proves that $\beta(B_1) = 1$.

Solution of Problem 2.12

We do the solution for $\gamma = \alpha$ and since $\beta \leqslant \alpha$ (see Definition 2.31), the result is also true for $\gamma = \beta$. For each $n \geqslant 1$ choose $u_n \in C_n$ and let $C = \{u_n\}_{n \geqslant 1}$. Since by hypothesis $\alpha(C_n) \longrightarrow 0$, given $\varepsilon > 0$, we can find $n_0 \geqslant 1$ such that

$$\alpha(\overline{C}_n) \;\leqslant\; \varepsilon \qquad \forall\, n \geqslant n_0. \tag{2.7}$$

We have

$$\begin{aligned} \alpha(C) \;&=\; \alpha\big(\{u_1, \ldots, u_{n_0}\} \cup \{u_n\}_{n \geqslant n_0+1}\big) \\ &=\; \alpha\big(\{u_n\}_{n \geqslant n_0+1}\big) \;\leqslant\; \alpha(C_{n_0}) \;\leqslant\; \varepsilon \end{aligned}$$

(see Proposition 2.32, (2.7) and recall that the sequence $\{C_n\}_{n \geqslant 1}$ is decreasing). Since $\varepsilon > 0$ is arbitrary, we let $\varepsilon \searrow 0$ and obtain $\alpha(C) = 0$, which means that $\overline{C}$ is compact. Therefore, by passing to a subsequence if necessary, we may assume that $u_n \longrightarrow u$ in X.

Evidently $u \in \overline{C}_n$ for all $n \geqslant 1$ and so $u \in \bigcap\limits_{n \geqslant 1} \overline{C}_n$. Therefore $\bigcap\limits_{n \geqslant 1} \overline{C}_n$ is nonempty. Moreover $\bigcap\limits_{n \geqslant 1} \overline{C}_n$ is closed and

$$\alpha\Big(\bigcap_{n \geqslant 1} \overline{C}_n\Big) \leqslant \alpha(\overline{C}_n) = \alpha(C_n) \qquad \forall\, n \geqslant 1,$$

so

$$\alpha\Big(\bigcap_{n \geqslant 1} \overline{C}_n\Big) = 0,$$

thus $\bigcap\limits_{n \geqslant 1} \overline{C}_n$ is compact.

Solution of Problem 2.13

We know that $\overline{A(C)}$ is compact. So, it suffices to show that $A(C) \subseteq Y$ is closed. Let $\{y_n\}_{n \geqslant 1} \subseteq A(C)$ be a sequence such that $y_n \longrightarrow y$ in Y. We have $y_n = A(u_n)$ with $u_n \in C$ for all $n \geqslant 1$. Since X is reflexive and $C \subseteq X$ is bounded, by the Eberlein–Smulian theorem (see Theorem I.5.78) and passing to a suitable subsequence if necessary, we may assume that $u_n \xrightarrow{w} u$ in X. So, $u \in \overline{C}^w$. But the convexity of C and the Mazur theorem (see Theorem I.5.58) imply that $u \in \overline{C} = C$. Also, since $A \in \mathcal{L}(X;Y)$, we have that $A \in \mathcal{L}(X_w;Y_w)$ (here X_w and Y_w denote the Banach spaces X and Y with their weak topologies). Therefore $A(u_n) \xrightarrow{w} A(u)$ in Y. Recall that $A(u_n) = y_n \longrightarrow y$ in Y. It follows that $y = A(u) \in A(C)$ and so $A(C) \subseteq Y$ is closed, hence compact.

Solution of Problem 2.14

Let $\{y_n\}_{n \geqslant 1} \subseteq Y$ be a bounded sequence. The continuity of L implies that the sequence $\{L(y_n)\}_{n \geqslant 1} \subseteq X$ is bounded. Since A is compact, it follows that the sequence $\{A(L(y_n))\}_{n \geqslant 1}$ admits a strongly convergent subsequence and this shows that $AL \in \mathcal{L}_c(Y)$ (see Problem 2.1).

Let $\{x_n\}_{n\geqslant 1} \subseteq X$ be a bounded sequence. Since $A \in \mathcal{L}_c(X;Y)$, the sequence $\{A(x_n)\}_{n\geqslant 1}$ admits a strongly convergent subsequence $\{A(x_{n_k})\}_{k\geqslant 1}$. But then $\{L(A(x_{n_k}))\}_{k\geqslant 1}$ is strongly convergent too and so by Problem 2.1, we conclude that $LA \in \mathcal{L}_c(X)$.

Solution of Problem 2.15

Hypothesis on A implies that A is injective. Let $V = A(X) \subseteq Y$. Then $A\colon X \longrightarrow V$ is bijective and so we can define $A^{-1}\colon V \longrightarrow X$ which is linear. Moreover, from the hypothesis on A, we have

$$\|A^{-1}(y)\|_X \leqslant \tfrac{1}{c}\|y\|_Y \qquad \forall\, y \in V,$$

so

$$A^{-1} \in \mathcal{L}(V;X).$$

If A is compact (see Definition 2.1(a)), then $I_X = A^{-1}A \in \mathcal{L}_c(X)$ (see Problem 2.14), hence X is finite dimensional (being locally compact; see Proposition I.5.9(a)).

On the other hand, if X is finite dimensional, then every $A \in \mathcal{L}(X;Y)$ is compact.

Solution of Problem 2.16

We argue indirectly. So, suppose that the claim of the problem is not true. Then we have

$$\inf_{u\in\partial B_1} \|A(u)\|_X > 0.$$

So, we can find $c > 0$ such that $\|A(u)\|_X \geqslant c\|u\|_X$ for all $u \in X$. From Problem 2.15, it follows that X is finite dimensional, a contradiction as $A \notin \mathcal{L}_f(X)$. This proves that $0 \in \overline{A(\partial B_1)}$.

Solution of Problem 2.17

No. Let e_n be the standard n-th basis element of l^2. Then $\|e_n\|_{l^2} = 1$ for all $n \geqslant 1$ and

$$\|S(e_n) - S(e_m)\|_{l^2} = \sqrt{2} \qquad \forall\, n,m \geqslant 1,\ n \neq m.$$

Therefore the sequence $\{S(e_n)\}_{n\geqslant 1}$ has no convergence subsequence and consequently $S \notin \mathcal{L}_c(X)$ (see Definition 2.1).

Solution of Problem 2.18

The hypothesis on A implies that we can find $u_1^*, \ldots, u_n^* \in X^*$ such that if

$$V = \big\{u \in X : \ |\langle u_k^*, u\rangle| < 1 \text{ for all } k = 1, \ldots, n\big\},$$

then $A(V) \subseteq B_1^Y = \{y \in Y : \|y\|_Y < 1\}$. Then we have $A|_{\bigcap\limits_{k=1}^{n} (u_k^*)^{-1}(0)} = 0$. Note that $\bigcap\limits_{k=1}^{n} (u_k^*)^{-1}(0)$ is finite codimensional. So, we can write

$$X = \Big(\bigcap_{k=1}^{n} (u_k^*)^{-1}(0)\Big) \oplus Z,$$

with $\dim Z < +\infty$. Observe that $A(X) = A(Z)$ and the latter is finite dimensional. Therefore $A \in \mathcal{L}_f(X;Y)$ (see Definition 2.1).

Solution of Problem 2.19

We know that

$$\|A\|_{\mathcal{L}} = \sup_{\|u\|_X \leqslant 1} \|A(u)\|_X.$$

From Problem 2.13, we know that the set $A(\overline{B}_1) \subseteq X$ is compact. Since the norm functional is continuous, from the Weierstrass theorem (see Theorem I.1.75), we know that we can find $u_0 \in X$ with $\|u_0\|_X \leqslant 1$ such that

$$\|A(u_0)\|_X = \sup_{\|u\|_X \leqslant 1} \|A(u)\|_X = \|A\|_{\mathcal{L}}.$$

Solution of Problem 2.20

Let $X = l^p$ with $1 \leqslant p < \infty$ and let $\{e_n\}_{n\geqslant 1}$ be the standard Schauder basis of l^p. Let $A \in \mathcal{L}(l^p)$ be defined by

$$A(u_1, u_2, \ldots) \;=\; (0, u_1, 0, u_3, 0, u_5, \ldots) \qquad \forall\ \{u_n\}_{n\geqslant 1} \in l^2.$$

Then $A(e_{2n-1}) = e_{2n}$ for all $n \geqslant 1$. Hence

$$\|A(e_n) - A(e_m)\|_{l^2} \;=\; 2^{\frac{1}{p}} \qquad \forall\ n, m \text{ odd},\ n \neq m,$$

so the sequence $\{A(e_n)\}_{n\geqslant 1}$ has no strongly convergent subsequence and thus A is not compact (see Definition 2.1(a)).

On the other hand,

$$A^2(u_1, u_2, \ldots) = A(0, u_1, 0, u_3, 0, u_5, \ldots) = (0, 0, \ldots) \qquad \forall\ \{u_n\}_{n\geqslant 1} \in l^2,$$

that is, $A^2 = 0$ and so it is trivially compact.

Solution of Problem 2.21

"**(a)** $\Longrightarrow$ **(b)**": Suppose that the implication is not true. So, we can find an orthonormal basis $\{e_\alpha\}_{\alpha\in J}$ and $\varepsilon > 0$ such that the set $\big\{\alpha \in J :\ |(A(e_\alpha), e_\alpha)_H| \geqslant \varepsilon\big\}$ is infinite. Thus, we can find an orthonormal sequence $\{e_n\}_{n\geqslant 1} \subseteq \{e_\alpha\}_{\alpha\in J}$ such that

$$|(A(e_n), e_n)_H| \;\geqslant\; \varepsilon \qquad \forall\ n \geqslant 1.$$

The compactness of A (see Definition 2.1(a)) implies that we can find a subsequence $\{e_{n_k}\}_{k\geqslant 1}$ of $\{e_n\}_{n\geqslant 1}$ such that $A(e_{n_k}) \longrightarrow u$ in H. By throwing away a finite number of elements of this sequence, we may assume that

$$\|A(e_{n_k}) - u\|_H \;<\; \tfrac{\varepsilon}{2} \qquad \forall\ k \geqslant 1,$$

so

$$\begin{aligned} \big|(A(e_{n_k}), e_{n_k})_H - (u, e_{n_k})_H\big| &\leqslant \|A(e_{n_k}) - u\|\,\|e_{n_k}\|_H \\ &= \|A(e_{n_k}) - u\|_H \;<\; \tfrac{\varepsilon}{2} \qquad \forall\ k \geqslant 1 \end{aligned}$$

and so

$$|(u, e_{n_k})_H| \;>\; \tfrac{\varepsilon}{2} \qquad \forall\ k \geqslant 1.$$

But this contradicts the Bessel inequality (see Theorem I.5.105(a)).

"**(b)** $\Longrightarrow$ **(c)**": Let $n \geqslant 1$ and let $\mathcal{Y}$ be the family of all orthonormal sets $\{e_\alpha\}_{\alpha \in J}$ in H such that

$$|(A(e_\alpha), e_\alpha)| \geqslant \tfrac{1}{n} \qquad \forall\, \alpha \in J.$$

By property **(b)**, J is finite. The family $\mathcal{Y}$ is partially ordered by inclusion. Let $\mathcal{D} \subseteq \mathcal{Y}$ be a chain (that is, a linear ordered subset of $\mathcal{Y}$). The union of the orthonormal set in $\mathcal{D}$ is still an element of $\mathcal{Y}$ and of course is an upper bound for $\mathcal{D}$. Thus invoking the Kuratowski–Zorn lemma (see Theorem 4.120), we infer that $\mathcal{Y}$ has a maximal element $\{e_\alpha\}_{\alpha \in L}$. Evidently L is finite. Let $Y = \operatorname{span}\{e_\alpha\}_{\alpha \in L}$. Then Y is finite dimensional and

$$|(A(u), u)_H| < \tfrac{1}{n} \qquad \forall\, u \in Y^\perp,\ \|u\|_H = 1$$

or otherwise $\{e_\alpha\}_{\alpha \in L} \cup \{u\} \in \mathcal{Y}$, which contradicts the maximality of $\{e_\alpha\}_{\alpha \in L}$. Let $P_n \in \mathcal{L}(H)$ be the orthogonal projection on Y (see Definition 2.27(c)) and let $u = (I_H - P_n)x$ with $x \in H$. We have

$$\left|(A(I_H - P_n)x, (I_H - P_n)x)_H\right| < \tfrac{1}{n},$$

so

$$\left|((I_H - P_n)A(I_H - P_n)x, x)_H\right| < \tfrac{1}{n} \qquad \forall\, x \in H,\ \|x\|_H \leqslant 1$$

(see Proposition I.5.161) and thus

$$\|(I_H - P_n)A(I_H - P_n)\|_{\mathcal{L}} < \tfrac{2}{n}.$$

The linear operator $A_n = AP_n + P_nA - P_nAP_n \in \mathcal{L}(H)$ has finite rank (i.e., $A_n \in \mathcal{L}_f(H)$; see Definition 2.1(c)) and from last inequality we see that $\|A - A_n\|_{\mathcal{L}} < \frac{2}{n}$.

"**(c)** $\Longrightarrow$ **(a)**": This implication follows from Problem 2.2 and Remark 2.2.

Solution of Problem 2.22
Evidently $L \in \mathcal{L}(L^2(S); L^2(T))$ and $\|L\|_{\mathcal{L}} \leqslant \|K\|_{L^2(T\times S)}$. Let $\{v_n\}_{n\geqslant 1}$ be an orthonormal basis for $L^2(T)$. For fixed $s \in S$, we expand $K(\cdot, s)$ with respect to this basis. So, we have

$$K(t,s) \;=\; \sum_{n\geqslant 1} v_n(t)h_n(s).$$

From the property of the Fourier coefficients, we have $h_n \in L^2(S)$. Using the Parseval relation (see Theorem I.5.105(b)), we have

$$\int_T |K(t,s)|^2\,ds \;=\; \sum_{n\geqslant 1} |h_n(s)|^2.$$

Integrating over S, we obtain

$$\int_S \int_T |K(t,s)|^2\,d\xi\,d\mu \;=\; \sum_{n\geqslant 1} \int_S |h_n(s)|^2\,d\mu.$$

We define

$$K_m(t,s) \;=\; \sum_{n=1}^{m} v_n(t)h_n(s)$$

and

$$L_m(u)(t) \;=\; \int_S K_m(t,s)u(s)\,ds \qquad \forall\; u \in L^2(S).$$

We see that $L_m \in \mathcal{L}_f(L^2(S); L^2(T))$ and we have

$$\|L - L_m\|_{\mathcal{L}}^2 \;\leqslant\; \int_T \int_S |K(t,s) - K_m(t,s)|^2\,d\mu\,d\xi \qquad \forall\; m \geqslant 1,$$

so $L_m \longrightarrow L$ in $\mathcal{L}(L^2(S); L^2(T))$. Thus operator L is compact (see Definition 2.1(a) and Problem 2.2).

Solution of Problem 2.23

Note that $hu \in C([0,1])$ for any $u \in C([0,1])$. So, A is well defined and of course linear. We have

$$\begin{aligned}\|A(u)\|_{C([0,1])} &= \sup_{t\in[0,1]} |A(u)(t)| = \sup_{t\in[0,1]} |h(t)u(t)| \\ &\leqslant \|h\|_{C([0,1])}\|u\|_{C([0,1])},\end{aligned}$$

so $A \in \mathcal{L}(C([0,1]), C([0,1]))$.

Suppose that A is compact (see Definition 2.1(a)). Then $A(\overline{B}_1) \subseteq C([0,1])$ is relatively compact (recall that $\overline{B}_1 = \{u \in C([0,1]) : \|u\|_{C([0,1])} \leqslant 1\}$). Let $\eta \in (0,1)$ and let $n \geqslant 1$ be such that $\frac{1}{n} < \eta$. We define

$$u_n(t) = \begin{cases} 0 & \text{if } 0 \leqslant t < \eta - \frac{1}{n}, \\ n(t-\eta)+1 & \text{if } \eta - \frac{1}{n} \leqslant t \leqslant \eta, \\ 1 & \text{if } \eta < t \leqslant 1, \end{cases}$$

for all $t \in [0,1]$. Then $hu_n \in A(\overline{B}_1)$ for all $n > \frac{1}{\eta}$. Since $A(\overline{B}_1) \subseteq C([0,1])$ is relatively compact, by the Arzela–Ascoli theorem (see Theorem I.2.181) it is equicontinuous (uniformly since $[0,1]$ is compact). So, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that for all $t, s \in [0,1]$ with $|t-s| \leqslant \delta$, we have

$$\big|h(t)u_n(t) - h(s)u_n(s)\big| < \varepsilon \qquad \forall\, n \geqslant 1,\ \tfrac{1}{n} < \eta.$$

We can find $n_\varepsilon \geqslant 1$ such that $\frac{1}{n} < \min\{\eta, \delta\}$ for all $n \geqslant n_\varepsilon$. For $t = \eta - \frac{1}{n}$, $s = \eta$, we have that $|t-s| = \frac{1}{n} < \delta$ and so

$$\big|h(\eta - \tfrac{1}{n})u_n(\eta - \tfrac{1}{n}) - h(\eta)u_n(\eta)\big| < \varepsilon \qquad \forall\, n \geqslant n_\varepsilon.$$

From the definition of $\{u_n\}_{n\geqslant 1}$ it follows that $|h(\eta)| < \varepsilon$ for all $\varepsilon > 0$. Let $\varepsilon \searrow 0$ to conclude that $h(\eta) = 0$ for $\eta \in (0,1)$. From the continuity of h, we have $h(0) = h(1) = 0$, that is, $h \equiv 0$, a contradiction to our hypothesis. Therefore $A(\overline{B}_1) \subseteq C([0,1])$ is not relatively compact which means that A is not a compact operator.

Solution of Problem 2.24
Let $u = \{u_n\}_{n\geqslant 1} \in l^1$. We have

$$\|u\|_{l^2} = \Big(\sum_{n\geqslant 1} u_n^2\Big)^{\frac{1}{2}} \leqslant \sum_{n\geqslant 1} |u_n| = \|u\|_{l^1}. \tag{2.8}$$

This shows that l^1 is embedded continuously into l^2. Let i denote the embedding operator. From (2.8) we see that $\|i\|_{\mathcal{L}} \leqslant 1$. On the other hand, if $e_1 = (1, 0, 0, \ldots)$, then

$$1 = \|i(e_1)\|_{l^2} \leqslant \|i\|_{\mathcal{L}}\|e_1\|_{l^1} = \|i\|_{\mathcal{L}}$$

and so we conclude that $\|i\|_{\mathcal{L}} = 1$.

Let $\{e_n\}_{n\geqslant 1}$ be the complete Schauder basis for l^1. We have

$$\{e_n\}_{n\geqslant 1} = \{i(e_n)\}_{n\geqslant 1} \subseteq i(\overline{B}_1)$$

(where $\overline{B}_1 = \big\{u \in l^1 : \|u\|_{l^1} \leqslant 1\big\}$), so i cannot be compact (see Definition 2.1(a) and Problem 2.1).

Solution of Problem 2.25
From the Schur property of l^1 (see Remark I.5.57), we know that a subset of l^1 is relatively w-compact if and only if it is norm totally bounded (see Definition I.1.70). Since X is reflexive, the closed unit ball $\overline{B}_1 = \big\{u \in X : \|u\| \leqslant 1\big\}$ is w-compact. We know that A is weak-to-weak continuous. Hence $A(\overline{B}_1) \subseteq l^1$ is w-compact (see Proposition I.2.82), thus by the Schur property mentioned above, the set $A(\overline{B}_1)$ is relatively norm compact. This implies that A is compact.

Solution of Problem 2.26
In the space

$$c_0 = \big\{\{u_n\}_{n\geqslant 1} : \{u_n\}_{n\geqslant 1} \text{ is a real sequence such that } u_n \longrightarrow 0\big\}$$

we consider the norm $\|\{u_n\}_{n\geqslant 1}\|_{c_0} = \sup\limits_{n\geqslant 1} |u_n|$. Given $\vartheta = \{\vartheta_n\}_{n\geqslant 1} \in c_0$, from Problem 2.8, we know that $A_\vartheta(u) = \{\vartheta_n u_n\}_{n\geqslant 1}$ for every

$u = \{u_n\}_{n \geqslant 1} \in l^2$ is a compact linear operator. Consider the map $\xi\colon c_0 \longrightarrow \mathcal{L}_c(l^2)$ defined by

$$\xi(\vartheta) = A_\vartheta \qquad \forall\, \vartheta \in c_0.$$

Clearly ξ is linear and for $\vartheta', \vartheta \in c_0$, we have

$$\begin{aligned} \|A_{\vartheta'} - A_\vartheta\|_{\mathcal{L}} &= \sup_{\|u\|_{l^2} \leqslant 1} \|A_{\vartheta'}(u) - A_\vartheta(u)\|_{l^2} \\ &= \sup_{\|u\|_{l^2} \leqslant 1} \Big(\sum_{n \geqslant 1} (\vartheta'_n - \vartheta_n)^2 u_n^2\Big)^{\frac{1}{2}} \\ &\leqslant \|\vartheta' - \vartheta\|_{c_0} \Big(\sup_{\|u\|_{l^2} \leqslant 1} \sum_{n \geqslant 1} u_n^2\Big)^{\frac{1}{2}} = \|\vartheta' - \vartheta\|_{c_0}, \end{aligned}$$

so $\xi \in \mathcal{L}(c_0; \mathcal{L}_c(l^2))$.

We claim that it is an isometry. To see this, it is enough to check that if $\|\vartheta\|_{c_0} = 1$, then $\|A_\vartheta\|_{\mathcal{L}} = 1$. To this end note that, if $\|u\|_{l^2} = 1$, then

$$\|A_\vartheta(u)\|_{l^2} = \Big(\sum_{n \geqslant 1} \vartheta_n^2 u_n^2\Big)^{\frac{1}{2}} \leqslant \|\vartheta\|_{c_0} \Big(\sum_{n \geqslant 1} u_n^2\Big)^{\frac{1}{2}} = \|\vartheta\|_{c_0} \|u\|_{l^2} = 1,$$

so $\|A_\vartheta\|_{\mathcal{L}} \leqslant 1$. On the other hand, given $\varepsilon > 0$, choose n_0 such that $|\vartheta_{n_0}| > 1 - \varepsilon$ (recall that $\|\vartheta\|_{c_0} = 1$). Let $e_{n_0} \in l^2$ be the standard basic element. Then $A_\vartheta(e_{n_0}) = \vartheta_{n_0} e_{n_0}$ and so

$$\|A_\vartheta(e_{n_0})\|_{l^2} = |\vartheta_{n_0}| > 1 - \varepsilon.$$

Let $\varepsilon \searrow 0$ to conclude that $\|A_\vartheta\|_{\mathcal{L}} \geqslant 1$.

Thus we have shown that $\|A_\vartheta\|_{\mathcal{L}} = 1$. This proves that ξ is an isometry. So, c_0 is isometrically isomorphic to a subspace of $\mathcal{L}_c(l^2)$. Since c_0 is not reflexive, we conclude that $\mathcal{L}_c(l^2)$ cannot be reflexive.

Solution of Problem 2.27

We argue indirectly. So, suppose that the conclusion of the problem is not true. Then we can find $c > 0$ such that

$$\|A(u)\|_Y \geqslant c\|u\|_X \qquad \forall\, u \in X.$$

Since A is compact, Problem 2.15 implies that X is finite dimensional, a contradiction to our hypothesis.

Solution of Problem 2.28

We know that $A^* \in \mathcal{L}(X^*; c_0^*)$. Since X^* is reflexive and $c_0^* = l^1$, from Problem 2.25, we infer that A^* is compact. Then Theorem 2.12 implies that A is compact.

Solution of Problem 2.29

Let $V = R(A)$. Then V is a Banach space and $A\colon X \longrightarrow V$ is surjective. So, by the open mapping theorem (see Theorem I.5.47), we can find $c > 0$ such that $cB_1^V \subseteq A(B_1^X)$ (where $B_1^X = \big(u \in X : \|u\|_X < 1\big)$ and $B_1^V = \big(u \in V : \|u\|_V < 1\big)$). Since A is compact, we have that B_1^V is relatively compact and so V is finite dimensional (being locally compact). This means that $A \in \mathcal{L}_f(X;Y)$. Let Z be a topological complement of $\ker A$ (i.e., Z is closed and $X = \ker A \oplus Z$). Then $\widehat{A} = A|_Z$ is bijective from Z onto $R(A)$. So, $\dim Z = \dim R(A) < +\infty$ (as proved earlier). Therefore we conclude that the Banach space X is finite dimensional.

Solution of Problem 2.30

"$\Longrightarrow$": Suppose that $R(A) \subseteq Y$ is closed. Then by the open mapping theorem (see Theorem I.5.47), A is an open map from X onto $V = R(A)$. Therefore, $A(\overline{B}_1^X)$ is a relatively w-compact set which contains an open neighborhood of the origin (namely the set $A(B_1^X)$). Therefore the set $\overline{B}_1^V = \big\{v \in V : \|v\|_V \leqslant 1\big\}$ is w-compact. So, by Theorem I.5.73, the space V is reflexive.

"$\Longleftarrow$": Recall that every reflexive subspace of Y is closed.

Solution of Problem 2.31

"**(a) $\Longrightarrow$ (b)**": Let

$$\overline{B}_1^X = \big\{u \in X : \|u\|_X \leqslant 1\big\} \quad \text{and} \quad \overline{B}_1^{X^{**}} = \big\{h \in X^{**} : \|h\|_{X^{**}} \leqslant 1\big\}.$$

From the Goldstine theorem (see Theorem I.5.70), we know that $\overline{\overline{B}_1^X}^{w^*} = \overline{B}_1^{X^{**}}$, with w^* being the w^*-topology on the space X^{**}. Recall that $A^{**} \in \mathcal{L}(X_{w^*}^{**}; Y_{w^*}^{**})$, where $X_{w^*}^{**}$ (respectively $Y_{w^*}^{**}$) denotes the space X^{**} (respectively Y^{**}) furnished with the w^*-topology. Also, the w^*-topology of Y^{**} restricted to Y, is the w-topology of Y. Therefore, since A is weakly compact (see Definition 2.1(d)), we have $\overline{A(\overline{B}_1^X)}^{w^*} \subseteq Y$, hence $A(\overline{\overline{B}_1^X}^{w^*}) = A(\overline{B}_1^{X^{**}}) \subseteq Y$ from which we conclude that $A(X^{**}) \subseteq Y$.

"**(b)** $\Longrightarrow$ **(c)**": Let $y_\alpha^* \xrightarrow{w^*} 0$ in Y^* and let $h \in X^{**}$. By hypothesis $A^{**}(h) \in Y$, so we have

$$\langle A^*(y_\alpha^*), h \rangle \;=\; \langle y_\alpha^*, A^{**}(h) \rangle \;\longrightarrow\; 0$$

and thus $A^*(y_\alpha^*) \xrightarrow{w} 0$ in X^*. This prove the continuity of $A^* \colon Y_{w^*}^* \longrightarrow X_w^*$. "**(c)** $\Longrightarrow$ **(d)**": From the Alaoglu theorem (see Theorem I.5.66), we know that $\overline{B}_1^{Y^*} = \{y^* \in Y^* : \ \|y^*\|_{Y^*} \leqslant 1\}$ is w^*-compact. So, by hypothesis **(c)**, we have that $A^*(\overline{B}_1^{Y^*}) \subseteq X^*$ is w-compact, which means that A^* is weakly compact.

"**(d)** $\Longrightarrow$ **(a)**": From the implications established thus far, we have that $A^{**} \colon X_{w^*}^{**} \longrightarrow Y_w^{**}$ is continuous. Therefore the set $A^{**}(\overline{B}_1^{X^{**}}) \subseteq Y^{**}$ is w-compact. The space Y is strongly closed in Y^{**} and so by the Mazur theorem (see Theorem I.5.58), it is also weakly closed in Y^{**}. It follows that the set $A^{**}(\overline{B}_1^{X^{**}}) \cap Y \subseteq Y^{**}$ is w-compact and since $w^* \subseteq w$ on Y^{**}, we infer that the set $A^{**}(\overline{B}_1^{X^{**}}) \cap Y \subseteq Y^{**}$ is w^*-compact. We know that

$$A(\overline{B}_1^X) \;=\; A^{**}(\overline{B}_1^X) \;\subseteq\; A^{**}(\overline{B}_1^{X^{**}}) \cap Y$$

and the w^*-topology of Y^{**} restricted on Y is the w-topology of Y.

So, we have that the set $A(\overline{B}_1^X) \subseteq Y$ is relatively w-compact, hence A is a weakly compact operator.

Solution of Problem 2.32

"$\Longrightarrow$": Assume that $A \in \mathcal{L}_c(X;Y)$. From Theorem 2.12 we have that $A^* \in \mathcal{L}_c(Y^*;X^*)$. Therefore $A^*(\overline{B}_1^{Y^*}) \subseteq X^*$ is relatively compact (recall that $\overline{B}_1^{Y^*} = \{y^* \in Y^* : \|y^*\|_{Y^*} \leqslant 1\}$). So, we can find a sequence $\{u_n^*\}_{n\geqslant 1} \subseteq X^*$ such that $\|u_n^*\|_{X^*} \longrightarrow 0$ and $A^*(\overline{B}_1^{Y^*}) \subseteq \overline{\operatorname{conv}}\,\{u_n^*\}_{n\geqslant 1}$ (see Theorem 2.163). For each $u \in X$, we have

$$\begin{aligned} \|A(u)\|_Y &= \sup_{\|y^*\|_{Y^*}\leqslant 1} \big|\,\langle y^*, A(u)\rangle\,\big| \\ &= \sup_{\|y^*\|_{Y^*}\leqslant 1} \big|\,\langle A^*(y^*), u\rangle\,\big| \;\leqslant\; \sup_{n\geqslant 1} \big|\,\langle u_n^*, u\rangle\,\big|. \end{aligned}$$

"$\Longleftarrow$": Consider $T \in \mathcal{L}(X;c_0)$ (see Problem 2.28), defined by

$$T(u) = \big\{\,\langle u_n^*, u\rangle\,\big\}_{n\geqslant 1}.$$

Let $h = \big\{\|u_n^*\|_{X^*}\big\}_{n\geqslant 1} \in c_0$. We have that $T(\overline{B}_1^X) \subseteq [-h,h]$ (here by $[-h,h]$ we denote the order interval in c_0 determined by $\pm h$). The order interval $[-h,h]$ is compact in c_0 (see Theorem 2.164). Therefore $T(\overline{B}_1^X) \subseteq c_0$ is relatively compact. Let $V = T(X)$ and define the operator $L\colon V \longrightarrow Y$ by

$$L(T(u)) = A(u).$$

This operator is well defined. Indeed, if $T(u) = T(x)$, then

$$\langle u_n^*, u - x\rangle = 0 \qquad \forall\, n \geqslant 1,$$

so

$$\|A(u) - A(x)\|_Y \;\leqslant\; \sup_{n\geqslant 1} \big|\,\langle u_n^*, u-x\rangle\,\big| \;=\; 0,$$

thus $A(u) = A(x)$. Moreover, note that

$$\|L(T(u))\|_Y = \|A(u)\|_Y \;\leqslant\; \sup_{n\geqslant 1} \big|\,\langle u_n^*, u\rangle\,\big| = \|T(u)\|_{c_0},$$

so $L \in \mathcal{L}(X;Y)$. Since $A(\overline{B}_1^X) = L(T(\overline{B}_1^X))$ and the latter is relatively compact in Y, we conclude that $A \in \mathcal{L}_c(X;Y)$.

Solution of Problem 2.33
Let $C \subseteq X^*$ be a compact set. We need to show that the set $f^{-1}(C) \subseteq X$ is compact. To this end let $\{u_n\}_{n \geqslant 1} \subseteq F^{-1}(C)$ be a sequence. For every $n \geqslant 1$, let $h_n = f(u_n) \in C$. Since $C \subseteq X^*$ is compact, passing to a subsequence if necessary, we may assume that $h_n \longrightarrow h$ in X^*. The coercivity of f implies that the sequence $\{u_n\}_{n \geqslant 1} \subseteq X$ is bounded. Because X is reflexive, by the Eberlein–Smulian theorem (see Theorem I.5.78), passing to another subsequence if necessary, we may assume that $u_n \xrightarrow{w} u$ in X. We have

$$\langle f(u_n) - f(u), u_n - u \rangle \;=\; \langle h_n - h, u_n - u \rangle \;\longrightarrow 0,$$

so, from the property of f, we have $u_n \longrightarrow u$ in X. Thus the set $f^{-1}(C)$ is compact, hence f is proper (see Definition 2.13).

Solution of Problem 2.34
"$\Longrightarrow$": By Problem 2.32, we can find a sequence $\{u_n^*\}_{n \geqslant 1} \subseteq X^*$ such that $u_n^* \longrightarrow 0$ in X^* and

$$\|A(u)\|_Y \;\leqslant\; \sup_{n \geqslant 1} |\,\langle u_n^*, u\rangle\,| \qquad \forall\, u \in X. \tag{2.9}$$

We can assume that $u_n^* \neq 0$ for every $n \geqslant 1$ and define

$$y_n^* \;=\; \frac{u_n^*}{\|u_n^*\|^{\frac{1}{2}}} \qquad \forall\, n \geqslant 1.$$

Then $\|y_n^*\|_{X^*} = \|u_n^*\|_*^{\frac{1}{2}} \longrightarrow 0$. So, we can define the linear operator $T\colon X \longrightarrow c_0$ by setting

$$T(u) \;=\; \big\{\, \langle y_n^*, u\rangle \,\big\}_{n \geqslant 1} \qquad \forall\, u \in X.$$

Note that

$$\|T(u)\|_{c_0} \;=\; \sup_{n \geqslant 1} |\,\langle y_n^*, u\rangle\,| \;\leqslant\; c\|u\|_X \qquad \forall\, u \in X,$$

for some $c > 0$, thus $T \in \mathcal{L}(X; c_0)$.

Moreover, Problem 2.32 implies that $T \in \mathcal{L}_c(X; c_0)$. We set $V = \overline{T(X)}$ and clearly we have $T \in \mathcal{L}_c(X; V)$. We consider the linear operator $L\colon T(X) \longrightarrow Y$ defined by $L(T(u)) = A(u)$. Note that, if $T(u) = T(v)$, then $\langle u_n^*, u - v \rangle = 0$ for all $n \geqslant 1$ and so from (2.9), we have that $A(u) = A(v)$, hence L is well defined. Moreover, we have

$$\|L(T(u))\|_Y = \|A(u)\|_Y \leqslant \sup_{n\geqslant 1} | \langle u_n^*, u \rangle | \leqslant \widehat{c} | \langle y_n^*, u \rangle | \leqslant \widehat{c} \|T(u)\|_{c_0}$$

with $\widehat{c} = \sup\limits_{n\geqslant 1} \|u_n^*\|_*^{\frac{1}{2}}$ (see (2.9) and recall the definition of T), so $L \in \mathcal{L}(T(X); Y)$. Therefore L has a continuous extension $\widehat{L}$ on $V = \overline{T(X)} \subseteq c_0$. Clearly $A = L \circ T$.

"$\Longleftarrow$": This is an immediate consequence of Problem 2.14.

Solution of Problem 2.35

From Problem 2.31 we know that $A\colon c_0 \longrightarrow X$ is weakly compact (see Definition 2.1(d)) if and only if $A^*\colon X^* \longrightarrow c_0^* = l^1$ is weakly compact. From the Schur property (see Remark I.5.57), it follows that $A^*\colon X^* \longrightarrow l^1$ is weakly compact if and only if it is compact. Invoking the Schauder theorem (see Theorem 2.12) we conclude that $A^*\colon X^* \longrightarrow l^1$ is compact if and only if $A\colon c_0 \longrightarrow X$ is compact.

Solution of Problem 2.36

Let $\{u_n\}_{n\geqslant 1} \subseteq C$ be a minimizing sequence, i.e.,

$$\|A(u_n) - y\|_Y \longrightarrow \inf_{u\in C} \|A(u) - y\|_Y .$$

From Problem 2.13 we know that the set $A(C) \subseteq Y$ is compact. So, we can find a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ of $\{u_n\}_{n\geqslant 1}$ and $u_0 \in C$ such that $A(u_{n_k}) \longrightarrow A(u_0)$ in Y. Then

$$\|A(u_{n_k}) - y\|_Y \longrightarrow \|A(u_0) - y\|_Y ,$$

so

$$\|A(u_0) - y\|_Y \;=\; \inf_{u \in C} \|A(u) - y\|_Y .$$

Solution of Problem 2.37
Suppose that $A \in \mathcal{L}_f(X;Y)$ and let $n = \dim R(A)$. Let $\{v_1, \dots v_n\}$ be a basis of $R(A)$. For every $u \in X$, we can write

$$A(u) \;=\; \sum_{k=1}^{n} \xi_k(u) v_k. \tag{2.10}$$

The coefficients $\xi_k(u)$ are uniquely determined and clearly ξ_k are linear functions of u, which are bounded since

$$|\xi_k(u)| \;\leqslant\; c\|A(u)\|_Y \;\leqslant\; c\|A\|_{\mathcal{L}}\|u\|_X$$

for some $c > 0$. Therefore, we can find $v_k^* \in X^*$ for $k = 1, \dots, n$ such that $\xi_k(u) = \langle v_k^*, u\rangle$. Hence (2.10) becomes

$$A(u) \;=\; \sum_{k=1}^{n} \langle v_k^*, u\rangle\, v_k.$$

For any $y^* \in Y^*$ we have

$$\begin{aligned} \langle A^*(y^*), u\rangle \;&=\; \langle y^*, A(u)\rangle \;=\; \sum_{k=1}^{n} \langle v_k^*, u\rangle\, \langle y^*, v_k\rangle \\ &=\; \Big\langle \sum_{k=1}^{n} \langle y^*, v_k\rangle\, v_k^*, u\Big\rangle. \end{aligned}$$

This is true for every $u \in X$. Therefore

$$A^*(y^*) \;=\; \sum_{k=1}^{n} \langle y^*, v_k\rangle\, v_k^*,$$

so $R(A^*)$ is spanned by $\{v_1^*, \dots, v_n^*\}$. This shows that $A^* \in \mathcal{L}_f(Y^*; X^*)$ and $\dim R(A^*) \leqslant \dim R(A)$.

In the above argument we replace A by A^* and obtain

$$A^{**} \;\in\; \mathcal{L}_f(X;Y) \quad \text{and} \quad \dim R(A^{**}) \;\leqslant\; \dim R(A^*).$$

Hence

$$\dim R(A^{**}) \leqslant \dim R(A).$$

But recalling that A is a restriction of A^{**}, we conclude that $\dim R(A) = \dim R(A^{*})$.

Solution of Problem 2.38

Let $C = \{A(e_n) : n \geqslant 1\}$. Then $C \subseteq A(\overline{B}_1^{l^1})$ (where $\overline{B}_1^{l^1} = \{u \in l^1 : \|u\|_{l^1} \leqslant 1\}$). So, if A is weakly compact (see Definition 2.1(d)), the set $A(\overline{B}_1^{l^1})$ is relatively weakly compact, hence the same is true for C.

Conversely, suppose that $C \subseteq Y$ is a relatively w-compact set. Let $u = \{u_n\}_{n \geqslant 1} \in l^1$ be such that $\|u\|_{l^1} \leqslant 1$. Let

$$h_n = \sum_{k=1}^{n} u_k A(e_k) \qquad \forall\, n \geqslant 1.$$

Then $\{h_n\}_{n \geqslant 1} \subseteq \overline{\operatorname{conv}}\, C$ and the latter is w-compact since the set $C \subseteq Y$ is relatively w-compact (see Theorem I.5.86). We have

$$h_n = A\Big(\sum_{k=1}^{n} u_k e_k\Big) \longrightarrow A(u) \quad \text{in } Y,$$

so $A(u) \in \overline{\operatorname{conv}}\, C$ and thus $A(\overline{B}_1^{l^1}) \subseteq \overline{\operatorname{conv}}\, C$, which implies that the set $A(\overline{B}_1^{l^1})$ is relatively w-compact. Therefore we conclude that A is weakly compact.

Solution of Problem 2.39

Note that there exists $M > 0$ such that $\|u_n\|_X \leqslant M$ for all $n \geqslant 1$. So, the series $\sum\limits_{n \geqslant 1} x_n u_n$ is norm convergent in X for each $x = \{x_n\}_{n \geqslant 1} \in l^1$. Therefore $A \in \mathcal{L}(l^1; X)$. We see that

$$A(e_n) = u_n \qquad \forall\, n \geqslant 1.$$

Therefore, we can apply Problem 2.38 to conclude that A is weakly compact (see Definition 2.1(d)).

Solution of Problem 2.40
Arguing by contradiction, suppose that $0 \notin \sigma(A)$ (see Definition 2.18). Then A is invertible and so by setting

$$|u| \;=\; \|A(u)\|_X \qquad \forall\, u \in X$$

we have defined an equivalent norm on X, whose open unit ball is $A(B_1)$. But this is relatively compact. Therefore $(X, |\cdot|)$ is locally compact, hence X is finite dimensional (see Proposition I.5.9(a)), a contradiction.

Solution of Problem 2.41
Let $\lambda \in \mathbb{C}$ and assume that for some $u \in L^2(0,1)$ we have

$$(\lambda - t)u(t) \;=\; 0 \quad \text{for a.a. } t \in (0,1).$$

Thus $u(t) = 0$ for almost all $t \in (0,1)$ and so $\lambda \notin \sigma_p(A)$ (see Definition 2.21).

Next, let $\lambda \in [0,1]$ and let $\varepsilon > 0$ be such that

$$[\lambda, \lambda+\varepsilon] \subseteq [0,1] \quad \text{or} \quad [\lambda-\varepsilon, \lambda] \subseteq [0,1].$$

To fix things, we assume that $[\lambda, \lambda+\varepsilon] \subseteq [0,1]$ (the analysis being analogous when the second inclusion holds). We define

$$u_\varepsilon(t) \;=\; \begin{cases} \frac{1}{\sqrt{\varepsilon}} & \text{if} \quad t \in [\lambda, \lambda+\varepsilon], \\ 0 & \text{if} \quad t \in [0,1] \setminus [\lambda, \lambda+\varepsilon]. \end{cases}$$

We have

$$\int_0^1 u_\varepsilon(t)^2\, dt \;=\; \int_\lambda^{\lambda+\varepsilon} \tfrac{1}{\varepsilon}\, dt \;=\; 1$$

and so $u_\varepsilon \in \partial B_1^{L^2} = \left\{u \in L^2(0,1) : \ \|u\|_2 = 1\right\}$. Then

$$\|(\lambda I_{L^2(0,1)} - A)(u_\varepsilon)\|_2^2 \;=\; \tfrac{1}{\varepsilon} \int_\lambda^{\lambda+\varepsilon} (\lambda - t)^2\, dt \;=\; \varepsilon,$$

so
$$(\lambda I_{L^2(0,1)} - A)(u_\varepsilon) \longrightarrow 0 \quad \text{in } L^2(0,1) \quad \text{as } \varepsilon \searrow 0.$$
This implies that $\lambda I_{L^2(0,1)} - A$ is not invertible. Indeed, otherwise we would have
$$\begin{aligned} \|u_\varepsilon\|_2 &= \|(\lambda I_{L^2(0,1)} - A)^{-1}(\lambda I_{L^2(0,1)} - A)(u_\varepsilon)\|_2 \\ &\leqslant \|(\lambda I_{L^2(0,1)} - A)^{-1}\|_{\mathcal{L}}\|(\lambda I_{L^2(0,1)} - A)(u_\varepsilon)\|_2 \longrightarrow 0 \quad \text{as } \varepsilon \searrow 0, \end{aligned}$$
a contradiction to the definition of u_ε. So $\lambda \in \sigma(A)$ (see Definition 2.18) and thus $[0,1] \subseteq \sigma(A)$

Solution of Problem 2.42

(a) Let $u \in H$ be such that $\|u\|_H \leqslant 1$. We have
$$|\,(A(u), u)_H\,| \leqslant \|A(u)\|_H\|u\|_H \leqslant \|A\|_{\mathcal{L}}\|u\|_H^2,$$
so
$$\sup_{\|u\|_H \leqslant 1} |\,(A(u), u)_H\,| \leqslant \|A\|_{\mathcal{L}}. \tag{2.11}$$
Now, let $\xi = \sup\limits_{\|u\|_H \leqslant 1} |\,(A(u), u)_H\,|$. We have
$$|\,(A(u), u)_H\,| \leqslant \xi\|u\|_H^2 \qquad \forall\, u \in H.$$
For $u \in H \setminus \{0\}$, we introduce $\lambda = \left(\frac{\|A(u)\|_H}{\|u\|_H}\right)^{\frac{1}{2}}$ and set $y = \frac{1}{\lambda}A(u)$. We have
$$\begin{aligned} &\|A(u)\|_H^2 \\ &\quad = (A(u), A(u))_H = \big(A(\lambda u), \tfrac{1}{\lambda}A(u)\big)_H = (A(\lambda u), y)_H \\ &\quad = \tfrac{1}{4}\big(\,(A(\lambda u + y), \lambda u + y)_H - (A(\lambda u - y), \lambda u - y)_H\,\big) \\ &\quad \leqslant \tfrac{1}{4}\xi\big(\|\lambda u + y\|_H^2 + \|\lambda u - y\|_H^2\big) \\ &\quad = \tfrac{1}{2}\xi\big(\|\lambda u\|_H^2 + \|y\|_H^2\big) = \tfrac{1}{2}\xi\big(\lambda^2\|u\|_H^2 + \tfrac{1}{\lambda^2}\|A(u)\|_H^2\big) \end{aligned} \tag{2.12}$$
(from the polarization identity since A is self-adjoint and from the parallelogram law; see Remark I.5.94). From the definition of $\lambda > 0$, we see that $\lambda\|u\|_H = \frac{1}{\lambda}\|A(u)\|_H$. Therefore
$$0 = \big(\lambda\|u\|_H - \tfrac{1}{\lambda}\|A(u)\|_H\big)^2,$$

so

$$\lambda^2\|u\|_H^2 + \tfrac{1}{\lambda^2}\|A(u)\|_H^2 \;=\; 2\|u\|_H\|A(u)\|_H.$$

Using this equality in (2.12), we obtain

$$\|A(u)\|_H^2 \;\leqslant\; \xi\|u\|_H\|A(u)\|_H,$$

so

$$\|A(u)\|_H \;\leqslant\; \xi\|u\|_H$$

and thus $\|A\|_{\mathcal{L}} \leqslant \xi$. From this and (2.11), we conclude that

$$\|A\|_{\mathcal{L}} \;=\; \sup_{\|u\|_H\leqslant 1} (A(u),u)_H\,.$$

(b) Since A is self-adjoint, for every $u \in H$, we have that $(A(u),u)_H \in \mathbb{R}$. Let λ be an eigenvalue of A with eigenfunction u (see Definition 2.21). We have

$$(A(u),u)_H \;=\; (\lambda u,u)_H \;=\; \lambda\|u\|_H^2,$$

so

$$\lambda \;=\; \tfrac{(A(u),u)_H}{\|u\|_H^2} \;\in\; \mathbb{R}.$$

Thus, every eigenvalue of A is real.
Next, let λ,μ be two distinct eigenvalues with corresponding eigenvectors $u,v \in H\setminus\{0\}$, respectively. Then

$$A(u) \;=\; \lambda u \quad\text{and}\quad A(v) \;=\; \mu v.$$

We have

$$(A(u),v)_H \;=\; \lambda\,(u,v)_H$$

and

$$(A(u),v)_H \;=\; (u,A(v))_H \;=\; \mu\,(u,v)_H$$

(since A is self-adjoint), so

$$(\lambda-\mu)\,(u,v)_H \;=\; 0,$$

thus $(u,v)_H = 0$. This shows that eigenvectors are orthogonal.

Solution of Problem 2.43

"$\Longrightarrow$": Let $\lambda \in \sigma(A)$ (see Definition 2.18).

It suffices to show that if $\inf\limits_{\|u\|_H=1} \|(\lambda I_H - A)(u)\|_H > 0$ then the operator $\lambda I_H - A$ is invertible. So, suppose that

$$\inf_{\|u\|_H=1} \|(\lambda I_H - A)(u)\|_H \;=\; c \;>\; 0.$$

Then

$$\|(\lambda I_H - A)(u)\|_H \;\geqslant\; c\|u\|_H \qquad \forall\, u \in H. \tag{2.13}$$

From (2.13) it is clear that $\lambda I_H - A$ is injective. So, if we show that $\lambda I_H - A$ is also surjective, then by the Banach theorem (see Theorem I.5.48), we would have that $\lambda I_H - A$ is invertible. We show surjectivity of $\lambda I_H - A$ in two steps. First we show that $(\lambda I_H - A)(H)$ is dense in H and then we show that $(\lambda I_H - A)(H)$ is closed.

To show that $(\lambda I_H - A)(H)$ is dense in H, we argue indirectly. So, suppose that $(\lambda I_H - A)(H)$ is not dense in H. Then we can find $\widehat{h} \in H \setminus \{\emptyset\}$ such that

$$\big((\lambda I_H - A)(u), \widehat{h}\big)_H \;=\; 0 \qquad \forall\, u \in H.$$

Exploiting the fact that A is self-adjoint (see Definition I.5.108(b)), we have

$$\big((\lambda I_H - A)(u), \widehat{h}\big)_H \;=\; \big(u, (\,\overline{\lambda} I_H - A)(\widehat{h})\big)_H \qquad \forall\, u \in H,$$

so

$$(\,\overline{\lambda} I_H - A)(\widehat{h}) \;=\; 0, \quad \text{with } \widehat{h} \neq 0,$$

thus $\overline{\lambda}$ is an eigenvalue of A (see Definition 2.21. But from Problem 2.42(b), we know that all eigenvalues of A are real. Hence $\overline{\lambda} \in \mathbb{R}$ and so

$$(\lambda I_H - A)(\widehat{h}) \;=\; 0, \quad \text{with } \widehat{h} \neq 0,$$

which contradicts (2.13). This proves the density of $(\lambda I_H - A)(H)$ in H. Next we show that $(\lambda I_H - A)(H) \subseteq H$ is closed. So, let $\{u_n\}_{n\geqslant 1} \subseteq H$ be a sequence such that

$$(\lambda I_H - A)(u_n) \;\longrightarrow\; h.$$

From (2.13), for all $n, m \geqslant 1$, we have

$$c\|u_n - u_m\|_H \leqslant \|(\lambda I_H - A)(u_n - u_m)\|_H \longrightarrow 0 \quad \text{as } n, m \to +\infty,$$

so $\{u_n\}_{n\geqslant 1} \subseteq H$ is a Cauchy sequence. Thus, we have $u_n \longrightarrow u$ in H for some $u \in H$. From the continuity of $\lambda I_H - A$, it follows that

$$(\lambda I_H - A)(u_n) \longrightarrow (\lambda I_H - A)(u) \quad \text{in } H,$$

so $h = (\lambda I_H - A)(u)$ and thus $h \in R(\lambda I_H - A)$. This proves that $R(\lambda I_H - A)$ is closed. Therefore $R(\lambda I_H - A) = H$.

"$\Longleftarrow$": If $\lambda \in \varrho(A)$, then $(\lambda I_H - A)^{-1} \in \mathcal{L}(H)$ and for $u \in \partial B_1$, we have

$$\begin{aligned} 1 &= \|u\|_H = \|(\lambda I_H - A)^{-1}(\lambda I_H - A)(u)\|_H \\ &\leqslant \|(\lambda I_H - A)^{-1}\|_{\mathcal{L}} \|(\lambda I_H - A)(u)\|_H \end{aligned}$$

(where $\partial B_1 = \{h \in H : \ \|h\|_H = 1\}$), so

$$\|(\lambda I_H - A)^{-1}\|_{\mathcal{L}}^{-1} \leqslant \|(\lambda I_H - A)(u)\|_H \qquad \forall\, u \in \partial B_1.$$

Solution of Problem 2.44

From Problem 2.42(b) we know that $\sigma(A) \in \mathbb{R}$. Let $\vartheta > 0$. We will show that $M + \vartheta \notin \sigma(A)$. According to Problem 2.43, it suffices to show that

$$\inf_{\|u\|_H = 1} \|((M+\vartheta)I_H - A)(u)\|_H > 0.$$

For $u \in H$, with $\|u\|_H = 1$, we have

$$((M+\vartheta)u - A(u), u)_H = (M+\vartheta) - (A(u), u)_H \tag{2.14}$$

$$\geqslant (M+\vartheta) - M = \vartheta > 0 \tag{2.15}$$

(recall that $\|u\|_H = 1$ and see the definition of M). Also we have

$$((M+\vartheta)u - A(u), u)_H \leqslant \|((M+\vartheta)I_H - A)(u)\|_H. \tag{2.16}$$

From (2.14) and (2.16) it follows that

$$0 \ < \ \vartheta \ \leqslant \ \inf_{\|u\|_H=1} \|((M+\vartheta)I_H - A)(u)\|_H,$$

so $M+\vartheta \notin \sigma(A)$ (see Problem 2.43).

In a similar fashion, we show that $m-\vartheta \notin \sigma(A)$. Therefore $\sigma(A) \subseteq [m, M]$.

Next we show that $M \in \sigma(A)$. Note that $\sigma(A+\vartheta I_H) = \sigma(A)+\vartheta$ and so by replacing A by $A+\vartheta I_H$, we may assume without only loss of generality that $0 \leqslant m \leqslant M$. Then by virtue of Problem 2.42(a), we have

$$\|A\|_{\mathcal{L}} \ = \ M. \tag{2.17}$$

According to Problem 2.43, it suffices to show that $\inf\limits_{\|u\|_H=1} \|(MI_H - A)(u)\|_H = 0$. Let $\{u_n\}_{n\geqslant 1} \subseteq H$ be a sequence such that $\|u_n\|_H = 1$ for all $n \geqslant 1$ and

$$(A(u_n), u_n)_H \ \longrightarrow \ M \ = \ \|A\|_{\mathcal{L}} \tag{2.18}$$

(see (2.17)). We have

$$\begin{aligned} 0 \ \leqslant \ \|(MI_H - A)(u_n)\|_H^2 \ &= \ (Mu_n - A(u_n), Mu_n - A(u_n))_H \\ &= \ M^2\|u_n\|_H^2 - 2M\,(A(u_n), u_n)_H + \|A(u_n)\|_H^2 \\ &\leqslant \ 2M^2 - 2M\,(A(u_n), u_n)_H \ \longrightarrow \ 0 \end{aligned}$$

(since A is self-adjoint and using (2.18)), so

$$\inf_{\|u\|_H=1} \|(MI_H - A)(u)\|_H \ = \ 0.$$

and thus $M \in \sigma(A)$.

Similarly we show that $m \in \sigma(A)$.

Solution of Problem 2.45

Let $Y = A(X)$ and $V = (I_X - A)(X)$. Then Y and V are nontrivial closed vector subspaces of X and $X = Y \oplus V$. If $y \in Y$ and $v \in V$ are both nonzero, then $A(y) = y$ and $A(v) = 0$, so 0 and 1 are eigenvalues of A (i.e., $\{0,1\} \subseteq \sigma_p(A)$; see Definition 2.21).

Next let $\lambda \in \mathbb{C} \setminus \{0,1\}$. If $u \in X$, then $u = y + v$ with $y \in Y$, $v \in V$ (in a unique way). Then

$$(\lambda I_X - A)(u) \;=\; \lambda u - A(u) \;=\; (\lambda - 1)y + \lambda v.$$

So, if $(\lambda I_X - A)(u) = 0$, then $(\lambda - 1)y = 0$ and $\lambda v = 0$. Since $\lambda \notin \{0,1\}$, we see that $y = v = 0$ and so $u = 0$. This proves that $\lambda I_X - A$ is injective.

Also, if $u = y + v$ with $y \in Y$, $v \in V$, then for $x = \frac{y}{\lambda - 1} + \frac{v}{\lambda}$ we have

$$(\lambda I_X - A)(x) \;=\; u,$$

so $\lambda I_X - A$ is surjective and thus $\lambda I_X - A$ is invertible (from the Banach theorem; see Theorem I.5.48). Thus $\sigma(A) \subseteq \{0,1\}$ and so $\sigma_p(A) = \sigma(A) = \{0,1\}$ (see Definitions 2.18 and 2.21).

Solution of Problem 2.46

Let

$$L(u) \;=\; \sum_{n \geqslant 0} \xi^{-(n+1)} A^n(u)$$

and

$$L_m(u) \;=\; \sum_{n=0}^{m} \xi^{-(n+1)} A^n(u) \qquad \forall\, u \in X,\ m \geqslant 1.$$

Clearly $L_m \in \mathcal{L}(X)$ and $L_m(u) \longrightarrow L(u)$ in X. Then by the uniform boundedness principle (see Corollary I.5.40), we infer that $L \in \mathcal{L}(X)$. For each $u \in X$, we have

$$(\xi I_X - A)L(u) \;=\; \lim_{m \to +\infty} \Big((\xi I_X - A) \sum_{n=0}^{m} \xi^{-(n+1)} A^n(u)\Big)$$

$$= \lim_{m \to +\infty} \Big(\sum_{n=0}^{m} \xi^{-n} A^n(u) - \sum_{n=0}^{m} \xi^{-(n+1)} A^{n+1}(u)\Big)$$

$$= \lim_{m \to +\infty} \big(u - \xi^{-(m+1)} A^{m+1}(u)\big).$$

By hypothesis $\lim\limits_{m \to +\infty} \xi^{-(m+1)} A^{m+1}(u) = 0$. Therefore it follows that

$$(\xi I_X - A)L(u) \;=\; u.$$

In a similar fashion we show that

$$L(\xi I_X - A)(u) \;=\; u.$$

From the two last relations we conclude that $(\xi I_X - A)^{-1} = L \in \mathcal{L}(X)$, hence $\xi \in \varrho(A)$ (see Definition 2.18).

Solution of Problem 2.47

Evidently A is well defined and linear. Also, by the Hölder inequality (see Theorem 1.3 and Problem 1.27), we have

$$\big|A(u)(t) - A(u)(\tau)\big| \;\leqslant\; |t-\tau|^{\frac{1}{p'}}\, \|u\|_p \qquad \forall\, t,\tau \in (0,1)$$

(with $\frac{1}{p} + \frac{1}{p'} = 1$). If $1 < p < +\infty$, then $p' < +\infty$ and so we can apply the Arzela–Ascoli theorem (see Theorem I.2.181) and conclude that $A \in \mathcal{L}_c(X)$.

If $p = 1$, then $p' = +\infty$ and then we apply the Kolomogorov–Riesz theorem (see Theorem 1.29) to conclude that $A \in \mathcal{L}_c(X)$.

Next we determine $\sigma(A)$ and $\sigma_p(X)$ (see Definitions 2.18 and 2.21). First we show that $0 \notin \sigma_p(A)$. To this end, assume that $A(u) = 0$. Then

$$\int_0^1 u(s)\chi_{[a,b]}(s)\, ds \;=\; 0 \qquad \forall\, a,b \in [0,1].$$

Exploiting the density of simple functions in $L^p(0,1)$ (with $1 \leqslant p \leqslant +\infty$), we conclude that $u \equiv 0$.

Also, let $\lambda \neq 0$ and $u \in C([0,1])$. We set $h = (\lambda I_X - A)^{-1}(u)$. Then $y(t) = \int_0^t h(s)\, ds$ satisfies

$$y \;\in\; C^1([0,1]) \quad \text{and} \quad y' = (\lambda I_X - A)^{-1}(u),$$

so

$$\lambda y' - A(y') \;=\; \lambda y' - y \;=\; u, \quad \text{with } y(0) = 0$$

and thus

$$h(t) \;=\; \tfrac{1}{\lambda} u(t) - \tfrac{1}{\lambda^2} \int_0^t e^{(t-s)\lambda} u(s)\, ds.$$

Exploiting the density of $C([0,1])$ in $L^p(0,1)$ (with $1 \leqslant p < +\infty$), we conclude that the last relation is in fact valid for all $u \in L^p(0,1)$. Therefore $\lambda \notin \sigma(A)$ and so

$$\sigma(A) = \{0\} \quad \text{and} \quad \sigma_p(A) = \emptyset.$$

Solution of Problem 2.48

Since A is an isomorphism, we see that $0 \notin \sigma(A)$ (see Definition 2.18). Then from Problem 2.43 we have

$$c\|u\|_H^2 \leqslant (A(u),u)_H \qquad \forall\, u \in H, \tag{2.19}$$

for some $c > 0$. Therefore $\langle A(u),u\rangle > 0$ for all $u \neq 0$ and it follows that $\langle \cdot,\cdot\rangle$ is a new inner product on H.

Let $|u| = \langle u,u\rangle^{\frac{1}{2}}$. Then

$$|u|^2 \leqslant \|A\|_{\mathcal{L}}\|u\|_H^2 \qquad \forall\, u \in H. \tag{2.20}$$

From (2.19) and (2.20) it follows that

$$c\|u\|_H^2 \leqslant |u|^2 \leqslant \|u\|_{\mathcal{L}}\|u\|_H^2 \qquad \forall\, u \in H,$$

so $\|\cdot\|_H$ and $|\cdot|$ are equivalent norms on H.

Solution of Problem 2.49

Note that $\ker L \subseteq \ker(S \circ L) = \ker(I_X + \widehat{K})$ and $\ker(I_X + \widehat{K})$ is finite dimensional (see Proposition 2.24). Also, we have

$$R(L) \supseteq R(L \circ T) = R(I_Y + K)$$

and $R(I_Y + K)$ is finite codimensional, hence so is $R(L)$ (see Proposition 2.24). This shows that $L \in \mathrm{Fred}\,(X;Y)$ (see Definition 2.29).

Solution of Problem 2.50

We can find $T \in \mathcal{L}(X;Y)$ such that

$$L \circ T \;=\; I_Y + K_1 \quad \text{and} \quad T \circ L \;=\; I_X + K_2,$$

with $K_1 \in \mathcal{L}_c(Y;Y)$ and $K_2 \in \mathcal{L}_c(X;X)$ (see Remark 2.30). Then we have

$$\begin{aligned} (L+K) \circ T &= I_Y + K_1 + K \circ T, \\ T \circ (L+K) &= I_X + K_2 + T \circ K. \end{aligned}$$

Since $K_1 + K \circ T \in \mathcal{L}_c(Y;Y)$ and $K_2 + T \circ K \in \mathcal{L}_c(X;X)$, from Problem 2.49, we infer that $L + K \in \operatorname{Fred}(X;Y)$. Then consider the map $[0,1] \ni t \longmapsto L(t) = L + tK$. From the first part of the solution, we have that $L(t) \in \operatorname{Fred}(X;Y)$ for all $t \in [0,1]$. Moreover, from Remark 2.30, it follows that $i(L) = i(L+K)$.

Solution of Problem 2.51

(a) "$\Longrightarrow$": Let φ be a lower semicontinuous function and let $V \subseteq \mathbb{R}$ be an open set. According to Proposition 2.37(d) we need to show that the set $E_\varphi^+(V)$ is open (where $E_\varphi^+(V) = \{u \in X : \; E_\varphi(u) \subseteq V\}$). Let $u \in E_\varphi^+(V)$. Then the inclusion $E_\varphi(u) \subseteq V$ means that $V \supseteq (\lambda, +\infty)$ for some $\lambda < \varphi(u)$. The lower semicontinuity of φ implies that we can find a neighborhood U of u such that $\lambda < \varphi(h)$ for all $h \in U$. Then $E_\varphi(h) \subseteq V$ for all $h \in U$ and so we conclude that $u \longmapsto E_\varphi(u)$ is upper semicontinuous.

"$\Longleftarrow$": Since E_φ is upper semicontinuous, for every $\lambda \in \mathbb{R}$, the set

$$\begin{aligned} E_\varphi^+((\lambda, +\infty)) &= \{u \in X : \; E_\varphi(u) \subseteq (\lambda, +\infty)\} \\ &= \{u \in X : \; \varphi(u) > \lambda\} \end{aligned}$$

is open. This means that φ is a lower semicontinuous function.

(b) "$\Longrightarrow$": Let φ be an upper semicontinuous function and let $V \subseteq \mathbb{R}$ be an open set. In this case, we need to show that $E_\varphi^-(V) =$

$\{u \in X : E_\varphi(u) \cap V \neq \emptyset\}$ is open (see Proposition 2.38(e)). Let $u \in E_\varphi^-(V)$. Then the relation $E_\varphi(u) \cap V \neq \emptyset$ means that we can find $\lambda \in V$ such that $\varphi(u) \leqslant \lambda$. In fact the openness of V implies that we can always choose $\lambda \in V$ such that $\varphi(u) < \lambda$. Then the upper semicontinuity of the function φ implies that we can find a neighborhood U of u such that $\varphi(h) < \lambda$ for all $h \in U$. Hence $E_\varphi(h) \cap V \neq \emptyset$ for all $h \in U$. This implies that the set $E_\varphi^-(V)$ is open, therefore E_φ is lower semicontinuous.

"$\Longleftarrow$": Since E_φ is lower semicontinuous, for every $\lambda \in \mathbb{R}$, the set

$$\begin{aligned} E_\varphi^-((-\infty,\lambda)) &= \{u \in X : E_\varphi(u) \cap (-\infty,\lambda) \neq \emptyset\} \\ &= \{u \in X : \varphi(u) < \lambda\} \end{aligned}$$

is open. This means that φ is an upper semicontinuous function.

Solution of Problem 2.52

First we consider the case of F being lower semicontinuous (see Definition 2.36(b)). So, let $C \subseteq X$ be a connected set and let $V_1, V_2 \subseteq X$ be two open sets such that

$$F(C) \subseteq V_1 \cup V_2, \quad F(C) \cap V_1 \neq \emptyset \quad \text{and} \quad F(C) \cap V_2 \neq \emptyset. \tag{2.21}$$

We need to show that $F(C) \cap V_1 \cap V_2 \neq \emptyset$. To this end, suppose that

$$C \cap F^-(V_1) \cap F^-(V_2) = \emptyset. \tag{2.22}$$

Since F is lower semicontinuous, the sets $F^-(V_1)$ and $F^-(V_2)$ are both open and from (2.21) we infer that

$$C \subseteq F^-(V_1) \cup F^-(V_2), \quad C \cap F^-(V_1) \neq \emptyset \quad \text{and} \quad C \cap F^-(V_2) \neq \emptyset. \tag{2.23}$$

Combining (2.22) and (2.23), we see that we have a contradiction to the connectedness of C. Therefore (2.22) cannot happen and so we have

$$C \cap F^-(V_1) \cap F^-(V_2) \neq \emptyset.$$

Let $u \in C \cap F^-(V_1) \cap F^-(V_2)$. We have

$$F(u) \subseteq V_1 \cup V_2, \quad F(u) \cap V_1 \neq \emptyset \quad \text{and} \quad F(u) \cap V_2 \neq \emptyset.$$

So, from the connectedness of $F(u)$ we infer that

$$F(u) \cap V_1 \cap V_2 \neq \emptyset,$$

so $F(C) \cap V_1 \cap V_2 \neq \emptyset$, which proves the connectedness of $F(C)$.

When F is upper semicontinuous, then the proof if similar, working this time with a “disconnection” consisting of closed sets.

Solution of Problem 2.53

Note that for every nonempty open set $V \subseteq Y$ we have

$$F(u) \cap V \neq \emptyset \quad \text{if and only if} \quad \overline{F(u)} \cap V \neq \emptyset.$$

Therefore F is lower semicontinuous if and only if $\overline{F}$ is lower semicontinuous (see Definition 2.36).

For upper semicontinuous multifunctions the result is not true. To see this, let $X = Y = \mathbb{R}$ and consider the multifunction

$$F(u) = (u-1, u+1) \qquad \forall\, u \in \mathbb{R}.$$

Then

$$F^+((-1,1)) = \{0\}$$

(see Definition 2.35(b)) and so F is not upper semicontinuous. On the other hand,

$$\overline{F}(u) = [u-1, u+1] \qquad \forall\, u \in \mathbb{R}$$

and for every $y \in \mathbb{R}$

$$\varphi_y(u) = \operatorname{dist}_Y(y, \overline{F}(u)) = \begin{cases} u-1-y & \text{if} \quad y \leqslant u-1, \\ 0 & \text{if} \quad u-1 < y < u+1, \\ y-(u+1) & \text{if} \quad u+1 \leqslant y. \end{cases}$$

Clearly φ_y is lower semicontinuous. So, Proposition 2.45 implies that $\overline{F}$ is upper semicontinuous.

Solution of Problem 2.54

Let $V \subseteq Y$ be an open set and let $y \in \operatorname{conv} F(u) \cap V$. Then we have

$$y = \sum_{k=1}^{n} \lambda_k y_k \in V, \quad \text{with } y_1, \ldots, y_n \in F(u),\ \lambda_1, \ldots, \lambda_n \subseteq [0,1],$$
$$\sum_{k=1}^{n} \lambda_k = 1.$$

Let $\varepsilon > 0$ be small such that

$$B_\varepsilon(y) = \left\{ y' \in Y : \|y' - y\|_Y < \varepsilon \right\} \subseteq V.$$

Note that $F(u) \cap B_\varepsilon(y_k) \neq \emptyset$ for all $k \in \{1, \ldots, n\}$. Then since F is lower semicontinuous, we can find $U_k \in \mathcal{N}(u)$ ($\mathcal{N}(u)$ being the filter of neighborhoods of u) such that for every $u' \in U_k$ we have $F(u') \cap B_\varepsilon(y_k) \neq \emptyset$, with $k \in \{1, \ldots, n\}$. We set $U = \bigcap\limits_{k=1}^{n} U_k$. For every $u' \in U$, let $y'_k \in F(u') \cap B_\varepsilon(y_k)$ and set $y' = \sum\limits_{k=1}^{n} \lambda_k y'_k$. Then

$$\|y' - y\|_Y = \Big\| \sum_{k=1}^{n} \lambda_k (y'_k - y) \Big\|_Y \leqslant \sum_{k=1}^{n} \lambda_k \|y'_k - y_k\|_Y < \varepsilon,$$

so $y' \in \operatorname{conv} F(u') \cap V$ and thus the set $\operatorname{conv} F^-(V)$ is open. Hence $u \longmapsto \operatorname{conv} F(u)$ is lower semicontinuous.

Then using Problem 2.53, we conclude that $u \longmapsto \overline{\operatorname{conv}}\, F(u)$ is lower semicontinuous too.

Solution of Problem 2.55

Since F has compact values, for every $u \in X$ we have $\overline{\operatorname{conv}}\, F(u) \in P_{kc}(Y)$ (by the Mazur theorem; see Theorem I.5.86). Then by virtue of Proposition 2.56(a), it suffices to show that F is h-upper semicontinuous (see Definition 2.53(a)). So, suppose that $\{x_\alpha\}_{\alpha\in J} \subseteq X$ is a net such that $x_\alpha \longrightarrow x$ in X. Then

$$h^* \left(\overline{\operatorname{conv}}\, F(u_\alpha), \overline{\operatorname{conv}}\, F(u)\right) \;\leqslant\; h^* \left(F(u_\alpha), F(u)\right)$$

(see Remark 2.50), so using the h-upper semicontinuity of F (see Proposition 2.56(a)), we have

$$h^* \left(\overline{\operatorname{conv}}\, F(u_\alpha), \overline{\operatorname{conv}}\, F(u)\right) \;\overset{\alpha\in J}{\longrightarrow}\; 0.$$

Thus the multifunction $u \longmapsto \overline{\operatorname{conv}}\, F(u)$ is h-upper semicontinuous, hence upper semicontinuous too.

Solution of Problem 2.56

(a) Let $\{y_\alpha\}_{\alpha\in J} \subseteq F(K)$ be a net. Then $y_\alpha \in F(u_\alpha)$ for all $\alpha \in J$ with some net $\{u_\alpha\}_{\alpha\in J} \subseteq K$. The compactness of K implies that we can find a subnet $\{u_\beta\}_{\beta\in I}$ of $\{u_\alpha\}_{\alpha\in J}$ such that $u_\beta \longrightarrow u \in K$. Note that $\{(u_\beta, y_\beta)\}_{\beta\in I} \subseteq \operatorname{Gr} F$. Invoking Proposition 2.42, we infer that $\{y_\beta\}_{\beta\in I}$ admits a cluster point in $F(u) \subseteq F(K)$. This proves the compactness of the set $F(K)$.

(b) According to Proposition 2.37, it suffices to show that for every closed set $C \subseteq Y$, we have that the set $H^-(C) = \{u \in X : \ H(u) \cap C \neq \emptyset\}$ is closed in X. To this end, let $\{u_\alpha\}_{\alpha\in J} \subseteq H^-(C)$ be a net such that $u_\alpha \longrightarrow u$ in X. Let

$$K \;=\; \{u\} \cup \{u_\alpha : \ \alpha \in J\} \;\in\; P_k(X).$$

From **(a)** we have that $G(K) \in P_k(Y)$. Let $y_\alpha \in H(u_\alpha) \cap C$. Then $\{y_\alpha\}_{\alpha\in J} \subseteq G(K)$ and so we can find a subnet $\{y_\beta\}_{\beta\in I}$ of $\{y_\alpha\}_{\alpha\in J}$ such that $y_\beta \longrightarrow y$ in Y. Since the set $C \subseteq Y$ is closed, we have that $y \in C$. Also, using Proposition 2.43, we see that the multifunction $u \longmapsto F(u) \cap G(u)$ is closed, hence $y \in F(u) \cap G(u)$. Therefore $y \in H(u) \cap C$ and so $u \in H^-(C)$. This shows that $H^-(C) \subseteq X$ is closed, hence the multifunction H is upper semicontinuous (see Proposition 2.37).

(c) No. To see this, let $X = Y = [0,1]$ and consider the multifunctions

$$F(u) = \begin{cases} \left\{\frac{1}{n} : n \geqslant 1\right\} & \text{if } u \neq 0, \\ \{0\} & \text{if } u = 0, \end{cases}$$

$$G(u) = \begin{cases} (0,1] \setminus \left\{\frac{1}{n} : n \geqslant 2\right\} & \text{if } u \neq 0, \\ [0,1] & \text{if } u = 0. \end{cases}$$

Both multifunctions are lower semicontinuous and

$$(F \cap G)(u) = \begin{cases} \{1\} & \text{if } u \neq 0, \\ \{0\} & \text{if } u = 0, \end{cases}$$

which is not lower semicontinuous (see Proposition 2.38) because for the closed set $C = \{1\} \subseteq X$, the set $F^+(C) = (0,1] \subseteq Y$ is not closed.

Solution of Problem 2.57

Evidently, it suffices to show that $\varphi(u) \leqslant \operatorname{dist}(u,D)$ for all $u \in X$. Arguing by contradiction, suppose that we can find $u \in X$ such that $\operatorname{dist}(u,D) < \varphi(u)$. Let $\varepsilon = \varphi(u) - \operatorname{dist}(u,D) > 0$ and choose $h \in D$ such that

$$d_X(u,h) < \operatorname{dist}(u,D) + \tfrac{\varepsilon}{3}.$$

From the hypothesis concerning φ, we can find $E \in P_f(X)$ such that

$$|\varphi(h) - \operatorname{dist}(h,E)| < \tfrac{\varepsilon}{3} \quad \text{and} \quad |\varphi(u) - \operatorname{dist}(u,E)| < \tfrac{\varepsilon}{3}.$$

We have

$$\begin{aligned} \varphi(u) &< \operatorname{dist}(u,E) + \tfrac{\varepsilon}{3} \leqslant d_X(u,h) + \operatorname{dist}(h,E) + \tfrac{\varepsilon}{3} \\ &\leqslant d_X(u,h) + \varphi(h) + \tfrac{2\varepsilon}{3} \\ &= d_X(u,h) + \tfrac{2\varepsilon}{3} < \operatorname{dist}(u,D) + \varepsilon \end{aligned}$$

(since $h \in D$), which contradicts the definition of ε. We conclude that

$$\varphi(u) = \operatorname{dist}(u,D) \qquad \forall\, u \in X.$$

Solution of Problem 2.58
The hypothesis that $C_n \xrightarrow{h} C$ (see Definition 2.86) implies that given $\varepsilon > 0$, we can find $n_0 = n_0(\varepsilon) \geqslant 1$ such that

$$C \subseteq (C_n)_\varepsilon \quad \text{and} \quad C_n \subseteq C_\varepsilon \qquad \forall\, n \geqslant n_0. \tag{2.24}$$

From the second inclusion in (2.24), we have that

$$\bigcup_{n \geqslant n_0} C_n \subseteq C_\varepsilon,$$

so

$$\overline{\bigcup_{n \geqslant n_0} C_n} \subseteq C_{2\varepsilon},$$

thus

$$\bigcap_{m \geqslant 1} \overline{\bigcup_{n \geqslant m} C_n} \subseteq C_{2\varepsilon} \qquad \forall\, \varepsilon > 0$$

and hence

$$\bigcap_{m \geqslant 1} \overline{\bigcup_{n \geqslant m} C_n} \subseteq \overline{C} = C \tag{2.25}$$

(since $C \in P_f(X)$). From the first inclusion in (2.24), we obtain

$$C \subseteq \bigcap_{\varepsilon > 0} \bigcup_{m \geqslant 1} \bigcap_{n \geqslant m} (C_n)_\varepsilon. \tag{2.26}$$

Finally let $u \in \bigcap\limits_{\varepsilon > 0} \bigcup\limits_{m \geqslant 1} \bigcap\limits_{n \geqslant m} (C_n)_\varepsilon$. This means that for every $\varepsilon > 0$, we can find $m_0 = m_0(\varepsilon) \geqslant 1$ such that for all $n \geqslant m_0$ we have $u \in (C_n)_\varepsilon \subseteq \big(\bigcup\limits_{n \geqslant m_0} C_n\big)_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we infer that

$$u \in \overline{\bigcup_{n \geqslant m_0} C_n},$$

so

$$\bigcap_{\varepsilon > 0} \bigcup_{m \geqslant 1} \bigcap_{n \geqslant m} (C_n)_\varepsilon \subseteq \bigcap_{m \geqslant 1} \overline{\bigcup_{n \geqslant m} C_n}. \tag{2.27}$$

Combining (2.25), (2.26) and (2.27), we reach the equality claimed by the problem.

Solution of Problem 2.59

Let $y \in X \setminus \big(\bigcup\limits_{C \in K} C\big)$. Then for every $C \in K$, $y \in X \setminus C$ and so $\operatorname{dist}(y, C) = m_C > 0$. This means that $h(\{y\}, C) \geqslant m_C > 0$. Consider the function $\xi \colon (P_f(X), h) \longrightarrow \mathbb{R}_+$ defined by

$$\xi(D) \;=\; h(\{h\}, D) \qquad \forall\, D \in P_f(X).$$

This function is continuous and so $\inf\limits_K \xi$ is attained. Therefore

$$h(\{y\}, C) \;\geqslant\; \varepsilon \;>\; 0 \qquad \forall\, C \in K,$$

for some $\varepsilon > 0$. This means that $y \notin \big(\bigcup\limits_{C \in K} C\big)_\varepsilon$, so

$$\operatorname{dist}(y, \bigcup_{C \in K} C) \;>\; 0.$$

Since $y \in X \setminus \big(\bigcup\limits_{C \in K} C\big)$ is arbitrary, we conclude that the set $\bigcup\limits_{C \in K} C$ is closed.

Solution of Problem 2.60

"$\Longrightarrow$": Let $\mathcal{F}$ is dense in $(P_f(X), h)$. Note that if $C \in P_f(X)$ and $F \subseteq X$ is finite, then F is an ε-net for the set C if and only if $C \subseteq F_\varepsilon$. Therefore, if $h(F, C) < \varepsilon$ (see Definition 2.49), then F is an ε-net for C. Let

$$B_h(C, \varepsilon) \;=\; \{D \in P_f(X) :\; h(D, C) < \varepsilon\}.$$

If $B_h(C, \varepsilon) \cap \mathcal{F} \neq \emptyset$ for all $\varepsilon > 0$, then C is totally bounded (see Definition I.1.70). In particular this is also true if $C = X$.

"$\Longleftarrow$": Since X is totally bounded, every $C \in P_f(X)$ is also totally bounded. Therefore, for every $C \in P_f(X)$ and every $\varepsilon > 0$, we can find $F \in \mathcal{F}$ such that $C \subseteq F_\varepsilon$. We can also have $F \subseteq C_\varepsilon$, because if this inclusion is not true, then we can find $\widehat{u} \in F$ such that $\operatorname{dist}(\widehat{u}, C) \geqslant \varepsilon$. This implies that $C \cap B_\varepsilon(\widehat{u}) = \emptyset$ (recall that $B_\varepsilon(\widehat{u}) = \{u \in X :\; d_X(u, \widehat{u}) < \varepsilon\}$) and so $F' = F \setminus \{\widehat{u}\}$ is still an ε-net for C and so, we can replace F by F'. Then $h(F, C) < \varepsilon$ and so $\mathcal{F}$ is dense in $(P_f(X), h)$.

Solution of Problem 2.61

Let $\{u_\alpha\}_{\alpha \in J} \subseteq X$ be a net such that $u_\alpha \longrightarrow u$ in X and let $v = F(u)$. According to Proposition 2.38(d), in order to establish the lower semicontinuity of the multifunction F, we need to find $v_\alpha \in F(u_\alpha)$ for all $\alpha \in J$ such that $v_\alpha \longrightarrow v$ in V. From the definition of F we have that

$$v = \varphi(u, y), \quad \text{with } y \in M(u).$$

Since by hypothesis M is lower semicontinuous, we can find $y_\alpha \in M(u_\alpha)$ for $\alpha \in J$ such that $y_\alpha \longrightarrow y$ in Y (see Proposition 2.38). The continuity of φ implies that

$$v_\alpha = \varphi(u_\alpha, y_\alpha) \longrightarrow \varphi(u, y) = v.$$

Since $v_\alpha = F(u_\alpha)$, we have proved the lower semicontinuity of F.

Solution of Problem 2.62

According to Proposition 2.37, it suffices to show that for every closed set $C \subseteq Y$, the set $F^-(C) = \{u \in X : F(u) \cap C \neq \emptyset\}$ is closed. To this end, let $\{u_n\}_{n \geqslant 1} \subseteq F^-(C)$ be a sequence such that $u_n \longrightarrow u$ in X. From the definition of F, we can find $y_n \in M(u_n)$ such that $\varphi(u_n, y_n) \in C$ for all $n \geqslant 1$. Note that $y_n \in M(\{u\} \cup \{u_n : n \geqslant 1\})$ and from Problem 2.56(a), we know that $M(\{u\} \cup \{u_n : n \geqslant 1\}) \in P_k(Y)$ and so we can find a subsequence $\{y_{n_k}\}_{k \geqslant 1}$ of $\{y_n\}_{n \geqslant 1}$ such that $y_{n_k} \longrightarrow y$ in Y. Invoking Proposition 2.43, we have that $y \in M(u)$. Also, the continuity of φ, implies that

$$\varphi(u_{n_k}, y_{n_k}) \longrightarrow \varphi(u, y) \in C \quad \text{in } V.$$

Since $y \in M(u)$, it follows that $u \in F^{-}(C)$ and so $F^{-}(C)$ is closed. This implies the upper semicontinuity of the multifunction F.

Solution of Problem 2.63

Let $\{u_n\}_{n \geqslant 1} \subseteq X$. For each $n \geqslant 1$, let $C_n = \{u_n\} \in P_f(X)$ for $n \geqslant 1$ and since by hypothesis $(P_f(X), h)$ is compact, we can find a subsequence $\{C_{n_k}\}_{k \geqslant 1}$ of $\{C_n\}_{n \geqslant 1}$ and a set $C \in P_f(X)$ such that

$$h(C_{n_k}, C) \longrightarrow 0$$

(see Definition 2.49). Given $\varepsilon > 0$, we can find $k_0 = k_0(\varepsilon) \geqslant 1$ such that

$$C \subseteq (C_{n_k})_\varepsilon \quad \text{and} \quad C_{n_k} \subseteq C_\varepsilon \qquad \forall\, k \geqslant k_0. \tag{2.28}$$

Note that $(C_{n_k})_\varepsilon = B_\varepsilon(u_{n_k})$. Hence $\operatorname{diam} C \leqslant 2\varepsilon$, so $C = \{u\}$ (i.e., C is singleton), thus

$$u_{n_k} \in B_\varepsilon(u) \qquad \forall\, k \geqslant k_0$$

(see (2.28)) and so $u_{n_k} \longrightarrow u$ in X. This proves the compactness of X.

Solution of Problem 2.64

We need to show that for every $\lambda \in \mathbb{R}$, the set $L_\lambda = \{u \in X : \ v(y) \leqslant \lambda\}$ is closed (see Definition I.2.46). So, let $\{u_\alpha\}_{\alpha \in J} \subseteq L_\lambda$ be a net such that $u_\alpha \longrightarrow u$ in X. Given any $\varepsilon > 0$, we can find $y \in F(u)$ such that

$$v(u) - \varepsilon \leqslant \varphi(u, y). \tag{2.29}$$

The lower semicontinuity of F implies that we can find $y_\alpha \in F(u_\alpha)$ for $\alpha \in J$ such that $y_\alpha \longrightarrow y$ in Y (see Proposition 2.38). We have

$$\varphi(u_\alpha, y_\alpha) \leqslant v(u_\alpha) \leqslant \lambda \qquad \forall\, \alpha \in J,$$

so

$$\varphi(u,y) \leqslant \liminf_{\alpha\in J} \varphi(u_\alpha,y_\alpha) \leqslant \lambda$$

(since φ is lower semicontinuous), thus $v(u)-\varepsilon \leqslant \lambda$ (see (2.29)).

Since $\varepsilon > 0$ was arbitrary, we let $\varepsilon \searrow 0$, to conclude that $v(u) \leqslant \lambda$, so $u \in L_\lambda$ and thus L_λ is closed. We conclude that v is a lower semicontinuous function.

Solution of Problem 2.65

Let $\{(u_\alpha,y_\alpha)\}_{\alpha\in J} \subseteq \operatorname{Gr} F$ be a net and assume that $u_\alpha \longrightarrow u$ in X. Since F is locally compact, we can find $U \in \mathcal{N}(u)$ (where $\mathcal{N}(u)$ is the filter of neighborhoods of u) such that $\overline{F(U)} \in P_k(Y)$. We can find $\alpha_0 \in J$ such that $u_\alpha \in U$ for all $\alpha \geqslant \alpha_0$. Then $\{y_\alpha\}_{\alpha\geqslant\alpha_0} \subseteq Y$ is relatively compact, hence $\{y_\alpha\}_{\alpha\in J}$ has a cluster point in $\overline{F(U)}$. We can find a subnet $\{y_\beta\}_{\beta\in I}$ of $\{y_\alpha\}_{\alpha\in J}$ and $y \in \overline{F(U)}$ such that $y_\beta \longrightarrow y$ in Y. Note that $\{(u_\beta,y_\beta)\}_{\beta\in J} \subseteq \operatorname{Gr} F$ and since $u_\beta \longrightarrow u$ in X, $y_\beta \longrightarrow y$ in Y and F is closed, we have $(u,y) \in \operatorname{Gr} F$, that is, $y \in F(u)$. So, invoking Proposition 2.42, we conclude that F is upper semicontinuous.

Solution of Problem 2.66

From Problem 2.64 we already know that v is lower semicontinuous. So, it suffices to show that v is upper semicontinuous. To this end, for every $\lambda \in \mathbb{R}$, let

$$U_\lambda = \{u \in X : v(u) \geqslant \lambda\}.$$

We need to show that U_λ is closed (see Definition I.2.46). So, let $\{u_\alpha\}_{\alpha\in J} \subseteq U_\lambda$ be a net and assume that $u_\alpha \longrightarrow u$ in X. Since φ is continuous and F is $P_k(Y)$-valued, for every $\alpha \in J$, we can find $y_\alpha \in F(u_\alpha)$ such that

$$v(u_\alpha) = \varphi(u_\alpha,y_\alpha) \geqslant \lambda. \tag{2.30}$$

Invoking Proposition 2.42, we can find a subnet $\{y_\beta\}_{\beta\in I}$ of $\{y_\alpha\}_{\alpha\in J}$ and $y \in F(u)$ such that $y_\beta \longrightarrow y$ in Y. Then the continuity of φ implies that

$$\varphi(u_\beta,y_\beta) \longrightarrow \varphi(u,y) \leqslant v(u),$$

so

$$\lambda \leqslant v(u)$$

(see (2.30)), thus $u \in U_\lambda$ and hence U_λ is closed. This proves that v is also upper semicontinuous, therefore it is continuous.

Clearly S is $P_k(Y)$-valued. For every $u \in X$, let

$$G(u) = \{y \in Y : v(u) = \varphi(u,y)\}.$$

The continuity of v and φ implies that G is closed (see Definition 2.40). We have

$$S(u) = G(u) \cap F(u).$$

Invoking Problem 2.56(b), we conclude that S is upper semicontinuous.

Solution of Problem 2.67

Let $X = [0,1]$ and $Y = \mathbb{R}^2_+$. Consider the multifunction $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ defined by

$$F(u) = \{(t, ut):\ t \geqslant 0\}.$$

Clearly F is lower semicontinuous (see Proposition 2.38(d)), but F is not h-lower semicontinuous (see Definitions 2.53(b) and 2.49(b)).

Next, let $X = [0,1]$ and $Y = \mathbb{R}$. Consider the multifunction $F\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ defined by

$$F(u) = \begin{cases} [0,1] & \text{if } u \in [0,1), \\ [0,1) & \text{if } u = 1. \end{cases}$$

Clearly F is h-upper semicontinuous (see Definitions 2.53(a) and 2.49(b)). However, note that $F^+((-1,1)) = \{1\}$ (see Definition 2.35(b)) and so F is not upper semicontinuous at 1 (see Definition 2.36(a) and Proposition 2.37).

Solution of Problem 2.68

Let $\{(u_\alpha, y_\alpha)\}_{\alpha\in J} \subseteq \operatorname{Gr} F$ be a net and assume that $(u_\alpha, y_\alpha) \longrightarrow (u, y)$ in $X \times Y$. We have

$$\operatorname{dist}_Y(y_\alpha, F(u)) \;\leqslant\; h^*(F(u_\alpha), F(u)) \;\longrightarrow\; 0$$

(see Definitions 2.53(a) and 2.49), so $\operatorname{dist}_Y(y, F(u)) = 0$, thus $y \in F(u)$ and hence F is closed (see Definition 2.40).

Solution of Problem 2.69

We need to show that for every $\lambda \geqslant 0$, the set

$$L_\lambda(y) \;=\; \{u \in X : \; \varphi_y(u) \leqslant \lambda\}$$

is closed. To this end, let $\{u_\alpha\}_{\alpha\in J} \subseteq L_\lambda(y)$ be a net such that $u_\alpha \longrightarrow u$ in X. For every $v_\alpha \in F(u_\alpha)$ we have

$$\begin{aligned} \operatorname{dist}_Y(y, F(u)) \;&\leqslant\; d_Y(y, v_\alpha) + \operatorname{dist}_Y(v_\alpha, F(u)) \\ &\leqslant\; d_Y(y, v_\alpha) + h^*(F(u_\alpha), F(u)) \end{aligned}$$

(see Definition 2.49), so

$$\varphi_y(u) \;\leqslant\; \varphi_y(u_\alpha) + h^*(F(u_\alpha), F(u)),$$

thus

$$\varphi_y(u) \;\leqslant\; \lambda + h^*(F(u_\alpha), F(u))$$

and from the h-upper semicontinuity of F, we get

$$\varphi_y(u) \;\leqslant\; \lambda.$$

This means that $u \in L_\lambda(u)$, which proves that the set $L_\lambda(u)$ is closed. Hence the function $u \longmapsto \varphi_y(u)$ is lower semicontinuous.

Solution of Problem 2.70

(a) This is a consequence of Proposition 2.56(a) and Problem 2.56(a).

(b) From the definition of the Hausdorff metric (see Definition 2.49), we have

$$\begin{aligned} h^*(R_F(K), R_F(K')) &= \sup_{h \in R_F(K)} \inf_{y \in R_F(K')} d_X(h, y) \\ = \sup_{h \in R_F(K)} \inf_{v \in K'} h^*(h, F(v)) &= \sup_{u \in K} \inf_{v \in K'} h^*(F(u), F(v)) \\ \leqslant \sup_{u \in K} \inf_{v \in K'} h(F(u), F(v)) &\leqslant k \sup_{u \in K} \inf_{v \in K'} d_X(u, v) = kh^*(K, K'). \end{aligned}$$

In a similar way, we show that

$$h^*(R_F(K'), R_F(K)) \leqslant kh^*(K', K).$$

Hence

$$h(R_F(K), R_F(K')) \leqslant kh(K, K')$$

(see Definition 2.49).

Solution of Problem 2.71

Let $V \subseteq Y$ be an open set. We need to show that $(F \cap G)^-(V) \subseteq X$ is open (see Proposition 2.38). So, let $u \in (F \cap G)^-(V)$ and $y \in F(u) \cap G(u) \cap V$. Then $(u, y) \in \operatorname{Gr} G \cap (X \times V)$. Since by hypothesis $\operatorname{Gr} G \subseteq X \times Y$ is open, we can find $U_1(u) \in \mathcal{N}(u)$ and $V_1(y) \in \mathcal{N}(y)$ (where $\mathcal{N}(u)$ and $\mathcal{N}(y)$ are the filters of neighborhoods of u in X and y in Y respectively) such that

$$U_1(x) \times V_1(y) \subseteq \operatorname{Gr} G \cap (X \times V). \tag{2.31}$$

Note that $F(u) \cap V_1(y) \neq \emptyset$ and recall that F is lower semicontinuous. So, we can find $U_2(u) \in \mathcal{N}(u)$ such that

$$F(u') \cap V_1(y) \neq \emptyset \qquad \forall\, u' \in U_2(u). \tag{2.32}$$

We set $U(u) = U_1(u) \cap U_2(u) \in \mathcal{N}(u)$. Then for all $u' \in U(u)$ we have

$$F(u') \cap V_1(y) \neq \emptyset \quad \text{and} \quad U(u) \times V_1(y) \subseteq \operatorname{Gr} G \cap (X \times Y)$$

(see (2.31) and (2.32)), so

$$F(u') \cap G(u') \cap V \neq \emptyset \qquad \forall\, u' \in U(u),$$

thus the set $(F \cap G)^{-}(V) \subseteq X$ is open and hence $u \longmapsto (F \cap G)(u)$ is lower semicontinuous.

Solution of Problem 2.72

Consider the multifunction $H\colon X \longrightarrow 2^Y \setminus \{\emptyset\}$ defined by

$$H(u) \;=\; F(u)_r \;=\; \{y \in Y:\ \operatorname{dist}_Y(y, F(u)) < r\} \qquad \forall\, u \in X.$$

Evidently H has convex and open values. We claim that H is lower semicontinuous (see Definition 2.36(b)). To this end, let $\{u_\alpha\}_{\alpha \in J} \subseteq X$ be a net such that $u_\alpha \longrightarrow u$ in X and let $y \in H(u)$. If $y \in F(u)$, then exploiting the lower semicontinuity of F (see Proposition 2.38) we can find $y_\alpha \in F(u_\alpha)$ for $\alpha \in J$ such that $y_\alpha \longrightarrow y$. Clearly $y_\alpha \in H(u_\alpha)$. If $y \notin F(u)$, then we can find $v \in F(u)$ such that $\|v - y\|_Y < r$. Let $v_\alpha \in F(u_\alpha)$ be such that $v_\alpha \longrightarrow v$ (again this is possible due to the lower semicontinuity of F). Then we can find $\alpha_0 \in J$ such that

$$\|v_\alpha - y\|_Y \;<\; r \qquad \forall\, \alpha \geqslant \alpha_0,$$

so

$$\operatorname{dist}_Y(y, F(u_\alpha)) \;<\; r \qquad \forall\, \alpha \geqslant \alpha_0,$$

thus $y \in H(u_\alpha)$ for all $\alpha \geqslant \alpha_0$ and hence H is lower semicontinuous (see Proposition 2.37). Invoking Proposition 2.60 (note that $\operatorname{int} H(u) \neq \emptyset$ for all $u \in X$), we can find a continuous function $h\colon X \longrightarrow Y$ such that

$$h(u) \;\in\; H(u) \qquad \forall\, u \in X,$$

so

$$\operatorname{dist}_Y(h(u), F(u)) \;<\; r \qquad \forall\, u \in X.$$

Solution of Problem 2.73

Recall that a compact Hausdorff space is paracompact (see Definition I.2.142 and Theorem I.2.144). Also, the collection $\{F^-(\{y\})\}_{y\in Y}$ is an open cover of X. The compactness of X implies that we can find a finite set $\{y_1,\ldots,y_n\} \subseteq Y$ such that $\{F^-(\{y_1\}),\ldots,F^-(\{y_1\})\}$ is an open cover of X. The paracompactness of X implies that there is a continuous partition of unity $\{s_1,\ldots,s_n\}$ subordinate to this finite open cover (see Definition I.2.146 and Theorem I.2.147). Let

$$f(u) \;=\; \sum_{k=1}^{n} s_k(u)y_k.$$

Then f is continuous. If $s_k(u) \neq 0$, then $u \in F^-(\{y_k\})$ and so $y_k \in F(u)$. The convexity of the values of F implies that $f(u) \in F(u)$ for all $u \in X$.

Solution of Problem 2.74

Let $G\colon Y \longrightarrow 2^X \setminus \{\emptyset\}$ be the multifunction defined by

$$G(y) \;=\; A^{-1}(y) \;=\; \{u \in X:\; A(u) = y\}. \qquad (2.33)$$

Since $A \in \mathcal{L}(X;Y)$ is surjective, by the open mapping theorem (see Theorem I.5.47), A is open. For every open set $V \subseteq X$, we have that the set $G^-(V) = A(V) \subseteq Y$ is open. So, by Proposition 2.38, G is lower semicontinuous. Also, from (2.33) it is clear that G has values in $P_{fc}(X)$. We can apply the Michael selection theorem (see Theorem 2.58) and find a continuous map $f\colon Y \longrightarrow X$ such that

$$f(y) \;\in\; G(y) \qquad \forall\, y \in Y,$$

so

$$A(f(y)) \;=\; y \qquad \forall\, y \in Y.$$

Solution of Problem 2.75

Suppose that $f\colon C \longrightarrow Y$ is a continuous selection of $F|_C$. Let $\widehat{F}\colon X \longrightarrow P_{fc}(Y)$ be the multifunction defined by

$$\widehat{F}(u) = \begin{cases} f(u) & \text{if } u \in C, \\ F(u) & \text{if } u \in X \setminus C. \end{cases}$$

Clearly $\widehat{F}$ is lower semicontinuous (see Proposition 2.38(d)). Also, it has values in $P_{fc}(Y)$. So, we can apply the Michael selection theorem (see Theorem 2.58) and conclude that there exists a continuous selection $\widehat{f}\colon X \longrightarrow Y$ of $\widehat{F}$. Evidently $\widehat{f}$ extends f on all of X.

Solution of Problem 2.76

For every $u \in X$, let $F(u) = (\varphi(u), \psi(u)) \subseteq \mathbb{R}$. We claim that F is lower semicontinuous. Indeed, let $u_n \longrightarrow u$ in X and let $y \in F(u)$. Exploiting the upper semicontinuity of φ and the lower semicontinuity of ψ, we see that we can find $n_0 \geqslant 1$ such that

$$y \in (\varphi(u_n), \psi(u_n)) \qquad \forall\, n \geqslant n_0,$$

so F is lower semicontinuous. Applying Proposition 2.60, we obtain a continuous selection $f\colon X \longrightarrow \mathbb{R}$ of F. Then $\varphi(u) < f(u) < \psi(u)$ for all $u \in X$.

Solution of Problem 2.77

According to Proposition 2.38, we need to show that for every open set $V \subseteq Y$, the set $H^-(V) = \{u \in X : H(u) \cap V \neq \emptyset\} \subseteq X$ is open. So, let $y_0 \in H(u_0) \cap V$. Then $y_0 \in (\varphi(u_0) + U) \cap V$. Thus there exists an open symmetric neighborhood W of the origin in V such that

$$y_0 + W + W \subseteq \big(\varphi(u_0) + U\big) \cap V,$$

so $y_0 + W \subseteq V$.

Let $D = \varphi^{-1}(\varphi(u_0) + W)$. Since φ is continuous, we see that $D \subseteq X$ is open and $u_0 \in D$.

Next, we will show that

$$y_0 + W \subseteq \varphi(u) + U \qquad \forall\, u \in D. \tag{2.34}$$

To prove this, let $w \in W$ and $u \in D$. Then

$$h = \varphi(u_0) - \varphi(u) \in W.$$

Since $y_0 + W + W \subseteq \varphi(u_0) + U$, we can find $\widehat{u} \in U$ such that $y_0 + w + h = \varphi(u_0) + \widehat{u}$ and so

$$y_0 + w = \varphi(u) + \widehat{u} \in \varphi(u) + U,$$

which proves (2.34). Since $y_0 \in F(u_0) \cap (y_0 + W)$, we have $u_0 \in F^-(y_0 + W)$ and since F is lower semicontinuous, we have that the set $F^-(y_0 + W)$ is open. Let

$$E = D \cap F^-(y_0 + W).$$

Then E is an open neighborhood of u_0. If $u \in E$, then

$$\begin{aligned} H(u) \cap V &= F(u) \cap \big(\varphi(u) + U\big) \cap V \\ &\supseteq F(u) \cap \big(\varphi(u) + U\big) \cap (y_0 + W) \\ &= F(u) \cap (y_0 + W) \neq \emptyset \end{aligned}$$

(see (2.34) and recall that $y_0 + W \subseteq V$), so H is lower semicontinuous.

Solution of Problem 2.78

We consider the multifunction $\Gamma \colon X \longrightarrow 2^{Y \times Y} \setminus \{\emptyset\}$ defined by

$$\Gamma(u) = F(u) \times G(u) \qquad \forall\, u \in X.$$

Let $V_1 \times V_2 \subseteq Y \times Y$ be a basic open set. Then

$$\Gamma^-(V_1 \times V_2) = F^-(V_1) \cap G^-(V_2),$$

so $\Gamma^-(V_1 \times V_2) \subseteq X$ is open (since both F and G are lower semicontinuous) and thus Γ is lower semicontinuous. Let $\varphi \colon Y \times Y \longrightarrow Y$ be defined by

$$\varphi(y_1, y_2) = y_1 + y_2 \qquad \forall\, y_1, y_2 \in Y.$$

We know that φ is continuous. Then we have $H = \varphi \circ \Gamma$. It is easily seen that the composition of two lower semicontinuous multifunctions is lower semicontinuous too. Therefore H is lower semicontinuous.

Solution of Problem 2.79

Given $\varepsilon > 0$, for every $u \in X$, we can find $V_u \in \mathcal{N}(u)$ such that

$$\bigcap_{u' \in V_u} F(u')_\varepsilon \neq \emptyset.$$

The collection $\{V_u\}_{u \in X}$ is an open cover of X. Since X is paracompact (see Definition I.2.142), there is a locally finite refinement $\{W_j\}_{j \in J}$ of $\{V_u\}_{u \in X}$ (see Definition I.2.146 and Theorem I.2.147). For each $j \in J$, let $u_j \in X$ be such that $W_j \subseteq V_{u_j}$. Also, let $\{\xi_j\}_{j \in J}$ be a corresponding continuous partition of unity subordinate to $\{W_j\}_{j \in J}$ (see Definition I.1.111). For every $j \in J$, we choose $y_j \in \bigcap\limits_{u' \in W_j} F(u')_\varepsilon$ and then define $f_\varepsilon \colon X \longrightarrow Y$ by setting

$$f_\varepsilon(u) = \sum_{j \in J} \xi_j(u) y_j.$$

Clearly $f_\varepsilon \in C(X;Y)$ and since F is convex valued, we have

$$f_\varepsilon(u) \in \operatorname{conv} F(u)_\varepsilon = F(u)_\varepsilon \qquad \forall\, u \in X.$$

Solution of Problem 2.80

Let $\widehat{F} \colon X \longrightarrow P_{fc}(Y)$ be the multifunction defined by

$$\widehat{F}(u) = \begin{cases} \{\widehat{y}\} & \text{if } u = \widehat{u}, \\ F(u) & \text{if } u \neq \widehat{u}. \end{cases}$$

It is easy to see that $\widehat{F}$ is lower semicontinuous (see Proposition 2.38) and has values in $P_{fc}(Y)$. So, we can apply the Michael selection theorem (see Theorem 2.58) and find a continuous function $\widehat{f} \colon X \longrightarrow Y$ such that $\widehat{f}(u) \in \widehat{F}(u)$ for all $u \in X$. Evidently $\widehat{f}$ is a continuous selector of $\widehat{F}$ such that $\widehat{f}(\widehat{u}) = \widehat{y}$.

Solution of Problem 2.81

For $u_0 \in D$, we have $F(u_0) \subseteq F(D) \subseteq \overline{F(D)}$. Suppose that $u_0 \in \overline{D} \setminus D$ and let $y \in F(u_0)$. For each open set $V \subseteq Y$ containing y, we have that $F^-(V) \subseteq X$ is an open set containing u_0 (see Proposition 2.38). Since $u_0 \in \overline{D}$, we can find $u \in D \cap F^-(V)$. Then $F(u) \cap V \neq \emptyset$ and so $F(D) \cap V \neq \emptyset$. This shows that $y \in \overline{F(D)}$ and so we conclude that $F(\overline{D}) \subseteq \overline{F(D)}$.

Solution of Problem 2.82

See the solution of Problem 2.78 for the Cartesian product of two lower semicontinuous multifunctions. The statement is not true for upper semicontinuous multifunctions. To see this let $X = Y = \mathbb{R}$ and consider the multifunction $F \colon \mathbb{R} \longrightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ defined by

$$F(u) = (-1, 1) \qquad \forall\, u \in \mathbb{R}.$$

Let $H \colon \mathbb{R} \longrightarrow 2^{\mathbb{R}\times\mathbb{R}} \setminus \{\emptyset\}$ be the multifunction defined by $H = I_{\mathbb{R}} \times F$. Clearly $I_{\mathbb{R}}$ and F are upper semicontinuous multifunctions. On the other hand, if $B_1 = \big\{h \in \mathbb{R}^2 : |h| < 1\big\}$, then $(I_{\mathbb{R}} \times F)(0) \subseteq B_1$. However, we cannot find a neighborhood U of 0 in $\mathbb{R}$ such that $(I_{\mathbb{R}} \times F)(U) \subseteq B_1$. Therefore the map $u \longmapsto H(u) = (u, F(u))$ is not upper semicontinuous.

Solution of Problem 2.83

We fix $\varepsilon > 0$. Because F is upper semicontinuous, for every $u \in X$, we can find $\delta = \delta(\varepsilon, u) \in (0, \frac{\varepsilon}{2})$ such that, if $u' \in B_\delta(u) = \{u' \in X : d_X(u', u) < \delta\}$, then $F(u') \subseteq F(u) + \frac{\varepsilon}{2} B_1$, where $B_1 = \big\{y \in Y : \|y\|_Y < \varepsilon\big\}$ (see Proposition 2.54). The family $\{B_{\frac{\delta}{4}}(u)\}_{u \in X}$ is an open cover of X. Recall that a metric space is paracompact (see Definition I.2.142 and Theorem I.2.144). So, we can find a locally finite refinement $\{U_\alpha\}_{\alpha \in J}$ of $\{B_{\frac{\delta}{4}}(u)\}_{u \in X}$ (see Definition I.2.146 and Theorem I.2.147). There exists a locally Lipschitz partition of unity $\{\xi_\alpha\}_{\alpha\in J}$ subordinate to this cover $\{U_\alpha\}_{\alpha\in J}$ (see Definition I.1.111). For each $\alpha \in J$, we choose $(u_\alpha, y_\alpha) \in \operatorname{Gr} F \cap (U_\alpha \times Y)$ and define

$$f_\varepsilon(u) \;=\; \sum_{\alpha\in J} \xi_\alpha(u) y_\alpha \qquad \forall\, u \in X.$$

Since the cover $\{U_\alpha\}_{\alpha\in J}$ is locally finite, f_ε is well defined and locally Lipschitz. Moreover, it is clear that

$$f_\varepsilon(u) \;\in\; \operatorname{conv} F(X) \qquad \forall\, u \in X.$$

Next, we fix $u \in X$. Recalling that $\{U_\alpha\}_{\alpha\in J}$ is locally finite, we have that $0 < \xi_\alpha(u)$ for all $\alpha \in J(u) \subseteq J$ with $J(u)$ finite. For every $\alpha \in J(u)$, let $u_\alpha \in X$ be such that $U_\alpha \subseteq B_{\frac{\delta_\alpha}{4}}(u_\alpha)$, with $\delta_\alpha = \delta(\varepsilon, u_\alpha) > 0$. Let $\beta \in J(u_\alpha)$ and set $\delta_\beta = \max\{\delta_\alpha : \ \alpha \in J(u)\}$. Then $u_\alpha \in B_{\frac{\delta_\beta}{4}}(u_\beta)$ and so $U_\alpha \subseteq B_{\delta_\beta}(u_\beta)$. Hence for every $\alpha \in J(u)$, we have

$$y_\alpha \;\in\; F(U_\alpha) \;\subseteq\; F(U_\beta) + \tfrac{\varepsilon}{2} B_1.$$

Since the latter set is convex, we have

$$f_\varepsilon(u) \;\in\; F(U_\beta) + \tfrac{\varepsilon}{2} B_1.$$

Thus we can find $y_\beta \in F(u_\beta)$ such that $\|f_\varepsilon(u) - y_\beta\|_Y < \frac{\varepsilon}{2}$ and so

$$d_{X\times Y}\left((u, f_\varepsilon(u)), (u_\beta, y_\beta)\right) \;=\; d_X(u, u_\beta) + \|f_\varepsilon(u) - y_\beta\|_Y \;<\; \varepsilon,$$

so $(u, f_\varepsilon(u)) \in \operatorname{Gr} F + \varepsilon B_1$. Since $u \in X$ is arbitrary, we conclude that $h^*(\operatorname{Gr} f_\varepsilon, \operatorname{Gr} F) < \varepsilon$ (see Definition 2.49).

Solution of Problem 2.84

We keep the notation from the solution of Problem 2.83. The family $\{B_\delta(u)\}_{u\in K}$ is an open cover of the compact set K. So, we can find a finite subcover $\{B_{\delta_1}(u_1), \ldots, B_{\delta_m}(u_m)\}$ with $\delta_i = \delta_i(\varepsilon, u_i) > 0$ for $i \in \{1, \ldots, m\}$. We can find a locally Lipschitz partition of unity $\{\xi_1, \ldots, \xi_m\}$ subordinate to this finite subcover (see Definition I.1.111). Also, let $y_i \in F(u_i)$ for $i \in \{1, \ldots, m\}$ and define

$$f_\varepsilon(u) \;=\; \sum_{i=1}^m \xi_i(u) y_i \qquad \forall\, u \in V_\varepsilon = \bigcup_{i=1}^m B_{\delta_i}(u_i).$$

Then $f_\varepsilon \colon V_\varepsilon \longrightarrow Y$ is locally Lipschitz and $f_\varepsilon(u) \in \operatorname{conv} F(K) \cap Y_\varepsilon$ with $Y_\varepsilon = \operatorname{span}\{y_1, \ldots, y_m\}$, hence f_ε has a finite dimensional range. As in the solution of Problem 2.83, we see that

$$f_\varepsilon(u) \in F(K \cap B_\varepsilon(u)) + \varepsilon B_1 \qquad \forall\, u \in V_\varepsilon.$$

Solution of Problem 2.85

Let $X_1 = F(X)$ and $X_{n+1} = F(X_n)$ for all $n \geqslant 1$. From Problem 2.56(a), we know that for each $n \geqslant 1$, the set X_n is compact. Also, since $X_1 \subseteq X$, we see that $X_{n+1} \subseteq X_n$ for all $n \geqslant 1$. Let $C = \bigcap\limits_{n \geqslant 1} X_n$. Since $\{X_n\}_{n \geqslant 1}$ has the finite intersection property (being decreasing; see Definition I.2.80 and Theorem I.2.81), it follows that $C \neq \emptyset$. We have

$$F(C) \subseteq \bigcap_{n \geqslant 1} F(X_n) \subseteq \bigcap_{n \geqslant 1} X_{n-1} = C$$

(setting $X_0 = X$). Suppose that $u_0 \in C \setminus F(C)$. Note that C is closed, hence $F(C) \subseteq X$ is closed too (see Problem 2.56(a)). Recall that a compact space is regular. So, we can find open sets $U_1, U_2 \subseteq X$ such that

$$u_0 \in U_1, \quad F(C) \subseteq U_2 \quad \text{and} \quad U_1 \cap U_2 = \emptyset. \tag{2.35}$$

Since F is upper semicontinuous, we have that the set $F^+(U_2) \subseteq X$ is open (see Proposition 2.37) and $C = \bigcap\limits_{n \geqslant 1} X_n \subseteq F^+(U_2)$. We have

$$X \setminus F^+(U_2) \subseteq \bigcup_{n \geqslant 1} (X \setminus X_n). \tag{2.36}$$

The set $X \setminus F^+(U_2) \subseteq X$ is closed, hence compact. From (2.36), it follows that we find a finite subcover of the open cover $\{(X \setminus X_n)\}_{n \geqslant 1}$. So, we can say that

$$X \setminus F^+(U_2) \subseteq \bigcup_{n=1}^{m} (X \setminus X_n),$$

so
$$\bigcap_{n=1}^{m} X_n \subseteq F^+(U_2),$$
thus
$$X_n \subseteq F^+(U_2) \qquad \forall\, n \geqslant m$$
hence
$$X_{n+1} = F(X_n) \subseteq U_2 \qquad \forall\, n \geqslant m$$
and so $C \subseteq U_2$. But $u_0 \in C$ and $u_0 \notin U_2$ (see (2.35)), we reach a contradiction. Therefore, we conclude that $F(C) = C$.

Solution of Problem 2.86

In the light of Problem 2.65, we need to consider a noncompact domain. So, let $X = Y = \mathbb{R}_+$ and consider the multifunction defined by
$$F(u) = \begin{cases} \{0\} & \text{if } \ u = 0, \\ [0,u] \cup \{\frac{1}{u}\} & \text{if } \ u > 0. \end{cases}$$
Evidently, for every $u \geqslant 0$, we have $F(u) \in P_k(Y)$. Also $\operatorname{Gr} F$ is closed. But F is not upper semicontinuous at 0. Indeed, note that $F(0) \subseteq [0,1)$, but $F^+([0,1)) = \{0\}$ which is not open in $Y = \mathbb{R}_+$. Therefore F is not upper semicontinuous (see Proposition 2.37).

Solution of Problem 2.87

We will consider the boundedly w-compact case. The other case can be established similarly. First note that for every $u \in X$, we have $\operatorname{proj}_K(u) \neq \emptyset$. Indeed, let $\{h_n\}_{n\geqslant 1} \subseteq K$ be a minimizing sequence, that is,
$$\|u - h_n\|_X \searrow \operatorname{dist}(u, K). \tag{2.37}$$
From (2.37) we see that the sequence $\{h_n\}_{n\geqslant 1} \subseteq K$ is bounded. So, we can find $r > 0$ such that $\{h_n\}_{n\geqslant 1} \subseteq K \cap \overline{B}_r$. Since K is by hypothesis boundedly w-compact, we have that the sequence $\{h_n\}_{n\geqslant 1} \subseteq X$ is relatively w-compact. So, by the Eberlein–Smulian theorem (see Theorem I.5.78), passing to a subsequence if necessary, we may assume that

$h_n \xrightarrow{w} h \in K \cap \overline{B}_r$. Exploiting the weak lower semicontinuity of the norm functional (see Proposition I.5.56(c)), we have

$$\|u-h\|_X \leqslant \liminf_{n\to+\infty} \|u-h_n\|_X,$$

so

$$\|u-h\|_X = \operatorname{dist}(u,K)$$

(see (2.37) and recall that $h \in K$), thus $h \in \operatorname{proj}_K(u)$ and hence $\operatorname{proj}_K(u) \neq \emptyset$.

Now let $C \subseteq X$ be a w-closed set. According to Proposition 2.37, we need to show that the set $\operatorname{proj}_K^-(C) = \{u \in X : \operatorname{proj}_K(u) \cap C \neq \emptyset\}$ is closed. So, let $\{u_n\}_{n\geqslant 1} \subseteq \operatorname{proj}_K^-(C)$ be a sequence such that $u_n \longrightarrow u$ in X. We have that $\operatorname{proj}_K(u_n) \cap C \neq \emptyset$ for all $n \geqslant 1$. We can find $h_n \in C$ such that

$$\operatorname{dist}(u_n,K) = \|u_n-h_n\|_X \qquad \forall\, n \geqslant 1. \tag{2.38}$$

We know that

$$\operatorname{dist}(u_n,K) \longrightarrow \operatorname{dist}(u,K). \tag{2.39}$$

From (2.38) and (2.39), it follows that the sequence $\{h_n\}_{n\geqslant 1} \subseteq K \cap C$ is bounded. So, there exists $r > 0$ such that $\{h_n\}_{n\geqslant 1} \subseteq K \cap \overline{B}_r$. Because K is boundedly w-compact, invoking the Eberlein Smulian theorem and by passing to a suitable subsequence if necessary, we can have

$$h_n \xrightarrow{w} h \in K \cap C \cap \overline{B}_r$$

(since C is w-closed). Also, the weak lower semicontinuity of the norm functional (see Proposition I.5.56(c)) implies that

$$\|u-h\|_X \leqslant \liminf_{n\to+\infty} \|u_n-h_n\|_X,$$

so

$$\|u-h\|_X = \operatorname{dist}(u,K)$$

(see (2.38), (2.39) and recall that $h \in K \cap C$), thus $u \in \operatorname{proj}_K^-(C)$ and hence proj_K is upper semicontinuous from X into X_w.

Solution of Problem 2.88

Note that $\mathcal{F}$ is closed from X into $X^*_{w^*}$. Indeed, let $\{(u_\alpha, u^*_\alpha)\}_{\alpha\in J} \subseteq \operatorname{Gr}\mathcal{F}$ be a net such that $u_\alpha \longrightarrow u$ in X and $u^*_\alpha \xrightarrow{w^*} u^*$ in X^*. Then $\|u_\alpha\|_X \longrightarrow \|u\|_X$ and

$$\|u_\alpha\|_X^2 \;=\; \|u^*_\alpha\|_*^2 \;=\; \langle u^*_\alpha, u_\alpha\rangle \;\longrightarrow\; \langle u^*, u\rangle\,,$$

so

$$\|u_\alpha\|_X^2 \;=\; \langle u^*, u\rangle \;\leqslant\; \|u^*\|_{X^*}\|u\|_X \tag{2.40}$$

On the other hand, the weak* lower semicontinuity of the norm functional of X^* (see Proposition I.5.65(c)) implies that

$$\|u^*\|_{X^*} \;\leqslant\; \liminf_{\alpha\in J}\|u^*_\alpha\|_{X^*} \;=\; \liminf_{\alpha\in J}\|u_\alpha\|_X \;=\; \|u\|_X. \tag{2.41}$$

From (2.40) and (2.41) it follows that

$$\|u^*\|_{X^*} \;=\; \|u\|_X \;=\; \langle u^*, u\rangle\,,$$

so $u^* \in \mathcal{F}(u)$ and hence $\mathcal{F}$ is closed.

From the definition of $\mathcal{F}$ (see Definition 2.112) and the Alaoglu theorem (see Theorem I.5.66), we see that $\mathcal{F}$ is locally compact from X into $X^*_{w^*}$. So, invoking Problem 2.65, we conclude that $\mathcal{F}$ is upper semicontinuous from X into $X^*_{w^*}$.

Solution of Problem 2.89

Since F is compact valued, according to Proposition 2.56(a), it suffices to show that F is h-upper semicontinuous (see Definition 2.53(a)). Arguing indirectly, suppose that F is not h-upper semicontinuous. Then we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ with $u_n \longrightarrow u$ in X and $\varepsilon > 0$ such that

$$F(u_n) \;\not\subseteq\; F(u) + \varepsilon B_1 \qquad \forall\, n \geqslant 1. \tag{2.42}$$

From (2.42) it follows that we can find $h_n \in F(u_n)$ such that

$$\operatorname{dist}(h_n, F(u)) \;\geqslant\; \varepsilon \qquad \forall\, n \geqslant 1.$$

Since F is lower semicontinuous and compact valued, it is also h-lower semicontinuous (see Proposition 2.56(b)). So, we can find $n_0 \in \mathbb{N}$ such that

$$F(u) \subseteq F(u_n) + \varepsilon B_1 \qquad \forall\, n \geqslant n_0. \tag{2.43}$$

Let $h \in F(u)$. From (2.43) we see that we can find $\widehat{h}_n \in F(u_n)$ for $n \geqslant n_0$ such that

$$\|\widehat{h}_n - h\|_X < \varepsilon \qquad \forall\, n \geqslant n_0,$$

so

$$\operatorname{dist}(\widehat{h}_n, F(u)) < \varepsilon \qquad \forall\, n \geqslant n_0. \tag{2.44}$$

Recall that F has connected values. Then the continuity of the map $y \longmapsto \operatorname{dist}(y, F(u))$ on $F(u_n)$ for $n \geqslant n_0$, together with (2.43) and (2.44) imply that there exists $h_n' \in F(u_n)$ such that

$$\operatorname{dist}(h_n', F(u)) = \varepsilon \qquad \forall\, n \geqslant n_0 \tag{2.45}$$

(the Bolzano theorem; see Theorem I.1.90). So, we have that

$$\{h_n'\}_{n \geqslant n_0} \subseteq F(u) + \varepsilon \overline{B}_1 \in P_k(\mathbb{R}^N).$$

By passing to a subsequence if necessary, we may assume that

$$h_n' \longrightarrow h' \quad \text{in } \mathbb{R}^N.$$

But $(u_n, h_n') \in \operatorname{Gr} F$ and the latter by hypothesis is closed. So $h' \in F(u)$. On the other hand, from (2.45), we have

$$\operatorname{dist}(h', F(u)) = \varepsilon > 0,$$

so $h' \notin F(u)$, a contradiction. Therefore F is h-upper semicontinuous, hence continuous.

Solution of Problem 2.90

Let $X = Y = \mathbb{R}$ and consider the multifunction $F\colon \mathbb{R} \longrightarrow P_{fc}(\mathbb{R})$ defined by

$$F(u) = \begin{cases} -1 & \text{if } \ u < 0, \\ [-1, 1] & \text{if } \ u = 0, \\ 1 & \text{if } \ u > 0 \end{cases}$$

(the sign multifunction). Clearly F is upper semicontinuous (see Definition 2.36(a)) but does not have a continuous selector.

Solution of Problem 2.91

From Theorem 2.63, we know that $\mathrm{Gr}\, F \in \mathcal{B}(T \times X)$. Let $\widehat{T}$ and $\widehat{X}$ be the metric completions of T and X respectively. Since by hypothesis T and X are Borel spaces, we have $\mathrm{Gr}\, F \in \mathcal{B}(\widehat{T} \times \widehat{X})$. Let $C \in P_f(X)$ and let $\widehat{C}$ be the $\widehat{X}$-closure of C. Then $C = \widehat{C} \cap X$. We have

$$\begin{aligned} F^-(C) &= \{t \in T : \ F(t) \cap C \neq \emptyset\} \\ &= \left\{t \in T : \ F(t) \cap \widehat{C} \neq \emptyset\right\} \\ &= \mathrm{proj}_{\widehat{T}} \big(\mathrm{Gr}\, F \cap (\widehat{T} \times \widehat{C})\big) \ \in \ \mathcal{B}(X) \end{aligned}$$

(by the Yankov–von Neumann–Aumann projection theorem; see Theorem I.4.65).

Solution of Problem 2.92

Consider the multifunction $F\colon \Omega \longrightarrow 2^X$ defined by

$$F(\omega) \ = \ \varphi^{-1}(\{\omega\}) \ = \ \{u \in X : \ \varphi(u) = \omega\}. \tag{2.46}$$

Therefore by hypothesis **(i)**, F has values in $P_f(X)$. Also, if $V \subseteq X$ is open, then

$$F^-(V) \ = \ \{\omega \in \Omega : \ F(\omega) \cap V \neq \emptyset\} \ = \ \varphi(V) \ \in \ \Sigma,$$

so F is measurable (see Definition 2.61). Invoking the Kuratowski–Ryll Nardzewski selection theorem (see Theorem 2.64), we can find a Σ-measurable map $f\colon \Omega \longrightarrow X$ such that

$$f(\omega) \ \in \ F(\omega) \qquad \forall\, \omega \in \Omega.$$

From (2.46) we see that $\varphi(f(\omega)) = \omega$ for all $\omega \in \Omega$.

Solution of Problem 2.93

"$\Longrightarrow$": Let $C \subseteq X$ be a closed set and let $U = X \setminus C$. We know that U is an F_σ-set. In particular, we have

$$U = \bigcup_{n \geqslant 1} \widehat{C}_n \quad \text{with } \widehat{C}_n = \left\{x \in X : \ \operatorname{dist}(x, C) \geqslant \tfrac{1}{n}\right\} \qquad \forall\, n \geqslant 1.$$

We have

$$F^-(C) = \Omega \setminus F^+(U) = \Omega \setminus F^+\Big(\bigcup_{n \geqslant 1} \widehat{C}_n\Big). \tag{2.47}$$

It is easy to see that we always have

$$\bigcup_{n \geqslant 1} F^+(\widehat{C}_n) \subseteq F^+\Big(\bigcup_{n \geqslant 1} \widehat{C}_n\Big). \tag{2.48}$$

We will show that the opposite inclusion also holds. Arguing indirectly, suppose that we can find $\omega \in F^+\Big(\bigcup\limits_{n \geqslant 1} \widehat{C}_n\Big)$ such that $F(\omega) \cap \widehat{C}_n^c \neq \emptyset$ for all $n \geqslant 1$. Let $u_n \in F(\omega) \cap \widehat{C}_n^c$ for all $n \geqslant 1$. Since $F(\omega) \in P_k(X)$, we may assume that $u_n \longrightarrow u \in F(\omega) \subseteq U$. We know that $u_n \in \widehat{C}_n^c$ for all $n \geqslant 1$. Hence we have

$$\operatorname{dist}(u_n, C) < \tfrac{1}{n} \qquad \forall\, n \geqslant 1$$

and so $\operatorname{dist}(u, C) = 0$, therefore $u \in C$, a contradiction. So, the inclusion in (2.48) is in fact an equality of the two sets and we have

$$F^+\Big(\bigcup_{n \geqslant 1} \widehat{C}_n\Big) = \bigcup_{n \geqslant 1} F^+(\widehat{C}_n) = \bigcup_{n \geqslant 1} \big(\Omega \setminus F^-(\widehat{C}_n^c)\big) \in \Sigma$$

(since F is measurable), thus $F^-(C) \in \Sigma$ (see (2.47)).

"$\Longleftarrow$": This follows from Theorem 2.63(a).

Solution of Problem 2.94

Since X is a separable metric space, it can be viewed as a dense subset of a compact metric space Y (in fact, X is homeomorphic to a subset of the Hilbert cube $[0,1]^{\mathbb{N}}$ which is compact by the Tichonov theorem; see Theorem I.2.91). Let $G\colon \Omega \longrightarrow P_k(Y)$ be defined by

$$G(\omega) \;=\; \overline{F(\omega)}^{Y} \qquad \forall\, \omega \in \Omega.$$

Then for every compact set $K \subseteq X$, we have

$$\begin{aligned} F^-(K) &= \{\omega \in \Omega:\; F(\omega) \cap K \neq \emptyset\} \\ &= \{\omega \in \Omega:\; G(\omega) \cap X \cap K \neq \emptyset\} \;=\; G^-(K). \end{aligned}$$

Since G is compact valued in Y and measurable, from Problem 2.93, we have that $G^-(K) \in \Sigma$ and so $F^-(K) \in \Sigma$.

Solution of Problem 2.95

We know that φ is jointly measurable. Let $C \subseteq X$ be a closed set and let $\{u_n\}_{n\geqslant 1} \subseteq C$ be a sequence dense in C. We have

$$\begin{aligned} F^-(C) &= \{\omega \in \Omega:\; F(\omega) \cap C \neq \emptyset\} \\ &= \{\omega \in \Omega:\; \varphi(\omega, u) \in U \text{ for some } u \in C\} \\ &= \{\omega \in \Omega:\; \varphi(\omega, u_{n_0}) \in U \text{ for some } n_0 \geqslant 1\} \\ &= \bigcup_{n\geqslant 1} \{\omega \in \Omega:\; \varphi(\omega, u_n) \in U\} \;\in\; \Sigma \end{aligned}$$

(since U is open and $\varphi(\omega,\cdot)$ is continuous), so F is measurable (see Theorem 2.63(a)).

Solution of Problem 2.96

Since X and Y are two σ-compact metric spaces, both are separable and

$$\begin{aligned} X &= \bigcup_{n\geqslant 1} C_n \quad \text{with } C_n \subseteq X \text{ compact for all } n \geqslant 1, \\ Y &= \bigcup_{m\geqslant 1} K_m \quad \text{with } K_m \subseteq Y \text{ compact for all } m \geqslant 1. \end{aligned}$$

Let $D \subseteq Y$ be a closed set. We have

$$F^-(D) = \bigcup_{n,m\geqslant 1} \operatorname{proj}_X\big((C_n \times (D \cap K_m)) \cap \operatorname{Gr} F\big).$$

Since F is a closed multifunction (see Definition 2.40), we have that the set

$$(C_n \times (D \cap K_m)) \cap \operatorname{Gr} F \subseteq X \times Y$$

is compact for all $n, m \geqslant 1$. Recall that proj_X is continuous. Hence the set $\operatorname{proj}_X\big((C_n \times (D \cap K_m)) \cap \operatorname{Gr} F\big)$ is compact for each $n, m \geqslant 1$, thus a Borel set. Therefore $F^-(D) \subseteq X$ is a Borel set and so F is measurable (see Theorem 2.63(a)).

Solution of Problem 2.97

Without any loss of generality, we may assume that $H(\omega) \neq \emptyset$ for all $\omega \in \Omega$. First we assume that all the multifunctions F_n are compact valued. We start by showing that the map $\omega \longmapsto H_2(\omega) = F_1(\omega) \cap F_2(\omega)$ is measurable. To this end, let $P(\omega) = F_1(\omega) \times F_2(\omega)$. Note that for every set $D \subseteq X \times Y$, we have

$$P^-(D) = F_1^-(\operatorname{proj}_1 D) \cap F_2^-(\operatorname{proj}_2 D),$$

with proj_1 (respectively proj_2) being the projection on the first (respectively second) factor of the product space. So, if $D \subseteq X \times Y$ is an open set, then $P^-(D) \in \Sigma$ as

$$P^-(D)= \big\{\omega \in \Omega : F_1(\omega) \cap \operatorname{proj}_1 D \neq \emptyset\big\} \cap \big\{\omega \in \Omega : F_2(\omega) \cap \operatorname{proj}_2 D \neq \emptyset\big\},$$

since the projections are open maps and F_1, F_2 are by hypothesis measurable multifunctions (see Definition 2.61(a)). So, P is measurable and compact valued.

Let $\Delta_2 \subseteq X \times X$ be the diagonal set, that is

$$\Delta_2 = \{(u,u) : u \in X\}.$$

For every closed set $\widehat{C} \subseteq X \times X$, we have

$$H_2^-(\widehat{C}) = \Big\{\omega \in \Omega : P(\omega) \cap \Delta_2 \cap \widehat{C} \neq \emptyset\Big\} \in \Sigma$$

(see Problem 2.93), so H_2 is measurable (see Theorem 2.63).

By induction, we see that the multifunction

$$\omega \longmapsto H_n(\omega) = \bigcap_{k=1}^{n} F_k(\omega)$$

is measurable for every $n \geqslant 1$. For every $\omega \in \Omega$, we have

$$h^*(H_n(\omega), H(\omega)) = \sup_{u \in H_n(\omega)} \operatorname{dist}(u, H(\omega))$$

(see Definition 2.49). Since $H_n(\omega) \in P_k(X)$, we can find $u_n \in H_n(\omega)$ such that

$$h^*(H_n(\omega), H(\omega)) = \operatorname{dist}(u_n, H(\omega)) \qquad \forall\, n \geqslant 1.$$

Note that $\{u_n\}_{n\geqslant 1} \subseteq F_1(\omega) \in P_k(X)$. Passing to a subsequence if necessary, we may assume that $u_n \longrightarrow u$ in X. Evidently $u \in H(\omega)$ and so

$$h^*(H_n(\omega), H(\omega)) \longrightarrow 0,$$

so

$$\operatorname{dist}(u, H_n(\omega)) \longrightarrow \operatorname{dist}(u, H(\omega)) \qquad \forall\, \omega \in \Omega,\ u \in X.$$

Since H_n is measurable, the map $\omega \longmapsto \operatorname{dist}(u, F_n(\omega))$ is Σ-measurable for every $n \geqslant 1$, hence the map $\omega \longmapsto \operatorname{dist}(u, H(\omega))$ is Σ-measurable for every $u \in X$. This implies the measurability of H, when all multifunctions F_n are compact valued.

Now, assume that at least on F_n is compact valued. Let $\widehat{X}$ be the metrizable compactification of X (recall that X being separable, is homeomorphic to a subset of the Hilbert cube $[0,1]^{\mathbb{N}}$). Let $\widehat{F}_n \colon \Omega \longrightarrow P_k(\widehat{X})$ be defined by

$$\widehat{F}_n(\omega) = \overline{F_n(\omega)} \qquad \forall\, \omega \in \Omega$$

(the closure in $\widehat{X}$). From the first part of the solution we have that, if

$$\widehat{H}(\omega) = \bigcap_{n\geqslant 1} \widehat{F}_n(\omega) \qquad \forall\, \omega \in \Omega,$$

then $\widehat{H}$ is measurable. Since $F_{n_0} = \widehat{F}_{n_0}$ for some $n_0 \geqslant 1$, it follows that $H = \widehat{H}$ and so H is measurable.

Solution of Problem 2.98

From Theorem 2.66, we know that there is a sequence $\{h_n\colon \Omega \longrightarrow X\}_{n\geqslant 1}$ of Σ-measurable selectors of U such that

$$U(\omega) \;=\; \overline{\{h_n(\omega)\}}_{n\geqslant 1} \qquad \forall\, u \in \Omega.$$

Let $V \subseteq Y$ be an open set. Then

$$\begin{aligned} G^{-}(V) \;&=\; \{\omega \in \Omega:\; \varphi(\omega, U(\omega)) \cap V \neq \emptyset\} \\ &=\; \bigcup_{n\geqslant 1} \{\omega:\; \varphi(\omega, h_n(\omega)) \in V\} \end{aligned} \tag{2.49}$$

(due to the continuity of $\varphi(\omega,\cdot)$). But φ being Carathéodory, is jointly measurable. Hence, for every $n \geqslant 1$, the map $\omega \longmapsto \varphi(\omega, h_n(\omega))$ is Σ-measurable. Therefore

$$\{\omega \in \Omega:\; \varphi(\omega, h_n(\omega)) \in V\} \;\in\; \Sigma \qquad \forall\, n \geqslant 1,$$

so $G^{-}(V) \in \Sigma$ (see (2.49)) and hence G is measurable (see Definition 2.61(a)).

Solution of Problem 2.99

Let

$$G(\omega) \;=\; \big\{u \in F(\omega):\; d_X(g(\omega), u) = \operatorname{dist}(g(\omega), F(\omega))\big\}.$$

Since F is $P_k(X)$-valued we see that $G(\omega) \in P_k(X)$ for all $\omega \in \Omega$. Also, let

$$\varphi(\omega, u) \;=\; d_X(g(\omega), u) - \operatorname{dist}(g(\omega), F(\omega)).$$

The measurability of F and the Σ-measurability of g imply that $\varphi(\omega, u)$ is a Carathéodory function. Let

$$L_n(\omega) \;=\; \big\{u \in X:\; \varphi(\omega, u) < \tfrac{1}{n}\big\}.$$

From Problem 2.95 we know that L_n is measurable, hence the multifunction $\omega \longmapsto \overline{L}_n(\omega) = \overline{L_n(\omega)}$ is measurable. Problem 2.97 implies that the multifunction $\omega \longmapsto \overline{L}_n(\omega) \cap F(\omega)$ is measurable and

$P_k(X)$-valued for every $n \geqslant 1$. Once again Problem 2.97 implies that the multifunction $\omega \longmapsto G(\omega) = \bigcap_{n \geqslant 1} (\overline{L}_n(\omega) \cap F(\omega))$ is measurable and $P_k(X)$-valued. Apply the Kuratowski–Ryll Nardzewski selection theorem (see Theorem 2.64), to find a Σ-measurable function $f \colon \Omega \longrightarrow X$ such that

$$f(\omega) \in G(\omega) \qquad \forall\, \omega \in \Omega,$$

so

$$f(\omega) \in F(\omega) \quad \text{and} \quad \operatorname{dist}(g(\omega), F(\omega)) = d_X(g(\omega), f(\omega)) \qquad \forall\, \omega \in \Omega.$$

Solution of Problem 2.100

Since X is σ-compact, we have

$$X = \bigcup_{n \geqslant 1} K_n \quad \text{with } K_n \in P_k(X) \qquad \forall\, n \geqslant 1.$$

Let $C \subseteq X$ be a closed set. Then

$$F^-(C) = F^-\Big(\bigcup_{n \geqslant 1} (C \cap K_n)\Big) = \bigcup_{n \geqslant 1} F^-(C \cap K_n) \in \Sigma$$

(since $C \cap K_n \in P_k(X)$ for all $n \geqslant 1$), so F is measurable (see Theorem 2.63(a)).

Solution of Problem 2.101

"$\Longrightarrow$": Let F be measurable. According to Problem 2.100, it suffices to show that for every $K \in P_k(Y)$, we have $G^-(K) \in \Sigma \times \mathcal{B}(X)$. Note that

$$\begin{aligned} G^-(K) &= \{(\omega, u) \in \Omega \times X : \ G(\omega, u) \cap K \neq \emptyset\} \\ &= \big\{(\omega, u) \in \Omega \times X : \ \text{there exists } y \in Y \\ &\qquad \text{such that } (u, y) \in F(\omega) \cap (X \times K)\big\}. \end{aligned}$$

Let $\Omega_K = \{\omega \in \Omega : \ F(\omega) \cap (X \times K) \neq \emptyset\}$. Since F is measurable, we have $\Omega_K \in \Sigma$ (see Theorem 2.63). Let $H \colon \Omega_K \longrightarrow P_f(X \times Y)$ be defined by

$$H(\omega) \ = \ F(\omega) \cap (X \times K) \qquad \forall\, \omega \in \Omega_K. \tag{2.50}$$

Then we have

$$\operatorname{Gr} H \ = \ \operatorname{Gr} F \cap (\Omega \times X \times K) \ \in \ \Sigma \times \mathcal{B}(X) \times \mathcal{B}(Y)$$

(see Theorem 2.63), so H is measurable (again by Theorem 2.63). Thus Theorem 2.66 implies that we can find $(\Sigma \cap \Omega_K)$-measurable maps

$$f_n \colon \Omega_K \longrightarrow X \quad \text{and} \quad g_n \colon \Omega_K \longrightarrow Y \qquad \forall\, n \geqslant 1,$$

such that

$$H(\omega) \ = \ \overline{\{f_n(\omega), g_n(\omega)\}}_{n \geqslant 1} \qquad \forall\, \omega \in \Omega.$$

Let $D_K = \big\{(\omega, u) \in \Omega_K \times X : \ u \in \overline{\{f_n(\omega)\}}_{n \geqslant 1}\big\}$.

We will show that $G^-(K) = D_K$. From the definitions of the two sets, we see that

$$G^-(K) \ \subseteq \ D_K.$$

Next let $(\omega, u) \in D_K$. Then we can find a subsequence $\{f_{n_k}(\omega)\}_{k \geqslant 1}$ of $\{f_n(\omega)\}_{n \geqslant 1}$ such that $f_{n_k}(\omega) \longrightarrow u$ in X. We consider the corresponding subsequence $\{g_{n_k}(\omega)\}_{k \geqslant 1}$ of $\{g_n(\omega)\}_{n \geqslant 1}$. From (2.50) we see that $\{g_{n_k}(\omega)\}_{k \geqslant 1} \subseteq K$ and so, passing to a further subsequence if necessary, we may assume that $g_{n_k}(\omega) \longrightarrow y \in K$ in Y. Therefore $(u, y) \in H(\omega)$, hence $(\omega, u) \in G^-(K)$. This shows that $D_K \subseteq G^-(K)$ so we get $G^-(K) = D_K$.

Note that

$$D_K \ = \ \big\{(\omega, u) \in \Omega \times X : \ \inf_{n \geqslant 1} d_X(u, f_n(\omega)) = 0\big\} \ \in \ \Sigma \times \mathcal{B}(X),$$

so $G^-(K) \in \Sigma \times \mathcal{B}(X)$ (as $G^-(K) = D_K$) and thus G is measurable (see Problem 2.100).

"$\Longleftarrow$": Let G be measurable. Note that

$$\operatorname{Gr} F \;=\; \operatorname{Gr} G \;\in\; \Sigma \times \mathcal{B}(X) \times \mathcal{B}(Y) \;=\; \Sigma \times \mathcal{B}(X \times Y).$$

So, by Theorem 2.63(a), F is measurable.

Solution of Problem 2.102

From the Yankov–von Neumann–Aumann selection theorem (see Theorem 2.68 and Remark 2.69), we can find a sequence $\{f_n \colon \Omega \longrightarrow X\}_{n \geqslant 1}$ of Σ-measurable selectors of F such that

$$F(\omega) \;\subseteq\; \overline{\{f_n(\omega)\}}_{n \geqslant 1} \qquad \forall\, \omega \in \Omega.$$

Then $\sigma_{F(\omega)}(u^*) \;=\; \sup\limits_{n \geqslant 1} \langle u^*, f_n(\omega) \rangle$ for all $\omega \in \Omega$ and so the function $\omega \longmapsto \sigma_{F(\omega)}(u^*)$ is Σ-measurable for every $u^* \in X^*$ (see Corollary I.3.69).

Solution of Problem 2.103

"$\Longrightarrow$": By the Mazur theorem (see Theorem I.5.58), the values of F are in $P_{fc}(X)$. So, according to Theorem 2.66, we can find a sequence $\{f_n \colon \Omega \longrightarrow X\}_{n \geqslant 1}$ of Σ-measurable selectors of F such that

$$F(\omega) \;=\; \overline{\{f_n(\omega)\}}_{n \geqslant 1} \qquad \forall\, \omega \in \Omega.$$

Then for every $u^* \in X^*$, we have

$$\sigma_{F(\omega)}(u^*) \;=\; \sup_{n \geqslant 1} \langle u^*, f_n(\omega) \rangle\,,$$

so the map $\omega \longmapsto \sigma_{F(\omega)}(u^*)$ in Σ-measurable (see Corollary I.3.69).

"$\Longleftarrow$": Let $\overline{B}_1^{X^*} \;=\; \{u^* \in X^* :\; \|u^*\|_{X^*} \leqslant 1\}$. Since X is separable, the set $\overline{B}_1^{X^*}$ furnished with the relative w^*-topology is compact (by the Alaoglu theorem; see Theorem I.5.66) and metrizable (see Theorem I.5.85). Note that

$$X^* = \bigcup_{n \geqslant 1} n\overline{B}_1^{X^*},$$

hence $X^*_{w^*}$ (the space X^* furnished with the w^*-topology) is separable. Then X^* is separable in all topologies τ such that $(X^*_\tau)^* = X$. In particular then this is the case for $\tau = m(X^*, X)$ (the Mackey topology on X^* for the pair (X^*, X), i.e., the strongest, locally convex topology τ on X^* for which we have $(X^*_\tau)^* = X$). Since F is $P_{wkc}(X)$-valued, the function $u^* \longmapsto \sigma_{F(\omega)}(u^*)$ is m-continuous (see Proposition 2.51(d)). From Proposition 2.51(c), we know that for all $u \in X$, we have

$$\operatorname{dist}(u, F(\omega)) = \sup\big\{ \langle u^*, u\rangle - \sigma_{F(\omega)}(u^*) : \ \|u^*\|_{X^*} \leqslant 1\big\}.$$

So, if $\{u_n^*\}_{n\geqslant 1} \subseteq \overline{B}_1^{X^*}$ is an m-dense sequence, then

$$\operatorname{dist}(u, F(\omega)) = \sup_{n\geqslant 1} \big(\langle u_n^*, u\rangle - \sigma_{F(\omega)}(u_n^*)\big),$$

thus the map $\omega \longmapsto \operatorname{dist}(u, F(\omega))$ is Σ-measurable and hence F is measurable (see Theorem 2.63(a)).

Solution of Problem 2.104

Let $\lambda \in \mathbb{R}$. Then, $m(\omega) < \lambda$ if and only if there exists $u \in F(\omega)$ such that $\varphi(\omega, u) < \lambda$. Therefore, we have

$$\{\omega \in \Omega : \ m(\omega) < \lambda\} = \operatorname{proj}_\Omega\big(\{(\omega, u) : \ \varphi(\omega, u) < \lambda\} \cap \operatorname{Gr} F\big).$$

The joint measurability of φ, the graph measurability of F and the Yankov–von Neumann–Aumann projection theorem (see Theorem I.4.65) imply that

$$\operatorname{proj}_\Omega\big(\{(\omega, u) : \ \varphi(\omega, u) < \lambda\} \cap \operatorname{Gr} F\big) \in \widehat{\Sigma} = \Sigma$$

(since Σ is complete), so

$$\{\omega \in \Omega : \ m(\omega) < \lambda\} \in \Sigma \qquad \forall\, \lambda \in \mathbb{R}$$

and thus the map $m\colon \Omega \longrightarrow \mathbb{R}^*$ is Σ-measurable (cf. Proposition I.3.63).

Solution of Problem 2.105

(a) From Theorem 2.66, we know that there exists a sequence $\{f_n\colon \Omega \longrightarrow X\}_{n\geqslant 1}$ of Σ-measurable selectors of F such that

$$F(\omega) \;=\; \overline{\{f_n(\omega)\}}_{n\geqslant 1} \qquad \forall\, \omega \in \Omega.$$

Also, let

$$\Delta \;=\; \Big\{\{r_k\}_{k\geqslant 1} :\; r_k \in \mathbb{Q},\; r_k \geqslant 0 \text{ and all but a finite number equal zero and } \sum_{k\geqslant 1} r_k = 1\Big\}.$$

Let

$$\Gamma \;=\; \Big\{h\colon \Omega \longrightarrow X :\; h(\omega) = \sum_{n\geqslant 1} r_n f_n(\omega) \text{ for all } \omega \in \Omega \text{ with } \{r_n\}_{n\geqslant 1} \in \Delta\Big\}.$$

Note that the set Δ is countable, hence so is Γ and each $h \in \Gamma$ is a Σ-measurable selector of $\omega \longmapsto \operatorname{conv} F(\omega)$. Moreover, for every $\omega \in \Omega$, we have

$$\overline{\Gamma(\omega)} \;=\; \overline{\operatorname{conv}}\, F(\omega),$$

so the multifunction $\omega \longmapsto \overline{\Gamma(\omega)}$ is measurable (see Theorem 2.66) and thus the multifunction $\omega \longmapsto \overline{\operatorname{conv}}\, F(\omega)$ is measurable. Since for every nonempty open set $U \subseteq X$, we have

$$\operatorname{conv} F(\omega) \cap U \;\neq\; \emptyset \quad \text{if and only if} \quad \overline{\operatorname{conv}}\, F(\omega) \cap U \;\neq\; \emptyset,$$

so it follows that the multifunction $\omega \longmapsto \operatorname{conv} F(\omega)$ is measurable too.

(b) This is proved as the measurability of $\omega \longmapsto \overline{\operatorname{conv}}\, F(\omega)$ in part **(a)**, using this time the Yankov–von Neumann–Aumann selection theorem (see Theorem 2.68).

Solution of Problem 2.106

Let $\{\widehat{u}_n\}_{n\geqslant 1}$ be a sequence which is dense in the open unit ball of X. Let

$$h_n(\omega) \;=\; u(\omega) + \varrho(\omega)\widehat{u}_n \qquad \forall\, \omega \in \Omega,\; n \geqslant 1.$$

Then for each $n \geqslant 1$, the map $\omega \longmapsto h_n(\omega)$ is Σ-measurable and

$$\overline{B_{\varrho(\omega)}(u(\omega))} = \overline{\{h_n(\omega)\}}_{n\geqslant 1} \qquad \forall\, \omega \in \Omega,$$

so the multifunction $\omega \longmapsto \overline{B_{\varrho(\omega)}(u(\omega))}$ is measurable (see Theorem 2.66) and thus the multifunction $\omega \longmapsto B_{\varrho(\omega)}(u(\omega))$ is measurable.

Solution of Problem 2.107
For every $n \geqslant 1$, let

$$F_n(\omega) = \left\{u \in X : |\varphi(\omega, u)| < \tfrac{1}{n}\right\}.$$

Problem 2.95 implies that for each $n \geqslant 1$, the multifunction $\omega \longmapsto F_n(\omega)$ is measurable, hence so is the multifunction $\omega \longmapsto \overline{F_n(\omega)} = \left\{u \in X : |\varphi(\omega, u)| \leqslant \tfrac{1}{n}\right\}$. Note that

$$F(\omega) = \bigcap_{n\geqslant 1} F_n(\omega),$$

so, invoking Problem 2.97, we conclude that F is measurable.

Solution of Problem 2.108
Let $\Gamma(\omega) = \{u \in F(\omega) : h(\omega, u) \in G(\omega)\}$. By hypothesis $\Gamma(\omega) \neq \emptyset$ for all $\omega \in \Omega$. Also, let $h_0 \colon \Omega \times X \longrightarrow \Omega \times T$ be defined by

$$h_0(\omega, u) = (\omega, h(\omega, u)) \qquad \forall\, (\omega, u) \in \Omega \times X.$$

Clearly h_0 is $(\Sigma \times \mathcal{B}(X), \Sigma \times \mathcal{T})$-measurable (see Definition I.3.53). We have

$$\operatorname{Gr}\Gamma = \operatorname{Gr} F \cap h_0^{-1}(\operatorname{Gr} G).$$

Since h_0 is measurable and F, G are both graph measurable (see Definition 2.61(b)), we have

$$\operatorname{Gr}\Gamma \in \Sigma \times \mathcal{B}(X).$$

So, we can apply the Yankov–von Neumann–Aumann selection theorem (see Theorem 2.68) and find a Σ-measurable function $f\colon \Omega \longrightarrow X$ such that

$$f(\omega) \ \in \ \Gamma(\omega) \qquad \forall\, \omega \in \Omega,$$

so

$$f(\omega) \ \in \ F(\omega) \quad \text{and} \quad h(\omega, f(\omega)) \ \in \ G(\omega) \qquad \forall\, \omega \in \Omega.$$

Solution of Problem 2.109

Let $U \subseteq X$ be an open set. Then

$$G^-(U) \ = \ \Big\{\omega \in \Omega : \bigcup_{n \geqslant 1} (F_n(\omega) \cap U) \neq \emptyset\Big\} \ = \ \bigcup_{n \geqslant 1} F_n^-(U) \ \in \ \Sigma,$$

so the multifunction $\omega \longmapsto G(\omega)$ is measurable.

The multifunction H has closed values and

$$\operatorname{Gr} H \ = \ \bigcap_{n \geqslant 1} \operatorname{Gr} F_n \ \in \ \Sigma \times \mathcal{B}(X),$$

so the multifunction H is measurable (see Theorem 2.67).

Solution of Problem 2.110

"$\Longrightarrow$": Obvious.

"$\Longleftarrow$": From the Yankov–von Neumann–Aumann selection theorem (see Theorem 2.68 and Remark 2.69), we can find a sequence $\{f_n\colon \Omega \longrightarrow X\}_{n \geqslant 1}$ of Σ_μ-measurable selectors of F such that

$$F(\omega) \ \subseteq \ \overline{\{f_n(\omega)\}_{n \geqslant 1}} \qquad \forall\, \omega \in \Omega.$$

Then

$$m(\omega) \ = \ \inf_{u \in F(\omega)} \|u\|_X \ = \ \inf_{n \geqslant 1} \|f_n(\omega)\|_X \qquad \forall\, \omega \in \Omega,$$

so the map m is Σ_μ-measurable and hence $m \in S_F^p$ (as $m \leqslant \vartheta$). Let $\varepsilon \in L^p(\Omega)_+$ with $\varepsilon(\omega) > 0$ for all $\omega \in \Omega$ and consider the multifunction

$$H_\varepsilon(\omega) \;=\; \{u \in F(\omega) : \; \|u\|_X \leqslant m(\omega) + \varepsilon(\omega)\}.$$

Then $\operatorname{Gr} H_\varepsilon \in \Sigma_\mu \times \mathcal{B}(X)$.

So, we can find a Σ-measurable map $f\colon \Omega \longrightarrow X$ such that $f(\omega) \in H_\varepsilon(\omega)$ for μ-almost all $\omega \in \Omega$ (see the Yankov–von Neumann–Aumann selection theorem; Theorem 2.68). Since

$$\|f(\omega)\|_X \;\leqslant\; m(\omega) + \varepsilon(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

we conclude that $f \in S_F^p$ (recall that $m + \varepsilon \in S_F^p$) and so $S_F^p \neq \emptyset$.

Solution of Problem 2.111

From Remark 2.89 we know that

$$H_l(\omega) \;=\; \liminf_{n\to+\infty} F_n(\omega) \;=\; \Big\{u \in X : \lim_{n\to+\infty} \operatorname{dist}(u, F_n(\omega)) = 0\Big\}, \tag{2.51}$$

$$H_u(\omega) \;=\; \limsup_{n\to+\infty} F_n(\omega) \;=\; \Big\{u \in X : \liminf_{n\to+\infty} \operatorname{dist}(u, F_n(\omega)) = 0\Big\}. \tag{2.52}$$

Both multifunctions are closed valued (see Remark 2.89). For every $n \geqslant 1$, the function

$$\varphi_n(\omega, u) \;=\; \operatorname{dist}(u, F_n(\omega)) \qquad \forall\, (\omega, u) \in \Omega \times X$$

is a Carathéodory function, hence jointly measurable (i.e., $\Sigma \times \mathcal{B}(X)$-measurable). Therefore

$$h_l(\omega, u) \;=\; \liminf_{n\to+\infty} \varphi_n(\omega, u) \quad \text{and} \quad h_u(\omega, u) \;=\; \limsup_{n\to+\infty} \varphi_n(\omega, u)$$

are both $\Sigma \times \mathcal{B}(X)$-measurable. From (2.51)-(2.51), we have

$$\begin{aligned} \operatorname{Gr} H_l &= \{(\omega,u) \in \Omega \times X : \ h_l(\omega,u) = h_u(\omega,u) = 0\} \ \in \ \Sigma \times \mathcal{B}(X), \\ \operatorname{Gr} H_u &= \{(\omega,u) \in \Omega \times X : \ h_l(\omega,u) = 0\} \ \in \ \Sigma \times \mathcal{B}(X), \end{aligned}$$

so H_l and H_u are both measurable multifunctions (see Theorem 2.67).

Solution of Problem 2.112

By the Yankov–von Neumann–Aumann selection theorem (see Theorem 2.68 and Remark 2.69), we can find a sequence $\{g_m\}_{m \geqslant 1}$ of Σ-measurable selectors of F such that

$$F(\omega) \ \subseteq \ \overline{\{g_m(\omega)\}}_{m \geqslant 1} \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Since the measure space is σ-finite, we can find a sequence $\{\Omega_n\}_{n \geqslant 1} \subseteq \Sigma$ of pairwise disjoint sets such that $\mu(\Omega_n) < +\infty$ for all $n \geqslant 1$ and $\Omega = \bigcup\limits_{n \geqslant 1} \Omega_n$. Let $h \in S_F^p$ and define

$$C_{kmn} \ = \ \{\omega \in \Omega : \ k-1 \leqslant g_m(\omega) < k\} \cap \Omega_n$$

and

$$f_{kmn} \ = \ \chi_{C_{kmn}} g_m + \chi_{C_{kmn}^c} h \ \in \ S_F^p.$$

Then, we have

$$F(\omega) \ \subseteq \ \overline{\{f_{kmn}(\omega)\}}_{k,m,n \geqslant 1} \qquad \forall\, \omega \in \Omega.$$

Solution of Problem 2.113

Let $\delta \in L^1(\Omega)$ be such that $\delta(\omega) > 0$ for all $\omega \in \Omega$ and

$$\int\limits_{\Omega} \delta(\omega)\, d\mu \ < \ \tfrac{2\varepsilon^p}{3}.$$

Let $\{C_n\}_{n\geqslant 1} \subseteq \Sigma$ be a sequence of pairwise disjoint sets such that $\Omega = N \cup \big(\bigcup\limits_{n\geqslant 1} C_n\big)$ with N being μ-null and

$$\|f(\omega) - f_n(\omega)\|_X^p \;<\; \delta(\omega) \qquad \forall\, \omega \in C_n.$$

As

$$\begin{aligned} &\sum_{k\geqslant 1}\int\limits_{C_k} \|f_k(\omega)\|_X^p\,d\mu \\ \leqslant\;\; & 2^{p-1}\sum_{k\geqslant 1}\int\limits_{C_k} \|f_k(\omega) - f(\omega)\|_X^p\,d\mu + 2^{p-1}\sum_{k\geqslant 1}\int\limits_{C_k} \|f(\omega)\|_X^p\,d\mu \\ \leqslant\;\; & 2^{p-1}\int\limits_{\Omega} \delta(\omega)\,d\mu + 2^{p-1}\int\limits_{\Omega} \|f(\omega)\|_X^p\,d\mu, \end{aligned}$$

we see that series $\sum\limits_{k\geqslant 1}\int\limits_{C_k} \|f_k(\omega)\|_X^p\,d\mu$ is convergent. Let $N \in \mathbb{N}$ be such that

$$\sum_{k\geqslant N+1}\int\limits_{C_k} \|f_k(\omega)\|_X^p\,d\mu \;<\; \tfrac{\varepsilon^p}{2^p\cdot 3}$$

and

$$\sum_{k\geqslant N+1}\int\limits_{C_k} \|f(\omega)\|_X^p\,d\mu \;<\; \tfrac{\varepsilon^p}{2^p\cdot 3}.$$

We define the finite Σ-partition of Ω by setting

$$A_1 \;=\; C_1 \cup \Big(\bigcup_{k\geqslant N+1} C_k\Big) \quad \text{and} \quad A_m \;=\; C_m \qquad \forall\, 2 \leqslant m \leqslant N.$$

We have

$$\Big\|f - \sum_{k=1}^{N} \chi_{A_k} f_k\Big\|_p^p \;\leqslant\; \int\limits_{\Omega} \delta\,d\mu + \sum_{k\geqslant N+1} 2^{p-1}\big(\|\chi_{C_k} f\|_p^p + \|\chi_{C_k} f_k\|_p^p\big) \;<\; \varepsilon^p.$$

Solution of Problem 2.114

According to Theorem 2.66, we can find two sequences $\{f_n\colon \Omega \longrightarrow X\}_{n\geqslant 1}$ and $\{g_m\colon \Omega \longrightarrow X\}_{m\geqslant 1}$ of Σ-measurable selectors of F and G respectively such that

$$F(\omega) \;=\; \overline{\{f_n(\omega)\}_{n\geqslant 1}} \quad\text{and}\quad G(\omega) \;=\; \overline{\{g_m(\omega)\}_{m\geqslant 1}} \qquad \forall\, \omega\in\Omega.$$

We have

$$H(\omega) \;=\; \overline{\{f_n(\omega)+g_m(\omega)\}_{n,m\geqslant 1}},$$

so the multifunction H is measurable and $P_f(X)$-valued (see Theorem 2.66).

Note that since H is closed valued, then S^p_H is closed. So, we have

$$\overline{S^p_F + S^p_G} \;\subseteq\; S^p_H. \tag{2.53}$$

We need to show that the opposite inclusion also holds. To this end, let $h\in S^p_H$ and $\varepsilon>0$. According to Problem 2.113, we can find a finite Σ-partition $\{A_1,\ldots,A_l\}$ of Ω and positive integers $n_1,\ldots,n_l$ and $m_1,\ldots,m_l$ such that

$$\Big\|h-\sum_{k=1}^{l}\chi_{A_k}(f_{n_k}+g_{n_k})\Big\|_p \;<\; \varepsilon,$$

so $h\in\overline{S^p_F+S^p_G}$. Thus, we have

$$S^p_H \;\subseteq\; \overline{S^p_F+S^p_G}. \tag{2.54}$$

From (2.53) and (2.54) we conclude that $S^p_H=\overline{S^p_F+S^p_G}$.

Solution of Problem 2.115

Clearly $S^p_{\overline{\operatorname{conv}}\,F}$ is closed, convex and so we have

$$\overline{\operatorname{conv}}\,S^p_F \;\subseteq\; S^p_{\overline{\operatorname{conv}}\,F}. \tag{2.55}$$

Suppose that the inclusion in (2.55) is strict. So, we can find $h\in S^p_{\overline{\operatorname{conv}}\,F}\setminus\overline{\operatorname{conv}}\,S^p_F$. The strong separation theorem for convex sets (see Theorem I.5.29) implies that we can find $h^*\in L^p(\Omega;X)^* = L^{p'}(\Omega;X^*_{w^*})$ (with $\frac{1}{p}+\frac{1}{p'}=1$; see Theorem 1.47) such that

$$\sigma_{\overline{\operatorname{conv}}\,S^p_F}(h^*) \;<\; \langle h^*,h\rangle \tag{2.56}$$

(see Definition 2.46), where by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(L^{p'}(\Omega; X^*_{w^*}), L^p(\Omega; X))$, i.e.,

$$\langle h^*, h \rangle = \int_\Omega \langle h^*(\omega), h(\omega) \rangle_X \, d\mu,$$

where by $\langle \cdot, \cdot \rangle_X$ we denote the duality brackets for the pair (X^*, X). We have

$$\begin{aligned}
& \sigma_{\overline{\operatorname{conv}} S^p_F}(h^*) = \sigma_{S^p_F}(h^*) = \sup_{f \in S^p_F} \langle h^*, f \rangle \\
= & \sup_{f \in S^p_F} \int_\Omega \langle h^*(\omega), f(\omega) \rangle_X \, d\mu = \int_\Omega \sup_{u \in F(\omega)} \langle h^*(\omega), u \rangle_X \, d\mu \\
= & \int_\Omega \sigma_{F(\omega)}(h^*(\omega)) \, d\mu = \int_\Omega \sigma_{\overline{\operatorname{conv}} F(\omega)}(h^*(\omega)) \, d\mu \\
< & \int_\Omega \langle h^*(\omega), h(\omega) \rangle_X \, d\mu
\end{aligned}$$

(see Theorem 2.74 and (2.56)), which contradicts the fact that $h \in S^p_{\overline{\operatorname{conv}} F}$. This proves that in (2.55) we have equality.

Solution of Problem 2.116
Since $S^p_F \subseteq L^p(\Omega; X)$ is closed, we have

$$\overline{S^p_F} \subseteq S^p_{\overline{F}}. \tag{2.57}$$

Let $f \in S^p_{\overline{F}}$ and for every $n \geqslant 1$ we define

$$V_n(\omega) = \left\{ u \in F(\omega) : \|u - f(\omega)\|_X < \tfrac{1}{n} \right\}.$$

By modifying $V_n(\omega)$ on a μ-null set, we may assume that $V_n(\omega) \neq \emptyset$ for all $\omega \in \Omega$. Note that, if $\varphi(\omega, u) = \|u - f(\omega)\|_X$, then φ is a Carathéodory function, hence jointly measurable and we have

$$\operatorname{Gr} V_n = \left\{ (\omega, u) \in \Omega \times X : \varphi(\omega, u) < \tfrac{1}{n} \right\} \cap \operatorname{Gr} F,$$

so $\operatorname{Gr} V_n \in \Sigma \times \mathcal{B}(X)$. Invoking the Yankov–von Neumann–Aumann selection theorem (see Theorem 2.68 and Remark 2.69), we can find a sequence $\{f_n \colon \Omega \longrightarrow X\}_{n \geqslant 1}$ of Σ-measurable functions such that

$$f_n(\omega) \in V_n(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega, \text{ all } n \geqslant 1,$$

so

$$\|f_n(\omega) - f(\omega)\|_X < \frac{1}{n} \quad \text{for } \mu\text{-a.a. } \omega \in \Omega, \text{ all } n \geqslant 1$$

and $f_n \in S_F^p$ for all $n \geqslant 1$, thus

$$f_n(\omega) \longrightarrow f(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega. \tag{2.58}$$

We have

$$\|f_n(\omega)\|_X \leqslant \|f(\omega)\|_X + 1 \quad \text{for } \mu\text{-a.a. } \omega \in \Omega, \text{ all } n \geqslant 1, \tag{2.59}$$

with $f \in L^p(\Omega; X)$. Because of (2.58) and (2.59) and the Lebesgue dominated convergence theorem, we have

$$f_n \longrightarrow f \quad \text{in } L^p(\Omega; X),$$

so $f \in \overline{S_F^p}$, hence we have equality in (2.57).

Solution of Problem 2.117

By working on each component of Ω of finite μ-measure, we see that we may assume that μ is finite. Let $f, g \in D$ and define $m \colon \Sigma \longrightarrow Y$ by setting

$$m(A) = T(\chi_A(f-g)) \qquad \forall\, A \in \Sigma.$$

We show that m is a vector measure (see Definition 1.43). To this end, let $\{A_n\}_{n \geqslant 1} \subseteq \Sigma$ be a sequence of pairwise disjoint sets and let $A = \bigcup\limits_{n \geqslant 1} A_n$. We set $C_n = \bigcup_{k \geqslant n+1} A_k \in \Sigma$. From the linearity of T, for every $n \geqslant 1$, we have

$$\begin{aligned} m(A) &= T(\chi_A(f-g)) = T\Big(\sum_{k=1}^n \chi_{A_k}(f-g) + \chi_{C_n}(f-g)\Big) \\ &= \sum_{k=1}^n T(\chi_{A_k}(f-g)) + T(\chi_{C_n}(f-g)), \end{aligned}$$

so

$$\begin{aligned}\Big\|m(A)-\sum_{k=1}^{n}T(\chi_{A_k}(f-g))\Big\|_Y &= \Big\|m(A)-\sum_{k=1}^{n}m(A_k)\Big\|_Y \\ &= \|T(\chi_{C_n}(f-g))\|_Y. \qquad (2.60)\end{aligned}$$

But $C_n \searrow \emptyset$ and so

$$\chi_{C_n}(f-g) \longrightarrow 0 \quad \text{in } L^p(\Omega;X).$$

Then the continuity of T implies that

$$\|T(\chi_{C_n}(f-g))\|_Y \longrightarrow 0,$$

so m is a vector measure (see (2.60) and Definition 2.70).

If $\mu(A)=0$, then $\chi_A(f-g)=0$ and so $m(A)=0$. This implies that $m \ll \mu$. Moreover, the nonatomicity of μ implies that m is nonatomic (see Definition I.3.40). Note that

$$\|m(A)\|_Y = \|T(\chi_A(f-g))\|_Y \leqslant \|T\|_{\mathcal{L}}\|\chi_A(f-g)\|_{L^p(\Omega;X)}. \qquad (2.61)$$

If

$$\vartheta(A) = \int_A \|f(\omega)-g(\omega)\|_X^p \, d\mu \qquad \forall\, A \in \Sigma,$$

then ϑ is a measure on (Ω,Σ) of bounded variation and from (2.61) we have

$$|m|(\Omega) \leqslant \|T\|_{\mathcal{L}}\vartheta(\Omega)^{\frac{1}{p}}$$

(see Definition 1.43), so m is a vector measure of bounded variation.

Therefore, so far we have that $m\colon \Sigma \longrightarrow Y$ is a nonatomic vector measure of bounded variation. Because Y has the RNP, we can apply the Lyapunov convexity theorem (see Theorem 2.165) and infer that $\overline{m(\Sigma)} \subseteq Y$ is convex and compact. Then the set

$$\overline{\bigcup_{A\in\Sigma} T(\chi_A(f-g))+T(g)} = \overline{\bigcup_{A\in\Sigma} T(\chi_A(f-g))+T(g)}$$

is convex. Using the linearity of T, we have

$$T(\chi_A(f-g))+T(g) = T(\chi_A f + \chi_{A^c} g).$$

The decomposability of D (see Definition 2.70) implies that $\chi_A f + \chi_{A^c} g \in D$ for all $A \in \Sigma$, so

$$\overline{\bigcup_{A\in\Sigma} T(\chi_A f + \chi_{A^c} g)} \subseteq \overline{T(D)}.$$

It follows that for all $\lambda \in [0,1]$, we have

$$T(\lambda f + (1-\lambda)g) \in \overline{T(D)},$$

so the set $\overline{T(D)}$ is convex (recall that $f, g \in D$).

Solution of Problem 2.118

By working on each component of Ω of finite μ-measure, we see that we may assume that μ is finite. Let $h \in \operatorname{conv} D$. Then

$$h = \sum_{k=1}^{n} \lambda_k f_k, \quad \text{with } \lambda_k \in [0,1],\ \sum_{k=1}^{n} \lambda_k = 1,\ f_k \in D \text{ for } k \in \{1,\ldots,n\}.$$

Recall that $L^p(\Omega;X)^* = L^{p'}(\Omega;X^*_{w^*})$ (with $\frac{1}{p} + \frac{1}{p'} = 1$; see Theorem 1.47) and let $\langle\cdot,\cdot\rangle$ denote the duality brackets for this dual pair. Let $U_w(h)$ be a basic weak neighborhood of h defined by

$$U_w(h) = \big\{ g \in L^p(\Omega;X) : |\langle u_k^*, g-h\rangle| < \varepsilon \text{ for all } k \in \{1,\ldots,m\}\big\},$$

with $m \geqslant 1$, $u_k^* \in L^{p'}(\Omega;X^*_{w^*})$ for all $k \in \{1,\ldots,m\}$ and $\varepsilon > 0$. We consider the continuous linear map $T\colon L^p(\Omega;X) \longrightarrow \mathbb{R}^m$ defined by

$$T(h) = \big(\langle u_1^*, h\rangle, \ldots, \langle u_m^*, h\rangle\big).$$

From Problem 2.117 we have that $T(D)$ is convex (no closure is needed since in finite dimensional Banach spaces the Lyapunov theorem is exact; see Theorem 2.165). Therefore, we can find $f \in D$ such that $T(f) = T(h)$ and so $D \cap U_w(h) \neq \emptyset$. Hence $h \in \overline{D}^w = D$ (since by definition D is w-closed). Then we conclude that $D = \operatorname{conv} D$.

Solution of Problem 2.119

Note that the set $\overline{S_F^p}^w$ remains decomposable (see Definition 2.70). Also, it is w-closed. Hence according to Problem 2.118 it is convex. Also, from Problem 2.115, we have $\overline{\text{conv}}\, S_F^p = S^p_{\overline{\text{conv}}\, F}$. Therefore

$$\overline{\text{conv}}\, S_F^p \ \subseteq\ \overline{S_F^p}^w. \tag{2.62}$$

But by the Mazur theorem (see Theorem I.5.58), a convex set is closed if and only if it is w-closed. So, from (2.62) it follows that $\overline{S_F^p}^w = \overline{\text{conv}}\, S_F^p$, thus

$$\overline{S_F^p}^w \ =\ S^p_{\overline{\text{conv}}\, F}.$$

Solution of Problem 2.120

We argue indirectly. So, suppose that $H(\omega) = F(\omega) \setminus G(\omega)$ is not empty on a set of positive μ-measure. Then

$$A \ =\ \operatorname{dom} H \ =\ \{\omega \in \Omega :\ H(\omega) \neq \emptyset\} \ \in\ \Sigma_\mu.$$

Since $H\colon A \longrightarrow 2^X \setminus \{\emptyset\}$ is graph measurable (see Definition 2.61(b)), form the Yankov–von Neumann–Aumann selection theorem (see Theorem 2.68), we can find a Σ_A-measurable map $h\colon A \longrightarrow X$ such that

$$h(\omega) \ \in\ H(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in A$$

(here $\Sigma_A = \Sigma \cap A$). Since μ is σ-finite, we can find a sequence $\{\Omega_n\}_{n\geqslant 1} \subseteq \Sigma$ with $\mu(\Omega_n) < +\infty$ for all $n \geqslant 1$ such that $\Omega = \bigcup\limits_{n\geqslant 1} \Omega_n$. Also for every $m \geqslant 1$, let

$$A_m \ =\ \{\omega \in A :\ m-1 \leqslant h(\omega) < m\} \ \in\ (\Sigma_\mu)_A.$$

We set $C_{mn} = A_m \cap \Omega_n$. Then $A = \bigcup\limits_{n,m\geqslant 1} C_{mn}$. Since $\mu(A) > 0$, we can find $m, n \geqslant 1$ such that $\mu(C_{mn}) > 0$. Let $f \in S_F^p$ and let $\vartheta = \chi_{C_{mn}} h + \chi_{C_{mn}^c} f \in S_F^p$ (by decomposability; see Definition 2.70) but $\vartheta \notin S_G^p$, a contradiction to the hypothesis that $S_F^p = S_G^p$.

Solution of Problem 2.121

(a) From Problem 2.116 we have $S_F^p = \overline{S_F^p} = S_{\overline{F}}^p$. Invoking Problem 2.120, we infer that

$$F(\omega) = \overline{F}(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

hence $F(\omega) \in P_f(X)$ for μ-almost all $\omega \in \Omega$.

(b) From the Mazur theorem (see Theorem I.5.58) and Problem 2.119, we have

$$S_F^p = \overline{S_F^p}^w = S_{\overline{\operatorname{conv}} F}^p,$$

so

$$F(\omega) = \overline{\operatorname{conv}} F(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega$$

(see Problem 2.120) and thus $F(\omega) \in P_{fc}(X)$ for μ-almost all $\omega \in \Omega$.

Solution of Problem 2.122

Let $T \in \mathcal{L}(L^1(\Omega; X); X)$ be defined by

$$T(h) = \int_\Omega h(\omega)\, d\mu \qquad \forall\, h \in L^1(\Omega; X).$$

From Definition 2.85 we know that

$$T(S_F^1) = \int_\Omega F(\omega)\, d\mu.$$

Problem 2.117 implies that $\overline{T(S_F^1)}$ is convex (note that in present case we do not need X to have the RNP, since the vector measure is from the definition of T in integral form). Therefore, we conclude that the set $\operatorname{cl} \int_\Omega F\, d\mu \subseteq X$ is convex.

Solution of Problem 2.123

Recall that $\int\limits_\Omega F(\omega)\,d\mu = T(S^1_F)$, where $T \in \mathcal{L}(L^1(\Omega;X);X)$ is defined by

$$T(h) = \int\limits_\Omega h(\omega)\,d\mu \qquad \forall\ h \in S^1_F$$

(see Definition 2.85). From Theorem 2.76, we have that $S^1_F \subseteq L^1(\Omega;X)$ is w-compact and convex. Hence

$$T(S^1_F) = \int\limits_\Omega F(\omega)\,d\mu \in P_{wkc}(X).$$

Solution of Problem 2.124

Let $\langle\cdot,\cdot\rangle$ denote the duality brackets for the pair

$$(L^\infty(\Omega;X^*_{w^*}) = L^1(\Omega;X)^*, L^1(\Omega;X))$$

(see Theorem 1.52). Also, by $\langle\cdot,\cdot\rangle_X$ we denote the duality brackets for the dual pair (X^*,X). According to Definition 2.46, we have

$$\begin{aligned}\sigma_{S^1_F}(h^*) &= \sup_{f\in S^1_F}\langle h^*,f\rangle = \sup_{f\in S^1_F}\int\limits_\Omega \langle h^*(\omega),f(\omega)\rangle_X\,d\mu\\ &= \int\limits_\Omega \sup_{u\in F(\omega)}\langle h^*(\omega),u\rangle_X = \int\limits_\Omega \sigma_{F(\omega)}(h^*(\omega))\,d\mu.\end{aligned}$$

(see Theorem 2.74)

Solution of Problem 2.125

We fix $\varepsilon > 0$ and let $\{\widehat{u}_n\}_{n\geqslant 1} \subseteq X$ be a sequence which is dense in X. Let

$$\varphi_n(v) = \operatorname{dist}(\widehat{u}_n, F(v)) \qquad \forall\ v \in V,\ n \geqslant 1.$$

By the Lusin theorem (see Theorem 2.166), we can find compact sets $K_n \subseteq V$ with $\mu(V \setminus K_n) < \frac{\varepsilon}{2^n}$ for all $n \geqslant 1$, such that $\varphi_n|_{K_n}$ is continuous. Let $K_\varepsilon = \bigcap\limits_{n\geqslant 1} K_n$. Evidently $K_\varepsilon \subseteq V$ is compact and $\mu(V \setminus K_\varepsilon) < \varepsilon$. Also, for every $n \geqslant 1$, $\varphi_n|_{K_\varepsilon}$ is continuous. Let $\{(v_m, u_m)\}_{m\geqslant 1} \subseteq \operatorname{Gr} F \cap (K_\varepsilon \times X)$ be a sequence such that $(v_m, u_m) \longrightarrow (\widetilde{v}, \widetilde{u})$. We fix $\delta > 0$ and choose $n \geqslant 1$ such that $d_X(\widehat{u}_n, \widetilde{u}) < \delta$. Then we can find $m_0 = m_0(n) \geqslant 1$ such that

$$\varphi_n(v_m) \;=\; \operatorname{dist}(\widehat{u}_n, F(v_m)) \;\leqslant\; d_X(\widehat{u}_n, u_m) \;<\; \delta \qquad \forall\, m \geqslant m_0.$$

The continuity of $\varphi_n|_{K_\varepsilon}$ implies that $\operatorname{dist}(\widehat{u}_n, F(\widetilde{v})) \leqslant \delta$ and so $\operatorname{dist}(\widetilde{u}, F(\widetilde{v})) < 2\delta$. Since $\delta > 0$ is arbitrary, we let $\delta \searrow 0$ and obtain $\operatorname{dist}(\widetilde{u}, F(\widetilde{v})) = 0$, thus $(\widetilde{v}, \widetilde{u}) \in \operatorname{Gr} F$ (recall that F has closed values) and this proves that $F|_{K_\varepsilon}$ is closed (see Definition 2.40).

Solution of Problem 2.126

Theorem 2.66 implies that there exists a sequence $\{f_n \colon V \longrightarrow X\}_{n\geqslant 1}$ of Borel measurable functions such that

$$F(v) \;=\; \overline{\{f_n(v)\}}_{n\geqslant 1} \qquad \forall\, v \in V. \tag{2.63}$$

By the Lusin theorem (see Theorem 2.166), for every $\varepsilon > 0$, we can find a compact set $K_\varepsilon \subseteq V$ with $\mu(V \setminus K_\varepsilon) < \varepsilon$ such that $f_n|_{K_\varepsilon}$ is continuous for every $n \geqslant 1$ (see the solution of Problem 2.125). Then because of (2.63) for every closed set $C \subseteq X$, we have that the set

$$F|_{K_\varepsilon}^{+}(C) \;=\; \bigcap_{n\geqslant 1} f_n|_{K_\varepsilon}^{-1}(C)$$

is closed. Therefore $F|_{K_\varepsilon}$ is lower semicontinuous (see Proposition 2.38).

Solution of Problem 2.127

By Theorem 2.66 there exists a sequence $\{f_n\}_{n\geqslant 1}$ of $\Sigma\times\mathcal{T}$-measurable selectors of F such that

$$F(\omega,t) \;=\; \overline{\{f_n(\omega,t)\}_{n\geqslant 1}} \qquad \forall\,(\omega,t)\in\Omega\times\mathcal{T}. \tag{2.64}$$

For every $n\geqslant 1$, let

$$\Omega_n \;=\; \{\omega\in\Omega:\; d_X(u_0(\omega),f_n(\omega,t_0))\leqslant\varepsilon\}\,. \tag{2.65}$$

From (2.64) we have $\Omega=\bigcup\limits_{n\geqslant 1}\Omega_n$. Let

$$D_1 \;=\; \Omega_1 \quad\text{and}\quad D_n \;=\; \Omega_n\setminus\bigcup_{k=1}^{n-1}D_k \qquad \forall\, n\geqslant 2.$$

We have $\{D_n\}_{n\geqslant 1}\subseteq\Sigma$. We set

$$\widehat{f}(\omega,t) \;=\; \sum_{n\geqslant 1}\chi_{D_n}(\omega)f_n(\omega,t).$$

Then $\widehat{f}$ is a $\Sigma\times\mathcal{T}$-measurable selector of F and

$$d_X(\widehat{f}(\omega,t_0),u_0(\omega)) \;\leqslant\; \varepsilon \qquad \forall\,\omega\in\Omega.$$

(see (2.65)).

Solution of Problem 2.128

By definition, we have

$$\begin{aligned} F^-(D) \;&=\; \{\omega\in\Omega:\; F(\omega)\cap D\neq\emptyset\} \\ &=\; \operatorname{proj}_\Omega\big(\operatorname{Gr}F\cap(\Omega\times D)\big) \;\in\; \Sigma \end{aligned}$$

(by the Yankov–von Neumann–Aumann projection theorem; see Theorem I.4.65).

Solution of Problem 2.129

Note that $\omega \longmapsto F(\omega)$ and $\omega \longmapsto F^c(\omega) = X \setminus F(\omega)$ are both graph measurable (see Definition 2.61(b)). Then from Problem 2.128 we see that they are measurable and then so are $\omega \longmapsto \overline{F(\omega)}$ and $\omega \longmapsto \overline{F^c(\omega)}$. Therefore the map $\omega \longmapsto \partial F(\omega) = \overline{F(\omega)} \cap \overline{F^c(\omega)}$ is graph measurable and then since ∂F is closed valued, from Theorem 2.67(d), we conclude that $\omega \longmapsto \partial F(\omega)$ is measurable.

Next note that $\operatorname{int} F(\omega) = F(\omega) \setminus \partial F(\omega)$, hence

$$\operatorname{Gr}(\operatorname{int} F) \;=\; \operatorname{Gr} F \cap (\operatorname{Gr} \partial F)^c \;\in\; \Sigma \times \mathcal{B}(X)$$

and thus $\omega \longmapsto \operatorname{int} F(\omega)$ is graph measurable.

Solution of Problem 2.130

We may assume that $A = \Omega$. Also, let $f \in S^1_F$. Then by replacing $F(\omega)$ by $F(\omega) - f(\omega)$, which is still graph measurable and open valued, we may assume that $0 \in F(\omega)$ for all $\omega \in \Omega$. Let $\xi_F(\omega) = \operatorname{dist}(0, F(\omega)^c)$. Since F is graph measurable (see Definition 2.61(b)), so is $\omega \longrightarrow F(\omega)^c$ which is closed valued, hence ξ_F is Σ_μ-measurable (see Theorem 2.67(a)). Note that $\xi_F(\omega) > 0$ for all $\omega \in \Omega$. So, we can find $\varepsilon > 0$ and $A \in \Sigma$ with $\mu(A) > 0$ such that $\xi_F(\omega) \geqslant \varepsilon$ for all $\omega \in A$. This means that, if $B_\varepsilon = \{u \in X : \|u\|_X < \varepsilon\}$ then

$$\mu(A) B_\varepsilon \;\subseteq\; \int_A F\, d\mu \;\subseteq\; \int_\Omega F\, d\mu$$

(since $0 \in F(\omega)$ for all $\omega \in \Omega$), so the set $\int_A F\, d\mu \subseteq X$ is open.

Solution of Problem 2.131

Without any loss of generality, we may assume that $\operatorname{int} F(\omega) \neq \emptyset$ for all $\omega \in \Omega$. We fix $A \in \Sigma$ with $\mu(A) > 0$ and we show that the set $\int\limits_A \operatorname{int} F\, d\mu$ is dense in $\int\limits_A F\, d\mu$. To this end let $f \in S^1_F$ and $\varepsilon > 0$. We introduce the multifunction $H\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ defined by

$$H(\omega) \;=\; \Big\{u \in \operatorname{int} F(\omega):\; \|f(\omega) - u\|_X < \tfrac{\varepsilon}{\mu(A)}\Big\}.$$

If $\varphi(\omega, u) = \|f(\omega) - u\|_X$, then φ is a Carathéodory function, hence jointly measurable and so

$$\operatorname{Gr} H \;=\; \operatorname{Gr} \operatorname{int} F \cap \Big\{(\omega, u) \in \Omega \times X:\; \varphi(\omega, u) < \tfrac{\varepsilon}{\mu(A)}\Big\} \;\in\; \Sigma_\mu \times \mathcal{B}(X)$$

(see Problem 2.129 and recall that Σ_μ stands for the μ-completion of Σ). Invoking the Yankov–von Neumann–Aumann selection theorem (see Theorem 2.68) we can find a Σ-measurable map $h\colon \Omega \longrightarrow X$ such that

$$h(\omega) \;\in\; H(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

We have

$$h(\omega) \;\in\; \operatorname{int} F(\omega) \quad \text{and} \quad \|f(\omega) - h(\omega)\|_X \;<\; \tfrac{\varepsilon}{\mu(A)} \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

so $h \in S^1_{\operatorname{int} F}$ (recall that $f \in S^1_F$) and

$$\Big\| \int\limits_A (f(\omega) - h(\omega))\, d\mu \Big\|_X \;<\; \varepsilon,$$

so $\int\limits_A \operatorname{int} F\, d\mu$ is dense in $\int\limits_A F\, d\mu$.

From Problem 2.130, we know that $\int\limits_A \operatorname{int} F\, d\mu$ is open and convex in X. So

$$\int\limits_A \operatorname{int} F\, d\mu \;=\; \operatorname{int} \overline{\int\limits_A \operatorname{inf} F\, d\mu} \;=\; \operatorname{int} \overline{\int\limits_A F\, d\mu} \;\supseteq\; \operatorname{int} \int\limits_A F\, d\mu. \tag{2.66}$$

On the other hand, the openness of $\int\limits_A \operatorname{int} F\, d\mu$ implies

$$\int\limits_A \operatorname{int} F\, d\mu \;\subseteq\; \operatorname{int} \int\limits_A F\, d\mu. \tag{2.67}$$

From (2.66) and (2.67) we conclude that $\operatorname{int} \int_A F\, d\mu = \int_A \operatorname{int} F\, d\mu$ for all $A \in \Sigma$.

Solution of Problem 2.132

Let $|D| = \{\|f(\cdot)\|_X : f \in D\} \subseteq L^1(\Omega)$ and let $h = \operatorname{ess\,sup} |D|$. We know that we can find a sequence $\{f_n\}_{n\geqslant 1} \subseteq D$ such that

$$h(\omega) = \sup_{n\geqslant 1} \|f_n(\omega)\|_X \quad \text{for } \mu\text{-a.a. } \omega \in \Omega$$

(see Proposition 2.168). The decomposability of D (see Definition 2.70) implies that we can have $\{\|f_n(\cdot)\|_X\}_{n\geqslant 1}$ to be increasing. Also, the boundedness of D and the Lebesgue monotone convergence theorem, imply that $h \in L^1(\Omega)_+$. Since

$$\|f(\omega)\|_X \leqslant h(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega \text{ and all } f \in D,$$

we conclude that D is uniformly integrable (see Definition 1.18).

Solution of Problem 2.133

From Theorem 2.76 we know that $S_F^1 \in P_{wkc}\big(L^1(\Omega;\mathbb{R}^N)\big)$. Therefore

$$V(t) = \int_0^t F(s)\, ds \in P_{kc}(\mathbb{R}^N) \qquad \forall\, t \in T$$

(see Definition 2.85). For $t, \tau \in [0,b]$ with $\tau < t$, we have

$$h(V(t), V(\tau)) = \sup_{|y^*|\leqslant 1} |\sigma_{V(t)}(y^*) - \sigma_{V(\tau)}(y^*)|$$

$$\leqslant \sup_{|y^*|\leqslant 1} \int_\tau^t \sigma_{F(s)}(y^*)\, ds \leqslant \int_\tau^t h(s)\, ds$$

(see the Hörmander formula (Proposition 2.51(b)) and recall that F is integrably bounded; see Definition 2.75) with $h \in L^1(T)$, so V

is h-continuous, hence continuous too (since is $P_{kc}(\mathbb{R}^N)$-valued; see Proposition 2.56(c)).

The Michael selection theorem (see Theorem 2.58) implies that $CS_V \neq \emptyset$. Let $\{v_n\}_{n\geqslant 1} \subseteq CS_V$ be a sequence. Then we can find a sequence $\{f_n\}_{n\geqslant 1} \subseteq S_F^1$ such that

$$v_n(t) \;=\; \int_0^t f_n(s)\,ds \qquad \forall\, t \in T,\; n \geqslant 1.$$

From Theorem 2.76 we know that by passing to a suitable subsequence, we may assume that

$$f_n \;\xrightarrow{w}\; f \quad \text{in } L^1(T;\mathbb{R}^N), \tag{2.68}$$

with $f \in S_F^1$. Since F is integrably bounded, we have

$$|F(t)| \;\leqslant\; h(t) \quad \text{for a.a. } t \in T,$$

with $h \in L^1(T)$. So, S_F^1 is uniformly integrable (see Definition 1.18). Moreover, using the Lusin theorem on h (see Theorem 2.166), we see that condition **(b)** of Definition 2.80 is satisfied. So, S_F^1 has property U (see Definition 2.80) and then by Proposition 2.82, the weak topology and the $\|\cdot\|_w$-topology (see Definition 2.78) coincide. Let

$$v(t) \;=\; \int_0^t f(s)\,ds \qquad \forall\, t \in T.$$

Then $v \in CS_V$ and we have

$$\begin{aligned}\|v_n - v\|_\infty \;&=\; \max_{t\in T} |v_n(t) - v(t)| \;=\; \max_{t\in T} \Big| \int_0^t (f_n(s) - f(s))\,ds \Big| \\ &\leqslant\; c\|f_n - f\|_w \qquad \forall\, n \geqslant 1,\end{aligned}$$

for some $c > 0$ (see Definition 2.78 and Remark 2.79). Because of (2.68) we have $\|f_n - f\|_w \longrightarrow 0$ and so $\|v_n - v\|_\infty \longrightarrow 0$, which proves the compactness of $CS_V \subseteq C(T;\mathbb{R}^N)$.

Remark. An alternative solution can be based on the Arzela–Ascoli theorem (see Theorem I.2.181).

Solution of Problem 2.134

If $s\text{–}\liminf\limits_{n\to+\infty} A_n = \emptyset$, then $\operatorname{dist}(u, s\text{–}\liminf\limits_{n\to+\infty} A_n) = +\infty$ and the desired inequality holds. So, we assume that $s\text{–}\liminf\limits_{n\to+\infty} A_n \neq \emptyset$. Let $v \in s\text{–}\liminf\limits_{n\to+\infty} A_n$. Then we can find a sequence $\{v_n\}_{n\geqslant 1}$ with $v_n \in A_n$ for all $n \geqslant 1$ such that $v_n \longrightarrow v$ in X (see Definition 2.88). We have

$$\operatorname{dist}(u, A_n) \ \leqslant \ \|u - v_n\|_X,$$

so

$$\limsup_{n\to+\infty} \operatorname{dist}(u, A_n)) \ \leqslant \ \|u - v\|_X. \tag{2.69}$$

Since $v \in s\text{–}\liminf\limits_{n\to+\infty} A_n$ is arbitrary, from (2.69) we conclude that

$$\limsup_{n\to+\infty} \operatorname{dist}(u, A_n) \ \leqslant \ \operatorname{dist}(u, s\text{–}\liminf_{n\to+\infty} A_n).$$

Solution of Problem 2.135

Let $u \in X$ and let $r = \liminf\limits_{n\to+\infty} \operatorname{dist}(u, A_n)$. If $r = +\infty$, then the desired inequality holds. So, let us assume that $r < +\infty$. Let $\{v_n\}_{n\geqslant 1}$ be a sequence such that $v_n \in A_n$ for all $n \geqslant 1$ and

$$\|u - v_n\|_X \ \leqslant \ \operatorname{dist}(u, A_n) + \tfrac{1}{n} \qquad \forall\, n \geqslant 1. \tag{2.70}$$

We have $\{v_n\}_{n\geqslant 1} \subseteq C \cap \overline{B}_{(r+\|u\|_X+1)} \in P_{wk}(X)$. So, by virtue of the Eberlein–Smulian theorem (see Theorem I.5.78), passing to a subsequence if necessary, we have

$$v_n \ \xrightarrow{w}\ v \quad \text{in } X \quad \text{and} \quad v \ \in \ w\text{–}\limsup_{n\to+\infty} A_n. \tag{2.71}$$

Exploiting the weak lower semicontinuity of the norm functional (see Proposition I.5.56(c)), we have

$$\|u - v\|_X \ \leqslant \ \liminf_{n\to+\infty} \|u - v_n\|_X \ \leqslant \ \liminf_{n\to+\infty} \operatorname{dist}(u, A_n)$$

(see (2.70)), so

$$\operatorname{dist}(u, w\text{–}\limsup_{n\to+\infty} A_n) \;\leqslant\; \liminf_{n\to+\infty} \operatorname{dist}(u, A_n)$$

(see (2.71)).

Solution of Problem 2.136

Because X is reflexive and $A \in P_{fc}(X)$, we can find $v \in A$ such that

$$\operatorname{dist}(u, A) \;=\; \|u - v\|_X. \tag{2.72}$$

Since by hypothesis $A_n \xrightarrow{M} A$, we can find $v_n \in A_n$ for all $n \geqslant 1$ such that $v_n \longrightarrow v$ in X (see Definitions 2.90 and 2.88). We have

$$\operatorname{dist}(u_n, A_n) \;\leqslant\; \|u_n - v_n\|_X \qquad \forall\, n \geqslant 1,$$

so

$$\limsup_{n\to+\infty} \operatorname{dist}(u_n, A_n) \;\leqslant\; \|u - v\|_X \;=\; \operatorname{dist}(u, A) \tag{2.73}$$

(see (2.72)). On the other hand, from triangle inequality, we have

$$\operatorname{dist}(u_n, A_n) \;\leqslant\; \|u - u_n\|_X + \operatorname{dist}(u, A_n) \qquad \forall\, n \geqslant 1. \tag{2.74}$$

The hypothesis that $A_n \xrightarrow{M} A$, the reflexivity of X and Problems 2.134 and 2.135 imply that

$$\operatorname{dist}(u, A_n) \;\longrightarrow\; \operatorname{dist}(u, A).$$

So, if in (2.74) we pass to the limit as $n \to +\infty$, then

$$\operatorname{dist}(u, A) \;\leqslant \liminf_{n\to+\infty} \operatorname{dist}(u_n, A_n). \tag{2.75}$$

From (2.73) and (2.75), we conclude that $\operatorname{dist}(u_n, A_n) \longrightarrow \operatorname{dist}(u, A)$.

Solution of Problem 2.137

From Remark 2.89, we know that the multifunction $\omega \longmapsto H(\omega) = \limsup\limits_{n\to+\infty} F_n(\omega)$ is $P_f(X)$-valued. So, according to Problem 2.100, it suffices to show that for all $K \in P_k(X)$, we have $H^-(K) \in \Sigma$. Note that

$$H^-(K) \;=\; \bigcap_{m\geqslant 1}\bigcap_{k\geqslant 1}\bigcup_{n\geqslant k} F_n^-(K_{\frac{1}{m}})$$

(see Remark 2.87), where $K_{\frac{1}{m}} = \big\{u \in X : \operatorname{dist}(u,K) < \frac{1}{m}\big\}$. But for each $n, m \geqslant 1$, we have $F_n^-(K_{\frac{1}{m}}) \in \Sigma$ (since F_n is measurable; see Definition 2.61(a)). Hence $H^-(K) \in \Sigma$ and so H is measurable.

Solution of Problem 2.138

Let $u \in A$. Then from the hypothesis we have $\operatorname{dist}(u, A_n) \longrightarrow 0$ and this by virtue of Definition 2.88 implies that $u \in s\text{–}\liminf\limits_{n\to+\infty} A_n$. So, we conclude that $A \subseteq s\text{–}\liminf\limits_{n\to+\infty} A_n$.

Solution of Problem 2.139

Since $A_n \subseteq C \in P_{wk}(X)$ for all $n \geqslant 1$, using the Eberlein–Smulian theorem (see Theorem I.5.78) and Definition 2.88 we conclude that $w\text{–}\limsup\limits_{n\to+\infty} A_n \neq \emptyset$.

Let $h \in w\text{–}\limsup\limits_{n\to+\infty} A_n$. Then we can find a subsequence $\{A_{n_k}\}_{k\geqslant 1}$ of $\{A_n\}_{n\geqslant 1}$ and $h_{n_k} \in A_{n_k}$ for all $k \geqslant 1$ such that $h_{n_k} \xrightarrow{w} h$ in X. For every $u^* \in X^*$, we have

$$\langle u^*, h_{n_k}\rangle \;\leqslant\; \sigma_{A_{n_k}}(u^*) \qquad \forall\, k \geqslant 1$$

(see Definition 2.46), so

$$\langle u^*, h\rangle \;\leqslant\; \limsup_{k\to+\infty} \sigma_{A_{n_k}}(u^*) \;\leqslant\; \limsup_{n\to+\infty} \sigma_{A_n}(u^*),$$

thus

$$\sigma_{\underset{n\to+\infty}{w\text{–}\limsup} A_n}(u^*) \leqslant \limsup_{n\to+\infty} \sigma_{A_n}(u^*). \tag{2.76}$$

On the other hand, given $u^* \in X^*$, we can find $h_n \in A_n$ such that

$$\sigma_{A_n}(u^*) - \tfrac{1}{n} \leqslant \langle u^*, h_n\rangle \qquad \forall\, n \geqslant 1 \tag{2.77}$$

(see Definition 2.46). We have $\{h_n\}_{n\geqslant 1} \subseteq C \in P_{wk}(X)$. The Eberlein–Smulian theorem implies that we can find a subsequence $\{h_{n_k}\}_{k\geqslant 1}$ of $\{h_n\}_{n\geqslant 1}$, such that

$$h_{n_k} \xrightarrow{w} h \quad \text{in } X.$$

It follows that $h \in \underset{n\to+\infty}{w\text{–}\limsup} A_n$. Then from (2.77) we have

$$\limsup_{n\to+\infty} \sigma_{A_n}(u^*) \leqslant \sigma_{\underset{n\to+\infty}{w\text{–}\limsup} A_n}(u^*). \tag{2.78}$$

From (2.76) and (2.78), we conclude that

$$\sigma_{\underset{n\to+\infty}{w\text{–}\limsup} A_n}(u^*) = \limsup_{n\to+\infty} \sigma_{A_n}(u^*) \qquad \forall\, u^* \in X^*.$$

Solution of Problem 2.140

Let $h \in \underset{n\to+\infty}{w\text{–}\limsup} A_n$. Then we can find a subsequence $\{A_{n_k}\}_{k\geqslant 1}$ of $\{A_n\}_{n\geqslant 1}$ and $h_{n_k} \in A_{n_k}$ for all $k \geqslant 1$ such that $h_{n_k} \xrightarrow{w} h$ in X. For every $u^* \in X^*$ we have

$$\langle u^*, h_{n_k}\rangle \longrightarrow \langle u^*, h\rangle\,,$$

so

$$\langle u^*, h\rangle \leqslant \limsup_{n\to+\infty} \sigma_{A_{n_k}}(u^*) \leqslant \sigma_{A}(u^*)$$

(see Definition 2.46), thus $h \in \overline{\operatorname{conv}}\, A$ and hence $\underset{n\to+\infty}{w\text{–}\limsup} A_n \subseteq \overline{\operatorname{conv}}\, A$.

Solution of Problem 2.141

From the Mazur theorem (see Theorem I.5.58), we know that

$$h(\omega) \in \overline{\text{conv}} \bigcup_{n \geqslant k} h_n(\omega) \qquad \forall\, k \geqslant 1.$$

So, for every $k \geqslant 1$, $u^* \in X^*$ and $\omega \in \Omega \setminus N$ with $\mu(N) = 0$, we have

$$\langle u^*, h(\omega) \rangle \leqslant \sigma_{\bigcup\limits_{n \geqslant k} h_n(\omega)}(u^*) = \sup_{n \geqslant k} \langle u^*, h_n(\omega) \rangle\,,$$

so

$$\langle u^*, h(\omega) \rangle \leqslant \limsup_{n \to +\infty} \langle u^*, h_n(\omega) \rangle = \sigma_{\substack{w\text{–}\limsup\{h_n(\Omega)\} \\ n\to+\infty}}(u^*)$$

(see Problem 2.139), thus

$$h(\omega) \in \overline{\text{conv}}\; w\text{–}\limsup_{n \to +\infty} \{h_n(\omega)\} \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Solution of Problem 2.142

Note that hypotheses **(i)** and **(ii)** do not imply the joint measurability of F (see Hu–Papageorgiou [8, p. 227]). So, the conclusion cannot be deduced from the Yankov–von Neumann–Aumann selection theorem (see Theorem 2.68). Let $v \in L^1(\Omega; X)$. We can find a sequence of simple functions $\{s_n\}_{n \geqslant 1} \subseteq L^1(\Omega; X)$ such that

$$\|v(\omega) - s_n(\omega)\|_X \longrightarrow 0 \quad \text{for } \mu\text{-a.a. } \omega \in \Omega$$

(see Definition 1.35(b)). Then because of hypothesis **(i)** for every $n \geqslant 1$, the multifunction $\omega \longmapsto F(\omega, s_n(\omega))$ is graph measurable (see Definition 2.61(b)). So, we can apply the Yankov–von Neumann–Aumann selection theorem (see Theorem 2.68) and obtain $h_n \in S^1_{F(\cdot, s_n(\cdot))}$. But from hypothesis **(iii)** (see Definitions 2.61(c) and 2.75) we know that $S^1_{F(\cdot, s_n(\cdot))} \subseteq S^1_G$ and the latter is w-compact in $L^1(\Omega; Y)$ (see Theorem 2.76). So, by passing to a suitable subsequence if necessary, we may assume that $h_n \xrightarrow{w} h$ in $L^1(\Omega; Y)$. From Problem 2.140, we have

$$\begin{aligned} h(\omega) &\in \overline{\text{conv}}\ w\text{–}\limsup_{n\to+\infty} \{h_n(\omega)\} \\ &\subseteq \overline{\text{conv}}\ w\text{–}\limsup_{n\to+\infty} F(\omega, s_n(\omega)) \\ &\subseteq \overline{\text{conv}}\, F(\omega, v(\omega)) \ = \ F(\omega, v(\omega)) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega \end{aligned}$$

(see hypothesis **(ii)**), so $h \in S^1_{F(\cdot,v(\cdot))}$ and thus $S^1_{F(\cdot,v(\cdot))} \neq \emptyset$.

Solution of Problem 2.143

Let m denote the Mackey topology $m(X^*, X)$ on X^* (i.e., the strongest, locally convex topology τ on X^* for which we have $(X^*_\tau)^* = X$). The space X^*_m (i.e., the space X^* furnished with the Mackey topology) is separable. Let $\{u^*_n\}_{n\geqslant 1} \subseteq X^*$ be an m-dense sequence. For every $u, v \in X$, we set

$$d_X(u,v) \ = \ \sum_{n\geqslant 1} \tfrac{|\langle u^*_n, u-v\rangle|}{1+|\langle u^*_n, u-v\rangle|}.$$

This is a metric on X and it can be easily verified that the d_X-topology is weaker than the weak topology and the two coincide on w-compact subsets of X (see Theorem I.2.84).

Let $H(\omega) = \overline{\{h_n(\omega)\}_{n\geqslant 1}}^w \in P_{wk}(X)$. Then $H(\omega) = \overline{h_n(\omega)}^d$ for μ-almost all $\omega \in \Omega$ and since X_d is separable, from Theorem 2.67 we infer that H is measurable into X_d. Then $\operatorname{Gr} H \in \Sigma \times \mathcal{B}(X_d) = \Sigma \times \mathcal{B}(X_w) = \Sigma \times \mathcal{B}(X)$ and so we can apply Theorem 2.76 and have that $S^1_H \subseteq L^1(\Omega; X)$ is w-compact. Therefore the sequence $\{h_n\}_{n\geqslant 1} \subseteq L^1(\Omega; X)$ is relatively w-compact.

Solution of Problem 2.144

We do the solution when condition **(i)** holds. The solution is similar if condition **(ii)** holds. Let $F(\omega) = [h(\omega), +\infty)$. Then F is a measurable multifunction from Ω into $P_{fc}(\mathbb{R})$. We have

$$\operatorname{dist}_{\mathbb{R}^N}(h_n(\omega), F(\omega)) \ = \ (h - h_n)^+(\omega) \ \longrightarrow \ 0 \quad \text{for } \mu\text{-a.a. } \omega \in \Omega$$

(due to condition **(i)**). Also we have

$$h(\omega) \in \operatorname{ext} F(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Therefore we can apply Proposition 2.97 and conclude that $h_n \longrightarrow h$ in $L^1(\Omega)$.

Solution of Problem 2.145

First we show that the set $w\text{–}\limsup\limits_{n\to+\infty} A_n$ need not be closed in X. Let X be a Banach space with a separable dual X^* (therefore, X is separable too; see Theorem I.5.82). Let $\overline{B}_1 = \{u \in X : \|u\|_X \leqslant 1\}$ and $S_1 = \partial B_1 = \{u \in X : \|u\|_X = 1\}$. We know that $0 \in \overline{S}_1^w$ and since the weak topology on $\overline{B}_1$ is metrizable, we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq S_1$ such that $u_n \xrightarrow{w} 0$ in X. Let $A_n = \{u_1, \ldots, u_n\}$ for all $n \geqslant 1$. Then

$$w\text{–}\limsup_{n\to+\infty} A_n = \{u_n\}_{n\geqslant 1} \cup \{0\}$$

and this set is not strongly closed.

Next we show that $w\text{–}\limsup\limits_{n\to+\infty} A_n$ need not be w-closed. To this end let $X = l^1$ and let $A_n = S_1$ for all $n \geqslant 1$. Then $w\text{–}\limsup\limits_{n\to+\infty} A_n$ is not w-closed. Indeed, recall that $0 \in \overline{S}_1^w$, but by the Schur property (see Remark I.5.57) there is no sequence in S_1 w-converging to 0 and this implies that $w\text{–}\limsup\limits_{n\to+\infty} A_n$ is not w-closed.

Solution of Problem 2.146

First we treat the "decreasing" case. Let $A = \bigcap\limits_{n\geqslant 1} A_n$. From Remark 2.89 we have that $A = s\text{–}\liminf\limits_{n\to+\infty} A_n$. If $w\text{–}\limsup\limits_{n\to+\infty} A_n = \emptyset$, then $A = \emptyset$ and we are done. So, assume that $w\text{–}\limsup\limits_{n\to+\infty} A_n \neq \emptyset$ and let $u \in w\text{–}\limsup\limits_{n\to+\infty} A_n$. We can find a subsequence $\{A_{n_k}\}_{k\geqslant 1}$ of $\{A_n\}_{n\geqslant 1}$

and $u_{n_k} \in A_{n_k}$ such that $u_{n_k} \xrightarrow{w} u$ in X. Since $\{A_n\}_{n\geqslant 1}$ is decreasing, we have $u \in \bigcap\limits_{k\geqslant 1} A_{n_k} = \bigcap\limits_{n\geqslant 1} A_n$, hence $u \in A$ and so $w\text{–}\limsup\limits_{n\to+\infty} A_n \subseteq A$. We conclude that

$$A_n \xrightarrow{M} \bigcap_{n\geqslant 1} A_n$$

(see Definition 2.90). For the "increasing" case, we have

$$s\text{–}\liminf_{n\to+\infty} A_n = \overline{\bigcup_{n\geqslant 1} A_n} \tag{2.79}$$

(see Remark 2.89). The set $\overline{\bigcup\limits_{n\geqslant 1} A_n}$ is closed, convex, hence it is w-closed, which implies that

$$w\text{–}\limsup_{n\to+\infty} A_n \subseteq \overline{\bigcup_{n\geqslant 1} A_n},$$

so

$$A_n \xrightarrow{M} \overline{\bigcup_{n\geqslant 1} A_n}$$

(see (2.79)).

Solution of Problem 2.147

Let $h \in L^1(\Omega; X)$ and $v \in L^1(\Omega; Y)$. Then

$$\begin{aligned}
&\operatorname{dist}_{L^1(\Omega;Y)}(v, S^1_{F(\cdot,h(\cdot))}) = \inf_{y\in S^1_{F(\cdot,h(\cdot))}} \|v-y\|_{L^1(\Omega;Y)} \\
&= \inf_{y\in S^1_{F(\cdot,h(\cdot))}} \int_\Omega \|v(\omega)-y(\omega)\|_Y\, d\mu = \int_\Omega \inf_{z\in F(\omega,h(\omega))} \|v(\omega)-z\|_Y\, d\mu \\
&= \int_\Omega \operatorname{dist}_Y(v(\omega), F(\omega,h(\omega)))\, d\mu
\end{aligned} \tag{2.80}$$

(see hypothesis **(i)** and Theorem 2.74). According to Proposition 2.44 it suffices to show that for every $v \in L^1(\Omega; Y)$, the function $h \longmapsto \operatorname{dist}_{L^1(\Omega;Y)}(v, S^1_{F(\cdot,h(\cdot))})$ is upper semicontinuous. So, for every $v \in L^1(\Omega; Y)$ and $\lambda \geqslant 0$ we need to show that the set

$$U_\lambda = \Big\{h \in L^1(\Omega; X) : \operatorname{dist}_{L^1(\Omega;Y)}(v, S^1_{F(\cdot,h(\cdot))}) \geqslant \lambda\Big\}$$

is closed (see Proposition I.2.54). To this end, let $\{h_n\}_{n\geqslant 1} \subseteq U_\lambda$ be a sequence such that $h_n \longrightarrow h$ in $L^1(\Omega; X)$. By passing to a suitable subsequence, we may assume that

$$h_n(\omega) \longrightarrow h(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega. \tag{2.81}$$

Then we have

$$\begin{aligned}
\lambda &\leqslant \limsup_{n\to+\infty} \operatorname{dist}_{L^1(\Omega;Y)}\big(v, S^1_{F(\cdot,h_n(\cdot))}\big) \\
&= \limsup_{n\to+\infty} \int_\Omega \operatorname{dist}_Y\big(v(\omega), S^1_{F(\omega,h_n(\omega))}\big)\, d\mu \\
&\leqslant \int_\Omega \limsup_{n\to+\infty} \operatorname{dist}_Y\big(v(\omega), F(\omega, h_n(\omega))\big)\, d\mu \\
&\leqslant \int_\Omega \operatorname{dist}_Y\big(v(\omega), F(\omega, h(\omega))\big)\, d\mu \\
&= \operatorname{dist}_{L^1(\Omega;Y)}\big(v, S^1_{F(\cdot,h(\cdot))}\big)
\end{aligned}$$

(by the Fatou lemma, hypotheses **(iii) and (ii)**, Proposition 2.44, (2.81), and (2.80)), so $h \in U_\lambda$ and thus the function $h \longmapsto \operatorname{dist}_{L^1(\Omega;Y)}(v, S^1_{F(\cdot,h(\cdot))})$ is upper semicontinuous on $L^1(\Omega; X)$. This proves that the multifunction $h \longmapsto S^1_{F(\cdot,h(\cdot))}$ is lower semicontinuous from $L^1(\Omega; X)$ into $L^1(\Omega; Y)$.

Solution of Problem 2.148

Let $\mathcal{Y} = \{E\colon X \supseteq D(E) \longrightarrow 2^{X^*} :\ E$ is monotone and $\operatorname{Gr} A \subseteq \operatorname{Gr} E\}$. Evidently $\mathcal{Y} \neq \emptyset$. We introduce a partial order $\preceq$ on $\mathcal{Y}$ as follows:

$$E_1 \preceq E_2 \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \operatorname{Gr} E_1 \subseteq \operatorname{Gr} E_2,$$

for all $E_1, E_2 \in \mathcal{Y}$. Let $\mathcal{C}$ be a chain in $\mathcal{Y}$. The map with graph equal to $\bigcup\limits_{E\in\mathcal{C}} \operatorname{Gr} E$ is an upper bound for $\mathcal{C}$. Invoking the Kuratowski–Zorn lemma (see Theorem 4.120), we can find a maximal element

$\widehat{A} \in \mathcal{Y}$. Clearly then $\widehat{A}\colon X \supseteq D(\widehat{A}) \longrightarrow 2^{X^*}$ is maximal monotone (see Definition 2.100) and $\operatorname{Gr} A \subseteq \operatorname{Gr} \widehat{A}$.

Solution of Problem 2.149

Let $\{(u_\alpha, u_\alpha^*)\}_{\alpha\in J} \subseteq \operatorname{Gr} A$ be a net and assume that

$$u_\alpha \xrightarrow{w} u \quad \text{in } X \quad \text{and} \quad u_\alpha^* \longrightarrow u^* \quad \text{in } X^*. \tag{2.82}$$

The monotonicity of A implies that

$$\langle u_\alpha^* - v^*, u_\alpha - v\rangle \geqslant 0 \qquad \forall\, (v, v^*) \in \operatorname{Gr} A$$

(see Definition 2.98(a)), so

$$\langle u^* - v^*, u - v\rangle \geqslant 0 \qquad \forall\, (v, v^*) \in \operatorname{Gr} A$$

(see (2.82)) and thus $(u, u^*) \in \operatorname{Gr} A$ (from the maximality of A; see Definition 2.100). This proves that $\operatorname{Gr} A$ is closed in $X_w \times X^*$.

Similarly, we show the closedness of $\operatorname{Gr} A$ in $X \times X^*_{w^*}$.

For the second part of the problem, the answer is No. To see this, let $X = l^2$ (a Hilbert space), let $C = \overline{B}_1$ and let p_C be the metric projection (that is, for any $u \in X$, $p_C(u) = \big\{h \in C :\ \|u-h\|_{l^2} = \inf\limits_{h'\in C} \|u-h'\|_{l^2}\big\}$). We know that p_C is single valued and nonexpansive (i.e., $\|p_C(u) - p_C(v)\|_{l^2} \leqslant \|u - v\|_{l^2}$ for all $u, v \in X$) (see Proposition I.5.99). We have that the map $u \longmapsto (I_X - p_C)(u)$ is continuous and monotone (see Definition 2.98(a)), since

$$\begin{aligned} &\langle (I_X - p_C(u) - (I_X - p_C))(v), u - v\rangle \\ \geqslant\ & \|u-v\|_{l^2}^2 - \|p_C(u) - p_C(v)\|_{l^2}\|u-v\|_{l^2} \\ \geqslant\ & \|u-v\|_{l^2}^2 - \|u-v\|_{l^2}^2 \;=\; 0. \end{aligned}$$

Invoking Corollary 2.111, we infer that $u \longmapsto (I_X - p_C)(u)$ is maximal monotone (see Definition 2.100). Let $\{e_n\}_{n\geqslant 1}$ be the standard orthonormal basis of $X = l^2$ and consider the sequence $\{u_n\}_{n\geqslant 1}$ defined by $u_n = e_1 + e_{2n}$ for all $n \geqslant 1$. Then

$$\big\{(u_n, (1 - \frac{1}{\sqrt{2}})u_n)\big\}_{n\geqslant 1} \subseteq \operatorname{Gr}(I_X - p_C)$$

and we have

$$(u_n, (1 - \tfrac{1}{\sqrt{2}})u_n) \xrightarrow{w} (e_1, (1 - \tfrac{1}{\sqrt{2}})e_1)$$

(recall that $e_n \xrightarrow{w} 0$). Note that $(e_1, (1 - \frac{1}{\sqrt{2}})e_1) \notin \operatorname{Gr} A$.

Solution of Problem 2.150

Let $u, v \in D(A)$. We have

$$\langle A(u) - A(v), u - v \rangle = \int_0^b \big(u'(t) - v'(t), u(t) - v(t)\big)_H \, dt$$

$$= \frac{1}{2} \int_0^b \frac{d}{dt} \|u(t) - v(t)\|_H^2 \, dt = \frac{1}{2}\big(\|u(b) - v(b)\|_H^2 - \|u(0) - v(0)\|_H^2\big)$$

$$= \frac{1}{2}\|u(b) - v(b)\|_H^2 \geqslant 0,$$

so A is monotone (see Definition 2.98(a)).

According to Theorem 2.116, in order to show the maximality of A, it suffices to show that $R(A + I_{L^2(T;H)}) = L^2(T;H)$. So, we fix $h \in L^2(T;H)$ and consider the abstract Cauchy problem

$$\begin{cases} u'(t) + u(t) = h(t) & \text{for a.a. } t \in T, \\ u(0) = u_0. \end{cases} \tag{2.83}$$

Let $g(t) = e^t h(t)$ for $t \in T$. Evidently $g \in L^2(T;H)$. We define

$$y(t) = u_0 + \int_0^t g(s) \, ds.$$

We have $y \in W^{1,2}(T;H) \subseteq C(T;H)$ and $y(0) = y_0$. Setting $u(t) = e^{-t}y(t)$ for $t \in T$, we have $u'(t) = -e^{-t}y(t) + e^{-t}y'(t) = -u(t) + h(t)$. Therefore $u \in W^{1,2}(T;H)$, $u(0) = u_0$ and it solves (2.83). This then proves the maximality of A.

Solution of Problem 2.151

Let $\{u_n\}_{n\geqslant 1} \subseteq X$ be a sequence such that $u_n \longrightarrow u$ in X. The map A is locally bounded (see Definition 2.102 and Proposition 2.104). Therefore the sequence $\{A(u_n)\}_{n\geqslant 1} \subseteq X^*$ is bounded and so it has a cluster point $h^* \in X^*$. The monotonicity of A implies that

$$\langle h^* - A(v), u - v \rangle \ \geqslant\ 0 \qquad \forall\, v \in X.$$

Let $v = ty + (1-t)u$ with $t \in [0,1]$ and $y \in X$. The hemicontinuity of A implies

$$\langle h^* - A(y), u - y \rangle \ \geqslant\ 0 \qquad \forall\, y \in X. \tag{2.84}$$

From Corollary 2.111, we know that A is maximal monotone (see Definition 2.100). So, (2.84) implies that $h^* = A(u)$. This means that for the original sequence, we have $A(u_n) \longrightarrow A(u)$ and so we conclude that A is continuous.

Solution of Problem 2.152

"$\Longrightarrow$": Assume that A is demicontinuous (see Definition 2.98(a)) and monotone. It is clear from Definition 2.110, that A is hemicontinuous too.

"$\Longleftarrow$": Assume that A is monotone and hemicontinuous. Let $\{u_n\}_{n\geqslant 1} \subseteq X$ be a sequence such that $u_n \longrightarrow u$ in X. Let $t_n = \|u_n - u\|_X^{\frac{1}{2}}$ and for $h \in X$, let $h_n = u + t_n h$. We have

$$\langle A(u_n), h \rangle \ =\ \tfrac{1}{t_n} \langle A(u_n), h_n - u_n \rangle + \tfrac{1}{t_n} \langle A(u_n), u_n - u \rangle\,. \tag{2.85}$$

Note that the sequence $\{A(u_n)\}_{n\geqslant 1} \subseteq X^*$ is bounded (see Proposition 2.104) and so we have

$$\tfrac{1}{t_n} \langle A(u_n), u_n - u \rangle \ =\ \Big\langle A(u_n), \tfrac{u_n - u}{\|u_n - u\|_X^{\frac{1}{2}}} \Big\rangle \ \longrightarrow\ 0. \tag{2.86}$$

Also, the hemicontinuity of A implies that

$$A(h_n) \ \xrightarrow{w^*}\ A(u) \quad \text{in } X^*,$$

so

$$\tfrac{1}{t_n} \langle A(h_n), h_n - u_n \rangle \ \longrightarrow\ \langle A(u), h \rangle\,. \tag{2.87}$$

From the monotonicity of A, we have

$$\langle A(u_n), h_n - u_n\rangle \leqslant \langle A(h_n), h_n - u_n\rangle \qquad \forall\, n \geqslant 1,$$

so

$$\limsup_{n\to+\infty} \tfrac{1}{t_n} \langle A(u_n), h_n - u_n\rangle \leqslant \langle A(u), h\rangle \tag{2.88}$$

(see (2.87)). Returning to (2.85), passing to the limit as $n \to +\infty$ and using (2.86) and (2.88), we obtain

$$\limsup_{n\to+\infty} \langle A(u_n), h\rangle \leqslant \langle A(u), h\rangle \qquad \forall\, h \in X,$$

so

$$\langle A(u_n), h\rangle \longrightarrow \langle A(u), h\rangle \qquad \forall\, h \in X$$

(see Proposition 2.104 and Problem 2.149), thus

$$A(u_n) \xrightarrow{w^*} A(u) \quad \text{in } X^*$$

and hence A is demicontinuous (see Definition 2.110).

Solution of Problem 2.153

For every $u \in W_0^{1,p}(\Omega)$, we have that $Du \in L^p(\Omega;\mathbb{R}^N)$ and so from the growth condition on a, we have

$$a(Du(\cdot)) \in L^{p'}(\Omega;\mathbb{R}^N).$$

From Theorem 1.140 it follows that $-\operatorname{div} a(Du(\cdot)) \in W^{-1,p'}(\Omega)$. We denote this operator by $A\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ and we have

$$\langle A(u), h\rangle = \langle -\operatorname{div} a(Du), h\rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\, dz \qquad \forall\, h \in W_0^{1,p}(\Omega)$$

(since $h|_{\partial\Omega} = 0$; see Definition 1.129). If $E \subseteq W_0^{1,p}(\Omega)$ is bounded, then

$$\text{the set } \{Du : u \in E\} \subseteq L^p(\Omega;\mathbb{R}^N) \text{ is bounded}$$

and so from the growth condition on a it follows that

$$\text{the set } \{a(Du):\ u \in E\} \subseteq L^{p'}(\Omega;\mathbb{R}^N) \text{ is bounded.} \tag{2.89}$$

So, we have

$$\begin{aligned} |\langle A(u),h\rangle| &= \Big|\int_\Omega (a(Du),Dh)_{\mathbb{R}^N}\Big| \leqslant \int_\Omega |a(Du)|\,|Dh|\,dz \\ &\leqslant c_1\|Dh\|_{L^p(\Omega;\mathbb{R}^N)} \qquad \forall\ h \in W_0^{1,p}(\Omega), \end{aligned}$$

for some $c_1 > 0$ (see (2.89)), so the set $A(E) \subseteq W^{-1,p'}(\Omega)$ is bounded.

Let $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega)$. Then for all $h \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned} |\langle A(u_n)-A(u),h\rangle| &\leqslant \int_\Omega |a(Du_n)-a(Du)|\,|Dh|\,dz \\ &\leqslant \|a(Du_n)-a(Du)\|_{L^{p'}(\Omega;\mathbb{R}^N)}\|Dh\|_{L^p(\Omega;\mathbb{R}^N)} \\ &= \|a(Du_n)-a(Du)\|_{L^{p'}(\Omega;\mathbb{R}^N)}\|h\|_{W_0^{1,p}(\Omega)} \end{aligned}$$

(see Theorem 1.131), so

$$\|A(u_n)-A(u)\|_{W^{-1,p'}(\Omega)} \leqslant \|a(Du_n)-a(Du)\|_{L^{p'}(\Omega;\mathbb{R}^N)}.$$

But by virtue of the Krasnoselskii theorem (see Problem 1.42), we have

$$\|a(Du_n)-a(Du)\|_{L^{p'}(\Omega;\mathbb{R}^N)} \longrightarrow 0,$$

so A is continuous. Finally for all $u,v \in W_0^{1,p}(\Omega)$, we have

$$\langle A(u)-A(v),u-v\rangle = \int_\Omega (a(Du)-a(Dv),Du-Dv)_{\mathbb{R}^N}\,dz \geqslant 0$$

(since a is monotone; see Definition 2.98(a)). Therefore A is continuous and monotone, thus according to Corollary 2.111, also maximal monotone (see Definition 2.100).

Solution of Problem 2.154
Let $A\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \langle -\mathrm{div}\,(Du), h\rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\, dz \qquad \forall\, h \in W_0^{1,p}(\Omega).$$

From Problem 2.153 we know that A is maximal monotone (see Definition 2.100). From the Poincaré inequality (see Theorem 1.131), we know that we can have

$$\|u\|_{1,p} \;=\; \|Du\|_{L^p(\Omega;\mathbb{R}^N)} \qquad \forall\, u \in W_0^{1,p}(\Omega).$$

From the assumption on a, we have

$$\tfrac{\langle A(u),u\rangle}{\|u\|_{1,p}} \;\geqslant\; \widehat{c}\tfrac{\|u\|_{1,p}^p}{\|u\|_{1,p}} \;=\; \widehat{c}\|u\|_{1,p}^{p-1},$$

so A is strongly coercive (see Definition 2.98(f)). Invoking Theorem 2.119, we conclude that there exists $u \in W_0^{1,p}(\Omega)$ such that $A(u) = g$.

Solution of Problem 2.155
"$\Longrightarrow$": This follows immediately from the definition of maximal monotonicity (see Remark 2.101).

"$\Longleftarrow$": Suppose that $\mathrm{Gr}\, A$ is maximal among all linear monotone graphs and suppose that it is not maximal monotone. Then according to Definition 2.100 there exists $(u, u^*) \in (X \times X^*) \setminus \mathrm{Gr}\, A$ such that

$$\langle u^* - h^*, u - h\rangle \;\geqslant\; 0 \qquad \forall\, (h, h^*) \in \mathrm{Gr}\, A.$$

Then the linear space span $(\mathrm{Gr}\, A \cup (u, u^*))$ is a linear monotone graph extending $\mathrm{Gr}\, A$, a contradiction to our hypothesis. So, A is a maximal monotone map.

Solution of Problem 2.156
Arguing indirectly suppose that $A \notin \mathcal{L}(X; X^*)$. Then we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ and $\varepsilon > 0$ such that

$$u_n \;\longrightarrow\; 0 \;\text{ in } X \quad\text{and}\quad \|A(u_n)\|_{X^*} \;\geqslant\; \varepsilon \qquad \forall\, n \geqslant 1. \tag{2.90}$$

Let $t_n = \frac{1}{\|u_n\|_X^{\frac{1}{2}}}$ for all $n \geqslant 1$ and set $h_n = t_n u_n$ for all $n \geqslant 1$. Clearly

$$\|h_n\|_X = \|u_n\|_X^{\frac{1}{2}} \longrightarrow 0.$$

Hence by the demicontinuity of A, we have that $A(h_n) \xrightarrow{w} 0$. Thus there exists $M > 0$ such that

$$\|A(h_n)\|_{X^*} \leqslant M \qquad \forall\, n \geqslant 1. \tag{2.91}$$

On the other hand, we have

$$\|A(h_n)\|_{X^*} = t_n \|A(u_n)\|_{X^*} \geqslant t_n \varepsilon \qquad \forall\, n \geqslant 1$$

(see (2.90)), so

$$\|A(h_n)\|_{X^*} \longrightarrow +\infty,$$

which contradicts (2.91).

Solution of Problem 2.157

We argue indirectly. So, suppose that we can find $u^*, v^* \in A(\widehat{u})$, $u^* \neq v^*$. Then there exists $u \in X$ such that $\varepsilon = \langle u^* - v^*, u \rangle > 0$. Let

$$\widehat{u}_n = \widehat{u} + \tfrac{1}{n} u \qquad \forall\, n \geqslant 1.$$

Then $\widehat{u}_n \longrightarrow \widehat{u}$ in X and because A is by hypothesis lower semicontinuous at $\widehat{u}$ into $X^*_{w^*}$, for some big $n_0 \geqslant 1$, we have that

$$S_n = A(\widehat{u}_n) \cap \left\{ h^* \in X^* : |\langle u^* - h^*, u \rangle| < \tfrac{\varepsilon}{2} \right\} \neq \emptyset \qquad \forall\, n \geqslant n_0$$

(see Definition 2.36). Let $h_n \in S_n$ for $n \geqslant n_0$. Then since A is monotone (see Definition 2.98(a)), we have

$$\begin{aligned} 0 &\leqslant \langle h_n^* - v^*, \widehat{u}_n - \widehat{u} \rangle = \tfrac{1}{n} \langle h_n^* - v^*, u \rangle \\ &= \tfrac{1}{n} \langle h_n^* - u^*, u \rangle + \tfrac{1}{n} \langle u^* - v^*, u \rangle \\ &< \tfrac{\varepsilon}{2n} - \tfrac{\varepsilon}{n} < 0 \end{aligned}$$

for big $n \geqslant 1$, a contradiction.

Solution of Problem 2.158

From Proposition 2.105 we know that A is upper semicontinuous from $\operatorname{int} D(A)$ into $X^*_{w^*}$ (where $X^*_{w^*}$ denotes the space X^* furnished with the w^*-topology). Moreover, from Propositions 2.104 and 2.106 and the Alaoglu theorem (see Theorem I.5.66), we see that $A|_{\operatorname{int} D(A)}$ has nonempty, convex, and w^*-compact values. Since X is separable, we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ which is dense in X. Then for $u^*, x^* \in X^*$, we set

$$d_{X^*}(u^*, v^*) \;=\; \sum_{n\geqslant 1} \tfrac{1}{2^n}\tfrac{|\langle u^*-v^*, u_n\rangle|}{1+|\langle u^*-v^*, u_n\rangle|}.$$

This is a metric on X^* and the topology it generates is weaker than the w^*-topology. So, we can apply Proposition 2.41(b) and infer that the set

$$\{u \in \operatorname{int} D(A) : \; A \text{ is not lower semicontinuous at } u\}$$

is of first category. Then Problem 2.157 implies that the set Γ is of first category.

Solution of Problem 2.159

"$\Longrightarrow$": We assume that the duality map $\mathcal{F}$ is linear (see Definition 2.112). Then for all $u, v \in X$, we have

$$\|u\pm v\|_X^2 \;=\; \langle \mathcal{F}(u\pm v), u\pm v\rangle \;=\; \|u\|_X^2 \pm \langle \mathcal{F}(u), v\rangle \pm \langle \mathcal{F}(v), u\rangle + \|v\|_X^2$$

(from the linearity of $\mathcal{F}$), so

$$\|u+v\|_X^2 + \|u-v\|_X^2 \;=\; 2\left(\|u\|_X^2 + \|v\|_X^2\right).$$

So, the space X satisfies the parallelogram law (see Remark I.5.94), hence X must be a Hilbert space.

"$\Longleftarrow$": A Hilbert space has a differentiable norm and we can easily check that for all $u, h \in X$, we have

$$\begin{aligned}\langle \mathcal{F}(u), h\rangle &= \tfrac{1}{2}\tfrac{d}{dt}\|u+th\|_X^2|_{t=0} \;=\; \tfrac{1}{2}\tfrac{d}{dt}\,(u+th, u+th)_X\,|_{t=0} \\ &= \tfrac{1}{2}\tfrac{d}{dt}\big(\|u\|_X^2 + 2t\,(u,h)_X + t^2\|h\|_X^2\big)|_{t=1} \;=\; (u,h)_X\,,\end{aligned}$$

so $\mathcal{F}(u) = u$ and so $\mathcal{F} = I_X$.

Solution of Problem 2.160

Invoking the Troyanski renorming theorem (see Theorem I.5.192 or Remark 2.115), without any loss of generality we may assume that both X and X^* are locally uniformly convex (see Definition I.5.168(b) and Remark I.5.169). As in the situation of Problem 2.148, using the Kuratowski–Zorn lemma (see Theorem 4.120), we can show that A admits a monotone extension $\widehat{A}$ which is maximal in the class of all monotone maps with domain in C. We will show that $\widehat{A}$ is in fact maximal monotone (see Definition 2.100). According to Theorem 2.116, it suffices to show that $R(A+\mathcal{F}) = X^*$. Let $h^* \in X^*$. From Theorem 3.117, we know that there exists $u \in C$ such that

$$\langle \mathcal{F}(u) + v^* - h^*, v - u \rangle \ \geqslant\ 0 \qquad \forall\ (v, v^*) \in \operatorname{Gr} \widehat{A}.$$

Since $\widehat{A}$ has no monotone extension on $C \times X^*$, we infer that

$$\text{if } u \in D(\widehat{A}) \text{ and } h^* - \mathcal{F}(u) \in \widehat{A}(u), \text{ then } h^* \in \widehat{A}(u) + \mathcal{F}(u).$$

Since $h^* \in X^*$ is arbitrary, we conclude that $R(\widehat{A}+\mathcal{F}) = X^*$ and so $\widehat{A}$ is maximal monotone.

Solution of Problem 2.161

From Theorem 2.119 we know that there exists $u_\lambda \in D(A)$ such that

$$0 \ \in\ \lambda A(u_\lambda) + \mathcal{F}(u_\lambda - u).$$

We show that this solution $u_\lambda \in D(A)$ is in fact unique. So, suppose that $u_\lambda, v_\lambda \in D(A)$ satisfy

$$\lambda u_\lambda^* + \mathcal{F}(u_\lambda - u) \ =\ 0, \quad \text{with } u_\lambda^* \in A(u_\lambda), \tag{2.92}$$

and

$$\lambda v_\lambda^* + \mathcal{F}(v_\lambda - u) \ =\ 0, \quad \text{with } v_\lambda^* \in A(v_\lambda). \tag{2.93}$$

From the monotonicity of A and $\mathcal{F}$ (see Proposition 2.114) and (2.92), (2.93), we have

$$0 \ \leqslant\ \lambda \langle u_\lambda^* - v_\lambda^*, u_\lambda - v_\lambda \rangle \ =\ \langle \mathcal{F}(v_\lambda - u) - \mathcal{F}(u_\lambda - u), u_\lambda - v_\lambda \rangle \ =\ 0,$$

so

$$\langle \mathcal{F}(u_\lambda - u) - \mathcal{F}(v_\lambda - u), u_\lambda - v_\lambda \rangle \;=\; 0. \tag{2.94}$$

But $\mathcal{F}$ is strictly monotone (see Definition 2.98(b) and Proposition 2.114(b)). Then from (2.94) we infer that $u_\lambda = v_\lambda$, which proves the uniqueness of the solution.

Solution of Problem 2.162

We do the solution under the assumption that

$$\limsup_{n,k\to+\infty} \langle u_n^* - u_k^*, u_n - u_k \rangle \;\leqslant\; 0. \tag{2.95}$$

The other case can be treated similarly.

The monotonicity of A and (2.95) imply that

$$\limsup_{n,k\to+\infty} \langle u_n^* - u_k^*, u_n - u_k \rangle \;=\; 0. \tag{2.96}$$

Let $\{(u_m, u_m^*)\}_{m\geqslant 1}$ be a subsequence of $\{(u_k, u_k^*)\}_{k\geqslant 1} \subseteq \operatorname{Gr} A$ such that

$$\lim_{m\to+\infty} \langle u_m^*, u_m \rangle \;=\; \limsup_{k\to+\infty} \langle u_k^*, u_k \rangle \;=\; \xi. \tag{2.97}$$

Let $\eta_{nm} = \langle u_n^* - u_m^*, u_n - u_m \rangle$ and $\eta_n = \langle u_n^*, u_n \rangle - \langle u_n^*, u \rangle - \langle u^*, u_n \rangle + \xi$. We have

$$\eta_{nm} \;\longrightarrow\; \eta_n \quad \text{as } m \to +\infty$$

(see (2.96)) and

$$\eta_n \;\longrightarrow\; 2\xi - 2\langle u^*, u \rangle$$

(see (2.97)). So, we can find an increasing (not necessarily strictly) sequence $\{n(m)\}_{m\geqslant 1}$ such that

$$\eta_{n(m)m} \;\longrightarrow\; 2\xi - 2\langle u^*, u \rangle \quad \text{as } m \to +\infty \tag{2.98}$$

(see Problem I.1.175). But from (2.96) we have

$$\eta_{n(m)m} \;\longrightarrow\; 0 \quad \text{as } m \to +\infty,$$

so $\xi = \langle u^*, u \rangle$ (see (2.98)), thus

$$\langle u_n^*, u_n \rangle \;\longrightarrow\; \langle u^*, u \rangle\,.$$

The monotonicity of A implies that

$$\langle u^* - h^*, u - h\rangle \;\geqslant\; 0 \qquad \forall\ (h, h^*) \in \operatorname{Gr} A,$$

so $(u, u^*) \in \operatorname{Gr} A$ (since A is maximal monotone; see Definition 2.100)).

Solution of Problem 2.163

By definition

$$A_\lambda(u) \;=\; \tfrac{1}{\lambda}\mathcal{F}(u - J_\lambda(u)) \qquad \forall\ u \in X.$$

So clearly A_λ is everywhere defined and single valued (see Proposition 2.114(a)). Moreover, from Problem 2.161, we see that

$$A_\lambda(u) \;\in\; A(J_\lambda(u)) \qquad \forall\ u \in X.$$

Now, let $u, v \in X$. We have

$$\begin{aligned} & \langle A_\lambda(u) - A_\lambda(v), u - v\rangle = \tfrac{1}{\lambda}\left\langle \mathcal{F}(u - J_\lambda(u)) - \mathcal{F}(v - J_\lambda(v)), u - v\right\rangle \\ = \;& \tfrac{1}{\lambda}\left\langle \mathcal{F}(u - J_\lambda(u)) - \mathcal{F}(v - J_\lambda(v)), u - J_\lambda(u) - (v - J_\lambda(v))\right\rangle \\ & + \left\langle A_\lambda(u) - A_\lambda(v), J_\lambda(u) - J_\lambda(v)\right\rangle, \end{aligned}$$

with $A_\lambda(u) \in A(J_\lambda(u))$ and $A_\lambda(v) \in A(J_\lambda(v))$. Therefore exploiting the monotonicity of A and $\mathcal{F}$ (see Proposition 2.114), we have

$$\langle A_\lambda(u) - A_\lambda(v), u - v\rangle \;\geqslant\; 0,$$

so A_λ is monotone (see Definition 2.98(a)).

For every $(h, h^*) \in \operatorname{Gr} A$, from the monotonicity of A, we have

$$\begin{aligned} & \langle h^*, J_\lambda(u) - h\rangle \;\leqslant\; \langle A_\lambda(u), J_\lambda(u) - h\rangle \\ & = -\tfrac{1}{\lambda}\left\langle \mathcal{F}(J_\lambda(u) - u), J_\lambda(u) - u\right\rangle - \tfrac{1}{\lambda}\left\langle \mathcal{F}(J_\lambda(u) - u), u - h\right\rangle \end{aligned} \tag{2.99}$$

(recall that $A_\lambda(u) \in A(J_\lambda(u))$ and see Problem 2.161), so

$$\|J_\lambda(u) - u\|_X^2 \;\leqslant\; -\lambda\left\langle h^*, J_\lambda(u) - h\right\rangle - \left\langle \mathcal{F}(J_\lambda(u) - u), u - h\right\rangle \tag{2.100}$$

(see Definition 2.112), thus

$$\|J_\lambda(u) - u\|_X \;\leqslant\; \lambda\|h^*\|_{X^*} + \|u - h\|_X,$$

hence

$$\|J_\lambda(u)\|_X \;\leqslant\; \lambda\|h^*\|_{X^*} + 2\|u\|_X + \|h\|_X,$$

which proves that J_λ is bounded (that is, maps bounded sets to bounded sets).

Since $A_\lambda(u) = \frac{1}{\lambda}\mathcal{F}(u - J_\lambda(u))$, it follows that A_λ is bounded too (see Definition 2.112).

Next let $\{u_n\}_{n\geqslant 1} \subseteq X$ be a sequence such that $u_n \longrightarrow u$ in X and set $y_n = J_\lambda(u_n)$ and $y_n^* = A_\lambda(u_n)$ for all $n \geqslant 1$. Then

$$\lambda y_n^* + \mathcal{F}(y_n - u_n) \;=\; 0 \qquad \forall\, n \geqslant 1. \tag{2.101}$$

So, we have

$$\begin{aligned}
& \langle \mathcal{F}(y_n - u_n) - \mathcal{F}(y_k - u_k), u_k - u_n \rangle \\
= \;& \langle \mathcal{F}(y_n - u_n) - \mathcal{F}(y_k - u_k), y_n - u_n - (y_k - u_k) \rangle \\
& + \lambda \langle y_n^* - y_k^*, y_n - y_k \rangle \qquad \forall\, n \geqslant 1.
\end{aligned} \tag{2.102}$$

Note that

$$\langle \mathcal{F}(y_n - u_n) - \mathcal{F}(y_k - u_k), u_k - u_n \rangle \;\longrightarrow\; 0 \quad \text{as } n, k \to +\infty$$

(see (2.101)). Moreover, both summands in the right-hand side of (2.102) are nonnegative (due to the monotonicity of $\mathcal{F}$ and A respectively). So, we obtain

$$\lim_{n,k\to+\infty} \langle \mathcal{F}(y_n - u_n) - \mathcal{F}(y_k - u_k), y_n - u_n - (y_k - u_k) \rangle \;=\; 0,$$

and

$$\lim_{n,k\to+\infty} \langle y_n^* - y_k^*, y_n - y_k \rangle \;=\; 0.$$

The boundedness of the maps J_λ, A_λ and $\mathcal{F}$ implies that by passing to a suitable subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } X, \quad y_n^* \xrightarrow{w} y^* \quad \text{in } X^* \quad \text{and} \quad \mathcal{F}(y_n - u_n) \xrightarrow{w} v^* \in X^*.$$

Then Problem 2.162 implies that

$$(y, y^*) \;\in\; \operatorname{Gr} A, \quad \mathcal{F}(y - u) \;=\; v^* \quad \text{and} \quad \lambda y^* + \mathcal{F}(y - u) \;=\; 0,$$

so

$$y \;=\; J_\lambda(u) \quad \text{and} \quad y^* \;=\; A_\lambda(u).$$

Therefore

$$J_\lambda(u_n) \xrightarrow{w} J_\lambda(u) \quad \text{in } X \quad \text{and} \quad A_\lambda(u_n) \xrightarrow{w} A_\lambda(u) \quad \text{in } X^*.$$

The second convergence proves the demicontinuity of A_λ. From the first convergence, Proposition 2.114 and the Kadec–Klee property, we have the continuity of J_λ.

Solution of Problem 2.164

From the solution of Problem 2.163 (see (2.99)), with $u = h$, we have

$$\|J_\lambda(u) - u\|_X \leqslant \lambda \|u^*\|_{X^*} \qquad \forall\ (u, u^*) \in \operatorname{Gr} A,$$

so

$$\|A_\lambda(u)\|_{X^*} = \tfrac{1}{\lambda}\|J_\lambda(u) - u\|_X \leqslant \|u^*\|_{X^*}$$

(see Remark after Problem 2.161 and Definition 2.112), thus

$$\|A_\lambda(u)\|_{X^*} \leqslant \|A^0(u)\|_{X^*} \qquad \forall\ \lambda > 0.$$

The reflexivity of X, implies that we can find a sequence $\lambda_n \searrow 0$ such that

$$A_{\lambda_n}(u) \xrightarrow{w} y^* \quad \text{in } X^*, \tag{2.103}$$

so

$$\|y^*\|_{X^*} \leqslant \|A^0(u)\|_{X^*}. \tag{2.104}$$

Also since

$$\|J_{\lambda_n}(u) - u\|_X \leqslant \sup_{n \geqslant 1} \lambda_n \|A^0(u)\|_{X^*} < +\infty, \tag{2.105}$$

we may assume that

$$\mathcal{F}(J_{\lambda_n}(u) - u) \xrightarrow{w} h^* \quad \text{in } X^*. \tag{2.106}$$

Then for all $v \in D(A)$ we have

$$\|J_{\lambda_n}(u) - u\|_X^2 \leqslant -\lambda_n \left\langle A^0(v), J_{\lambda_n}(u) - u \right\rangle - \langle \mathcal{F}(J_{\lambda_n}(u) - u), u - v \rangle$$

(cf. (2.100) in the solution of Problem 2.163), so

$$\limsup_{n\to+\infty} \|J_{\lambda_n}(u)-u\|_X^2 \leqslant \langle h^*, u-v\rangle$$

(see (2.105) and (2.106)). Evidently this is true for every $u, v \in \overline{\operatorname{conv}}\, D(A)$. Set $u = v$, to have

$$J_{\lambda_n}(u) \longrightarrow u \quad \text{in } X. \tag{2.107}$$

From the monotonicity of A, we have

$$0 \leqslant \langle w^* - A_{\lambda_n}(u), w - J_{\lambda_n}(u)\rangle \qquad \forall\ (w, w^*) \in \operatorname{Gr} A$$

(as $A_{\lambda_n}(u) \in A(J_{\lambda_n}(u))$; see Problem 2.163) so

$$0 \leqslant \langle w^* - y^*, w - u\rangle \qquad \forall\ (w, w^*) \in \operatorname{Gr} A$$

(see (2.103) and (2.107)), thus $y^* \in A(u)$ (since A is maximal monotone; see Definition 2.100)), hence

$$\|A^0(u)\|_{X^*} \leqslant \|y^*\|_{X^*} \leqslant \|A^0(u)\|_{X^*}$$

(see (2.104)), which implies that

$$\|A^0(u)\|_{X^*} = \|y^*\|_{X^*} \leqslant \|A^0(u)\|_{X^*}$$

and so $y^* = A^0(u)$. We conclude that $A_\lambda(u) \longrightarrow A^0(u)$ as $\lambda \searrow 0$ for all $u \in D(A)$.

Solution of Problem 2.165

Since the notion of maximal monotonicity is independent of the particular norms on X and X^*, invoking the Troyanski renorming theorem (see Theorem I.5.192 or Remark 2.115) without any loss of generality, we may assume that both X and X^* are locally uniformly convex (see Definition I.5.168(b) and Remark I.5.169). From Problem 2.164 we know that for every $u \in \overline{\operatorname{conv}}\, D(A)$, we have

$$J_\lambda(u) \longrightarrow u \quad \text{in } X \quad \text{as } \lambda \searrow 0.$$

Recall that $J_\lambda(u) \in D(A)$. Therefore $u \in \overline{D(A)}$. Hence $\overline{D(A)} = \overline{\operatorname{conv}}\, D(A)$ and thus the set $\overline{D(A)}$ is convex.

Note that $R(A) = D(A^{-1})$ and A^{-1} remains maximal monotone (recall that $A^{-1}(u^*) = \{u \in X : (u, u^*) \in \operatorname{Gr} A\}$). Therefore $\overline{R(A)} = \overline{D(A^{-1})} \subseteq X^*$ is convex too.

Solution of Problem 2.166

We argue indirectly. So, suppose that the conclusion does not hold. Then we can find a sequence $\{u_n\}_{n \geqslant 1} \subseteq X$ and $M > 0$ such that

$$|u_n| \longrightarrow +\infty \quad \text{and} \quad |A(u_n)| \leqslant M \qquad \forall\, n \geqslant 1.$$

We may assume that $A(u_n) \longrightarrow h^*$ in $\mathbb{R}^N$. Set $y_n = \frac{u_n}{|u_n|}$ for all $n \geqslant 1$. Then $|y_n| = 1$ for all $n \geqslant 1$ and so we may assume that $y_n \longrightarrow y$ in $\mathbb{R}^N$, $|y| = 1$. From the monotonicity of A, we have

$$0 \leqslant \big(A(u_n) - A(h), \tfrac{u_n}{|u_n|} - \tfrac{h}{|u_n|}\big)_{\mathbb{R}^N} \qquad \forall\, h \in \mathbb{R}^N,\ n \geqslant 1,$$

so

$$0 \leqslant (h^* - A(h), y)_{\mathbb{R}^N} \qquad \forall\, h \in \mathbb{R}^N. \tag{2.108}$$

Since by hypothesis A is surjective, for every $n \geqslant 1$, we can find $v_n \in \mathbb{R}^N$ such that $A(v_n) = ny$. Then from (2.108), we have

$$0 \leqslant (h^* - A(v_n), y)_{\mathbb{R}^N} \qquad \forall\, n \geqslant 1,$$

so

$$0 \leqslant (h^* - ny, y)_{\mathbb{R}^N} \qquad \forall\, n \geqslant 1,$$

thus

$$n|y|^2 \leqslant (h^*, y)_{\mathbb{R}^N} \leqslant |h^*||y|$$

and hence

$$n|y| \leqslant |h^*| \qquad \forall\, n \geqslant 1,$$

a contradiction since $|y| = 1$. Therefore $\lim\limits_{|u| \to +\infty} |A(u)| = +\infty$.

Solution of Problem 2.167

Arguing by contradiction, let $\{u_n\}_{n\geqslant 1} \subseteq \overline{B}_r$ be a sequence such that $u_n \longrightarrow u$ in $\mathbb{R}^N$ and $|A(u_n)| \longrightarrow +\infty$. Let $y_n = \frac{A(u_n)}{|A(u_n)|}$ for $n \geqslant 1$. Then $|y_n| = 1$ for all $n \geqslant 1$ and so we may assume that $y_n \longrightarrow y$ in $\mathbb{R}^N$, with $|y| = 1$. The monotonicity of A implies that

$$0 \leqslant \left(\frac{A(u_n)}{|A(u_n)|} - \frac{A(v)}{|A(u_n)|}, u_n - v\right)_{\mathbb{R}^N} = \left(y_n - \frac{A(v)}{|A(u_n)|}, u_n - v\right)_{\mathbb{R}^N}$$

for all $v \in B_{r+\varepsilon}$ and all $n \geqslant 1$, so

$$0 \leqslant (y, u - v)_{\mathbb{R}^N} \qquad \forall\, v \in B_{r+\varepsilon}$$

and thus

$$(y, v)_{\mathbb{R}^N} \leqslant (y, u)_{\mathbb{R}^N} \qquad \forall\, v \in B_{r+\varepsilon}.$$

Let $v = u + \frac{\varepsilon}{2} y$. Then we reach a contradiction. So, the set $A(\overline{B}_r) \subseteq \mathbb{R}^N$ is bounded.

Solution of Problem 2.168

From Problem 2.150 we know that A is maximal monotone (see Definition 2.100). So, J_1^A can be defined (see Definition 2.122) and $\operatorname{dom} J_1^A = L^2(T;H)$ (see Proposition 2.123(a)). Moreover, from the solution of Problem 2.150, we know that for every $u \in L^2(T;H)$, if

$$v(t) = e^{-t}u_0 + \int_0^t e^{t-s}u(s)\,ds \qquad \forall\, t \in [0,b],$$

then $v \in W^{1,2}(T;H)$, $v(0) = u_0$ and $u(t) = v(t) + v'(t)$ for almost all $t \in T$. Hence $u = (I_{L^2(T;H)} + A)v$, so $J_1^A(u) = v$ (see Definition 2.122 and Proposition 2.123).

Solution of Problem 2.169

Since A is everywhere defined and uniformly monotone (see Definition 2.98(d)), we have

$$\langle A(u) - A(v), u - v\rangle \geqslant \vartheta(|u - v|)|u - v| \qquad \forall\, u, v \in X, \quad (2.109)$$

with a strictly increasing function $\vartheta\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\vartheta(0) = 0$ and $\vartheta(r) \longrightarrow +\infty$ as $r \to +\infty$. Then $\vartheta^{-1}\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ exists and it is strictly increasing too, with $\vartheta^{-1}(0) = 0$. Let

$$u^* \;=\; A(u) \quad \text{and} \quad v^* \;=\; A(v) \quad \text{with } u, v \in X.$$

Then from (2.109), we have

$$\|A^{-1}(u^*) - A^{-1}(v^*)\|_X \;\leqslant\; \vartheta^{-1}(\|u^* - v^*\|_*),$$

so A^{-1} is continuous on $R(A)$. Let $B_r^* = \{u^* \in X^* :\ \|u^*\|_* < r\}$. Then for all $u^* \in B_r^*$ we have

$$\|A^{-1}(u^*)\|_X \;\leqslant\; \vartheta^{-1}(r) + \|A^{-1}(0)\|_X,$$

so A^{-1} is bounded from X^* into X. Since A is maximal monotone (see Corollary 2.111) we can apply Theorem 2.118 to conclude that A is surjective.

Solution of Problem 2.170

Let V be a finite dimensional subspace of X. Let $u \in D(A) \cap V$ and let U be a weakly open set in X^* containing $A(u)$. We need to show that $A|_V^+(U) = \{y \in V :\ A(y) \subseteq U\}$ is open in V (see Definition 2.36). Arguing by contradiction, suppose that $A|_V^+(U)$ is not open in V. Then we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq V$ such that $u_n \longrightarrow u$ and $A(u_n) \cap U^c \neq \emptyset$ for all $n \geqslant 1$. Let $u_n^* \in A(u_n) \cap U^c$. The boundedness of A implies that the sequence $\{u^*\}_{n\geqslant 1} \subseteq X^*$ is bounded. Then the reflexivity of X (hence of X^* too) and the Eberlein–Smulian theorem (see Theorem I.5.78) imply that at least for a subsequence, we have $u_n^* \xrightarrow{w} u^*$ in X^*. Then

$$\langle u_n^*, u_n - u\rangle \;\longrightarrow\; 0,$$

so $(u, u^*) \in \operatorname{Gr} A$ (see the condition which is assumed on A). Also U^c is w-closed in X^*, hence $u^* \in U^c$. Therefore $u^* \in A(u) \cap U^c$, a contradiction to the fact that $A(u) \subseteq U$. This proves the desired upper semicontinuity property of the map A.

Solution of Problem 2.171

Let $u \in X$ and $u^* \in X^*$ be such that

$$0 \leqslant \langle u^* - h^*, u - h \rangle \qquad \forall\, (h, h^*) \in \operatorname{Gr} A. \tag{2.110}$$

Given $v^* \in (A + C)(v)$, we have

$$v^* = h^* + C(v), \tag{2.111}$$

for some $h^* \in A(v)$. Then we have

$$\begin{aligned} \langle u^* + C(u) - v^*, u - v \rangle &= \langle u^* + C(u) - h^* - C(v), u - v \rangle \\ = \langle u^* - h^*, u - v \rangle + \langle C(u) - C(v), u - v \rangle &\geqslant 0 \end{aligned} \tag{2.112}$$

(see (2.111) and (2.110)). Because by hypothesis $A + C$ is a maximal monotone map (see Definition 2.100), from (2.112) we infer that

$$u \in D(A + C) \quad \text{and} \quad u^* + C(u) \in (A + C)(u).$$

This implies that $u^* \in A(u)$ and so we have the maximal monotonicity of A (see (2.110)).

Solution of Problem 2.172

The monotonicity of $\widehat{A}$ is a direct consequence of the monotonicity of A. Then according to Theorem 2.116, in order to show maximality of $\widehat{A}$, it suffices to show that $R(\widehat{A} + I_{L^2(\Omega;H)}) = L^2(\Omega; H)$. So, let $h \in L^2(\Omega; H)$ and exploiting the maximal monotonicity of A, set

$$v(\omega) = (I_H + A)^{-1}(h(\omega)) \qquad \forall\, \omega \in \Omega.$$

Evidently v is Σ-measurable. Also, by hypothesis $(0, 0) \in \operatorname{Gr} A$, we have $J_1^A(0) = 0$ and so from Proposition 2.123(b), it follows that

$$\|v(\omega)\|_H \leqslant \|h(\omega)\|_H \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

so $v \in L^2(\Omega; H)$. Therefore $R(A + I_H) = L^2(\Omega; H)$ and we have proved the maximality of $\widehat{A}$. From the above argument it is clear that for every $\lambda > 0$, we have

$$J_\lambda^{\widehat{A}}(u)(\cdot) = J_\lambda^A(u(\cdot)) \qquad \forall\, u \in L^2(\Omega; H).$$

Hence we infer that

$$\widehat{A}_\lambda(u)(\cdot) \;=\; A_\lambda(u(\cdot)) \qquad \forall\, u \in L^2(\Omega; H)$$

(see Definition 2.122). Finally, if $\mu(\Omega) < +\infty$, then constant functions belong to $L^2(\Omega; H)$ and so from the fact that J_1^A is nonexpansive (see Proposition 2.123(b)), we have that $v \in L^2(\Omega; H)$. So, in that case we can drop the requirement that $(0,0) \in \operatorname{Gr} A$.

Solution of Problem 2.173

No. We always have $R(A+C) \subseteq R(A) + R(C)$ and the inclusion can be strict. For example, let $H = \mathbb{R}^2$ and let $A, C \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the operators that perform rotation by $\frac{\pi}{2}$ and by $-\frac{\pi}{2}$ respectively. Then $R(A+C) = \{0\}$, while $R(A) + R(C) = \mathbb{R}^2$.

Solution of Problem 2.174

Without any loss of generality we may assume that both X and X^* are locally uniformly convex (by the Troyanski renorming theorem; see Theorem I.5.192 or Remark 2.115) and so the duality map $\mathcal{F} \colon X \longrightarrow X^*$ is a homeomorphism (see Proposition 2.114). Let $\xi \colon X \times X^* \longrightarrow X^* \times X$ be the map defined by $\xi(u, u^*) = (u^* + \mathcal{F}(u), u)$. Then ξ is a homeomorphism. For every $\omega \in \Omega$, we have

$$\xi(\operatorname{Gr} A(\omega)) \;=\; \operatorname{Gr}\,(A(\omega) + \mathcal{F})^{-1}.$$

Then property **(a)** is equivalent to saying that

$$\begin{aligned} &\big\{(\omega, u^*, u) \in \Omega \times X^* \times X : u = (A(\omega) + \mathcal{F})^{-1}(u^*)\big\} \\ &\qquad \in \Sigma \times \mathcal{B}(X^*) \times \mathcal{B}(X) \;=\; \Sigma \times \mathcal{B}(X^* \times X), \end{aligned}$$

which is precisely property **(b)**.

Solution of Problem 2.175

(1) Let $E \in \mathcal{B}(X)$. Then $E \times X^* \in \mathcal{B}(X) \times \mathcal{B}(X^*) = \mathcal{B}(X \times X^*)$. We have

$$\begin{aligned} D^-(E) &= \{\omega \in \Omega : \ D(\omega) \cap E \neq \emptyset\} \\ &= \{\omega \in \Omega : \ \operatorname{Gr} A(\omega) \cap (E \times X^*) \in \Sigma\} \end{aligned}$$

(see property **(a)** in Problem 2.174).

(2) Let $E^* \in \mathcal{B}(X^*)$. Then $\{u\} \times E^* \in \mathcal{B}(X) \times \mathcal{B}(X^*) = \mathcal{B}(X \times X^*)$ and so

$$\begin{aligned} &\{\omega \in \Omega : \ A(\omega)(u) \cap E^* \neq \emptyset\} \\ &= \{\omega \in \Omega : \ \operatorname{Gr} A(\omega) \cap (\{u\} \times E^*) \neq \emptyset\} \ \in \ \Sigma, \end{aligned}$$

so the multifunction $\omega \longmapsto A(\omega)(u)$ is measurable from Ω into X^*.

(3) Fix $u \in X$. We have

$$\Omega(u) \ = \ \{\omega \in \Omega : \ A(\omega)(u) \cap X^* \neq \emptyset\},$$

so $\Omega(u) \in \Sigma$ (see part **(2)**). Then the multifunction $\omega \longmapsto A(\omega)(u)$ is measurable form $\Omega(u)$ into $P_{fc}(X^*)$. So, Theorem 2.66 implies the existence of a sequence $\{v_n^* \colon \Omega(u) \longrightarrow X^*\}_{n \geqslant 1}$ of $\Sigma \cap \Omega(u)$-measurable functions such that

$$A(\omega)(u) \ = \ \overline{\{v_n^*(\omega)\}}_{n \geqslant 1} \qquad \forall\, \omega \in \Omega.$$

Then we have

$$\|A^0(\omega)(u)\|_* \ = \ \inf_{n \geqslant 1} \|v_n^*(\omega)\|_* \qquad \forall\, \omega \in \Omega,$$

so the map $\omega \longmapsto \|A^0(\omega)(u)\|_*$ is $\Sigma \cap \Omega(u)$-measurable. Note that

$$\begin{aligned} \operatorname{Gr} A^0(\cdot)(u) \ &= \ \big\{(\omega, u^*) \in \Omega(u) \times X^* : \ u^* \in A(\omega)(u), \\ &\qquad \|u^*\|_* = \|A^0(\omega)(u)\|_*\big\} \\ &\in \ (\Sigma \cap \Omega(u)) \times \mathcal{B}(X^*) \end{aligned}$$

(see part **(2)**), so $\omega \longmapsto A^0(\omega)(u)$ is $\Sigma \cap \Omega(u)$-measurable (since Σ is complete).

Solution of Problem 2.176

Let $\{u_n^*\}_{n\geqslant 1} \subseteq A(C)$ be a sequence such that $u_n^* \longrightarrow u^*$ in X^*. We have that $u_n^* \in A(u_n)$ with $u_n \in C$. Recall that A^{-1} is maximal monotone (see Definition 2.100) from $R(A) = X^*$ (recall that A is surjective) onto X. Proposition 2.104 implies that the sequence $\left\{A^{-1}(u_n^*)\right\}_{n\geqslant 1}$ is bounded. Hence the sequence $\{u_n\}_{n\geqslant 1} \subseteq C$ is bounded and because X is reflexive, by passing to a suitable subsequence if necessary, we may assume that $u_n \xrightarrow{w} u$ in X. Since the set $C \subseteq X$ is closed and convex, it is also weakly closed (by the Mazur theorem; see Theorem I.5.58) and so $u \in C$. Also, by virtue of Problem 2.149, we have that $(u, u^*) \in \operatorname{Gr} A$ and so finally $u^* \in A(C)$. This proves that the set $A(C) \subseteq X^*$ is closed.

Solution of Problem 2.177

For any $\widehat{u} \in D(A) \cap D(C)$, the elements $h_\lambda^* \in (A + C_\lambda + \mathcal{F})(\widehat{u})$ for $\lambda > 0$, are bounded in $\lambda \in (0, 1]$ since

$$\|C_\lambda(\widehat{u})\|_* \leqslant \|C^0(\widehat{u})\|_*$$

(see the solution of Problem 2.164). Exploiting the monotonicity of the maps A and C_λ (see Problem 2.163), we have

$$(\|u_\lambda\|_X - \|\widehat{u}\|_X)^2 \leqslant \langle h^* - h_\lambda^*, u_\lambda - \widehat{u}\rangle ,$$

so

$$\|u_\lambda\|_X^2 \leqslant (\|h^* - h_\lambda^*\|_* + 2\|\widehat{u}\|_X)\,\|u_\lambda\|_X + (\|h^* - h_\lambda^*\|_* - \|\widehat{u}\|_X)\,\|\widehat{u}\|_X,$$

hence the family $\{u_\lambda\}_{\lambda>0}$ remains bounded in X as $\lambda \searrow 0$.

Solution of Problem 2.178

"**(a) $\Longrightarrow$ (b)**": Let $\lambda > 0$ and let $u_\lambda^* \in \mathcal{F}(u + \lambda y)$. We set $h_\lambda^* = \frac{u_\lambda^*}{\|u_\lambda^*\|_*}$. We have

$$\begin{aligned} \|u\|_X &\leqslant \|u + \lambda y\|_X = \frac{\langle u_\lambda^*, u+\lambda y\rangle}{\|u_\lambda^*\|_*} = \langle h_\lambda^*, u + \lambda y\rangle \\ &= \langle h_\lambda^*, u\rangle + \lambda \langle h_\lambda^*, y\rangle \leqslant \|u\|_X + \lambda \langle h_\lambda^*, y\rangle . \end{aligned} \tag{2.113}$$

Since $\|h_\lambda^*\|_* = 1$, the net $\{h_\lambda^*\}_{\lambda>0}$ admits a w^*-cluster point h^*, with $\|h^*\|_* \leqslant 1$. Also, we have

$$\|u\|_X \leqslant \langle h^*, u\rangle \quad \text{and} \quad \langle h^*, y\rangle \geqslant 0 \tag{2.114}$$

(see (2.113)). Therefore

$$\|u\|_X \leqslant \langle h^*, u\rangle \leqslant \|u\|_X$$

(see (2.114) and recall that $\|h^*\|_* \leqslant 1$), so $\|u\|_X = \langle h^*, u\rangle$ and thus $\|h^*\|_* = 1$. Then

$$u^* = \|u\| h^* \in \mathcal{F}(u) \quad \text{and} \quad \langle u^*, y\rangle \geqslant 0.$$

"**(b)** $\Longrightarrow$ **(a)**": For every $\lambda > 0$, we have

$$\begin{aligned} \|u\|_X^2 &= \langle u^*, u\rangle \leqslant \langle u^*, u\rangle + \lambda \langle u^*, y\rangle = \langle u^*, u+\lambda y\rangle \\ &\leqslant \|u^*\|_* \|u+\lambda y\|_X = \|u\|_X \|u+\lambda y\|_X, \end{aligned}$$

so $\|u\|_X \leqslant \|u+\lambda y\|_X$ for all $\lambda > 0$.

Solution of Problem 2.179

"$\Longrightarrow$": For every $u \in C$ and every $\lambda \in [0,1]$, we have

$$\|u_0\|_X \leqslant \|u_0 + \lambda(u-u_0)\|_X$$

(recall that C is convex). Invoking Problem 2.178 and since $\mathcal{F}$ is single valued (see Proposition 2.114(a)), we have

$$\langle \mathcal{F}(u_0), u - u_0\rangle \geqslant 0 \qquad \forall\, u \in C,$$

so $\langle \mathcal{F}(u_0), u_0\rangle \leqslant \langle \mathcal{F}(u_0), u\rangle$ for all $u \in C$.

"$\Longleftarrow$": We have

$$\|u_0\|_X^2 = \langle \mathcal{F}(u_0), u_0\rangle \leqslant \langle \mathcal{F}(u_0), u\rangle \leqslant \|u_0\|_X \|u\|_X \qquad \forall\, u \in C,$$

so

$$\|u_0\|_X \leqslant \|u\|_X \qquad \forall\, u \in C.$$

Solution of Problem 2.180

From Problem 2.161, we know that

$$(J_{\lambda_n}(u_n), A_{\lambda_n}(u_n)) \in \operatorname{Gr} A \qquad \forall\, n \geqslant 1.$$

Note that the boundedness of the sequence $\{A_{\lambda_n}(u_n)\}_{n\geqslant 1} \subseteq X^*$ implies that

$$\|J_{\lambda_n}(u_n) - u_n\|_X \longrightarrow 0$$

(see the solution of Problem 2.164). Hence Problem 2.162 implies that

$$(u, y) \in \operatorname{Gr} A \quad \text{and} \quad \limsup_{n,m\to+\infty} \langle A_{\lambda_n}(u_n) - A_{\lambda_m}(u_m), u_n - u_m\rangle = 0.$$

Solution of Problem 2.181

From Problem 2.177 and the hypothesis, we know that we can find $M > 0$ and $\lambda^* > 0$ such that

$$\|C_\lambda(u_\lambda)\|_* \leqslant M \quad \text{and} \quad \|u_\lambda\|_* \leqslant M \qquad \forall\, \lambda \in (0, \lambda^*). \tag{2.115}$$

So, if $\lambda_n \searrow 0$, then because of (2.115) and the reflexivity of X, we may assume that

$$C_n(u_n) = C_{\lambda_n}(u_{\lambda_n}) \xrightarrow{w} v^* \quad \text{in } X^*, \quad \mathcal{F}(u_n) \xrightarrow{w} g^* \quad \text{in } X^* \tag{2.116}$$

and

$$u_n = u_{\lambda_n} \xrightarrow{w} u \quad \text{in } X. \tag{2.117}$$

We have

$$h^* = u_n^* + C_n(u_n) + \mathcal{F}(u_n) \quad \text{with } u_n^* \in A(u_n) \qquad \forall\, n \geqslant 1. \tag{2.118}$$

For $n, m \geqslant 1$, we have

$$\langle u_n^* - u_m^*, u_n - u_m\rangle + \langle C_n(u_n) - C_m(u_m), u_n - u_m\rangle + \langle \mathcal{F}(u_n) - \mathcal{F}(u_m), u_n - u_m\rangle = 0. \tag{2.119}$$

Because of the monotonicity of the duality map (see Proposition 2.114) and the monotonicity of A (by hypothesis), we have

$$\limsup_{n,m\to+\infty} \langle C_n(u_n) - C_m(u_m), u_n - u_m\rangle \leqslant 0. \tag{2.120}$$

Then from (2.120) and Problem 2.180, we have

$$(u, v^*) \in \operatorname{Gr} C \quad \text{and} \quad \lim_{n,m\to+\infty} \langle C_n(u_n) - C_m(u_m), u_n - u_m\rangle = 0. \tag{2.121}$$

From (2.119) and (2.121), it follows that

$$\lim_{n,m\to+\infty} \langle u_n^* + \mathcal{F}(u_n) - (u_m^* + \mathcal{F}(u_m)), u_n - u_m\rangle = 0,$$

so

$$\limsup_{n,m\to+\infty} \langle \mathcal{F}(u_n) - \mathcal{F}(u_m), u_n - u_m\rangle \leqslant 0 \tag{2.122}$$

(since A is monotone). From (2.122) and Problem 2.162, we infer that

$$g^* = \mathcal{F}(u) \quad \text{and} \quad \lim_{n,m\to+\infty} \langle u_n^* - u_m^*, u_n - u_m\rangle = 0. \tag{2.123}$$

From (2.115) and (2.118), we see that the sequence $\{u_n^*\}_{n\geqslant 1} \subseteq X^*$ is bounded and so, passing to a subsequence if necessary, we may assume that

$$u_n^* \xrightarrow{w} u^* \quad \text{in } X^*. \tag{2.124}$$

Using (2.123), (2.124) and Problem 2.162, we have

$$(u, u^*) \in \operatorname{Gr} A. \tag{2.125}$$

Passing to the limit as $n \to +\infty$ in (2.118) and using (2.121), (2.123) and (2.125), we obtain

$$h^* = u^* + v^* + \mathcal{F}(u),$$

with $u^* \in A(u)$, $v^* \in C(u)$. Therefore $h^* \in R(A + C + \mathcal{F})$.

Solution of Problem 2.182

We need to show that for every $\lambda \geqslant 0$, the sublevel set

$$L_\lambda = \{u \in X : m(u) \leqslant \lambda\}$$

is closed. To this end, let $\{u_n\}_{n\geqslant 1} \subseteq L_\lambda$ be a sequence such that $u_n \longrightarrow u$ in X. Since A is maximal monotone (see Definition 2.100) for every $n \geqslant 1$, we have $A(u_n) \in P_{fc}(X^*)$ (see Corollary 2.107). Since

X^* is reflexive (recall that X is reflexive), we can find $u_n^* \in A(u_n)$ such that $m(u_n) = \|u_n^*\|_*$ for all $n \geqslant 1$. We have $\|u_n^*\|_* \leqslant \lambda$ for all $n \geqslant 1$ and so by passing to a suitable subsequence if necessary, we may assume that $u_n^* \xrightarrow{w} u^*$ in X^*. From Problem 2.149 we have $(u, u^*) \in \operatorname{Gr} A$, from the weak lower semicontinuity of the norm functional (see Proposition I.5.56(c)), we have

$$\|u^*\|_* \leqslant \liminf_{n\to+\infty} \|u_n^*\|_* = \liminf_{n\to+\infty} m(u_n) \leqslant \lambda,$$

so $m(u) \leqslant \lambda$ (since $u^* \in A(u)$), thus $u \in L_\lambda$ and hence L_λ is closed. This proves lower semicontinuity of m.

Solution of Problem 2.183

Let $u \in X$ and $u^* \in X^*$ be such that

$$0 \leqslant \langle u^* - h^*, u - h\rangle \qquad \forall\ (h, h^*) \in \operatorname{Gr} A. \tag{2.126}$$

From (2.126) and the monotonicity of C, we have

$$0 \leqslant \langle u^* + C(u) - (h^* + C(h)), u - h\rangle \qquad \forall\ (h, h^*) \in \operatorname{Gr} A. \tag{2.127}$$

Let $v^* \in X^*$, $t > 0$, $u_t^* = u^* + C(u) + tv^*$, $u_t = (A + C)^{-1}(u_t^*)$. Since $R(A + C) = X^*$, we can find $y_t^* \in A(u_t)$ such that $y_t^* + C(u_t) = u_t^*$. In (2.127) we choose $h = u_t$ and $h^* = y_t^*$. Then

$$0 \leqslant \langle v^*, u_t - u\rangle\,.$$

But by hypothesis $(A + C)^{-1}$ is continuous. So, if $t \searrow 0$, then

$$0 \leqslant \langle v^*, u_0 - u\rangle \quad \text{with } u_0 = (A + C)^{-1}(u^* + C(u)). \tag{2.128}$$

Since $v^* \in X^*$ is arbitrary, from (2.128) we infer that $u_0 = u$. Therefore $u^* \in A(u)$, which by virtue of (2.126) implies the maximal monotonicity of A.

Solution of Problem 2.184

"$\Longrightarrow$": If X is reflexive, then $\overline{B}_1 = \{u \in X : \|u\|_X \leqslant 1\}$ is w-compact. So, given $u^* \in X^*$, we can find $u_0 \in \overline{B}_1$ such that

$$\langle u^*, u_0 \rangle \;=\; \sup\{\langle u^*, u\rangle : \|u\|_X \leqslant 1\} \;=\; \|u^*\|_*,$$

so $u^* \in \mathcal{F}(\|u^*\|_* u_0)$ (see Definition 2.112), hence $R(\mathcal{F}) = X^*$.

"$\Longleftarrow$": Suppose that X is not reflexive Then $\overline{B}_1$ is not w-compact and so by the James theorem (see Theorem I.5.74) we can find $u_0^* \in X^*$ such that $\sup\big\{\langle u_0^*, u\rangle : \|u\|_X \leqslant 1\big\}$ is not attained. Then $u_0^* \notin R(\mathcal{F})$ and so $R(\mathcal{F}) \neq X^*$.

Solution of Problem 2.185

From Proposition 2.114, we know that $\mathcal{F}$ is single valued. Let $u_n \xrightarrow{w} u$ in X and assume that

$$\limsup_{n\to+\infty} \langle \mathcal{F}(u_n), u_n - u\rangle \;\leqslant\; 0. \tag{2.129}$$

We have

$$\left(\|u_n\|_X - \|u\|_X\right)^2 \;\leqslant\; \langle \mathcal{F}(u_n) - \mathcal{F}(u), u_n - u\rangle\,,$$

so

$$\|u_n\|_X \;\longrightarrow\; \|u\|_X$$

(see (2.129) and recall that $u_n \xrightarrow{w} u$ in X), thus $u_n \longrightarrow u$ (by the Kadec–Klee property; see Corollary 1.26) and hence $\mathcal{F}$ is of type $(S)_+$ (see Definition 2.137).

Solution of Problem 2.186

According to Definition 2.131 we consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $u_n \xrightarrow{w} u$ in X, $T(u_n) \xrightarrow{w} u^*$ in X^* and $\limsup\limits_{n\to+\infty} \langle T(u_n), u_n - u\rangle \leqslant 0$.

Since K is compact, the sequence $\{K(u_n)\}_{n\geqslant 1} \subseteq X^*$ is relatively compact and so by passing to a suitable subsequence if necessary, we may assume that

$$K(u_n) \longrightarrow v^* \quad \text{in } X^*. \tag{2.130}$$

Then we have

$$A(u_n) \xrightarrow{w} u^* - v^* \quad \text{in } X^*.$$

Also

$$\langle K(u_n), u_n - u\rangle \longrightarrow 0$$

(see (2.130)) and

$$\limsup_{n\to+\infty} \langle A(u_n), u_n - u\rangle \leqslant 0 \tag{2.131}$$

(from the choice of the sequence $\{u_n\}_{n\geqslant 1}$), so

$$u_n \longrightarrow u \quad \text{in } X$$

(since A is of type $(S)_+$; see Definition 2.137 and (2.131)). Then we have $K(u_n) \longrightarrow K(u)$ (since K is compact, hence continuous too) and $A(u_n) \xrightarrow{w} A(u)$ in X^* (since A is demicontinuous). Therefore $v^* = A(u) + K(u) = T(u)$ and we have that

$$\langle T(u_n), u_n\rangle \longrightarrow \langle T(u), u\rangle\,,$$

that is, T is generalized pseudomonotone (see Definition 2.131).

Solution of Problem 2.187

Let $T = A + K$. By hypothesis we have that $u_n \xrightarrow{w} u$ in X, $T(u_n) \xrightarrow{w} u^*$ in X^* and

$$\limsup_{n\to+\infty} \langle T(u_n), u_n - u\rangle \leqslant 0. \tag{2.132}$$

From the hypothesis on K, we have that $K(u_n) \xrightarrow{w} K(u)$ in X^*. Then

$$A(u_n) \xrightarrow{w} u^* - K(u) \quad \text{in } X^*. \tag{2.133}$$

Also we have

$$\begin{aligned}\limsup_{n\to+\infty}\langle A(u_n),u_n\rangle &= \limsup_{n\to+\infty}\big(\langle T(u_n),u_n\rangle-\langle K(u_n),u_n\rangle\big)\\ &\leqslant \limsup_{n\to+\infty}\langle T(u_n),u_n\rangle-\liminf_{n\to+\infty}\langle K(u_n),u_n\rangle\\ &\leqslant \langle u^*-K(u),u\rangle\end{aligned}$$

(see (2.132) and use the semicontinuity of φ), so

$$\begin{aligned}\limsup_{n\to+\infty}\langle A(u_n),u_n-u\rangle &\leqslant \limsup_{n\to+\infty}\langle A(u_n),u_n\rangle-\liminf_{n\to+\infty}\langle A(u_n),u\rangle\\ &\leqslant \langle u^*-K(u),u\rangle-\langle u^*-K(u),u\rangle \;=\; 0\end{aligned}$$

(see (2.133)). But recall that A is generalized pseudomonotone. Hence from the above inequality and Definition of 2.131, it follows that $A(u)=u^*-K(u)$, so $u^*=(A+K)(u)$.

Solution of Problem 2.188

Let $\{u_n^*\}_{n\geqslant 1}\subseteq A(C)$ be a sequence such that $u_n^*\longrightarrow u^*$ in X^*. Then we have

$$u_n^* \;\in\; A(u_n) \quad \text{with } u_n\in C \qquad \forall\, n\geqslant 1.$$

The set C is w-compact in X since the space X is reflexive. So, by the Eberlein–Smulian theorem (see Theorem I.5.78) we may assume that $u_n \xrightarrow{w} u$ in X. We have $u\in C$ and

$$\lim_{n\to+\infty}\langle u_n^*,u_n-u\rangle \;=\; 0.$$

Since A is generalized pseudomonotone (see Definition 2.131), we have $(u,u^*)\in \operatorname{Gr} A$, hence $u^*\in A(u)$ with $u\in C$, which proves that the set $A(C)\subseteq X$ is closed.

Solution of Problem 2.189

According to Problem 2.186, it suffices to show that the identity map I_H is of type $(S)_+$ (see Definition 2.137). So, let $\{u_n\}_{n\geqslant 1} \subseteq H$ be a sequence such that $u_n \xrightarrow{w} u$ in H and

$$\limsup_{n\to+\infty} (u_n, u_n - u)_H \ \leqslant\ 0.$$

So, using also the weak lower semicontinuity of the norm functional (see Proposition I.5.56(c)), we have

$$\|u\|_H^2 \ \leqslant\ \liminf_{n\to+\infty} \|u_n\|_H^2 \ \leqslant\ \limsup_{n\to+\infty} \|u_n\|_H^2 \ \leqslant\ \|u\|_H^2,$$

thus $\|u_n\|_H \longrightarrow \|u\|_H$ and from the Kadec–Klee property in Hilbert spaces (see Corollary 1.26), we get $u_n \longrightarrow u$. Therefore I_H is of type $(S)_+$ and so we can use Problem 2.186 to conclude that the map $u \longmapsto u + K(u)$ is generalized pseudomonotone (see Definition 2.131).

Solution of Problem 2.190

No. Let $X = H = l^2$. This is a Hilbert space and $X^* = H = l^2$. We consider the map $K\colon l^2 \longrightarrow l^2$ defined by

$$K(\widehat{u}) \ =\ (2 - \|\widehat{u}\|_{l^2}, 0, 0, \ldots) \qquad \forall\, \widehat{u} = \{u_k\}_{k\geqslant 1} \in l^2.$$

Evidently K is of finite rank (see Definition 2.1(c)) and so it is compact (see Remark 2.2). Let $\widehat{u}_n = \{u_k^n\}_{k\geqslant 1} = \{\delta_{nk}\}_{k\geqslant 1}$, with δ_{nk} being the Kronecker symbol, i.e.,

$$\delta_{nk} \ =\ \begin{cases} 1 & \text{if} \quad n = k, \\ 0 & \text{if} \quad n \neq k, \end{cases} \qquad n, k \geqslant 1.$$

Then $\widehat{u}_n \in l^2$ for all $n \geqslant 1$ and $\widehat{u}_n \xrightarrow{w} \widehat{u} = 0$ in l^2. Also

$$K(\widehat{u}_n) \ \longrightarrow\ \widehat{h} = (1, 0, 0, \ldots) \quad \text{in } l^2$$

and

$$(K(\widehat{u}_n), \widehat{u}_n)_H \ \longrightarrow 0.$$

However, $\widehat{h} \neq K(\widehat{u}) = (2,0,0,\ldots)$ and so K is not generalized pseudomonotone (see Definition 2.131).

Solution of Problem 2.191

Let $u_n \xrightarrow{w} u$ in X. The monotonicity of A implies that

$$\langle A(u), u_n - u\rangle \leqslant \langle A(u_n), u_n - u\rangle \qquad \forall\, n \geqslant 1,$$

so

$$0 \leqslant \liminf_{n\to+\infty} \langle A(u_n), u_n - u\rangle\,. \tag{2.134}$$

From the sequential weak continuity of A, we have

$$\liminf_{n\to+\infty} \langle A(u_n), u_n - u\rangle = \liminf_{n\to+\infty} \langle A(u_n), u_n\rangle - \langle A(u), u\rangle\,,$$

so

$$\varphi(u) \leqslant \liminf_{n\to+\infty} \varphi(u_n)$$

(see (2.134)), thus φ is sequentially weakly lower semicontinuous.

Solution of Problem 2.192

The maximal monotonicity of A can be established similarly as in the solution of Problem 2.153. So, we need to show that A is of type $(S)_+$ (see Definition 2.137). To this end, let $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ be such that

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n\to+\infty} \langle A(u_n), u_n - u\rangle \leqslant 0. \tag{2.135}$$

From (2.135) we have

$$\limsup_{n\to+\infty} \langle A(u_n) - A(u), u_n - u\rangle \leqslant 0. \tag{2.136}$$

The monotonicity of A implies that

$$\liminf_{n\to+\infty} \langle A(u_n) - A(u), u_n - u\rangle \geqslant 0. \tag{2.137}$$

From (2.136) and (2.137), we have

$$\lim_{n\to+\infty} \langle A(u_n) - A(u), u_n - u\rangle \;=\; 0. \tag{2.138}$$

Let

$$\xi_n(z) \;=\; (a(z, Du_n(z)) - a(z, Du(z)), Du_n(z) - Du(z))_{\mathbb{R}^N}\,.$$

From the convexity of $G(z,\cdot)$, we have that $a(z,\cdot)$ is monotone (see Definition 2.98(a)) and so

$$\xi_n(z) \;\geqslant\; 0 \quad \text{for a.a. } z\in\Omega,\ \text{all } n\geqslant 1.$$

Also, from (2.138), we have

$$\int\limits_\Omega \xi_n(z)\,dz \;\longrightarrow\; 0,$$

so

$$\xi_n \;\longrightarrow\; 0 \quad \text{in } L^1(\Omega). \tag{2.139}$$

From (2.139) and by passing to a suitable subsequence if necessary, we may assume that

$$\xi_n(z) \;\longrightarrow\; 0 \quad \text{for a.a. } z\in\Omega \tag{2.140}$$

and

$$\xi_n(z) \;\leqslant\; k(z) \quad \text{for a.a. } z\in\Omega, \tag{2.141}$$

with $k\in L^1(\Omega)$. From the convexity of $G(z,\cdot)$, we have

$$(a(z,y),y)_{\mathbb{R}^N} \;\geqslant\; G(z,y) \;\geqslant\; \tfrac{c_0}{p}|y|^p \quad \text{for a.a. } z\in\Omega,\ \text{all } y\in\mathbb{R}^N \tag{2.142}$$

(see hypothesis **(v)**). Using (2.140)–(2.142) and hypothesis **(iii)**, for almost all $z\in\Omega$ and all $n\geqslant 1$, we have

$$\begin{aligned} k(z) \;&\geqslant\; \xi_n(z) \;=\; (a(z,Du_n(z)) - a(z,Du(z)), Du_n(z) - Du(z))_{\mathbb{R}^N} \\ &\geqslant\; \tfrac{c_0}{p}\left(|Du_n(z)|^p + |Du(z)|^p\right) \\ &\quad - \big(\widehat{a}(z)(1+|Du(z)|^{p-1})\big)\,|Du_n(z)| \\ &\quad - \big(\widehat{a}(z)(1+|Du_n(z)|^{p-1})\big)\,|Du(z)|. \end{aligned} \tag{2.143}$$

From (2.143) it follows that for all $z\in\Omega\setminus D$ with $|D|_N=0$ (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$), the sequence $\{Du_n(z)\}_{n\geqslant 1}\subseteq$

$\mathbb{R}^N$ is bounded. Hence, by passing to a suitable subsequence if necessary (in general the subsequence depends on $z \in \Omega \setminus D$), we will have

$$Du_n(z) \longrightarrow \eta(z) \quad \text{in } \mathbb{R}^N \qquad \forall\, z \in \Omega \setminus D,$$

so

$$a(z, Du_n(z)) \longrightarrow a(z, \eta(z)) \qquad \forall\, z \in \Omega \setminus D_1,$$

with $D_1 \supseteq D$, $|D_1|_N = 0$. Passing to the limit as $n \to +\infty$ and using (2.140)–(2.141), we obtain

$$(a(z, \eta(z)) - a(z, Du(z)), \eta(z) - Du(z))_{\mathbb{R}^N} = 0 \qquad \forall\, z \in \Omega \setminus D_1. \tag{2.144}$$

Since $G(z, \cdot)$ is strictly convex, we have that $a(z, \cdot)$ is strictly monotone. So, from (2.144) it follows that $\eta(z) = Du(z)$ for almost all $z \in \Omega$. So, by the Urysohn criterion for the convergence of sequences (see Problem I.1.3), for the original sequence, we have

$$Du_n(z) \longrightarrow Du(z) \quad \text{for a.a. } z \in \Omega. \tag{2.145}$$

Let $E \subseteq \Omega$ be a Lebesgue measurable set. Then from (2.143), using the Hölder inequality (see Theorem 1.3 and Problem 1.27) and the boundedness of the sequence $\{Du_n\}_{n\geqslant 1} \subseteq L^p(\Omega; \mathbb{R}^N)$ (see (2.135)), we have

$$\frac{c_0}{p} \int\limits_E |Du_n|^p \, dz \leqslant c_2\Big(\int\limits_E |Du| \, dz + |C|_N^{\frac{1}{p'}} + \Big(\int\limits_E |Du|^p \, dz \Big)^{\frac{1}{p}}$$
$$+ \Big(\int\limits_E |Du|^p \, dz \Big)^{\frac{1}{p'}} \Big)$$

for some $c_2 >$ (here $\frac{1}{p} + \frac{1}{p'} = 1$). So the sequence $\{|Du_n|^p\}_{n\geqslant 1} \subseteq L^1(\Omega)$ is uniformly integrable (see Definition 1.18 and Problem 1.6).

This fact, (2.145) and the Vitali theorem (the extended dominated convergence theorem; see Theorem I.3.128 and Proposition I.3.130) imply that

$$\|Du_n\|_p \longrightarrow \|Du\|_p. \tag{2.146}$$

Recall that

$$Du_n \xrightarrow{w} Du \quad \text{in } L^p(\Omega; \mathbb{R}^N) \tag{2.147}$$

(see (2.135)). Since $L^p(\Omega;\mathbb{R}^N)$ is uniformly convex, it has the Kadec–Klee property (see Corollary 1.26). So, from (2.146) and (2.147) it follows that

$$Du_n \longrightarrow Du \quad \text{in } L^p(\Omega;\mathbb{R}^N). \tag{2.148}$$

On the other hand, from (2.135) and the Rellich–Kondrachov embedding theorem (see Theorem 1.135), we have

$$u_n \longrightarrow u \quad \text{in } L^p(\Omega). \tag{2.149}$$

From (2.148) and (2.149) we conclude that $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$ and so we infer that A is of type $(S)_+$.

Solution of Problem 2.193

According to Theorem 2.116, it suffices to show that $R(A+C+I_H)=H$. Let $h \in H$ and

$$E(u) \;=\; (A+I_H)^{-1}(h-C(u)) \qquad \forall\, u \in D(C).$$

We know that $(A+I_H)^{-1}$ is nonexpansive (see Proposition 2.123(b)). So, we have

$$\|E(u)-E(y)\|_H \;\leqslant\; k\|u-y\|_H \qquad \forall\, u,y \in D(C).$$

Since by hypothesis $D(C) \subseteq H$ is closed, we can apply the Banach fixed point theorem (see Theorem I.1.49) and find $\widehat{v} \in D(C)$ such that $E(\widehat{v}) = \widehat{v}$. Then $h \in (A+C+I_H)(\widehat{v})$. Because $h \in H$ is arbitrary, we infer that $R(A+C+I_H) = H$ and so we conclude that $A+C$ is maximal monotone (see Definition 2.100).

Solution of Problem 2.194

Let $\{u_n\}_{n\geqslant 1} \subseteq X$ be a sequence such that $u_n \xrightarrow{w} u$ in X and

$$\limsup_{n\to+\infty} \langle A(u_n), u_n - u\rangle \;\leqslant\; 0.$$

Since A is of type $(S)_+$ (see Definition 2.137), we have that $u_n \longrightarrow u$ in X (see Definition 2.137). Because of the demicontinuity of A we have $A(u_n) \xrightarrow{w} A(u)$ in X^*. Hence for every $y \in X$, we have

$$\langle A(u), u - y \rangle \leqslant \liminf_{n \to +\infty} \langle A(u_n), u_n - y \rangle,$$

so A is pseudomonotone (see Definition 2.129).

Solution of Problem 2.195

From Problem 2.194, we know that A is pseudomonotone. Then strong coercivity of A and Theorem 2.136 imply that A is surjective.

Solution of Problem 2.196

Let $\{u_n\}_{n\geqslant 1} \subseteq H_0^1(\Omega)$ be a sequence such that $u_n \xrightarrow{w} u$ in $H_0^1(\Omega)$. Since $N \leqslant 3$, we have that $H_0^1(\Omega)$ is embedded compactly in $L^4(\Omega)$ (see the Rellich–Kondrachov embedding theorem; Theorem 1.135). So, we have $u_n \longrightarrow u$ in $L^4(\Omega)$. We need to show that $A(u_n) \longrightarrow A(u)$ in $H^{-1}(\Omega)$. Arguing by contradiction, suppose that the last convergence in $H^{-1}(\Omega)$ is not true. Then we can find $\varepsilon > 0$ and a sequence $\{v_n\}_{n\geqslant 1} \subseteq H_0^1(\Omega)$ such that

$$\|v_n\|_{H_0^1(\Omega)} \leqslant 1 \quad \text{and} \quad \langle A(u_n) - A(u), v_n \rangle \geqslant \varepsilon \qquad \forall\, n \geqslant 1. \tag{2.150}$$

From (2.150) we see that by passing to a subsequence if necessary, we may assume that $v_n \xrightarrow{w} v$ in $H_0^1(\Omega)$, so

$$v_n \longrightarrow v \quad \text{in } L^4(\Omega).$$

For all $n \geqslant 1$ and $k \in \{1, \ldots, N\}$, we have

$$\begin{aligned} & (\sin u_n)(D_k u_n)v_n - (\sin u)(D_k u)v_n \\ = \ & (\sin u_n - \sin u)(D_k u_n)v_n + (\sin u)(D_k u_n)(v_n - v) \\ & + (\sin u)(D_k u_n - D_k u)v + (\sin u)(D_k u)(v - v_n). \end{aligned} \tag{2.151}$$

Using the mean value theorem, we have

$$|\sin u_n - \sin u| \leqslant |u_n - u|.$$

Then invoking the generalized Hölder inequality (see Theorem I.3.105), we have

$$\begin{aligned} \Big| \int_\Omega (\sin u_n - \sin u)(D_k u_n) v_n \, dz \Big| &\leqslant \|u_n - u\|_4 \|D_k u_n\|_2 \|v_n\|_4 \\ &\leqslant c \|u_n - u\|_4 \|u_n\|_{H_0^1(\Omega)} \|v_n\|_{H_0^1(\Omega)}, \end{aligned}$$

with $c > 0$, so

$$\int_\Omega (\sin u_n - \sin u)(D_k u_n) v_n \, dz \longrightarrow 0.$$

Also, since $v_n \xrightarrow{w} v$ in $H_0^1(\Omega)$ and $D_k u_n \xrightarrow{w} D_k u$ in $L^2(\Omega)$, then

$$\begin{aligned} \int_\Omega (\sin u)(D_k u_n)(v_n - v) \, dz &\longrightarrow 0, \\ \int_\Omega (\sin u)(D_k u_n - D_k u) v \, dz &\longrightarrow 0, \\ \int_\Omega (\sin u)(D_k u)(v - v_n) \, dz &\longrightarrow 0. \end{aligned}$$

We return to (2.151), integrate over Ω (for every $k \in \{1, \ldots, N\}$) and obtain

$$\langle A(u_n) - A(u), v_n \rangle \longrightarrow 0,$$

which contradicts (2.150). This proves the complete continuity of A (see Definition 2.1(b)).

Solution of Problem 2.197

"$\Longrightarrow$": Suppose that A is self-adjoint (see Definition I.5.108(b)). According to Theorem 2.116, it suffices to show that $R(A+I_H) = H$. First we show that $R(A+I_H)$ is dense in H. So, let $h \in R(A+I_H)^{\perp}$. Then

$$(h,u)_H + (h,A(u))_H \;=\; 0 \qquad \forall\, u \in D(A),$$

so

$$(h,A(u))_H \;=\; -\,(h,u)_H \qquad \forall\, u \in D(A)$$

and thus $h \in D(A^*)$ and $A^*(h) = -h$. Since by hypothesis A is self-adjoint (i.e., $A = A^*$), we have $h \in D(A)$ and $A(h) = -h$. The monotonicity of A implies that

$$0 \;\leqslant\; (A(h),h)_H \;=\; -\|h\|_H^2,$$

so $h = 0$ and so $R(A+I_H)$ is dense in H.

Next we show that $R(A+I_H)$ is closed. To this end, let $\{h_n\}_{n\geqslant 1} \subseteq R(A+I_H)$ and suppose that $h_n \longrightarrow h$ in H. Let $\{u_n\}_{n\geqslant 1} \subseteq D(A)$ be such that

$$h_n \;=\; A(u_n) + u_n \qquad \forall\, n \geqslant 1, \tag{2.152}$$

so

$$\|u_n - u_m\|_H \;\leqslant\; \|h_n - h_m\|_H \qquad \forall\, n,m \geqslant 1$$

(see Proposition 2.123(b)), so $\{u_n\}_{n\geqslant 1} \subseteq H$ is a Cauchy sequence and thus $u_n \longrightarrow u$ in H. From (2.152), we have

$$A(u_n) \;\longrightarrow\; h-u \quad \text{in } H.$$

But A being self-adjoint, is closed and so $A(u) = h-u$, so $h = A(u)+u$, thus $h \in R(A+I_H)$ and hence the set $R(A+I_H) \subseteq H$ is closed. We conclude that $R(A+I_H) = H$ and so A is maximal monotone (see Definition 2.100).

"$\Longleftarrow$": Suppose that A is maximal monotone. First we show that A is densely defined (i.e., $\overline{D(A)} = H$). So, let $h \in D(A)^{\perp}$. Then $(h,u)_H = 0$ for all $u \in D(A)$. Since A is maximal monotone, we have $R(A+I_H) = H$ (see Theorem 2.116). Hence we can find $u \in D(A)$ such that $h = A(u)+u$. We have

$$\|u\|_H^2 \;\leqslant\; (A(u),u) + \|u\|_H^2 \;=\; (h,u)_H \;=\; 0$$

(recall that A is monotone), so $u = 0$ and thus $h = 0$. This shows that the operator A is densely defined.

Since by hypothesis A is symmetric, we have $A \subseteq A^*$ (i.e., $\operatorname{Gr} A \subseteq \operatorname{Gr} A^*$). Therefore, in order to have that A^* is self-adjoint, it is enough to show that $D(A^*) \subseteq D(A)$. We fix $v \in D(A^*)$. Since $R(A+I_H) = H$ (from the maximal monotonicity of A), we can find $u, z \in D(A)$ such that

$$A(u) + u \;=\; A^*(v) + v \quad \text{and} \quad A(z) + z \;=\; v - u. \tag{2.153}$$

Since A is symmetric, then so is $A + I_H$ and we have

$$\begin{aligned} & (A^*(v) + v, z)_H \;=\; (A(u) + u, z)_H \\ =\; & (A(z) + z, u)_H \;=\; (v - u, u)_H \end{aligned} \tag{2.154}$$

(see (2.153) and use the symmetry of $A + I_H$). From the definition of the adjoint operator, we have

$$(A^*(v), z)_H \;=\; (A(z), v)_H \,.$$

It follows that

$$\begin{aligned} (v - u, v)_H \;&=\; (A(z) + z, v)_H \;=\; (z, A^*(v) + v)_H \\ &=\; (z, A(u) + u)_H \;=\; (v - u, u)_H \end{aligned}$$

(see (2.153), (2.154) and use the symmetry of $A+I_H$), so $\|v-u\|_H^2 = 0$, thus $v = u \in D(A)$ and hence $D(A^*) \subseteq D(A)$. This proves that $D(A) = D(A^*)$, which shows that A is self-adjoint.

Solution of Problem 2.198

From Problem 2.197 we have that A is maximal monotone (see Definition 2.100). Since T is monotone and $\operatorname{Gr} A \subseteq \operatorname{Gr} T$, we must have $\operatorname{Gr} A = \operatorname{Gr} T$ (see Remark 2.101) and so $A = T$.

Solution of Problem 2.199

From Proposition 2.114(a) we know that the duality map $\mathcal{F}\colon X \longrightarrow X^*$ is single valued. The maximality of A (see Definition 2.139(b)) implies that

$$A(u) \;=\; \bigcap_{(x,y)\in \operatorname{Gr} A} \{v \in X :\ \langle \mathcal{F}(u-x), v-y\rangle \geqslant 0\}$$

(recall that A is accretive; see Definition 2.139(a)). In this intersection, every set is closed and convex. Therefore $A(u) \in P_{fc}(X)$.

Solution of Problem 2.200

(a) Let $\{(u_n, v_n)\}_{n\geqslant 1} \subseteq \operatorname{Gr} A$ be a sequence such that $u_n \longrightarrow u$ and $v_n \longrightarrow v$ in X. According to Definition 2.139(a), for every $(x, y) \in \operatorname{Gr} A$, we can find $h_n^* \in \mathcal{F}(u_n - x)$ such that

$$\langle h_n^*, v_n - y\rangle \;\geqslant\; 0. \tag{2.155}$$

From Problem 2.88 we know that the duality map is upper semicontinuous from X into $X^*_{w^*}$ (where $X^*_{w^*}$ denotes the Banach space X^* furnished with the w^*-topology). Then since $K = \{u_n - x :\ n \geqslant 1\} \cup \{u - x\} \in P_k(X)$, from Problem 2.56(a), we have that the set $\mathcal{F}(K) \subseteq X^*$ is w^*-compact. Because $\{h_n^*\}_{n\geqslant 1} \subseteq \mathcal{F}(K)$, we can find a subnet $\{h_\alpha^*\}_{\alpha\in J}$ of $\{h_n^*\}_{n\geqslant 1}$ such that $h_n^* \xrightarrow{w^*} h^*$ in X^* and $h^* \in \mathcal{F}(u - x)$. From (2.155) we obtain

$$\langle h^*, v - x\rangle \;\geqslant\; 0 \qquad \forall\ (x, y) \in \operatorname{Gr} A.$$

The maximality of A implies that $(u, v) \in \operatorname{Gr} A$ and so $\operatorname{Gr} A$ is closed in $X \times X$.

(b) Since the space X^* is locally uniformly convex (see Definition I.5.168), it is also strictly convex (see Remark I.5.169). By Proposition 2.114, the duality map $\mathcal{F}\colon X \longrightarrow X^*$ is single valued and continuous. Let $\{(u_\alpha, v_\alpha)\}_{\alpha\in J} \subseteq \operatorname{Gr} A$ be a net such that

$$u_\alpha \;\longrightarrow\; u \quad \text{in } X \quad \text{and} \quad v_\alpha \;\xrightarrow{w}\; v \quad \text{in } X. \tag{2.156}$$

For every $(x, y) \in \operatorname{Gr} A$, we have

$$\langle \mathcal{F}(u_\alpha - x), v_\alpha - y \rangle \geqslant 0,$$

so

$$\langle \mathcal{F}(u - x), v - y \rangle \geqslant 0$$

(see (2.156)) and thus $(u, v) \in \operatorname{Gr} A$ (due to maximality of A).

Solution of Problem 2.201

First note that X being uniformly convex, it is reflexive (see the Milman–Pettis theorem; Theorem I.5.89). We will show that for every sequence $\{\lambda_n\}_{n\geqslant 1}$ such that $\lambda_n \searrow 0$, the sequence $\{A_{\lambda_n}(u)\}_{n\geqslant 1}$ admits a subsequence strongly convergent to $A^0(u)$. By the Urysohn criterion for the convergence of subsequences (see Problem I.1.3), this implies that $A_\lambda(u) \longrightarrow A^0(u)$ as $\lambda \searrow 0$.

From Proposition 2.143, we see that the sequence $\{A_{\lambda_n}(u)\}_{n\geqslant 1} \subseteq X$ is bounded. The reflexivity of X implies that the sequence $\{A_{\lambda_n}(u)\}_{n\geqslant 1}$ is relatively w-compact in X. Then the Eberlein–Smulian theorem (see Theorem I.5.78) implies that we can find a subsequence $\{A_{\lambda_{n_k}}(u)\}_{k\geqslant 1}$ of $\{A_{\lambda_n}(u)\}_{n\geqslant 1}$ such that

$$A_{\lambda_{n_k}}(u) \xrightarrow{w} v \quad \text{in } X. \tag{2.157}$$

From Proposition 2.143(b), we know that

$$A_{\lambda_{n_k}}(u) \in A(J_{\lambda_{n_k}}(u)) \qquad \forall\, k \geqslant 1. \tag{2.158}$$

From Proposition 2.143(d), we have

$$J_{\lambda_{n_k}}(u) \longrightarrow u \quad \text{in } X. \tag{2.159}$$

Because A is m-accretive (see Definition 2.139(c)) it is also maximal accretive (see Proposition 2.144). So, from (2.157), (2.158), (2.159) and Problem 2.200(b), we have $(u, v) \in \operatorname{Gr} A$. Then

$$\|A^0(u)\|_X \leqslant \|v\|_X. \tag{2.160}$$

Also, we have

$$\|v\|_X \leqslant \liminf_{k\to+\infty} \|A_{\lambda_{n_k}}(u)\|_X \leqslant \limsup_{k\to+\infty} \|A_{\lambda_{n_k}}(u)\|_X \leqslant \|A^0(u)\|_X \tag{2.161}$$

(see Proposition 2.143(c)). From (2.160) and (2.161), we have $\|v\|_X = \|A^0(u)\|_X$. From Problem 2.199 we know that $A(u) \in P_{fc}(X)$. Therefore $v = A^0(u)$ and so

$$\|A_{\lambda_{n_k}}(u)\|_X \longrightarrow \|A^0(u)\|_X \quad \text{and} \quad A_{\lambda_{n_k}}(u) \xrightarrow{w} A^0(u) \quad \text{in } X,$$

so

$$A_{\lambda_{n_k}}(u) \longrightarrow A^0(u) \quad \text{in } X$$

(by the Kadec–Klee property; see Corollary 1.26)). Using once more Proposition 2.143, we have

$$\|A^0(J_\lambda(u))\|_X \leqslant \|A_\lambda(u)\|_X \leqslant \|A^0(u)\|_X \qquad \forall\, u \in D(A),\ \lambda > 0.$$

Then the same argument as above, via the Kadec–Klee property, gives

$$A^0(J_\lambda(u)) \longrightarrow A^0(u) \quad \text{as } \lambda \searrow 0 \qquad \forall\, u \in D(A).$$

Solution of Problem 2.202

We argue indirectly. So, suppose that A is not upper semicontinuous as claimed by the problem. Then we can find a weakly open set $U \subseteq X$ such that

$$A^+(U) = \{u \in D(A) : A(u) \subseteq U\}$$

is not open in $D(A)$ (see Proposition 2.37). This means that we can find $u \in D(A) \cap A^+(U)$ and a sequence $\{u_n\}_{n\geqslant 1} \subseteq D(A)$ such that $u_n \longrightarrow u$ in X and $v_n \in A(u_n)$ such that $v_n \notin U$ for all $n \geqslant 1$. By hypothesis A is locally bounded (see Definition 2.102). So, the sequence $\{v_n\}_{n\geqslant 1} \subseteq X$ is bounded and due to the reflexivity of X, by passing to a subsequence if necessary, we may assume that $v_n \xrightarrow{w} v \in X \setminus U$. Since $(u_n, v_n) \in \operatorname{Gr} A$ for every $n \geqslant 1$, Problem 2.200(b) implies that $v \in A(u) \subseteq U$, a contradiction.

Solution of Problem 2.203

For every $t, s \in \mathbb{R}$ and every $u^* \in \mathcal{F}(u(t))$, we have

$$\langle u^*, u(t)\rangle = \|u(t)\|_X^2 \quad \text{and} \quad \langle u^*, u(s)\rangle \leqslant \|u(t)\|_X \|u(s)\|_X.$$

It follows that

$$\langle u^*, u(s) - u(t)\rangle \leqslant \|u(t)\|_X \left(\|u(s)\|_X - \|u(t)\|_X\right).$$

Let $t \in \mathbb{R}$ be a point of differentiability of $\|u(\cdot)\|_X$ and of weak differentiability of u. Dividing by $s - t$ and letting $s \to t$, we obtain

$$\left\langle u^*, u'(t)\right\rangle \leqslant \|u(t)\|_X \tfrac{d}{dt}\|u(t)\|_X \quad \text{as } s \to t^+$$

and

$$\left\langle u^*, u'(t)\right\rangle \geqslant \|u(t)\|_X \tfrac{d}{dt}\|u(t)\|_X \quad \text{as } s \to t^-.$$

So, finally we have

$$\left\langle u^*, u'(t)\right\rangle = \|u(t)\|_X \tfrac{d}{dt}\|u(t)\|_X \quad \text{for a.a. } t \in T, \text{ all } u^* \in \mathcal{F}(u(t)).$$

Solution of Problem 2.204

According to Definition 2.139(c), we need to show that for every $h \in X$, the operator equation $u + A(u) = h$ admits a solution $u \in X$.

Let

$$\widehat{A}(u) = (I_X + A)(u) + h \qquad \forall\, u \in X.$$

Evidently $\widehat{A}$ is continuous accretive (see Definition 2.139) with $D(\widehat{A}) = X$. For fixed $u_0 \in X$, we consider the following abstract Cauchy problem

$$\begin{cases} \frac{du}{dt}(t) + \widehat{A}(u(t)) = 0, \\ u(0) = u_0. \end{cases} \tag{2.162}$$

From Theorem 2.158, we know that problem (2.162) has a unique solution $u \in C^1((0,+\infty); X)$. We have

$$\tfrac{d}{dt}\left(u(t+h) - u(t)\right) = -A\left(u(t+h)\right) + A\left(u(t)\right) - u(t+h) + u(t) \qquad \forall\, t > 0,$$

so

$$\|u(t+h) - u(t)\|_X \tfrac{d}{dt}\|u(t+h) - u(t)\|_X \leqslant -\|u(t+h) - u(t)\|_X^2 \qquad \forall\, t \geqslant 0$$

(see Problem 2.203 and recall that A is accretive), thus

$$\|u(t+h)-u(t)\|_X \leqslant e^{-t}\|u(h)-u_0\|_X \qquad \forall\, t \geqslant 0. \tag{2.163}$$

Similarly, from the equation

$$\tfrac{d}{dt}\left(u(t)-u_0\right) = -A(u(t))-u(t)+h,$$

we obtain

$$\begin{aligned} &\|u(t)-u_0\|_X \tfrac{d}{dt}\|u(t)-u_0\|_X \\ \leqslant\ & -\|u(t)-u_0\|_X^2+\|u(t)-u_0\|_X\left(\|A(u_0)\|_X+\|u_0\|_X+\|h\|_X\right), \end{aligned}$$

so

$$\|u(t)-u_0\|_X \leqslant (1+e^{-t})\left(\|A(u_0)\|_X+\|u_0\|_X+\|h\|_X\right) \qquad \forall\, t \geqslant 0. \tag{2.164}$$

From (2.163) and (2.164), we infer that $\lim\limits_{t\to+\infty} u(t) = \widehat{u} \in X$ exists and $\lim\limits_{t\to+\infty} \frac{du}{dt}(t) = 0$. Therefore, if in (2.162) we pass to the limit as $t \to +\infty$, we obtain $(I_X+A)\widehat{u} = h$, so $R(I_X+A) = X$ and thus A is m-accretive (see Definition 2.139).

Solution of Problem 2.205

We argue by contradiction. So, suppose that A is not maximal accretive (see Definition 2.139(b)). Then we can find $(\widehat{u},\widehat{v}) \notin \operatorname{Gr} A$ such that

$$0 \leqslant \langle \mathcal{F}(x-\widehat{u}), y-\widehat{v}\rangle \qquad \forall\, (x,y) \in \operatorname{Gr} A. \tag{2.165}$$

The set $A(\widehat{u})$ is nonempty, closed, and convex. Since $\widehat{v} \notin A(\widehat{u})$, we can find $h^* \in X^*$, $h^* \neq 0$ such that

$$\sup_{v\in A(\widehat{u})} \langle h^*, v\rangle < \langle h^*, \widehat{v}\rangle .$$

The duality map $\mathcal{F}$ is monotone, continuous and coercive (see Proposition 2.114). So, Theorem 2.119 implies that $R(\mathcal{F}) = X^*$. Therefore we can find $h \in X$ such that $\mathcal{F}(h) = h^*$. So, we have

$$\langle \mathcal{F}(h), v\rangle < \langle \mathcal{F}(h), \widehat{v}\rangle \qquad \forall\, v \in A(\widehat{u}). \tag{2.166}$$

Set $U = \{y \in X : \langle \mathcal{F}(h), y - \widehat{v}\rangle < 0\}$. Evidently $U \subseteq X$ is weakly open and $A(\widehat{u}) \subseteq U$ (see (2.166)). Let $t \in (0,1)$ and let $\widehat{u}_t = \widehat{u} + th$, $\widehat{v}_t \in A(\widehat{u}_t)$. By hypothesis A is upper semicontinuous from X into X_w. So, for $t \in (0,1)$ small, $\widehat{v}_t \in U$. Then

$$\langle \mathcal{F}(h), \widehat{v}_t - \widehat{v}\rangle \ < \ 0. \tag{2.167}$$

On the other hand, from (2.165), we have

$$0 \ \leqslant \ \langle \mathcal{F}(\widehat{u}_t - \widehat{u}), \widehat{v}_t - \widehat{v}\rangle \ = \ t\,\langle \mathcal{F}(h), \widehat{v}_t - \widehat{v}\rangle\,,$$

so

$$\langle \mathcal{F}(h), \widehat{v}_t - \widehat{v}\rangle \ \geqslant \ 0,$$

a contradiction to (2.167). This shows that A is maximal accretive.

Solution of Problem 2.206

Arguing indirectly, suppose that we can find $u_0 \in D(A)$ and a sequence $\{t_n\}_{n\geqslant 1} \subseteq (0, +\infty)$ such that

$$\|S(t_n)u_0\|_X \ \geqslant \ n \qquad \forall\, n \geqslant 1.$$

Since $t \longmapsto S(t)u_0$ is continuous, we must have $t_n \longrightarrow +\infty$. Then from the hypothesis of the problem, we have

$$\|AS(t_n)u_0\|_X \ \longrightarrow \ +\infty.$$

On the other hand, from Theorem 2.158, we know that the map $t \longmapsto \|AS(t)u_0\|_X$ is nonincreasing, a contradiction. So, we conclude that $\sup\limits_{t\geqslant 0} \|S(t)u_0\|_X < +\infty$.

Solution of Problem 2.207
We have

$$\begin{aligned}\Big\|S(t)u - \tfrac{1}{t}\int_0^t S(\tau)u\,d\tau\Big\|_X &= \tfrac{1}{t}\Big\|\int_0^t \big(S(t)u - S(\tau)u\big)\,d\tau\Big\|_X \\ \leqslant \tfrac{1}{t}\int_0^t \big\|S(\tau)(S(t-\tau)u) - S(\tau)u\big\|_X\,d\tau &\leqslant \tfrac{1}{t}\int_0^t \|S(t-\tau)u - u)\|_X\,d\tau \\ = \tfrac{1}{t}\int_0^t \|S(r)u - u\|_X\,dr \qquad (2.168)\end{aligned}$$

(see Definitions 2.156(b) and (a)). Using the triangle inequality, we have

$$\begin{aligned}\|S(t)u - u\|_X &\leqslant \Big\|S(t)u - \tfrac{1}{t}\int_0^t S(\tau)u\,d\tau\Big\|_X + \tfrac{1}{t}\Big\|\int_0^t (S(\tau)u - u)\,d\tau\Big\|_X \\ &\leqslant \tfrac{2}{t}\int_0^t \|S(\tau)u - u\|_X\,d\tau\end{aligned}$$

(see (2.168)).

Solution of Problem 2.208
From the triangle inequality, for every $u \in \overline{D(A)}$, $x \in D(A)$, $y \in A(x)$ and $t > 0$, we have

$$\begin{aligned}\|S(t)u - u\|_X &\leqslant \|S(t)u - S(t)x\|_X + \|S(t)x - x\|_X + \|y - x\|_X \\ &\leqslant 2\|u - x\|_X + t\|y\|_X \qquad (2.169)\end{aligned}$$

(see Definition 2.156 and Theorem 2.158). In (2.169), we let $x = J_\lambda(u)$ and $y = A_\lambda(u)$ (recall that $A_\lambda(u) \in A(J_\lambda(u))$). So, we obtain

$$\|S(t)u - u\|_X \leqslant \left(2 + \tfrac{t}{\lambda}\right)\|J_\lambda(u) - u\|_X$$

(see Definition 2.142). For all $x \in D(A)$, all $y \in A(x)$ and all $t > 0$, we have

$$\|S(t)u - x\|_X \leqslant \|u - x\|_X + \frac{1}{\lambda} \int_0^t \big(\|S(\tau)u - x - \lambda y\|_X - \|S(\tau)u - x\|_X\big)\, d\tau \quad \forall\, \lambda > 0.$$

Choosing $x = J_\lambda(u)$ and $y = A_\lambda(u)$, we obtain

$$\|S(t)u - J_\lambda(u)\|_X \leqslant \|u - J_\lambda(u)\|_X + \frac{1}{\lambda} \int_0^t \big(\|S(\tau)u - u\|_X - \|S(\tau)u - J_\lambda(u)\|_X\big)\, d\tau,$$

so

$$\|u - J_\lambda(u)\|_X \leqslant \frac{\lambda}{t}\|S(t)u - u\|_X + \frac{2}{t} \int_0^t \|S(\tau)u - u\|_X\, d\tau$$

and thus

$$\|u - J_\lambda(u)\|_X \leqslant \frac{2}{t}\left(1 + \frac{\lambda}{t}\right) \int_0^t \|S(\tau)u - u\|_X\, d\tau$$

(see Problem 2.207).

Solution of Problem 2.209

"**(a)** $\Longrightarrow$ **(b)**": Suppose that the nonlinear semigroup $\{S(t)\}_{t \geqslant 0}$ is compact (see Definition 2.159(a)). From Remark 2.160, we know that $\{S(t)\}_{t \geqslant 0}$ is equicontinuous. Recall that $J_\lambda \colon X \longrightarrow D(A)$ (see Definition 2.142). So, from Theorem 2.158, for all $u \in X$, we have

$$\|S(t)J_\lambda(u) - J_\lambda(u)\|_X \leqslant t\,|A(J_\lambda(u))| \leqslant t\|A_\lambda(u)\|_X = \frac{t}{\lambda}\|u - J_\lambda(u)\|_X$$

(recall that $A_\lambda(u) \in A(J_\lambda(u))$ and see Definition 2.142), so

$$\lim_{t \searrow 0} S(t)J_\lambda = J_\lambda \text{ uniformly on bounded subsets of } X$$

(see Proposition 2.143). But $S(t)J_\lambda$ is compact. Hence so is the limit J_λ.

"**(b)** $\Longrightarrow$ **(a)**": Suppose that for every $\lambda > 0$, the map J_λ is compact and $\{S(t)\}_{t \geqslant 0}$ is equicontinuous (see Definition 2.159(b)). From Problem 2.208, for all $u \in \overline{D(A)}$, all $t > 0$ and all $\lambda > 0$, we have

$$\|J_\lambda(S(t)u) - S(t)u\|_X \ \leqslant\ \tfrac{4}{\lambda} \int\limits_0^\lambda \|S(t+\tau)u - S(t)u\|_X \, d\tau.$$

The equicontinuity hypothesis on $\{S(t)\}_{t \geqslant 0}$ implies that for every bounded set $B \subseteq \overline{D(A)}$, we can find a function $\xi_B \colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\xi_B(r) \longrightarrow 0$ as $r \searrow 0$ and for all $\tau \geqslant 0$, all $u \in B$, we have

$$\|S(t+\tau)u - S(t)u\|_X \ \leqslant\ \xi_B(\tau)$$

(see Definition 2.159(b)). So, for all $u \in B$, we have

$$\|J_\lambda(S(t)u) - S(t)u\|_X \ \leqslant\ 4 \sup_{r \in [0,\lambda]} \xi_B(r),$$

thus

$$J_\lambda(S(t)u) \ \longrightarrow\ S(t)u \quad \text{as } \lambda \searrow 0 \text{ uniformly on bounded sets of } \overline{D(A)}.$$

But $J_\lambda \circ S$ is compact for every $\lambda > 0$. So, Problem 2.2 implies that $S(t)$ is compact for all $t > 0$.

Solution of Problem 2.210

For every $n \geqslant 1$, we have

$$\begin{aligned} \|u - J_{\lambda_n}(u_n)\|_X \ &\leqslant\ \|u - u_n\|_X + \|u_n + J_{\lambda_n}(u_n)\|_X \\ \leqslant\ \|u - u_n\|_X + \lambda_n \|A_{\lambda_n}(u_n)\|_X \ &\leqslant\ \|u - u_n\|_X + \lambda c, \end{aligned}$$

for some $c > 0$ independent of $n \geqslant 1$ (see Definition 2.142 and recall that the sequence $\{A_{\lambda_n}(u_n)\}_{n \geqslant 1} \subseteq X$ is bounded), so

$$\|u - J_{\lambda_n}(u_n)\|_X \ \longrightarrow\ 0. \tag{2.170}$$

We know that $A_{\lambda_n}(u_n) \in A(J_{\lambda_n}(u_n))$ for all $n \geqslant 1$ (see Proposition 2.143). From (2.170) and since $A_{\lambda_n}(u_n) \longrightarrow h$ in X, invoking Problem 2.200(a), we conclude that $(u,h) \in \operatorname{Gr} A$. The last part of the problem is a consequence of Problem 2.200(b).

Solution of Problem 2.211

"**(a) $\Longrightarrow$ (b)**": From Proposition 2.143(c), we know that

$$\|A_\lambda(u)\|_X \;=\; \tfrac{1}{\lambda}\|u - J_\lambda(u)\|_X \;\leqslant\; \eta \qquad \forall\, u \in L_\eta,\ \lambda > 0,$$

so

$$\|u - J_\lambda(u)\|_X \;\leqslant\; \lambda\eta,$$

thus

$$\lim_{\lambda \searrow 0} J_\lambda \;=\; I_X \quad \text{uniformly on } L_\eta.$$

Since by hypothesis J_λ is compact, it follows that $I_X|_{L_\eta}$ is compact (see Problem 2.2) and so L_η is relatively compact in X.

"**(b) $\Longrightarrow$ (a)**": Let $B \subseteq X$ be a bounded set. Then $J_\lambda(B)$ is bounded for every $\lambda > 0$ (recall that J_λ in nonexpansive). For any $u \in X$ and $\lambda > 0$, we have

$$\begin{aligned} \|J_\lambda(u)\|_X + |A(J_\lambda(u))| \;&\leqslant\; \|J_\lambda(u)\|_X + \|A_\lambda(u)\|_X \\ &=\; \|J_\lambda(u)\|_X + \tfrac{1}{\lambda}\|u - J_\lambda(u)\|_X. \end{aligned}$$

So, there exists $\eta > 0$ such that $J_\lambda(B) \subseteq L_\eta$, hence $\overline{J_\lambda(B)}$ is compact in X. Since $B \subseteq X$ is an arbitrary bounded set, we conclude that J_λ is compact.

Solution of Problem 2.212

To show the maximal monotonicity of A, it suffices to show that

$$R(A + I_{L^2_{2\pi}(\mathbb{R})}) \;=\; L^2_{2\pi}(\mathbb{R})$$

(see Theorem 2.116). So, for every $h \in L^2_{2\pi}(\mathbb{R})$, we consider the periodic problem

$$\begin{cases} u'(t) + u(t) = h(t) & \text{for a.a. } t \in T, \\ u(0) = u(2\pi). \end{cases} \tag{2.171}$$

It is well known that (2.171) has a unique solution $u \in W^{1,2}(0, 2\pi)$ (variation of constant formula). Therefore A is maximal monotone (see Definition 2.100). The Hille–Yosida theorem (see Theorem 2.153) implies that A is the infinitesimal generator of the C_0-semigroup defined by

$$S(t)u(\tau) \;=\; u(t-\tau) \qquad \forall\, u \in L^2_{2\pi}(\mathbb{R}),\ t \geqslant 0.$$

so, for each $t \geqslant 0$, $S(t)$ is an isometry and so it cannot be compact for $t > 0$.

Solution of Problem 2.213

Without any loss of generality, we may assume that $\eta > 0$. Let $\varepsilon \in (0, \eta)$. If $\overline{B}^*_{\eta-\frac{\varepsilon}{2}} = \left\{ v^* \in X^* :\ \|v^*\|_* \leqslant \eta - \frac{\varepsilon}{2} \right\}$, then

$$\overline{B}^*_{\eta-\frac{\varepsilon}{2}} \cap F(u) \;=\; \emptyset \qquad \forall\, u \in E.$$

From the separation theorem for convex sets (see Theorem I.5.29), we see that for all $u \in E$, we can find $\xi_u \in X$ such that $\|\xi_u\|_X = 1$ and

$$\langle h^*, \xi_u \rangle \;\leqslant\; \langle v^*, \xi_u \rangle \qquad \forall\, h^* \in \overline{B}^*_{\eta-\frac{\varepsilon}{2}},\ v^* \in F(u).$$

Then

$$\eta - \varepsilon \;\leqslant\; \langle v^*, \xi_u \rangle\,.$$

The upper semicontinuity of F implies that we can find an open neighborhood $U(u)$ of u such that

$$\langle v^*, \xi_u \rangle \;>\; \eta - \varepsilon \qquad \forall\, u' \in U(u),\ v^* \in F(u').$$

Since the set E is compact and $\{U(u)\}_{u \in E}$ is an open cover of E, we can find a finite subcover $\{U(u_1), \ldots, U(u_m)\}$. Let $\{v_1, \ldots, v_m\} \subseteq X$ be such that

$$\|v_k\|_X \;=\; 1 \quad \text{and} \quad \langle v^*, v_k \rangle \;>\; \eta - \varepsilon$$

for all $k \in \{1, \ldots, m\}$, $u \in U(u_k)$ and $v^* \in F(u)$. If $\lambda_k(u) = \operatorname{dist}_E(u, \partial U(u_k))$, we define

$$\mu_k(u) \;=\; \frac{\lambda_k(u)}{\sum\limits_{k=1}^{m} \lambda_k} \quad \text{and} \quad \xi(u) = \sum_{k=1}^{m} \mu_k(u) v_k.$$

This is the desired continuous map $\xi \colon E \longrightarrow X$.

Solution of Problem 2.214

We argue by contradiction. So, suppose that we can find $\varepsilon_0 > 0$ and a sequence $\{u_n\}_{n \geqslant 1} \subseteq X$ such that

$$\|L(u_n)\|_Y \;>\; \varepsilon_0 \|u_n\|_X + n\|u_n\|_Z. \tag{2.172}$$

Replacing u_n with $\frac{u_n}{\|u_n\|_X}$ if necessary, we may assume that $\|u_n\|_X = 1$ for all $n \geqslant 1$. Since X is reflexive, passing to a subsequence if necessary, we may assume that

$$u_n \;\xrightarrow{w}\; u \quad \text{in } X,$$

so

$$u_n \;\xrightarrow{w}\; u \quad \text{in } Z \tag{2.173}$$

(since the embedding $X \hookrightarrow Z$ is continuous). As $L \in \mathcal{L}_c(X;Y)$, from Proposition 2.3, we have

$$L(u_n) \;\longrightarrow\; L(u) \quad \text{in } Y. \tag{2.174}$$

From (2.172), we have

$$\|u_n\|_Z \;<\; \tfrac{1}{n}\|L(u_n)\|_Y,$$

so $\|u\|_Z \leqslant 0$ (see (2.173) and (2.174)), thus $u = 0$. So, we have $L(u_n) \longrightarrow 0$ in Y, while from (2.172), we have $\varepsilon_0 \leqslant \liminf \|L(u_n)\|_Y$, a contradiction. This completes the solution of the problem.

Solution of Problem 2.215

Let $X = W^{1,p}(\Omega)$, $Y = L^q(\partial\Omega)$, $Z = L^p(\Omega)$ and let $L = \gamma \in \mathcal{L}_c(W^{1,p}(\Omega), L^q(\partial\Omega))$ (the trace map; see Definition 1.129). So, we can use Problem 2.214 and then given $\varepsilon > 0$, we can find $\widehat{c}_\varepsilon > 0$ such that for all $u \in W^{1,p}(\Omega)$, we have

$$\|u\|_{L^q(\partial\Omega)} \leqslant \varepsilon\big(\|u\|_p + \|Du\|_p\big) + \widehat{c}\|u\|_p = \varepsilon\|Du\|_p + c_\varepsilon\|u\|_p,$$

with $c_\varepsilon = \widehat{c}_\varepsilon + \varepsilon > 0$.

Solution of Problem 2.216

Let $(v, v^*) \in U \times X^*$ and suppose that

$$\langle v^* - u^*, v - u\rangle \geqslant 0 \qquad \forall\, (u, u^*) \in \operatorname{Gr} A. \tag{2.175}$$

Suppose that $v^* \notin A(v)$. Then by the strong separation theorem (see Theorem I.5.29 and recall that $A(v)$ is nonempty, convex, and w^*-closed), we can find $h \in X \setminus \{0\}$ such that

$$F(v) \subseteq \{y^* \in X^* : \langle y^*, h\rangle < \langle v^*, h\rangle\} = W. \tag{2.176}$$

The set W is w^*-open. Since F is upper semicontinuous from X into $X^*_{w^*}$, we can find a norm open neighborhood D of v such that $A(D) \subseteq W$ (see Definition 2.36(a)). For $t > 0$ small we have $v+th \in D$ and so $A(v + th) \subseteq W$. Returning to (2.175) and choosing $(u, u^*) = (v + th, u^*)$ with $u^* \in A(v + th)$ we have

$$0 \leqslant \langle v^* - u^*, -th\rangle = -t\,\langle v^* - u^*, h\rangle\,,$$

so

$$\langle v^*, h\rangle \leqslant \langle u^*, h\rangle$$

and thus $u^* \notin W$, a contradiction. So, $v^* \in A(v)$ and this proves the maximal monotonicity of A (see Definition 2.100).

Solution of Problem 2.217

First note that A has w^*-closed values. Also, since A is locally bounded, by the Alaoglu theorem (see Theorem I.5.66), it is locally compact (see Proposition 2.45). So, we can use Problem 2.65 and infer that A is upper semicontinuous from X into $X^*_{w^*}$. Then we can invoke Problem 2.216 and conclude that A is maximal monotone.

Solution of Problem 2.218

Let $u^* \in X^*$ and let $\{u_n^*\}_{n \geqslant 1} \subseteq X^*$ be a sequence such that $u_n^* \longrightarrow u^*$ in X^*. Let $u_n \in D(A)$ be such that $u_n^* \in A(u_n)$ for all $n \geqslant 1$. Since A is strongly coercive (see Definition 2.98(f)), we can find a function $\eta\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$ with $\eta(r) \longrightarrow +\infty$ as $r \to +\infty$ such that

$$\langle u_n^*, u_n \rangle \;\geqslant\; \eta(\|u_n\|)\|u_n\|_X \qquad \forall\, n \geqslant 1,$$

so

$$\eta(\|u_n\|_X) \;\leqslant\; \|u_n^*\|_* \;\leqslant\; c_1 \qquad \forall\, n \geqslant 1,$$

for some $c_1 > 0$, thus

$$\|u_n\|_X \;\leqslant\; c_2 \qquad \forall\, n \geqslant 1,$$

for some $c_2 > 0$ and hence A^{-1} is locally bounded at $u^* \in X^*$ (see Definition 2.102). Since $u^* \in X^*$ was arbitrary, we conclude that A^{-1} is locally bounded.

Bibliography

[1] Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis. A Hitchhiker's Guide. Springer, Berlin (2006)

[2] Aubin, J.-P., Frankowska, H.: Set-Valued Analysis. Birkhäuser, Boston (2009)

[3] Barbu, V.: Nonlinear Semigroups and Differential Equations in Banach Spaces. Noordhoff International Publishing, Leiden (1976)

[4] Brézis, H.: Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert. American Elsevier Publishing Co., Inc., New York (1973)

[5] Browder, F.E.: Nonlinear Operators and Nonlinear Equations of Evolution in Banach Spaces. In: Proc. Sympos. Pure Math., vol. XVIII, Part 2, Chicago, 1968. American Mathematical Society, Providence (1976)

[6] Denkowski, Z., Migórski, S., Papageorgiou, N.S.: An Introduction to Nonlinear Analysis: Applications. Kluwer Academic, Boston (2003)

[7] Gasiński L., Papageorgiou, N.S.: Nonlinear Analysis. Chapman & Hall/CRC, Boca Raton (2006)

[8] Hu, S., Papageorgiou, N.S.: Handbook of Multivalued Analysis. Vol. I. Theory. Kluwer Academic, Dordrecht (1997)

[9] Hu, S., Papageorgiou, N.S.: Handbook of Multivalued Analysis. Vol. II. Applications. Kluwer Academic, Dordrecht (2000)

[10] Kato, T.: Perturbation Theory for Linear Operators. Springer, Berlin (1995)

[11] Klein, E., Thompson, A.C.: Theory of Correspondences. Wiley, New York (1984)

[12] Papageorgiou, N.S., Kyritsi-Yiallourou, S.Th.: Handbook of Applied Analysis. Springer, New York (2009)

[13] Pascali, D., Sburlan, S.: Nonlinear Mappings of Monotone Type. Sijthoff & Noordhoff International Publishers, Alphen aan den Rijn (1978)

[14] Reed, M., Simon, B.: Methods of Modern Mathematical Physics. Vol. I. Functional Analysis. Academic, New York (1980)

[15] Showalter, R.E.: Monotone Operators in Banach Space and Nonlinear Partial Differential Equations. American Mathematical Society, Providence, RI (1997)

[16] Vainberg, M.M.: Variational Method and Method of Monotone Operators in the Theory of Nonlinear Equations. Wiley, New York (1973)

[17] Zeidler, E.: Nonlinear Functional Analysis and Its Applications, vols. II/A and II/B. Springer, New York (1990)

http://www.springer.com/978-3-319-27815-5

Exercises in Analysis
Part 2: Nonlinear Analysis
Gasiński, L.; Papageorgiou, N.S.
2016, VIII, 1062 p., Hardcover
ISBN: 978-3-319-27815-5