Chapter 2
Theoretical Foundation on Conservation Equations of Laminar Mixed Convection

Abstract In this chapter, the basic conservation equations corresponding to laminar free and forced mixed convection are introduced. For this purpose, the related general laminar conservation equations on continuity, momentum and energy equations are presented by means of theoretical and mathematical derivation for deeply understanding fluid convection. From these general conservation equations of convection, the fluid's variable physical properties are fully taken into account. By using the quantity grade analysis, the mass, momentum, and energy equations of laminar mixed convection boundary layer are obtained with consideration of variable physical properties.

Keywords Control volume · Variable physical properties · Cartesian forms · Continuity equation · Navier-Stokes Equation · Energy equation · Mixed convection · Quantitative grade analysis · Conservation equations

2.1 Continuity Equation

The conceptual basis for the derivation of the continuity equation of fluid flow is the mass conservation law. The control volume for the derivation of continuity equation is shown in Fig. 2.1 in which the mass conservation principle is stated as

$$\dot{m}_{increment} = \dot{m}_{in} - \dot{m}_{out} \tag{2.1}$$

where $\dot{m}_{increment}$ expresses the mass increment per unit time in the control volume, $\dot{m}_{in}$ represents the mass flowing into the control volume per unit time, and $\dot{m}_{out}$ is the mass flowing out of the control volume per unit time. The dot notation signifies a unit time.

In the control volume, the mass of fluid flow is given by $\rho \mathrm{d}x\mathrm{d}y\mathrm{d}z$, and the mass increment per unit time in the control volume can be expressed as

$$\dot{m}_{increment} = \frac{\partial \rho}{\partial \tau} dxdydz. \tag{2.2}$$

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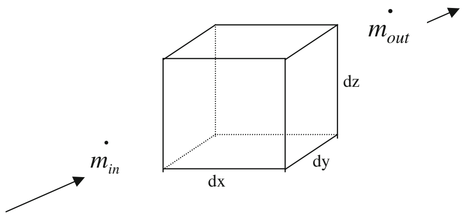

Fig. 2.1 Control volume for the derivation of the continuity equation

The mass flowing into the control volume per unit time in the x direction is given by $\rho w_x \mathrm{d}y\mathrm{d}z$. The mass flowing out of the control volume in a unit time in the x direction is given by $[\rho w_x + \partial(\rho w_x)/\partial x \cdot dx]dydz$. Thus, the mass increment per unit time in the x direction in the control volume is given by $\frac{\partial(\rho w_x)}{\partial x}dxdydz$. Similarly, the mass increments in the control volume in the y and z directions per unit time are given by $\frac{\partial(\rho w_y)}{\partial y}dydxdz$ and $\frac{\partial(\rho w_z)}{\partial z}dzdxdy$ respectively. Thus, we obtain

$$\dot{m}_{\mathrm{out}} - \dot{m}_{\mathrm{in}} = \left(\frac{\partial(\rho w_x)}{\partial x} + \frac{\partial(\rho w_y)}{\partial y} + \frac{\partial(\rho w_z)}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{2.3}$$

Combining Eq. (2.1) with Eqs. (2.2) and (2.3) we obtain the following continuity equation in Cartesian coordinates:

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial(\rho w_x)}{\partial x} + \frac{\partial(\rho w_y)}{\partial y} + \frac{\partial(\rho w_z)}{\partial z} = 0 \tag{2.4}$$

or in the vector notation

$$\frac{\partial \rho}{\partial \tau} + \nabla \cdot (\rho \vec{W}) = 0 \tag{2.5}$$

or

$$\frac{D\rho}{D\tau} + \rho \nabla \cdot (\vec{W}) = 0 \tag{2.6}$$

where $\vec{W} = iw_x + jw_y + kw_z$ is the fluid velocity.

For steady state, the vector and Cartesian forms of the continuity equation are given by

$$\frac{\partial}{\partial x}(\rho w_x) + \frac{\partial}{\partial y}(\rho w_y) + \frac{\partial}{\partial z}(\rho w_z) = 0 \tag{2.7}$$

or

$$\nabla \cdot (\rho \vec{W}) = 0 \tag{2.8}$$

2.2 Momentum Equation (Navier-Stokes Equation)

The control volume for the derivation of the momentum equation of fluid flow is shown in Fig. 2.2. Meanwhile, take an enclosed surface A that includes the control volume. According to momentum law, the momentum increment of the fluid flow per unit time equals the sum of the mass force and surface force acting on the fluid. The relationship is shown as below:

$$\dot{G}_{increment} = \vec{F_m} + \vec{F_s} \tag{2.9}$$

where $\vec{F_m}$ and $\vec{F_s}$ denote mass force and surface force respectively.

In the system the momentum increment $\dot{G}_{increment}$ of the fluid flow per unit time can be described as

$$\dot{G}_{increment} = \frac{D}{D\tau}\int_V \rho \vec{W}\, dV \tag{2.10}$$

In the system the sum of mass force F_m and surface force F_s acting on the fluid is expressed as

$$\vec{F_m} + \vec{F_s} = \int_V \rho \vec{F}\, dV + \int_A \vec{\tau_n}\, dA \tag{2.11}$$

where V and A are volume and surface area of the system respectively, and $\vec{\tau_n}$ is surface force acting on unit area.

Combining Eq. (2.9) with Eqs. (2.10) and (2.11), we have the following equation:

$$\frac{D}{D\tau}\int_V \rho \vec{W}\, dV = \int_V \rho \vec{F}\, dV + \int_A \vec{\tau_n}\, dA \tag{2.12}$$

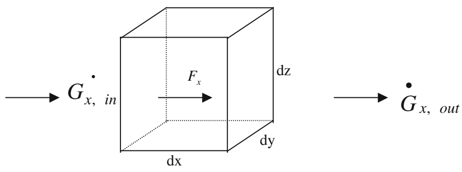

Fig. 2.2 Control volume for the derivation of momentum equation

According to tensor calculation, the right side of Eq. (2.12) is changed into the following form:

$$\int_V \rho \vec{F}\, dV + \int_A \vec{\tau_n}\, dA = \int_V \rho \vec{F}\, dV + \int_V \nabla \cdot [\tau] dV \tag{2.13}$$

where $\nabla \cdot [\tau]$ is divergence of the shear force tensor.

The left side of Eq. (2.12) can be rewritten as

$$\frac{D}{D\tau}\int_V \rho \vec{W}\, dV = \int_V \frac{D(\rho \vec{W})}{D\tau} dV \tag{2.14}$$

With Eqs. (2.13) and (2.14), Eq. (2.12) can be simplified as

$$\int_V \{\frac{D(\rho \vec{W})}{D\tau} - \rho \vec{F} - \nabla \cdot [\tau]\} dV = 0 \tag{2.15}$$

Therefore,

$$\frac{D(\rho \vec{W})}{D\tau} = \rho \vec{F} + \nabla \cdot [\tau] \tag{2.16}$$

This is the Navier-Stokes equations of fluid flow. For Cartesian Coordinates, Eq. (2.16) can be expressed as

$$\frac{D(\rho w_x)}{D\tau} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x \tag{2.17}$$

$$\frac{D(\rho w_y)}{D\tau} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y \tag{2.18}$$

$$\frac{D(\rho w_z)}{D\tau} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho g_z \tag{2.19}$$

where

$$\frac{D(\rho w_x)}{D\tau} = \frac{\partial(\rho w_x)}{\partial \tau} + \frac{(\partial \rho w_x)}{\partial x} w_x + \frac{(\partial \rho w_x)}{\partial y} w_y + \frac{(\partial \rho w_x)}{\partial z} w_z$$
$$\frac{D(\rho w_y)}{D\tau} = \frac{\partial(\rho w_y)}{\partial \tau} + \frac{(\partial \rho w_y)}{\partial x} w_x + \frac{(\partial \rho w_y)}{\partial y} w_y + \frac{(\partial \rho w_y)}{\partial z} w_z$$
$$\frac{D(\rho w_z)}{D\tau} = \frac{\partial(\rho w_z)}{\partial \tau} + \frac{(\partial \rho w_z)}{\partial x} w_x + \frac{(\partial \rho w_z)}{\partial y} w_y + \frac{(\partial \rho w_z)}{\partial z} w_z$$

In Eqs. (2.17)–(2.19), g_x, g_y and g_z are gravity accelerations in x, y, and z directions respectively, while, the related shear forces are given below:

$$
\begin{aligned}
\tau_{xx} &= -\left[p + \frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}\right)\right] + 2\mu\frac{\partial w_x}{\partial x} \\
\tau_{yy} &= -[p + \frac{2}{3}\mu(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z})] + 2\mu\frac{\partial w_y}{\partial y} \\
\tau_{zz} &= -[p + \frac{2}{3}\mu(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z})] + 2\mu\frac{\partial w_z}{\partial z} \\
\tau_{xy} &= \tau_{yx} = \mu(\frac{\partial w_y}{\partial x} + \frac{\partial w_x}{\partial y}) \\
\tau_{yz} &= \tau_{zy} = \mu(\frac{\partial w_z}{\partial y} + \frac{\partial w_y}{\partial z}) \\
\tau_{zx} &= \tau_{xz} = \mu(\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x})
\end{aligned}
$$

Then, Eqs. (2.17)–(2.19) are rewritten as follows respectively:

$$
\begin{aligned}
&\frac{D(\rho w_x)}{D\tau} \\
&= -\frac{\partial p}{\partial x} + 2\frac{\partial}{\partial x}(\mu\frac{\partial w_x}{\partial x}) + \frac{\partial}{\partial y}[\mu(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x})] + \frac{\partial}{\partial z}[\mu(\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x})] \\
&\quad - \frac{\partial}{\partial x}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z})] + \rho g_x
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
&\frac{D(\rho w_y)}{D\tau} \\
&= -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}[\mu(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x})] + 2\frac{\partial}{\partial y}(\mu\frac{\partial w_y}{\partial y}) + \frac{\partial}{\partial z}[\mu(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y})] \\
&\quad - \frac{\partial}{\partial y}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z})] + \rho g_y
\end{aligned} \tag{2.21}
$$

$$
\begin{aligned}
&\frac{D(\rho w_z)}{D\tau} \\
&= -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}[\mu(\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x})] + \frac{\partial}{\partial y}[\mu(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y})] + 2\frac{\partial}{\partial z}(\mu\frac{\partial w_z}{\partial z}) \\
&\quad - \frac{\partial}{\partial z}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z})] + \rho g_z
\end{aligned} \tag{2.22}
$$

For steady state, the momentum Eqs. (2.20)–(2.22) are given as follows respectively:

$$\begin{aligned}&\rho(\frac{\partial w_x}{\partial x}w_x+\frac{\partial w_x}{\partial y}w_y+\frac{\partial w_x}{\partial z}w_z)+w_x(w_x\frac{\partial \rho}{\partial x}+w_y\frac{\partial \rho}{\partial y}+w_z\frac{\partial \rho}{\partial z})\\&=-\frac{\partial p}{\partial x}+2\frac{\partial}{\partial x}(\mu\frac{\partial w_x}{\partial x})+\frac{\partial}{\partial y}[\mu(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x})]+\frac{\partial}{\partial z}[\mu(\frac{\partial w_x}{\partial z}+\frac{\partial w_z}{\partial x})]\\&\quad-\frac{\partial}{\partial x}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}+\frac{\partial w_z}{\partial z})]+\rho g_x\end{aligned}\tag{2.23}$$

$$\begin{aligned}&\rho(\frac{\partial w_y}{\partial x}w_x+\frac{\partial w_y}{\partial y}w_y+\frac{\partial w_y}{\partial z}w_z)+w_y(w_x\frac{\partial \rho}{\partial x}+w_y\frac{\partial \rho}{\partial y}+w_z\frac{\partial \rho}{\partial z})\\&=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}[\mu(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x})]+2\frac{\partial}{\partial y}(\mu\frac{\partial w_y}{\partial y})+\frac{\partial}{\partial z}[\mu(\frac{\partial w_y}{\partial z}+\frac{\partial w_z}{\partial y})]\\&\quad-\frac{\partial}{\partial y}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}+\frac{\partial w_z}{\partial z})]+\rho g_y\end{aligned}\tag{2.24}$$

$$\begin{aligned}&\rho(\frac{\partial w_z}{\partial x}w_x+\frac{\partial w_z}{\partial y}w_y+\frac{\partial w_z}{\partial z}w_z)+w_z(w_x\frac{\partial \rho}{\partial x}+w_y\frac{\partial \rho}{\partial y}+w_z\frac{\partial \rho}{\partial z})\\&=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}[\mu(\frac{\partial w_x}{\partial z}+\frac{\partial w_z}{\partial x})]+\frac{\partial}{\partial y}[\mu(\frac{\partial w_y}{\partial z}+\frac{\partial w_z}{\partial y})]+2\frac{\partial}{\partial z}(\mu\frac{\partial w_z}{\partial z})\\&\quad-\frac{\partial}{\partial z}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}+\frac{\partial w_z}{\partial z})]+\rho g_z\end{aligned}\tag{2.25}$$

Let us compare term $\rho(\frac{\partial w_x}{\partial x}w_x+\frac{\partial w_x}{\partial y}w_y+\frac{\partial w_x}{\partial z}w_z)$ with term $w_x(w_x\frac{\partial \rho}{\partial x}+w_y\frac{\partial \rho}{\partial y}+w_z\frac{\partial \rho}{\partial z})$. In general, derivatives $\frac{\partial w_x}{\partial x}$, $\frac{\partial w_x}{\partial y}$ and $\frac{\partial w_x}{\partial z}$ are much larger than the derivatives $\frac{\partial \rho_x}{\partial x}$, $\frac{\partial \rho_x}{\partial y}$ and $\frac{\partial \rho_x}{\partial z}$ respectively. In this case, the term $w_x(w_x\frac{\partial \rho}{\partial x}+w_y\frac{\partial \rho}{\partial y}+w_z\frac{\partial \rho}{\partial z})$ is omitted, and (2.23) is rewritten as generally

$$\begin{aligned}&\rho(\frac{\partial w_x}{\partial x}w_x+\frac{\partial w_x}{\partial y}w_y+\frac{\partial w_x}{\partial z}w_z)\\&=-\frac{\partial p}{\partial x}+2\frac{\partial}{\partial x}(\mu\frac{\partial w_x}{\partial x})+\frac{\partial}{\partial y}[\mu(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x})]+\frac{\partial}{\partial z}[\mu(\frac{\partial w_x}{\partial z}+\frac{\partial w_z}{\partial x})]\\&\quad-\frac{\partial}{\partial x}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}+\frac{\partial w_z}{\partial z})]+\rho g_x\end{aligned}\tag{2.23a}$$

Similarly, in general, (2.24) and (2.25) are rewritten as respectively

$$\rho(\frac{\partial w_y}{\partial x} w_x + \frac{\partial w_y}{\partial y} w_y + \frac{\partial w_y}{\partial z} w_z)$$
$$= -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}[\mu(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x})] + 2\frac{\partial}{\partial y}(\mu\frac{\partial w_y}{\partial y}) + \frac{\partial}{\partial z}[\mu(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y})] \tag{2.24a}$$
$$-\frac{\partial}{\partial y}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z})] + \rho g_y$$

$$\rho(\frac{\partial w_z}{\partial x} w_x + \frac{\partial w_z}{\partial y} w_y + \frac{\partial w_z}{\partial z} w_z)$$
$$= -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}[\mu(\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x})] + \frac{\partial}{\partial y}[\mu(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y})] + 2\frac{\partial}{\partial z}(\mu\frac{\partial w_z}{\partial z}) \tag{2.25a}$$
$$-\frac{\partial}{\partial z}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z})] + \rho g_z$$

2.3 Energy Equation

The control volume for derivation of the energy equation of fluid flow is shown in Fig. 2.3. Meanwhile, take an enclosed surface A that includes the control volume. According to the first law of thermodynamics, we have the following equation:

$$\Delta\dot{E} = \dot{Q} + \dot{W}_{out} \tag{2.26}$$

where $\Delta\dot{E}$ is energy increment in the system per unit time, $\dot{Q}$ is heat increment in the system per unit time, and $\dot{W}_{out}$ denotes work done by the mass force and surface force on the system per unit time.

The energy increment per unit time in the system is described as

$$\Delta\dot{E} = \frac{D}{D\tau}\int_V \rho(e + \frac{W^2}{2})\,dV \tag{2.27}$$

where τ denotes time, $\frac{W^2}{2}$ is the fluid kinetic energy per unit mass, W is fluid velocity, and the symbol e represents the internal energy per unit mass.

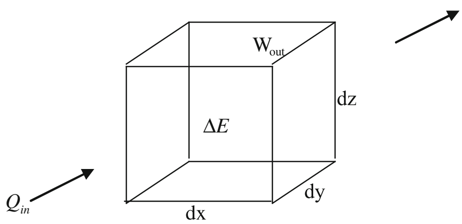

Fig. 2.3 Control volume for derivation of the energy equations of fluid flow

The work done by the mass force and surface force on the system per unit time is expressed as

$$\dot{W}_{out} = \int_V \rho \vec{F} \cdot \vec{W} \, dV + \int_A \vec{\tau_n} \cdot \vec{W} \, dA \tag{2.28}$$

where $\vec{F}$ is the mass force per unit mass, and $\vec{\tau_n}$ is surface force acting on unit area.

The heat increment entering into the system per unit time through thermal conduction is described by using Fourier's law as follows:

$$\dot{Q} = \int_A \lambda \frac{\partial t}{\partial n} dA \tag{2.29}$$

where n is normal line of the surface, t is temperature and here the heat conduction is considered only.

With Eqs. (2.27)–(2.29), Eq. (2.26) is rewritten as

$$\frac{D}{D\tau} \int_V \rho(e + \frac{W^2}{2}) \, dV = \int_V \rho \vec{F} \cdot \vec{W} \, dV + \int_A \vec{\tau_n} \cdot \vec{W} \, dA + \int_A \lambda \frac{\partial t}{\partial n} dA \tag{2.30}$$

where

$$\frac{D}{D\tau} \int_V \rho(e + \frac{W^2}{2}) \, dV = \int_V \frac{D}{D\tau} [\rho(e + \frac{W^2}{2})] \, dV \tag{2.31}$$

$$\int_A \vec{\tau_n} \cdot \vec{W} \, dA = \int_A \vec{\vec{n}} [\tau] \cdot \vec{W} \, dA = \int_A \vec{\vec{n}} ([\tau] \cdot \vec{W}) \, dA = \int_V \nabla \cdot ([\tau] \cdot \vec{W}) \, dV \tag{2.32}$$

$$\int_A \lambda \frac{\partial t}{\partial n} dA = \int_V \nabla \cdot (\lambda \nabla t) \, dV \tag{2.33}$$

With Eqs. (2.31) to (2.33), Eq. (2.30) is rewritten as

$$\int_V \frac{D}{D\tau} [\rho(e + \frac{W^2}{2})] \, dV = \int_V \rho \vec{F} \cdot \vec{W} \, dV + \int_v \nabla \cdot ([\tau] \cdot \vec{W}) \, dV + \int_V \nabla \cdot (\lambda \nabla t) \, dV. \tag{2.34}$$

Then,

$$\frac{D}{D\tau} [\rho(e + \frac{W^2}{2})] = \rho \vec{F} \cdot \vec{W} + \nabla \cdot ([\tau] \cdot \vec{W}) + \nabla \cdot (\lambda \nabla t) \tag{2.35}$$

where $[\tau]$ denotes tensor of shear force.

Equation (2.35) is the energy equation.

Through tensor and vector analysis, Eq. (2.35) can be further derived into the following form:

$$\frac{D(\rho e)}{D\tau} = [\tau]\cdot[\varepsilon] + \nabla\cdot(\lambda\nabla t) \tag{2.36}$$

Equation (2.36) is another form of the energy equation. Here, $[\tau]\cdot[\varepsilon]$ is the scalar quantity product of force tensor $[\tau]$ and deformation rate tensor $[\varepsilon]$, and represents the work done by fluid deformation surface force. The physical significance of Eq. (2.36) is that the internal energy increment of fluid with unit volume during the unit time equals the sum of the work done by deformation surface force of fluid with unit volume, $[\tau]\cdot[\varepsilon]$, and the heat entering the system.

The general Newtonian law is expressed as

$$[\tau] = 2\mu[\varepsilon] - (p + \frac{2}{3}\mu\nabla\cdot\vec{W})[I] \tag{2.37}$$

where $[I]$ is unit tensor.

According to Eq. (2.37) the following equation can be obtained:

$$[\tau]\cdot[\varepsilon] = -p\nabla\cdot\vec{W} - \frac{2}{3}\mu(\nabla\cdot\vec{W})^2 + 2\mu[\varepsilon]^2 \tag{2.38}$$

Then, Eq. (2.36) can be rewritten as

$$\frac{D(\rho e)}{D\tau} = -p\nabla\cdot\vec{W} + \Phi + \nabla\cdot(\lambda\nabla t) \tag{2.39}$$

where $\Phi = -\frac{2}{3}\mu(\nabla\cdot\vec{W})^2 + 2\mu[\varepsilon]^2$ is viscous dissipation function, which is further described as

$$\Phi = \mu\{2(\frac{\partial w_x}{\partial x})^2 + 2(\frac{\partial w_y}{\partial y})^2 + 2(\frac{\partial w_z}{\partial z})^2 + (\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x})^2 + (\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y})^2 + (\frac{\partial w_z}{\partial x} + \frac{\partial w_x}{\partial z})^2 - \frac{2}{3}[div(\vec{W})]^2\} \tag{2.40}$$

Equation (2.6) can be rewritten as

$$\nabla\cdot\vec{W} = -\frac{1}{\rho}\frac{D\rho}{D\tau} = \rho\frac{D}{D\tau}(\frac{1}{\rho})$$

With the above equation, Eq. (2.39) is changed into the following form:

$$[\frac{D(\rho e)}{D\tau} + p\rho\frac{D}{D\tau}(\frac{1}{\rho})] = \Phi + \nabla\cdot(\lambda\nabla t) \tag{2.41}$$

According to thermodynamics equation of fluid

$$\frac{D(\rho h)}{D\tau} = \frac{D(\rho e)}{D\tau} + p\rho\frac{D}{D\tau}(\frac{1}{\rho}) + \frac{Dp}{D\tau} \tag{2.42}$$

Equation (2.41) can be expressed as the following enthalpy form:

$$\frac{D(\rho h)}{D\tau} = \frac{Dp}{D\tau} + \Phi + \nabla \cdot (\lambda \nabla t) \tag{2.43}$$

or

$$\frac{D(\rho c_p t)}{D\tau} = \frac{Dp}{D\tau} + \Phi + \nabla \cdot (\lambda \nabla t) \tag{2.44}$$

where $h = c_p t$, while c_p is specific heat.

In Cartesian form, the energy Eq. (2.44) can be rewritten as

$$\begin{aligned}&\frac{\partial(\rho c_p t)}{\partial \tau} + w_x\frac{\partial(\rho c_p t)}{\partial x} + w_y\frac{\partial(\rho c_p t)}{\partial y} + w_z\frac{\partial(\rho c_p t)}{\partial z}\\ &= \frac{Dp}{D\tau} + \frac{\partial}{\partial x}(\lambda\frac{\partial t}{\partial x}) + \frac{\partial}{\partial y}(\lambda\frac{\partial t}{\partial y}) + \frac{\partial}{\partial z}(\lambda\frac{\partial t}{\partial z}) + \Phi\end{aligned} \tag{2.45}$$

For steady state and nearly constant pressure processes, the viscous dissipation can be ignored, and then the Cartesian form of the energy equation (2.45) is changed into

$$w_x\frac{\partial(\rho c_p t)}{\partial x} + w_y\frac{\partial(\rho c_p t)}{\partial y} + w_z\frac{\partial(\rho c_p t)}{\partial z} = \frac{\partial}{\partial x}(\lambda\frac{\partial t}{\partial x}) + \frac{\partial}{\partial y}(\lambda\frac{\partial t}{\partial y}) + \frac{\partial}{\partial z}(\lambda\frac{\partial t}{\partial z}) \tag{2.46}$$

Above equation is usually approximately rewritten as

$$\rho[w_x\frac{\partial(c_p t)}{\partial x} + w_y\frac{\partial(c_p t)}{\partial y} + w_z\frac{\partial(c_p t)}{\partial z}] = \frac{\partial}{\partial x}(\lambda\frac{\partial t}{\partial x}) + \frac{\partial}{\partial y}(\lambda\frac{\partial t}{\partial y}) + \frac{\partial}{\partial z}(\lambda\frac{\partial t}{\partial z}) \tag{2.46a}$$

or

$$\rho c_p[w_x\frac{\partial t}{\partial x} + w_y\frac{\partial t}{\partial y} + w_z\frac{\partial t}{\partial z}] = \frac{\partial}{\partial x}(\lambda\frac{\partial t}{\partial x}) + \frac{\partial}{\partial y}(\lambda\frac{\partial t}{\partial y}) + \frac{\partial}{\partial z}(\lambda\frac{\partial t}{\partial z}) \tag{2.46b}$$

In fact, in Eq. (2.46a) the temperature-dependent density is ignored, and in (2.46b) both the temperature-dependent density and specific heat are ignored.

2.4 Conservation Equations of Laminar Mixed Convection Boundary Layer

2.4.1 Continuity Equation

Based on the Eq. (2.7), the steady state three-dimensional continuity equation is given by

$$\frac{\partial}{\partial x}(\rho w_x) + \frac{\partial}{\partial y}(\rho w_y) + \frac{\partial}{\partial z}(\rho w_z) = 0. \tag{2.47}$$

While, the steady state two-dimensional continuity equation is given by

$$\frac{\partial}{\partial x}(\rho w_x) + \frac{\partial}{\partial y}(\rho w_y) = 0 \tag{2.48}$$

In Eqs. (2.47) and (2.48) variable fluid density with temperature is considered.

Before the quantitative grade analysis on the above equations, it is necessary to define its analytical standard. A normal quantitative grade is regarded as $\{1\}$, i.e. unit quantity grade, a very small quantitative grade is regarded as $\{\delta\}$, even very small quantitative grade is regarded as $\{\delta^2\}$, and so on. The ration of the quantities is easily defined, and some examples of ratios are introduced as follows:

$$\frac{\{1\}}{\{1\}} = \{1\}, \frac{\{\delta\}}{\{\delta\}} = \{1\}, \frac{\{1\}}{\{\delta\}} = \{\delta^{-1}\}, \frac{\{1\}}{\{\delta^2\}} = \{\delta^{-2}\}$$

According to the theory of laminar free boundary layer, the quantities of the velocity component w_x and the coordinate x can be regarded as unity, i.e. $\{w_x\} = \{1\}$ and $\{x\} = \{1\}$. However, the quantities of the velocity component w_y and the coordinate y should be regarded as δ, i.e. $\{w_y\} = \{\delta\}$ and $\{y\} = \{\delta\}$.

For the terms of Eq. (2.48) the following ratios of quantity grade are obtained: $\frac{\{\rho w_x\}}{\{x\}} = \frac{\{1\}}{\{1\}} = \{1\}$ and $\frac{\{\rho w_y\}}{\{y\}} = \frac{\{\delta\}}{\{\delta\}} = \{1\}$. Therefore both the two terms of Eq. (2.48) should be kept, and Eq. (2.48) can be regarded as the continuity equation of the steady state laminar two-dimensional boundary layers. Of course, Eq. (2.48) is also suitable for the steady state two-dimensional boundary layers with laminar free convection.

2.4.2 Momentum Equation (Navier-Stokes Equation)

According to Eqs. (2.23a) and (2.24a), the momentum equations for steady two-dimensional convection are

$$\rho(w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y})$$
$$= -\frac{\partial p}{\partial x} + 2\frac{\partial}{\partial x}(\mu \frac{\partial w_x}{\partial x}) + \frac{\partial}{\partial y}[\mu(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x})] - \frac{\partial}{\partial x}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y}] + \rho g_x \tag{2.49}$$

$$\rho(w_x \frac{\partial w_y}{\partial x} + w_y \frac{\partial w_y}{\partial y})$$
$$= -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}[\mu(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x})] + 2\frac{\partial}{\partial y}(\mu \frac{\partial w_y}{\partial y}) - \frac{\partial}{\partial y}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y})] + \rho g_y \tag{2.50}$$

According to the theory of boundary layer, the quantity grade of the pressure gradient $\frac{\partial p}{\partial x}$ can be regarded as unity, i.e. $\{\frac{\partial p}{\partial x}\} = \{1\}$, but the quantity grade of the pressure gradient $\frac{\partial p}{\partial y}$ is only regarded as very small quantity grade, i.e. $\{\frac{\partial p}{\partial y}\} = \{\delta\}$.

The quantity grades of the terms of Eqs. (2.49) and (2.50) are expressed as follows respectively:

$$\begin{aligned} \rho(w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y}) &= -\frac{\partial p}{\partial x} + 2\frac{\partial}{\partial x}(\mu \frac{\partial w_x}{\partial x}) \\ &\quad + \frac{\partial}{\partial y}[\mu(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x})] - \frac{\partial}{\partial x}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y})] + \rho g_x \\ \{1\}(\{1\}\frac{\{1\}}{\{1\}} + \{\delta\}\frac{\{1\}}{\{\delta\}}) &= -\{1\} + \frac{\{1\}}{\{1\}}\{\delta^2\}\frac{\{1\}}{\{1\}} + \frac{\{1\}}{\{\delta\}}\{\delta^2\}(\frac{\{1\}}{\{\delta\}} + \frac{\{\delta\}}{\{1\}}) \\ &\quad - \frac{\{1\}}{\{1\}}\delta^2(\frac{\{1\}}{\{1\}} + \frac{\{\delta\}}{\{\delta\}}) + \{1\}\{1\} \end{aligned} \tag{2.49a}$$

$$\begin{aligned} \rho(w_x \frac{\partial w_y}{\partial x} + w_y \frac{\partial w_y}{\partial y}) &= -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}[\mu(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x})] \\ &\quad + 2\frac{\partial}{\partial y}(\mu \frac{\partial w_y}{\partial y}) - \frac{\partial}{\partial y}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y})] + \rho g_y \\ \{1\}(\{1\}\frac{\{\delta\}}{\{1\}} + \{\delta\}\frac{\{\delta\}}{\{\delta\}}) &= -\{\delta\} + \frac{\{1\}}{\{1\}}\{\delta^2\}\ (\frac{\{1\}}{\{\delta\}} + \frac{\{\delta\}}{\{1\}}) + \frac{\{1\}}{\{\delta\}}\{\delta^2\}\frac{\{\delta\}}{\{\delta\}} \\ &\quad - \frac{\{1\}}{\{\delta\}}\{\delta^2\}(\frac{\{1\}}{\{1\}} + \frac{\{\delta\}}{\{\delta\}}) + \{1\}\{\delta\} \end{aligned} \tag{2.50a}$$

The quantity grades of Eqs. (2.49a) and (2.50a) are simplified as follows respectively:

$$\begin{aligned}\rho(w_x\frac{\partial w_x}{\partial x}+w_y\frac{\partial w_x}{\partial y}) &= -\frac{\partial p}{\partial x}+2\frac{\partial}{\partial x}(\mu\frac{\partial w_x}{\partial x})\\ &\quad+\frac{\partial}{\partial y}[\mu(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x})]-\frac{\partial}{\partial x}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y})]+\rho g_x\\ \{1\}(\{1\}+\{1\}) &= -\{1\}+\{\delta^2\}+\{1\}+\{\delta^2\}\\ &\quad-(\{\delta^2\}+\{\delta^2\})+\{1\}\end{aligned} \tag{2.49b}$$

$$\begin{aligned}\rho(w_x\frac{\partial w_y}{\partial x}+w_y\frac{\partial w_y}{\partial y}) &= -\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}[\mu(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x})]\\ &\quad+2\frac{\partial}{\partial y}(\mu\frac{\partial w_y}{\partial y})-\frac{\partial}{\partial y}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y})]+\rho g_y\\ \{1\}(\{\delta\}+\{\delta\}) &= -\{\delta\}+(\{\delta\}+\{\delta^3\})+\{\delta\}\\ &\quad-(\{\delta\}(\{1\}+\{1\}))+\{\delta\}\end{aligned} \tag{2.50b}$$

Observing the quantity grades in Eq. (2.49b) it is found that the terms $2\frac{\partial}{\partial x}(\mu\frac{\partial w_x}{\partial x})$, and $\frac{\partial w_y}{\partial x}$ in term $\frac{\partial}{\partial y}[\mu(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x})]$, and $\frac{\partial}{\partial x}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y})]$ are very small and can be ignored from Eq. (2.49). Then, Eq. (2.49) is simplified as follows:

$$\rho(w_x\frac{\partial w_x}{\partial x}+w_y\frac{\partial w_x}{\partial y}) = -\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}(\mu\frac{\partial w_x}{\partial y})+\rho g_x \tag{2.51}$$

Comparing the quantity grades of Eq. (2.49b) with those of Eq. (2.50b), it is found that the quantity grades of Eq. (2.50b) are very small. Then, Eq. (2.50) can be ignored, and only Eq. (2.51) is taken as the momentum equation of two-dimensional boundary layer convection.

For consideration of the buoyancy term ρg_x, Eq. (2.51) actually the momentum equation of free convection. If free convection is located on inclined plate the gravity acceleration component g_x is expressed as

$$g_x = g\cdot\cos\alpha \tag{2.52}$$

where g is gravity acceleration and α is the inclined angle of the plate.

With Eq. (2.52), Eq. (2.51) is rewritten as

$$\rho(w_x\frac{\partial w_x}{\partial x}+w_y\frac{\partial w_x}{\partial y}) = -\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}(\mu\frac{\partial w_x}{\partial y})+\rho g\cdot\cos\alpha \tag{2.53}$$

Suppose the direction of $g \cdot \cos\alpha$ is reverse to that of the velocity component w_x, Eq. (2.53) can be rewritten as

$$\rho(w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y}) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}(\mu \frac{\partial w_x}{\partial y}) - \rho g \cdot \cos\alpha \tag{2.54}$$

Beyond the boundary layer, where the effects of viscosity can be ignored, the momentum equation (2.54) is simplified into the following equation:

$$-\frac{dp}{dx} = \rho_\infty g \cdot \cos\alpha + \rho_\infty w_{x,\infty} \frac{dw_{x,\infty}}{dx} \tag{2.55}$$

where ρ_∞ and $w_{x,\infty}$ are fluid density and velocity component beyond the boundary layer.

With Eq. (2.55), Eq. (2.54) becomes

$$\rho(w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y}) = \frac{\partial}{\partial y}(\mu \frac{\partial w_x}{\partial y}) + g(\rho_\infty - \rho)\cos\alpha + \rho_\infty w_{x,\infty} \frac{dw_{x,\infty}}{dx} \tag{2.56}$$

For constant $w_{x,\infty}$ the Eq. (2.56) transforms to

$$\rho(w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y}) = \frac{\partial}{\partial y}(\mu \frac{\partial w_x}{\partial y}) + g(\rho_\infty - \rho)\cos\alpha \tag{2.57}$$

This is the momentum equation of two-dimensional boundary layer on an inclined plate with laminar convection.

Equation (2.57) can be rewritten as

$$\rho(w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y}) = \frac{\partial}{\partial y}(\mu \frac{\partial w_x}{\partial y}) + g|\rho_\infty - \rho|\cos\alpha \tag{2.57a}$$

In (2.57a), the absolute value of buoyancy factor $|\rho_\infty - \rho|$ shows that the buoyancy term $g|\rho_\infty - \rho|\cos\alpha$ has always positive sign no matter which one is larger between ρ and ρ_∞. In this case, the buoyancy term $g|\rho_\infty - \rho|\cos\alpha$ and the velocity component w_x have same sign.

For the free convection of a perfect gas (ideal gas) the following simple power law can be used:

$\frac{\rho_\infty}{\rho} = \frac{T}{T_\infty}$ where T denotes absolute temperature. In fact, for general real gas, this relation is also available. Therefore,

$$|\rho_\infty - \rho|\cos\alpha = \rho\left|\frac{T}{T_\infty} - 1\right|\cos\alpha \tag{2.58}$$

Thus, for the laminar free convection of a perfect gas, Eq. (2.57) can be changed into

$$\rho(w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y}) = \frac{\partial}{\partial y}(\mu \frac{\partial w_x}{\partial y}) + g\rho \left| \frac{T}{T_\infty} - 1 \right| \cos\alpha \tag{2.59}$$

If the temperature difference $|T_w - T_\infty|$ is very small, which will lead to a very small density difference $|\rho_\infty - \rho_w|$, the Boussinesq approximation can be applied. In this case, buoyancy factor in Eq. (2.57a) becomes $|\rho_\infty - \rho| = \rho\beta|T - T_\infty|$, and then, Eq. (2.57a) is changed to

$$w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y} = \nu \frac{\partial^2 w_x}{\partial y^2} + g\beta|T - T_\infty| \cos\alpha \tag{2.57b}$$

2.4.3 Energy Equation

According to Eq. (2.46a), the energy equation for steady two-dimensional convection is shown as follows:

$$\rho[w_x \frac{\partial(c_p t)}{\partial x} + w_y \frac{\partial(c_p t)}{\partial y}] = \frac{\partial}{\partial x}(\lambda \frac{\partial t}{\partial x}) + \frac{\partial}{\partial y}(\lambda \frac{\partial t}{\partial y}) \tag{2.60}$$

With the quantity grade analysis similar to that mentioned above, Eq. (2.60) can be changed into the following form for energy equation of two-dimensional boundary layer convection.

$$\rho[w_x \frac{\partial(c_p t)}{\partial x} + w_y \frac{\partial(c_p t)}{\partial y}] = \frac{\partial}{\partial y}(\lambda \frac{\partial t}{\partial y}) \tag{2.61}$$

Up to now it is the time to summarize the basic governing equations for description of mass, momentum, and energy conservation of two-dimensional boundary layers with laminar steady state mixed free and forced convection as follows:

$$\frac{\partial}{\partial x}(\rho w_x) + \frac{\partial}{\partial y}(\rho w_y) = 0 \tag{2.48}$$

$$\rho(w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y}) = \frac{\partial}{\partial y}(\mu \frac{\partial w_x}{\partial y}) + g(\rho_\infty - \rho) \cos\alpha \tag{2.57}$$

$$\rho[w_x \frac{\partial(c_p t)}{\partial x} + w_y \frac{\partial(c_p t)}{\partial y}] = \frac{\partial}{\partial y}(\lambda \frac{\partial t}{\partial y}) \tag{2.61}$$

with convection boundary condition equations

$$y = 0 : w_x = 0, w_y = 0, t = t_w \tag{2.62}$$

$$y \to \infty : w_x = w_{x,\infty}, t = t_\infty \tag{2.63}$$

For rigorous solutions of the governing equations, the fluid temperature-dependent properties, such as density ρ in mass equation and in buoyancy factor of momentum equation, absolute viscosity μ, specific heat c_p, and thermal conductivity λ will be considered in the successive chapters of this book. In addition, t_w is plate temperature, t_∞ is the fluid temperature beyond the boundary layer, and $w_{x,\infty}$ denotes the fluid velocity component in x-direction beyond the boundary layer. Here, the boundary conditions are the isothermal plate boundary conditions.

It is known from the above governing equations and boundary conditions that the laminar mixed convection with two-dimensional boundary layer belongs to two-point boundary value problem.

2.5 Summary

Up to now the related governing partial differential conservation equations can be summarized as follows:

2.5.1 Governing Partial Differential Conservation Equations of Laminar Convection with Consideration of Variable Physical Properties

Governing partial differential conservation equations in rectangular coordinate system for laminar convection with consideration of variable physical properties are

Mass equation:

$$\frac{\partial}{\partial x}(\rho w_x) + \frac{\partial}{\partial y}(\rho w_y) + \frac{\partial}{\partial z}(\rho w_z) = 0$$

Momentum equation:

$$\begin{aligned}
&\rho(\frac{\partial w_x}{\partial x} w_x + \frac{\partial w_x}{\partial y} w_y + \frac{\partial w_x}{\partial z} w_z) \\
&= -\frac{\partial p}{\partial x} + 2\frac{\partial}{\partial x}(\mu \frac{\partial w_x}{\partial x}) + \frac{\partial}{\partial y}[\mu(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x})] + \frac{\partial}{\partial z}[\mu(\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x})] \\
&\quad - \frac{\partial}{\partial x}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z})] + \rho g_x
\end{aligned}$$

$$\rho(\frac{\partial w_y}{\partial x}w_x + \frac{\partial w_y}{\partial y}w_y + \frac{\partial w_y}{\partial z}w_z)$$
$$= -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}[\mu(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x})] + 2\frac{\partial}{\partial y}(\mu\frac{\partial w_y}{\partial y}) + \frac{\partial}{\partial z}[\mu(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y})]$$
$$- \frac{\partial}{\partial y}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z})] + \rho g_y$$

$$\rho(\frac{\partial w_z}{\partial x}w_x + \frac{\partial w_z}{\partial y}w_y + \frac{\partial w_z}{\partial z}w_z)$$
$$= -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}[\mu(\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x})] + \frac{\partial}{\partial y}[\mu(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y})] + 2\frac{\partial}{\partial z}(\mu\frac{\partial w_z}{\partial z})$$
$$- \frac{\partial}{\partial z}[\frac{2}{3}\mu(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z})] + \rho g_z$$

Energy equation:

$$\rho[w_x\frac{\partial(c_p \cdot t)}{\partial x} + w_y\frac{\partial(c_p \cdot t)}{\partial y} + w_z\frac{\partial(c_p \cdot t)}{\partial z}] = \frac{\partial}{\partial x}(\lambda\frac{\partial t}{\partial x}) + \frac{\partial}{\partial y}(\lambda\frac{\partial t}{\partial y}) + \frac{\partial}{\partial z}(\lambda\frac{\partial t}{\partial z}) + \Phi$$

$$\Phi = \mu\{2(\frac{\partial w_x}{\partial x})^2 + 2(\frac{\partial w_y}{\partial y})^2 + 2(\frac{\partial w_z}{\partial z})^2 + (\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x})^2$$
$$+ (\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y})^2 + (\frac{\partial w_z}{\partial x} + \frac{\partial w_x}{\partial z})^2] - \frac{2}{3}[div(\vec{W})]^2\}$$

2.5.2 Governing Partial Differential Equations of Laminar Mixed Convection Boundary Layer with Consideration of Variable Physical Properties

Governing partial differential equations in rectangular coordinate system for two-dimensional laminar mixed convection boundary layer with consideration of variable physical properties are

Mass equation:

$$\frac{\partial}{\partial x}(\rho w_x) + \frac{\partial}{\partial y}(\rho w_y) = 0$$

Momentum equation:

$$\rho(w_x\frac{\partial w_x}{\partial x}+w_y\frac{\partial w_x}{\partial y})=\frac{\partial}{\partial y}(\mu\frac{\partial w_x}{\partial y})+g|\rho_\infty-\rho|\cos\alpha$$

Energy equation:

$$\rho[w_x\frac{\partial(c_pt)}{\partial x}+w_y\frac{\partial(c_pt)}{\partial y}]=\frac{\partial}{\partial y}(\lambda\frac{\partial t}{\partial y})$$

Boundary conditions:

$$y=0: w_x=0, w_y=0, t=t_w$$

$$y\to\infty: w_x=w_{x,\infty}, t=t_\infty$$

2.5.3 Governing Partial Differential Equations of Laminar Mixed Convection Boundary Layer for Boussinesq Approximation

Governing partial differential equations in rectangular coordinate system for two-dimensional laminar mixed convection boundary layer for Boussinesq approximation are

Mass equation:

$$\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}=0$$

Momentum equation:

$$w_x\frac{\partial w_x}{\partial x}+w_y\frac{\partial w_x}{\partial y}=v\frac{\partial^2 w_x}{\partial y^2})+g\beta|T-T_\infty|\cos\alpha$$

Energy equation:

$$w_x\frac{\partial t}{\partial x}+w_y\frac{\partial t}{\partial y}=\frac{v}{\mathrm{Pr}}\frac{\partial^2 t}{\partial y^2}$$

Boundary conditions:

$$y=0: w_x=0, w_y=0, t=t_w$$

$$y\to\infty: w_x=w_{x,\infty}, t=t_\infty$$

2.6 Remarks

In this chapter, the theoretical foundation of conservation equations on laminar mixed convection are introduced. For this purpose, first, the conservation equations of general laminar convection on continuity, momentum and energy are derived with a series of theoretical and mathematical approaches. On this basis, the corresponding conservation equations of mass, momentum, and energy for laminar mixed convection boundary layer are obtained by the quantity grade analysis. In these derived conservation equations, variable physical properties are fully taken into account. These equations are important theoretical foundation of the study for this book.

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