

Chapter 2

Background

2.1 Basic Definitions

Denote the set of integers (reals) as $\mathbb{I}$ ($\mathbb{R}$), the set of non-negative integers (reals) as $\mathbb{I}_{\geq 0}$ ($\mathbb{R}_{\geq 0}$) and the set of positive integers (reals) as $\mathbb{I}_{> 0}$ ($\mathbb{R}_{> 0}$). The integers from 0 to N are denoted as $\mathbb{I}_{[0, N]}$. Given a square matrix A and the scalar λ_i denoting the i th eigenvalue of A , then $\lambda_{\max}(A) = \max_i |\lambda_i|$, $\lambda_{\min}(A) = \min_i |\lambda_i|$. The notation $\text{BlockDiag}(A, B)$ denotes a block diagonal matrix, and $\otimes$ denotes the Kronecker product. The identity matrix of size N is denoted by I_N , while $\underline{1}_m$ is a unit column vector of m elements. A block of zeros in a matrix is simply denoted by 0. Matrix dimensions are omitted if they can be inferred from the context. In the thesis, $A \succ 0$ (respectively, $\succeq$) denote that the matrix A is positive (semi)definite while $\geq$ (respectively, $>$) is used for scalar (in)equalities. The scalar “positive-wrapping” operator is defined as

$$(a)_+ = \max(a, 0). \quad (2.1.1)$$

The following definitions are used throughout the thesis:

Definition 2.1.1 ($\mathcal{D}_{++}$ -matrix) A diagonal matrix S is said to be of class $\mathcal{D}_{++}$ if its entries are strictly positive.

Definition 2.1.2 (Affine function) A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *affine* if it has the form linear plus constant, $f(x) = Ax + b$.

Definition 2.1.3 ((Generalised) quadratic function) A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a (generalised) *quadratic* if it has the form

$$F(x) = x^T P x + 2d^T x + e. \quad (2.1.2)$$

Definition 2.1.4 A (generalised) quadratic function of the form (2.1.2) is *non-degenerate* iff $P = P^T \succ 0$, $e - d^T P^{-1} d < 0$.

2.2 Convex Sets

Convex sets are generally used to represent convex constraints, the domain of convex functions and level sets. In the following, a set of definition will be given for convex sets and functions, as in Hindi (2004). For an extensive discussion refer to Boyd and Vandenberghe (2004), Hindi (2004).

Definition 2.2.1 (*Convex set*) A set $\mathcal{S} \subseteq \mathbb{R}^n$ is *convex* if it contains the line segment joining any two of its points, that is

$$x, y \in \mathcal{S}, \quad \theta \in [0, 1] \Rightarrow \theta x + (1 - \theta)y \in \mathcal{S}. \quad (2.2.1)$$

Definition 2.2.2 (*Convex cone*) A set $\mathcal{S} \subseteq \mathbb{R}^n$ is a *convex cone* if it contains all rays passing through its points, which emanate from the origin, as well as all line segments joining any points of those rays, i.e.

$$x, y \in \mathcal{S}, \quad \alpha, \beta \geq 0 \Rightarrow \alpha x + \beta y \in \mathcal{S}. \quad (2.2.2)$$

Definition 2.2.3 (*Norm ϵ -ball*) A *norm ϵ -ball* is a convex set, $\mathcal{B}_{p,\epsilon}(x_c) = \{x \mid \|x - x_c\|_p \leq \epsilon\}$, $\epsilon > 0$, where $\|\cdot\|_p$ is an ℓ_p -norm.

Definition 2.2.4 (*Norm cone*) $\mathcal{C}_p = \{(x, t) \mid \|x\|_p \leq t\}$, where $\|\cdot\|_p$ is an ℓ_p -norm.

Definition 2.2.5 (*Closed set*) A set $\mathcal{S} \subseteq \mathbb{R}^n$ is *closed* if every point not in $\mathcal{S}$ has a neighbourhood disjoint from $\mathcal{S}$, that is

$$\forall x \notin \mathcal{S}, \quad \exists \epsilon > 0 : \mathcal{B}_{p,\epsilon}(x) \cap \mathcal{S} = \emptyset. \quad (2.2.3)$$

Definition 2.2.6 (*Bounded set*) A set $\mathcal{S} \subseteq \mathbb{R}^n$ is *bounded* if it is contained inside some ball, $\mathcal{B}_{p,r}(x)$, of finite radius r , that is

$$\exists r > 0, \quad x \in \mathbb{R}^n : \mathcal{S} \subseteq \mathcal{B}_{p,r}(x). \quad (2.2.4)$$

Definition 2.2.7 (*Compact set*) A set $\mathcal{S} \subseteq \mathbb{R}^n$ is *compact* if it is closed and bounded.

Definition 2.2.8 (*C-set*) A set $\mathcal{S} \subset \mathbb{R}^n$ is said to be a *C-set* if it is convex and compact.

2.2.1 Set Operators

Proofs of stated lemmas are omitted for brevity. Refer to Blanchini (1999).

Definition 2.2.9 (*Convex Hull*) Given a set $\mathcal{S} \subseteq \mathbb{R}^n$, its *convex hull*, $\mathbf{conv}(\mathcal{S}) \subset \mathbb{R}^n$, is the smallest convex set containing $\mathcal{S}$.

Definition 2.2.10 (*Set union*) Given two sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$, their *union*, $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$, verifies $x \in \mathcal{C} \Leftrightarrow x \in \mathcal{A} \vee x \in \mathcal{B}$.

Definition 2.2.11 (*Set intersection*) Given two sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$, their *intersection*, $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$, verifies $x \in \mathcal{C} \Leftrightarrow x \in \mathcal{A} \wedge x \in \mathcal{B}$.

Lemma 2.2.1 *Convexity is preserved under intersection. The intersection of convex sets of a given type provides a convex set also of the same type.*

Definition 2.2.12 (*Set scaling*) Given the convex set $\mathcal{S} \subseteq \mathbb{R}^n$ and the scalar $\alpha \geq 0$, then $\alpha\mathcal{S} = \{x = \alpha y, y \in \mathcal{S}\}$.

Definition 2.2.13 (*Set difference*) Given two C-sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$, their *set difference* is $\mathcal{A} \setminus \mathcal{B} = \{x \mid x \in \mathcal{A}, x \notin \mathcal{B}\}$.

Definition 2.2.14 (*Minkowski sum*) Given two sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$, their *Minkowski sum* is $\mathcal{A} \oplus \mathcal{B} = \{x = a + b \in \mathbb{R}^n \mid a \in \mathcal{A}, b \in \mathcal{B}\}$.

Definition 2.2.15 (*Set erosion or Pontryagin difference*) The *Pontryagin difference* of two sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$ is $\mathcal{A} \sim \mathcal{B} = \{x \in \mathbb{R}^n \mid x + b \in \mathcal{A} \forall b \in \mathcal{B}\}$.

Lemma 2.2.2 *The set erosion and Minkowski sum satisfy*

$$(\mathcal{A} \sim \mathcal{B}) \oplus \mathcal{B} \subseteq \mathcal{A}.$$

Definition 2.2.16 (*Projection on a subset*) Given two sets $\mathcal{A} \subseteq \mathbb{R}^n, \mathcal{B} \subseteq \mathbb{R}^n$, the *projection* of $\mathcal{A}$ on $\mathcal{B}$ is

$$\pi_{\mathcal{B}}(\mathcal{A}) = \{b \in \mathcal{B} : \exists a \in \mathcal{A} \mid a = b + c, c^T b = 0\}.$$

Definition 2.2.17 (*Projection on a lower dimensional space*) Given a set $\mathcal{A} \subseteq \mathbb{R}^n$, the *projection* of $\mathcal{A}$ on $\mathbb{R}^m$, with $m < n$, is

$$\pi_m(\mathcal{A}) = \{b \in \mathbb{R}^m : \exists c \in \mathbb{R}^{n-m} \mid (b, c) \in \mathcal{A}\}.$$

Lemma 2.2.3 *The operations of Set scaling, Minkowski sum, Pontryagin difference and Projection on a subset or on a subspace preserve convexity, if the involved sets are convex.*

2.2.2 Commonly Referred Sets

Definition 2.2.18 (*Halfspace*) A closed *halfspace* in $\mathbb{R}^n$ is a convex set, $\mathcal{S} = \{x \mid a^T x \leq b\}$, with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. If $b = 0$, then $\mathcal{S}$ is also a convex cone.

Definition 2.2.19 (*Polytope (Polyhedron)*) A *polytope* is a *bounded polyhedron*, namely a bounded intersection of a finite number of halfspaces, $\mathbb{X} = \{x \mid Gx \leq \underline{1}\}$. A polytope can also be represented by its vertices, $v \in \mathcal{V}(\mathbb{X})$.

Definition 2.2.20 (*Polyhedral partition*) The collection of sets $\mathbf{part}^{\mathcal{P}} = \{\mathcal{P}_j\}_{j=1}^N$ with $N \in \mathbb{I}_{>0}$ is called a *polyhedral partition* of a set $\mathcal{P} \subseteq \mathbb{R}^n$ if all $\mathcal{P}_j \subseteq \mathbb{R}^n$ are polyhedra, $\cup_{j=1}^N \mathcal{P}_j = \mathcal{P}$ and $\mathbf{int}\mathcal{P}_i \cap \mathbf{int}\mathcal{P}_j = \emptyset \forall i \neq j, i, j \in \mathbb{I}_{[1,N]}$.

Definition 2.2.21 (*Ellipsoid*) An *ellipsoid* is a convex set, $\mathcal{E} = \{x \mid (x - x_c)^T E^{-1} (x - x_c) \leq 1\}$. It can also be expressed as a sublevel set of a non-degenerate generalised quadratic function

$$\mathcal{E} = \{x \mid F(x) \leq 0\}, \quad F(x) = x^T P x + 2d^T x + e, \quad P = P^T \succ 0, \quad e - d^T P^{-1} d < 0. \quad (2.2.5)$$

Definition 2.2.22 (*First-order cone*) A *first-order cone* is the norm cone associated with the ℓ_1 -norm, $\mathcal{C}_1 = \{(x, t) \mid \|x\|_1 \leq t\}$.

Definition 2.2.23 (*Second-order cone*) A *second-order cone* is the cone associated with the Euclidean norm, $\mathcal{C}_2 = \{(x, t) \mid \|x\|_2 \leq t\}$.

2.2.2.1 Quadratic Constraints and LMIs

A couple of commonly referred Lemmas are stated. Proofs are omitted for brevity. The interested reader can refer to Boyd and Vandenberghe (2004), Hindi (2004).

Lemma 2.2.4 *The second-order cone constraint (SOCC)*

$$\|Ax + b\|_2 \leq c^T x + d, \quad (2.2.6)$$

with $A \in \mathbb{R}^{k \times n}$, is convex. Moreover, it is equivalent to $(Ax + b, c^T x + d) \in \mathcal{S}_c$, where $\mathcal{S}_c$ is the second-order cone in $\mathbb{R}^{k+1}$.

Lemma 2.2.5 *The solution set of the Linear Matrix Inequality (LMI):*

$$F(x) = A_0 + x_1 A_1 + \cdots + x_m A_m \geq 0, \quad (2.2.7)$$

is convex. In particular, any SOCC can be expressed as an LMI, and vice versa.

2.3 Functions and Norms

The following definitions and Lemmas are given according to Boyd and Vandenberghe (2004), Lerner (2010), Robbins (n.d.), Walker (n.d.):

Definition 2.3.1 (*Extended function*) A function, $f : \mathbf{dom} f \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, is *extended* if it is assumed that $f(x) = \infty, \forall x \notin \mathbf{dom} f$.

Definition 2.3.2 (*Continuous function*) A function $f : \mathbf{dom} f \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is *continuous* on $\mathcal{X} \subseteq \mathbf{dom} f$ iff

$$\forall x_0 \in \mathcal{X}, \quad \forall \epsilon > 0, \quad \exists \delta(x_0, \epsilon) > 0 : \forall x \in \mathcal{X}, |x - x_0| < \delta(x_0, \epsilon) \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

Definition 2.3.3 (*Uniformly continuous function*) A function $f : \mathbf{dom} f \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is *uniformly continuous* on $\mathcal{X} \subseteq \mathbf{dom} f$ iff

$$\forall \epsilon > 0, \quad \exists \delta(\epsilon) > 0 : \forall x, x_0 \in \mathcal{X}, |x - x_0| < \delta(\epsilon) \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

Definition 2.3.4 (*Lipschitz continuous function*) A function $f : \mathbf{dom} f \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is *Lipschitz continuous* on $\mathcal{X} \subseteq \mathbf{dom} f$, with *Lipschitz constant* $L \in \mathbb{R}$, iff

$$\forall x, x_0 \in \mathcal{X}, \quad |f(x) - f(x_0)| < L|x - x_0|.$$

Lemma 2.3.1 A *Lipschitz continuous function* on $\mathcal{X}$ is also *uniformly continuous* on $\mathcal{X}$.

Lemma 2.3.2 A *uniformly continuous function* on $\mathcal{X}$ is also *continuous* on $\mathcal{X}$.

Theorem 2.3.3 (Heine–Cantor) A *function that is continuous on a compact set*, $\mathcal{X} \subset \mathbb{R}^n$ is *uniformly continuous* on $\mathcal{X}$.

Definition 2.3.5 (*Epigraph*) The *epigraph* of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the set of points above the function, namely

$$\mathbf{epi} f = \{(x, \alpha) \in \mathbf{dom} f \times \mathbb{R} : \alpha \geq f(x)\}. \quad (2.3.1)$$

Different definitions of convexity are given. The purpose of these definitions will be clarified in Sect. 2.4.1.

Definition 2.3.6 (*(Strictly) convex function*) A function $f : \mathbf{dom} f \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is *(strictly) convex* if its domain, $\mathcal{X} = \mathbf{dom} f$, is convex and if $\forall x, y \in \mathcal{X}, \forall \theta \in [0, 1]$

$$f(\theta x + (1 - \theta)y) \leq (<) \theta f(x) + (1 - \theta)f(y). \quad (2.3.2)$$

Lemma 2.3.4 A function $f : \mathbf{dom} f \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is *convex* iff $\mathbf{epi} f$ is a *convex set*.

Definition 2.3.7 (*Strongly convex function*) A function $f : \mathbf{dom} f \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is *strongly convex* on the closed convex set $\mathcal{X} \subseteq \mathbf{dom} f$ with respect to the ℓ_p -norm $\|\cdot\|_p$ if $\exists \sigma > 0 : \forall x, y \in \mathcal{X}, \forall \theta \in [0, 1]$

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y) - \frac{1}{2}\sigma\theta(1 - \theta)\|x - y\|_p^2, \quad (2.3.3)$$

where σ is called the *convexity parameter* of $f(\cdot)$.

Lemma 2.3.5 *Given a strongly convex function $f : \text{dom } f \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, and a convex function $g : \text{dom } g \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, on the set $\mathcal{X}$, then $h(x) = f(x) + g(x)$ is strongly convex on $\mathcal{X}$.*

Proof The result follows trivially from application of the respective definitions. ■

The following definition is based on Kakade et al. (2012). Refer to Vinter (2000) for a more general discussion, concerning a broader class of functions.

Definition 2.3.8 (*Sub-differential of a convex function*) Given a convex (extended) function, $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$, it is *sub-differential* at $x \in \mathcal{X}$ if

$$\partial f(x) = \{\mathbf{v} \in \mathcal{X} \mid \forall z, f(x+z) \geq f(x) + \mathbf{v}^T z\}.$$

Moreover, if $x \in \mathcal{X}$, then $\mathbf{v} \in \partial f(x)$ is said to be a *subgradient* of $f(\cdot)$ at x .

Definition 2.3.9 (*Sub-differential of the ℓ_1 and ℓ_∞ -norm*) The p -norm sub-differential, evaluated at u , is denoted $\partial \|u\|_p$. For $p = 1$ any subgradient $\mathbf{v} \in \partial \|u\|_1$ satisfies (Osborne et al. 2000)

$$\mathbf{v}_i = \begin{cases} \text{sign}(u_i) & \text{if } |u_i| > 0, \\ \alpha_i \in [-1, 1] & \text{if } u_i = 0. \end{cases} \quad (2.3.4)$$

For $p = \infty$, we have (using the arguments of Sect. 3.3 of Boyd and Vandenberghe 2007)

$$\partial \|u\|_\infty = \text{conv}(\cup \{\partial |u_i| \mid |u_i| = \|u\|_\infty\}). \quad (2.3.5)$$

Remark 2.3.1 It can be shown, for $p = 1, \infty$, that any $\mathbf{v} \in \partial \|u\|_p$ satisfies $\mathbf{v}^T u = \|u\|_p$.

Definition 2.3.10 (*Differentiable function*) An extended function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ is *differentiable* at $x \in \mathcal{X}$ iff its sub-differential at x is a *singleton*, namely $\partial f(x) = \{\nabla f(x)\}$. If this happens $\forall x \in \mathcal{X}$, then f is *differentiable on $\mathcal{X}$* .

Convexity of differentiable functions can be checked using the following conditions (Boyd and Vandenberghe 2004; Hindi 2004):

Definition 2.3.11 (*First-order convexity condition*) A differentiable function $f : \mathcal{X} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is *convex* if and only if, for all $x, x_0 \in \mathcal{X}$:

$$f(x) \geq f(x_0) + \nabla f(x_0)^T (x - x_0), \quad (2.3.6)$$

where $\partial f(x) = \{\nabla f(x)\}$, $\forall x \in \mathcal{X}$. Namely, the first-order Taylor approximation of f is a *global underestimator*.

Definition 2.3.12 (*Second-order (strict) convexity condition*) A twice differentiable function $f : \mathcal{X} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is (strictly) *convex* if and only if, for all $x \in \mathbf{dom} f$, $\nabla^2 f(x) \succeq 0$ ($\succ 0$), where $\partial \nabla f(x) = \{\nabla^2 f(x)\}$. In other words, the Hessian of f is positive semidefinite (positive definite) on $\mathcal{X}$.

Definition 2.3.13 (*Piece-wise affine (quadratic) function*) A function $f : \mathcal{P} \rightarrow \mathbb{R}^m$, with $\mathcal{P} \subseteq \mathbb{R}^n$ is piece-wise affine (PWA), respectively, piece-wise quadratic (PWQ) with $m = 1$, if given a partition **part** $\mathcal{P}$ it holds that, respectively,

$$\begin{aligned} f(x) &= A_j x + b_j, \quad \Leftarrow x \in \mathcal{P}_j, \quad \forall \mathcal{P}_j \in \mathbf{part} \mathcal{P}, \\ f(x) &= x^T P_j x + 2d_j^T x + e_j, \quad \Leftarrow x \in \mathcal{P}_j, \quad \forall \mathcal{P}_j \in \mathbf{part} \mathcal{P}, \end{aligned}$$

where $A_j \in \mathbb{R}^{m \times n}$, $b_j \in \mathbb{R}^m$, $P_j \in \mathbb{R}^{n \times n}$, $d_j \in \mathbb{R}^n$, $e_j \in \mathbb{R}$, $\forall j \in \mathbb{I}_{[1, N]}$.

Proofs of the following Lemmas are omitted for brevity. The interested reader can refer to Boyd and Vandenberghe (2004), Hindi (2004).

Definition 2.3.14 (*Functions composition*) Given two functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $g : \mathbb{R}^p \rightarrow \mathbb{R}^n$, then $f \circ g(x) = f(g(x))$.

Lemma 2.3.6 *The composition of two convex functions is convex.*

Lemma 2.3.7 *Affine transformation of domain preserves convexity, namely f convex $\Rightarrow f(Ax + b)$ convex.*

Definition 2.3.15 (*Set image under a map*) Given a set $\mathcal{A} \subseteq \mathbb{R}^n$ and a map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, the *image* of $\mathcal{A}$ under f is $f(\mathcal{A}) = \{y = f(x) \in \mathbb{R}^m \mid x \in \mathcal{A}\}$.

Similarly, one can define the set *preimage*, given an invertible map.

Lemma 2.3.8 *The image (or preimage) of a convex set under affine transformation is convex. If $\mathcal{S}$, $\mathcal{T}$ are convex, then so are*

$$f(\mathcal{T}) = \{Ax + b \mid x \in \mathcal{T}\}, \quad f^{-1}(\mathcal{S}) = \{x \mid Ax + b \in \mathcal{S}\}.$$

Lemma 2.3.9 *The minimum over x of a function $f(x, y)$ that is convex in both x and y , denoted as $g(y) = \min_x f(x, y)$, is a convex function of y .*

The following norm identity can be verified from Horn and Johnson (2010), p. 314:

Lemma 2.3.10 *Given $x \in \mathbb{R}^n$, and $p > r > 0$, then*

$$\|x\|_p \leq \|x\|_r \leq n^{(1/r - 1/p)} \|x\|_p. \quad (2.3.7)$$

2.4 Convex Optimisation

A brief review of convex optimisation is given, based on Hindi (2004, 2006) and on Boyd and Vandenberghe (2004). Convex optimisation can be described as a fusion of three disciplines: convex analysis, optimisation and numerical computation.

Several difficulties may arise in solving a general optimisation problem: presence of local minima, difficult characterisation of feasible domain, stopping criteria, convergence rates and numerical problems. For convex optimisation problems, the first three difficulties are naturally overcome, by definition of convexity. Development of interior point algorithms, first introduced by Karmarkar (1984), provided a drastic improvement in terms of solution time to the point that, at the present time, convex MPC can be applied to many problems in real time (Richter et al. 2013, Wang and Boyd 2010).

The objectives of this section are: to review some convex problems related to MPC, to provide a brief review of duality and optimality conditions and to introduce exact penalty functions.

2.4.1 Conic Programming

Conic programs or conic problems are convex problems in which inequalities are specified in terms of affine functions and generalised inequalities. Geometrically, the inequalities are feasible if the range of the affine mappings intersects the cone of the inequality.

A *linear program* (LP) has the form

$$\begin{aligned} \min_{\chi} \quad & g^T \chi + d \\ \text{s.t.} \quad & \Phi \chi = \phi, \\ & \Omega \chi \leq b. \end{aligned} \tag{2.4.1}$$

This family of problems can be solved using several algorithms, for instance, the *simplex* and its most recent extensions (Boyd and Vandenberghe 2004).

A *quadratic program* (QP) has the form

$$\begin{aligned} \min_{\chi} \quad & \frac{1}{2} \chi^T H \chi + g^T \chi + d \\ \text{s.t.} \quad & \Phi \chi = \phi, \\ & \Omega \chi \leq b. \end{aligned} \tag{2.4.2}$$

where the Hessian is at least positive semidefinite, $H \succeq 0$. More general conic programs include *quadratically constrained quadratic program* (QCQP), *second-*

order cone program (SOCP) and *semidefinite program* (SDP). Refer to Boyd and Vandenberghe (2004), Hindi (2004) for details.

2.4.2 Duality

Duality, or *Lagrangian duality*, is a theory used for simplifying and solving optimisation problems. The basic idea in Lagrangian duality is, given a general nonlinear program, to augment the objective function in order to account for constraint violation as a weighted extra cost.

Given a general nonlinear program

$$\begin{aligned} f^* &= \min_{\chi} f(\chi) \\ \text{s.t. } F(\chi) &\leq 0, \\ h(\chi) &= 0. \end{aligned} \quad (2.4.3)$$

where $F \in \mathbb{R}^p$ and $h \in \mathbb{R}^n$ have entries F_i, h_i . Define the *Lagrangian* function, as

$$L(\chi, \lambda, \nu) = f(\chi) + \sum_{i=1}^m \nu_i h_i(\chi) + \sum_{i=1}^p \lambda_i F_i(\chi). \quad (2.4.4)$$

The *Lagrange dual function* can be defined as

$$\mathcal{L}(\lambda, \nu) = \inf_{\chi \in \mathcal{D}} L(\chi, \lambda, \nu), \quad (2.4.5)$$

where $\mathcal{D}$ is the domain defined by the intersection of inequality and equality constraints.

Definition 2.4.1 (*Lagrange dual problem*) The *Lagrange dual problem* is a convex program, defined as

$$d^* = \max_{\lambda, \nu} \mathcal{L}(\lambda, \nu) \quad (2.4.6)$$

$$\text{s.t. } \lambda \geq 0. \quad (2.4.7)$$

Note that, if $\lambda < 0$, then $\mathcal{L}(\lambda, \nu) = -\infty$. If $\mathcal{L}(\lambda, \nu) > -\infty$ and the pair (λ, ν) is called *dual feasible* (Hindi 2006; Boyd and Vandenberghe 2004). The optimal solution (λ^*, ν^*) of the dual problem is referred to as the *dual optimal*. The original problem is referred to as the *primal* problem.

Definition 2.4.2 (*Weak duality*) Denote d^* the optimal cost of the Lagrange dual problem, and f^* the optimal cost of the primal problem. Then, d^* is the tightest lower bound on f^* . This property is called *weak duality* and it holds even if the primal problem is not convex.

Remark. Weak duality still holds when f^* and d^* are infinite. This provides a tool for feasibility inspection: unbounded dual (primal) corresponds to infeasible primal (dual).

The *optimal duality gap* is defined as

$$D_{gap}^* = f^* - d^* \geq 0. \quad (2.4.8)$$

Definition 2.4.3 (*Strong duality*) If the equality

$$d^* = f^*$$

holds, namely the duality gap is zero, the property of *strong duality* holds.

Strong duality does not hold in the general case. For a convex problem, strong duality can be guaranteed, for instance, under *Slater's condition* (Boyd and Vandenberghe 2004; Hindi 2006)

Theorem 2.4.1 (Slater's condition) *If a convex (primal) problem is strictly feasible, in other words, if $\exists \bar{\chi} : F(\bar{\chi}) \leq 0$, $h(\bar{\chi}) = 0$, then strong duality holds.*

Duality can therefore be used to provide bounds for non-convex problems, and stopping criteria for optimisation algorithms. For instance, in convex programming, the duality gap can be evaluated at each iteration, and a stopping criterion can be $D_{gap} \leq \epsilon$.

2.4.3 Optimality Conditions

This section provides a set of conditions for optimality. These are then particularised for convex problems.

Definition 2.4.4 (*First-order optimality (KKT) conditions*) Consider the nonlinear program (2.4.3). Assume that the cost and constraint functions are differentiable and that *strong duality* holds. Define the *Lagrangian* function as (2.4.4). Define χ^* as the optimal solution of (2.4.3), λ^* and ν^* as the vectors of *Lagrange multipliers* at the optimiser for, respectively, inequality and equality constraints. The following first-order conditions, called the *Karush–Kuhn–Tucker conditions*, are necessary (and sufficient, for a convex problem) optimality conditions, (Hindi 2006)

$$\begin{aligned} F_i(\chi^*) &\leq 0, \forall i \in \{1, \dots, p\} \\ h_i(\chi^*) &= 0, \forall i \in \{1, \dots, n\} \\ \lambda^* &\geq 0, \\ \lambda_i^* F_i(\chi^*) &= 0, \forall i \in \{1, \dots, p\} \\ \nabla f(\chi^*) + (\nabla h(\chi^*))^T \nu^* + (\nabla F(\chi^*))^T \lambda^* &= 0. \end{aligned} \quad (2.4.9)$$

The last condition in (2.4.9) says that the gradient of the Lagrangian must vanish at the optimum, while $\lambda_i^* F_i(\chi^*) = 0$, known as *complementary slackness*, is related to inequality constraint satisfaction and dual optimality.

For a QP in the form of (2.4.2), the KKT conditions specialise to the following:

Definition 2.4.5 (*First-order optimality (KKT) conditions* (Quadratic Program)) Given a convex QP in the form of (2.4.2), the necessary and sufficient conditions for global optimality, namely the *KKT conditions*, are (Boyd and Vandenberghe 2004; Maciejowski 2002)

$$\begin{aligned} -\Omega \chi^* - t &= -b, \\ -\Phi \chi^* &= -\phi, \\ \nu^* &\geq 0, \\ t &\geq 0, \\ t^T \lambda^* &= 0, \\ H \chi^* + \Phi^T \nu^* + \Omega^T \lambda^* &= -g. \end{aligned} \tag{2.4.10}$$

An overview of convex optimisation techniques commonly used in MPC can be found in Boyd and Vandenberghe (2004), Nocedal and Wright (1999). The most commonly used, in the field of MPC, are *active set* and *interior point* methods (Karmarkar 1984). These algorithms have polynomial complexity.¹ Recent developments provided encouraging results for real-time applications (Richter et al. 2013; Wang and Boyd 2010).

Most convex optimisation problems presented in this thesis have been solved using a free academic version of the multi-purpose solver IBM ILOG CPLEX (IBM 2013), available online through the IBM Academic Initiative.

2.4.4 Degeneracy and Multiple Optima

The *active set* of a QP is the set of constraints that are active at the optimum. A QP is said to be *primal degenerate* (Bemporad et al. 2002) if its active set contains more constraints than the number of decision variables. In this case, the optimal Lagrange multiplier vector may not be unique. A QP is said to be *dual degenerate* (Bemporad et al. 2002) if its Lagrange dual is primal degenerate. A QP is dual degenerate if its cost function is not strictly positive definite, and its solution may not be unique (Bemporad et al. 2002).

A result that will often be recalled throughout the thesis is the following:

Lemma 2.4.2 (Nesterov 2009, Lemma 6) *Consider a function $f : \text{dom } f \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, strongly convex with respect to an ℓ_p -norm, with convexity parameter σ . Consider*

¹In the worst case a complexity of $\mathcal{O}(N^3 m^3)$, where N is the MPC prediction horizon length and m is the number of control inputs.

also a closed convex set $\mathcal{X} \subseteq \text{dom } f$. Then, the problem

$$\min_{\chi \in \mathcal{X}} f(\chi) \quad (2.4.11)$$

is solvable and its solution, χ^* , is unique. Moreover, for any $\chi \in \mathcal{X}$ we have

$$f(\chi) \geq f(\chi^*) + \frac{1}{2}\sigma\|\chi - \chi^*\|_p^2. \quad (2.4.12)$$

2.4.5 Exact Penalty Functions

This section is based on the definitions given in Chap. 7 of Kerrigan (2000), which follows Fletcher (1987). Consider the nonlinear program (2.4.3) and, without loss of generality, assume only the inequality constraints.² Consider then the following unconstrained non-smooth program

$$\chi_s^* = \arg \min_{\chi} f(\chi) + \rho \|F(\chi)^+\|, \quad (2.4.13)$$

where $\|\cdot\|$ can be any norm, $F(\chi)^+$ contains the magnitude of the violation of the equality constraint of (2.4.3) for a given χ , namely $F(\chi)_j^+ = \max(F(\chi)_j, 0)$, and $\rho > 0$ is the *constraint violation penalty weight*.

Definition 2.4.6 Define the *dual norm* of $\|\cdot\|$ as

$$\|u\|_D = \max_{\|v\| \leq 1} u^T v. \quad (2.4.14)$$

Remark 2.4.1 It can be shown that $\|\cdot\|_1$ is the dual of $\|\cdot\|_\infty$ and vice versa, and that $\|\cdot\|_2$ is the dual of itself (see A.1.6 of Boyd and Vandenberghe 2004).

Theorem 2.4.3 Exact penalty function (Fletcher 1987, Theorem 14.3.1) *Denote χ^* and λ^* as, respectively, the minimiser and Lagrange multiplier vector of (2.4.3), and χ_s^* as the minimiser of (2.4.13). If the penalty weight $\rho > \|\lambda^*\|_D$ and $F(\chi_s^*) \leq 0$, then $\chi^* = \chi_s^*$. In this case, the constraint violation penalty is said to be an exact penalty function.*

The above result will be used in Chaps. 6 and 7.

²Equality constraints can be, for instance, accommodated as back-to-back inequalities, $[h(\chi)^T, -h(\chi)^T]^T \leq 0$. This is generally not done in practise, as it makes the optimisation problem primal degenerate and more difficult to solve.

2.5 Constrained Control Systems in Discrete-Time Domain

The theoretical development in this thesis concerns the control of systems that are time-invariant and that evolve in discrete-time. The focus is on constrained Linear Time-Invariant (LTI) systems, however, closed-loop stability is investigated for the class of nonlinear systems with differentiable vector field and with the origin as equilibrium. The numerical tools currently available are reviewed, however, while the concepts upon which stability is based are becoming quite standard for constrained LTI systems, research is still active for the considered nonlinear systems class where computational complexity is still a strong limitation.

The considered nonlinear systems are of the form:

$$x(k+1) = f(x(k), u(k), w(k)), \text{ with } f(0, 0, 0) = 0, \quad (2.5.1)$$

where $x(k)$ is the system's *state vector*, $u(k)$ is the *input vector*, $w(k)$ is a *disturbance* or *uncertainty vector* at time k . The *vector field* $f : \mathbb{R}^{n \times m \times q} \rightarrow \mathbb{R}^n$ is assumed to be continuous and differentiable. The variable $x(k)$ is assumed to be measured at each k . Most theoretical results in this thesis will concern a particular subclass of (2.5.1), the LTI systems:

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k). \quad (2.5.2)$$

Systems (2.5.1) and (2.5.2) are subject to convex constraints containing the origin of the form:

$$u(k) \in \mathbb{U} \subset \mathbb{R}^m, \quad x(k) \in \mathbb{X} \subset \mathbb{R}^n, \quad w(k) \in \mathbb{W} \subset \mathbb{R}^q, \quad \forall k \in \mathbb{I}_{\geq 0}. \quad (2.5.3)$$

The closed-loop expressions of (2.5.1) and (2.5.2) under a control law $u(k) = \kappa(x(k))$ are, respectively,

$$x(k+1) = f(x(k), \kappa(x(k)), w(k)), \quad (2.5.4)$$

$$x(k+1) = Ax(k) + B\kappa(x(k)) + B_w w(k). \quad (2.5.5)$$

The nominal dynamics of (2.5.1) and (2.5.2) are, respectively,

$$x(k+1) = \bar{f}(x(k), u(k)) = f(x(k), u(k), 0), \quad (2.5.6)$$

$$x(k+1) = Ax(k) + Bu(k). \quad (2.5.7)$$

Depending on the context, the notation $\bar{x}$ will occasionally be used to denote the nominal evolution of x . The closed-loop expressions of (2.5.6) and (2.5.7) under a control law $u(k) = \kappa(x(k))$ are, respectively,

$$x(k+1) = \bar{f}(x(k), \kappa(x(k))), \quad (2.5.8)$$

$$x(k+1) = Ax(k) + B\kappa(x(k)). \quad (2.5.9)$$

Definition 2.5.1 (*Nominal i th step evolution under a policy*) The i th step evolution of the nominal system (2.5.6) from an initial state $x(0)$, under an admissible input sequence $\mathbf{u}_{[i-1]} = \{u(k)\}_0^{i-1} \in \mathbb{U}^i$, is defined as

$$\bar{\phi}(i, x(0), \mathbf{u}_{[i-1]}) = \circ_{k=0}^{i-1} \bar{f}(x(k), u(k)), \quad u(k) \in \mathbf{u}_{[i-1]} \quad \forall k \in (0, i-1), \quad (2.5.10)$$

where $\circ_{k=0}^{i-1}$ represents composition of $\bar{f}$ with itself and $u(k)$ from time 0 to $i-1$, namely

$$\bar{\phi}(i, x(0), \mathbf{u}_{[i-1]}) = \bar{f}(\bar{\phi}(i-1, x(0), \mathbf{u}_{[i-2]}), u(i-1)), \quad \bar{\phi}(0, x(0), \cdot) = x(0).$$

2.5.1 Stability Definitions

This section concerns fundamental notions for studying stability of the dynamic systems of interest. The results are widely used throughout the thesis.

Remark 2.5.1 For the sake of brevity, in the following statements it will be implicitly assumed that for a considered set $\mathcal{X} \subseteq \mathbb{X}$ the considered control law $\kappa(x)$ is *admissible*, that is $\kappa(x) \in \mathbb{U}$, $\forall x \in \mathcal{X}$.

Definition 2.5.2 (*Positively invariant (PI) set*) A set $\mathcal{X} \subseteq \mathbb{R}^n$ is said to be *positively invariant* for (2.5.6) under a control law $u(k) = \kappa(x(k))$ if

$$\bar{f}(x, \kappa(x)) \in \mathcal{X}, \quad \forall x \in \mathcal{X}.$$

Definition 2.5.3 (*Robustly positively invariant (RPI) set*) A set $\mathcal{X} \subseteq \mathbb{R}^n$ is said to be *robustly positively invariant* for (2.5.1) under a control law $u(k) = \kappa(x(k))$ if

$$f(x, \kappa(x), w) \in \mathcal{X}, \quad \forall x \in \mathcal{X}, \quad \forall w \in \mathbb{W}.$$

The following definitions are based on Vidyasagar (1993):

Definition 2.5.4 ($\mathcal{K}$ -function) A continuous function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be a $\mathcal{K}$ -function ($\alpha \in \mathcal{K}$) if it is strictly increasing and if $\alpha(0) = 0$.

Definition 2.5.5 ($\mathcal{K}_\infty$ -function) A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be a $\mathcal{K}_\infty$ -function ($\alpha \in \mathcal{K}_\infty$) if it is a $\mathcal{K}$ -function and if it is radially unbounded, that is $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Definition 2.5.6 ($\mathcal{KL}$ -function) A continuous function $\beta : \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}_{\geq 0}$ is said to be a $\mathcal{KL}$ -function ($\beta \in \mathcal{KL}$) if it is a $\mathcal{K}$ -function in the first argument, if it is positive definite and non-increasing in the second argument, and if $\beta(r, t) \rightarrow 0$ as $t \rightarrow \infty$.

The following statements refer to the system's origin, but they can be extended to arbitrary steady states by a coordinate shift (Limón et al. 2006; Rawlings and Mayne 2009):

Definition 2.5.7 (*Lyapunov stability*) Given a PI set $\mathcal{X}$, $0 \in \text{int}\mathcal{X}$, the origin of system (2.5.8) is said to be *Lyapunov stable* in $\mathcal{X}$ if

$$\forall \epsilon > 0 \exists \delta > 0 : \forall x(0) \in \mathcal{X}, \|x(0)\| \leq \delta \Rightarrow \|x(k)\| < \epsilon, \quad \forall k \in \mathbb{I}_{\geq 0}.$$

Definition 2.5.8 (*Attractivity*) Given a PI set $\mathcal{X}$, $0 \in \text{int}\mathcal{X}$, the origin of system (2.5.8) is said to be *attractive* in $\mathcal{X}$ if

$$\lim_{k \rightarrow \infty} \|x(k)\| = 0, \quad \forall x(0) \in \mathcal{X}.$$

Definition 2.5.9 (*Asymptotic stability (AS)*) Given a PI set $\mathcal{X}$, $0 \in \text{int}\mathcal{X}$, the origin of system (2.5.8) is said to be *asymptotically stable* (AS) in $\mathcal{X}$ if it is Lyapunov stable and attractive in $\mathcal{X}$, equivalently, if there exists a $\mathcal{KL}$ -function $\beta(\cdot, \cdot)$ such that

$$\|x(k)\| \leq \beta(\|x(0)\|, k), \quad \forall x(0) \in \mathcal{X},$$

the PI set $\mathcal{X}$ is then called a *domain of attraction*.

Definition 2.5.10 (*Practical stability (pS)*) Given a PI set $\mathcal{X}$, $0 \in \text{int}\mathcal{X}$, the origin of system (2.5.8) is said to be *practically stable* (pS) in $\mathcal{X}$ if there exists a $\mathcal{KL}$ -function $\beta(\cdot, \cdot)$ and a $\zeta > 0$ such that

$$\|x(k)\| \leq \beta(\|x(0)\|, k) + \zeta, \quad \forall x(0) \in \mathcal{X}.$$

Remark 2.5.2 Regarding practical stability, a slight abuse of notation will be taken in this thesis by referring to the PI set $\mathcal{X}$ as a domain of attraction.

Definition 2.5.11 (*(practical) Lyapunov function*) Let $\mathcal{X}$ be a compact PI set for system (2.5.8) containing a compact neighbourhood $\mathcal{N}$ of the origin in its interior and let $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ be $\mathcal{K}_\infty$ -functions (and $\delta > 0$). A function $V : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ with $V(0) = 0$ is a *practical Lyapunov function* in $\mathcal{X}$ if

$$V(x) \geq \alpha_1(\|x\|), \quad \forall x \in \mathcal{X}, \quad (2.5.11)$$

$$V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathcal{N}, \quad (2.5.12)$$

$$V(\bar{f}(x, \kappa(x))) - V(x) \leq -\alpha_3(\|x\|) + \delta, \quad \forall x \in \mathcal{X}. \quad (2.5.13)$$

Moreover, if (2.5.13) holds with $\delta = 0$, then $V(\cdot)$ is a *Lyapunov function*.

Theorem 2.5.1 (*Asymptotic stability*) If system (2.5.8) admits a Lyapunov function in $\mathcal{X}$, then the system origin is AS with domain of attraction $\mathcal{X}$.

Proof of the above theorem can be found, for instance, in Rawlings and Mayne (2009), Lazar (2006).

Theorem 2.5.2 (Practical stability) *If system (2.5.8) admits a practical Lyapunov function in $\mathcal{X}$, then the system origin is pS with domain of attraction $\mathcal{X}$.*

Definition 2.5.12 (Control Lyapunov Function (CLF)) Let $\mathcal{X}$ be a compact set. A function $V : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ with $V(0) = 0$ is a *control Lyapunov function* in $\mathcal{X}$ for system (2.5.6) iff $\exists \kappa(x)$ such that $V(\cdot)$ is a Lyapunov function in $\mathcal{X}$ for (2.5.8).

Remark 2.5.3 An important property of (control) Lyapunov functions is that their *level sets* are positively (*control*) *invariant sets* for the considered system (Blanchini 1999).

Define the disturbance sequence $\mathbf{w}_{[k]} = \{w_j\}_0^k$. The following definition generalises the concept of stability to uncertain systems (Limón et al. 2006).

Definition 2.5.13 (*Input-to-state (practical) stability*) Given a RPI set $\mathcal{X}$, $0 \in \text{int}\mathcal{X}$, the origin of system (2.5.4) is said to be (*locally*) *input-to-state practically stable*, or ISpS, in $\mathcal{X}$ if there exists a $\mathcal{KL}$ -function $\beta(\cdot, \cdot)$ and a $\mathcal{K}_\infty$ -function γ (and a $\zeta > 0$) such that

$$\begin{aligned} \|x(k)\| &\leq \beta(\|x(0)\|, k) + \gamma(\|\mathbf{w}_{[k-1]}\|) + \zeta, \quad \forall x(0) \in \mathcal{X}, \\ \forall \mathbf{w} : w_j &\in \mathbb{W} \quad \forall j \in \mathbb{I}_{\geq 0}, \quad \forall k \in \mathbb{I}_{\geq 0}, \end{aligned} \quad (2.5.14)$$

where $\|\mathbf{w}_{[k-1]}\| = \max_{j \in \mathbb{I}_{[0, k-1]}} \{\|w_j\|\}$.

If (2.5.14) holds with $\zeta = 0$, then the origin is (*locally*) *input-to-state stable*, or ISS.

Definition 2.5.14 (*IS(p)S Lyapunov function*) Let $\mathcal{X}$ be a compact RPI set for system (2.5.4) containing a compact neighbourhood of the origin $\mathcal{N}$ in its interior and let $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ be $\mathcal{K}_\infty$ -functions, $\sigma(\cdot)$ be a $\mathcal{K}$ -function and $\delta \geq 0$. A function $V : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ with $V(0) = 0$ is an *IS(p)S Lyapunov function* in $\mathcal{X}$ if

$$V(x) \geq \alpha_1(\|x\|), \quad \forall x \in \mathcal{X}, \quad (2.5.15)$$

$$V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathcal{N}, \quad (2.5.16)$$

$$V(f(x, \kappa(x), w)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|w\|) + \delta, \quad \forall x \in \mathcal{X}, \quad \forall w \in \mathbb{W}. \quad (2.5.17)$$

If (2.5.17) holds with $\delta = 0$, then $V(\cdot)$ is an *ISS Lyapunov function*.

Theorem 2.5.3 (Input-to-state stability) *If system (2.5.4) admits an ISS Lyapunov function in $\mathcal{X}$ with respect to $\mathbb{W}$, then the system origin is ISS in $\mathcal{X}$ with respect to $\mathbb{W}$ with ISS gain $\sigma(\cdot)$.*

Theorem 2.5.4 (Input-to-state practical stability) *If system (2.5.4) admits an ISpS Lyapunov function in $\mathcal{X}$ with respect to $\mathbb{W}$, then the system origin is ISpS in $\mathcal{X}$, with respect to $\mathbb{W}$, with ISpS gain $\sigma(\cdot)$.*

The following results are adapted from Limón et al. (2006):

Lemma 2.5.5 (Uniform continuity and $\mathcal{K}$ -functions) *Given a uniformly continuous function $V(\cdot)$, there exists a $\mathcal{K}$ -function $\sigma(\cdot)$, such that*

$$|V(x + w) - V(x)| \leq \sigma(\|w\|), \quad \forall x, w \in \mathbf{dom} V. \quad (2.5.18)$$

Theorem 2.5.6 (Uniform continuity and ISS) *Consider the C-sets $\mathcal{X}, \mathbb{W}$. Assume (2.5.4) to be continuous in w , $\forall x \in \mathcal{X}, \forall w \in \mathbb{W}$.³ Assume (2.5.8) to be AS in $\mathcal{X}$, then (2.5.4) is ISS in an RPI set $\mathbb{X}^r \subseteq \mathcal{X}$ if $\mathbb{X}^r \neq \emptyset$ and if, either,*

1. *There exists a continuous Lyapunov function in $\mathcal{X}$ for system (2.5.8),*
2. *The function $\tilde{f}(x, \kappa(x))$, in (2.5.8), is continuous $\forall x \in \mathcal{X}$.*

2.5.2 Set-Theoretic Control

The following definitions and theorems are based on Blanchini (1999), Blanchini and Miani (2008):

Definition 2.5.15 (Control invariant and λ -contractive set) A C-set $\mathcal{X}$ is λ -contractive for (2.5.6) if there exists a $\lambda \in [0, 1)$, and u such that $x \in \mathcal{X} \Rightarrow \tilde{f}(x, u) \in \lambda\mathcal{X}$. If this holds for $\lambda = 1$, then the set $\mathcal{X}$ is simply *control invariant*.

If the above condition holds (with $\lambda \in [0, 1)$) for system (2.5.1) $\forall w \in \mathbb{W}$, then the set $\mathcal{X}$ is a λ -contractive set for (2.5.1). If $\lambda = 1$, then the set is simply *robustly control invariant*.

Definition 2.5.16 (Minkowski function) Given a C-set $\mathcal{X}$ its *Minkowski function* or *gauge function* is

$$\psi_{\mathcal{X}}(x) = \min_{\gamma} \{\gamma \mid x \in \gamma\mathcal{X}\}. \quad (2.5.19)$$

Theorem 2.5.7 (Blanchini 1999, Theorem 3.3) *The following statements are equivalent:*

- *The C-set $\mathcal{X}$, with $0 \in \mathbf{int}(\mathcal{X})$, is λ -contractive, for (2.5.6),*
- *The Minkowski function of $\mathcal{X}$, $\psi_{\mathcal{X}}(x)$, is a CLF in $\mathcal{X}$ for (2.5.7). Hence, $\exists \kappa(x)$ such that (2.5.8) is AS in $\mathcal{X}$.*

Definition 2.5.17 (Ultimate boundedness) Let $\mathcal{S}$ be a neighbourhood of the origin ($0 \in \mathbf{int}(\mathcal{S})$). System (2.5.8) is *ultimately bounded* in $\mathcal{S}$ if $\exists \mathcal{X} \supset \mathcal{S}$ such that,

$$\forall x(0) \in \mathcal{X}, \exists \bar{k} \in \mathbb{I}_{\geq 0} : x(k) \in \mathcal{S}, \quad \forall k \geq \bar{k}.$$

³Uniform continuity is implied by continuity if all the involved sets are bounded, e.g. C-sets, (see Theorem 2.3.3).

Definition 2.5.18 (*Lyapunov function outside a set*) Let $\mathcal{S}$ be a neighbourhood of the origin. A function $V(\cdot)$ is a *Lyapunov function outside $\mathcal{S}$* if $\exists c : \mathcal{B}_{V,c} = \{x \mid V(x) \leq c\} \subseteq \mathcal{S}$ and if $\exists \mathcal{X} \supset \mathcal{S}$ such that $V(\cdot)$ is a Lyapunov function in $\mathcal{X} \setminus \mathcal{B}_{V,c}$.

Theorem 2.5.8 (Ultimate boundedness (Blanchini and Miani 2008, Theorem 2.24)) *If system (2.5.8) admits a Lyapunov function outside $\mathcal{S}$ (from $\mathcal{X}$), then it is ultimately bounded in $\mathcal{S}$ (for all $x(0) \in \mathcal{X}$).*

Remark 2.5.4 Robust versions of the above statements can be obtained by imposing the corresponding conditions to hold $\forall w \in \mathbb{W}$.

For the case of LTI systems and polytopic constraints, the following holds:

Theorem 2.5.9 (Blanchini and Miani 2008, Corollary 4.43) *A polyhedral C -set $\mathcal{X}$ is λ -contractive for system (2.5.2) iff for each of its vertices $v \in \mathcal{V}(\mathcal{X})$ there exists a control $u_{\{v\}}$ such that, $\forall w \in \mathbb{W}$, we have $Av + Bu_{\{v\}} + B_w w \in \lambda\mathcal{X}$, where $\lambda \in [0, 1)$.*

Corollary 2.5.1 *Theorem 2.5.9 applies to the nominal system (2.5.7), by taking $\mathbb{W} = \{0\}$.*

Set-theoretic control makes use of a set of operators. For the study of predictive controllers, the following notions are required (Kerrigan 2000; Kerrigan and Maciejowski 2000a):

Definition 2.5.19 (*1-step operator*) Given the system (2.5.6), subject to constraints (2.5.3), the *1-step operator* is defined as

$$\mathcal{Q}(\mathcal{X}) = \{x \in \mathbb{R}^n : \exists u \in \mathbb{U} \mid \bar{f}(x, u) \in \mathcal{X}\}. \quad (2.5.20)$$

Recalling Definition 2.5.1, we can introduce the following (Kerrigan 2000; Kerrigan and Maciejowski 2000a):

Definition 2.5.20 (*Controllable set*) Given the system (2.5.6), subject to constraints (2.5.3), the *N -step controllable set* from $\mathcal{X} \subseteq \mathbb{R}^n$ to $\mathbb{X}_f \subseteq \mathbb{R}^n$ is defined as

$$\mathcal{K}_N(\mathcal{X}, \mathbb{X}_f) = \{x \in \mathcal{X} : \exists \mathbf{u}_{[N-1]} \in \mathbb{U}^N \mid \hat{x}(j) = \bar{f}^j(x, \mathbf{u}_{[j-1]}) \in \mathcal{X} \forall j \in \mathbb{I}_{[0, N]}, \hat{x}(N) \in \mathbb{X}_f\}. \quad (2.5.21)$$

Remark 2.5.5 As shown in Kerrigan (2000), Kerrigan and Maciejowski (2000a), the N -step controllable set can be computed by recursive application of the 1-step operator. Namely,

$$\mathcal{K}_{i+1}(\mathcal{X}, \mathbb{X}_f) = \mathcal{K}_1(\mathcal{X}, \mathcal{K}_i(\mathcal{X}, \mathbb{X}_f)), \quad (2.5.22)$$

$$\mathcal{K}_1(\mathcal{X}, \mathbb{X}_f) = \mathcal{Q}(\mathbb{X}_f) \cap \mathcal{X}. \quad (2.5.23)$$

In the LTI case, the $\mathcal{Q}$ operator can be implemented using polytopic projections (Kerrigan 2000).

Definition 2.5.21 (*Stabilisable set*) Given the system (2.5.6), subject to constraints (2.5.3), and given a positively invariant set $\mathbb{X}_f \subseteq \mathbb{X}$, the *N-step stabilisable set* from $\mathcal{X} \subseteq \mathbb{X}$, associated with $\mathbb{X}_f$, is defined as

$$\mathcal{S}_N(\mathcal{X}, \mathbb{X}_f) = \mathcal{K}_N(\mathcal{X}, \mathbb{X}_f). \quad (2.5.24)$$

Note that the above definitions are for the *nominal system*. Robust cases are similarly defined in (Kerrigan 2000), with the difference that the above conditions must hold $\forall w(k) \in \mathbb{W}, \forall k \in \mathbb{I}_{[0, N]}$.

2.6 Model Predictive Control

Model Predictive Control (MPC) is a popular optimal control strategy, in which a constrained finite-horizon optimal problem is solved at each iteration on the basis of the available measurement. A plant model is used to compute N future predictions, and an input sequence is generated optimising a performance index. The first control move of the optimal sequence is then applied to the plant. The main strength of predictive control is the ability to handle physical and operational constraints on the plant behaviour as well as on the inputs. MPC is capable of operating the system safely within constraints and can improve performance by allowing operating points to be close to constraint boundaries (Maciejowski 2002). Operation points are implicitly determined by the constraints, the prediction horizon length N and by the performance index or *cost function*.

It is becoming standard, among researchers, to use a *terminal constraint* in MPC (Bemporad et al. 2002; Chen and Allgöwer 1998; Chmielewski and Manousiouthakis 1996; Goodwin et al. 2005; Kerrigan and Maciejowski 2000a; Maciejowski 2002; Mayne et al. 2000; Rawlings and Mayne 2009). In particular, the state prediction at the end of the horizon is constrained to be into a (nominal or robust) control invariant set (thus called the *terminal set*). The advantage of using this *safe set* is that recursive feasibility is guaranteed for the optimisation problem. Consequently, the feasible region is (nominally or robustly) positively invariant for the closed-loop system (Kerrigan and Maciejowski 2000a). Additionally, a particular CLF is used to penalise the state prediction (or the control error) at the end of the horizon (thus called the *terminal cost*). These two terminal ingredients are used in most MPC literature to satisfy a set of neat sufficient conditions for (nominal and robust) closed-loop stability (Mayne et al. 2000). This setting is considered for the majority of the thesis.

2.6.1 Standard MPC Formulation

Most MPC implementations are based on a quadratic cost, mainly because quadratic functions are convex and twice differentiable (hence relatively easy to minimise) and

partly because of the possibility of inheriting properties from the LQR (Chmielewski and Manousiouthakis 1996). The aim of most common MPC strategies is to steer the nominal system (2.5.6) to the origin of the state space (or to a different set point), while keeping its evolution within constraints and minimising the time summation of N stage costs and of a terminal cost (or cost-to-go). A standard MPC with quadratic cost, or *quadratic MPC*, can be formulated as follows:

Definition 2.6.1 (*Quadratic MPC*) Consider the following finite-horizon constrained optimal control problem (FHCOC):

$$\begin{aligned} V_N^o(x) = \min_{\underline{u}} \left\{ V_N(x, \underline{u}) \triangleq F(\hat{x}_N) + \sum_{j=0}^{N-1} \ell(\hat{x}_j, \hat{u}_j) \right\} \\ \text{s.t. } \hat{x}_{j+1} = \hat{f}(\hat{x}_j, \hat{u}_j), \\ \hat{u}_j \in \mathbb{U}, \quad \forall j \in \mathbb{I}_{[0, N-1]}, \quad \hat{x}_j \in \mathbb{X}, \quad \forall j \in \mathbb{I}_{[0, N]}, \\ \hat{x}_0 = x, \quad \hat{x}_N \in \mathbb{X}_f \subseteq \mathbb{X}, \end{aligned} \quad (2.6.1)$$

with stage cost

$$\ell(x, u) = x^T Q x + u^T R u + 2u^T T x, \quad (2.6.2)$$

and terminal cost (cost-to-go)

$$F(x) = x^T P x, \quad (2.6.3)$$

where $\hat{f} : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^n$ is a continuous prediction model, $\underline{u}^T = [\hat{u}_0^T, \dots, \hat{u}_{N-1}^T]$. At each iteration k , the MPC applies to the plant the first move of the optimal policy, $u(k) = \hat{u}_0^*$, obtained by online solution of (2.6.1) and (2.6.2), at the current state, $x(k)$. The resulting implicit control law is referred to as $K_N(x) \equiv \hat{u}_0^*(x)$. The closed-loop evolution of (2.5.1) under quadratic MPC is

$$x(k+1) = f(x(k), K_N(x(k)), w(k)). \quad (2.6.4)$$

The one-step evolution of (2.6.4) for a given x is simply denoted as $x^+ = f(x, K_N(x), w)$.

Formulations based on 1 and ∞ -norm costs, $\ell(x, u) = \|x\|_p + \|u\|_p$, $p \in \{1, \infty\}$, have also been investigated in the MPC literature, being the most common alternatives to Quadratic MPC (Maciejowski 2002; Rao and Rawlings 2000; Rawlings and Mayne 2009). Although they may be useful in some cases, for instance, when an LP solver is available instead of a QP solver, formulations based on 1 and ∞ norm often display less desirable performance (Rao and Rawlings 2000). Throughout this thesis, quadratic, ℓ_1 and ℓ_∞ MPC will be compared to ℓ_{asso} -MPC.

2.6.2 Considered Set-up

It is common in the MPC literature to assume that $\hat{f}(\cdot, \cdot) \equiv \tilde{f}(\cdot, \cdot)$, namely that the prediction model corresponds to the nominal plant model. In this thesis, the following is assumed:

Assumption 2.6.1

(H0) $\hat{f}(\cdot, \cdot), \ell(\cdot, \cdot), F(\cdot)$ are continuous, $\hat{f}$ differentiable, $\hat{f}(0, 0) = 0, \ell(0, 0) = 0, F(0) = 0$.

The constraints and the initial state are assumed to satisfy

Assumption 2.6.2

(H1) $\mathbb{X}, \mathbb{X}_f$, and $\mathbb{U}$ are C-sets, containing the origin in their interior,

(H2) $x(0) \in \mathbb{X}_N = \{x \in \mathbb{X} \mid \exists \underline{u} \in \mathbb{U}^N, V_N(x, \underline{u}) < \infty\}$.

In Assumption 2.6.2 the MPC cost function is implicitly assumed to be *extended*. The following result is standard in the MPC literature:

Proposition 2.6.1 (Rawlings and Mayne 2009) *Assume Assumptions 2.6.1 and 2.6.2. Then, a solution exists for the MPC (2.6.1).*

From Proposition 2.6.1, a solution exists for problem (2.6.1), when a differentiable $\hat{f}$ is used. Finding the optimal solution, however, requires to solve a non-convex minimisation, which could be subject to local optima.

Remark 2.6.1 As it is common in the MPC literature, in this thesis it is assumed that the global optimum is found at each MPC iteration. The implications of suboptimal solutions for non-convex MPC are discussed, for instance, in Lazar and Heemels (2009), Pannocchia et al. (2011), Scokaert et al. (1999). In particular, under some conditions, suboptimal solutions can be tolerated, with small loss of performance.

Remark 2.6.2 Continuity of $\hat{f}$ (or f) is not strictly necessary for using MPC, however, theoretical development for discontinuous systems is beyond the scope of this thesis. The interested reader can refer, for instance, to Lazar and Heemels (2009), Lazar et al. (2006) and reference therein.

2.6.3 Recursive Feasibility and Invariance

In order to safely implement an MPC control law, it is required that the associated optimisation is *feasible for all time*, according to the following (Kerrigan and Maciejowski 2000a):

Definition 2.6.2 (*Feasible for all time*) The MPC problem is *feasible for all time* $k \in \mathbb{I}_{\geq 0}$ if and only if (H2) holds and all future evolutions of the state belong to the feasible set, namely $x(k) \in \mathbb{X}_N, \forall k \in \mathbb{I}_{\geq 0}$.

Define $C_\infty(\mathbb{X}) = \mathcal{K}_\infty(\mathbb{X}, \mathbb{X})$ as the infinite-time control invariant set in $\mathbb{X}$ (Kerrigan and Maciejowski 2000a). The following result provides sufficient conditions for guaranteeing that the MPC problem is feasible for all time:

Theorem 2.6.2 (Kerrigan and Maciejowski 2000a) *The MPC problem is feasible for all time if and only if $x(0) \in \mathbb{X}_N \cap C_\infty(\mathbb{X})$ and $\forall x(k) \in \mathbb{X}_N \cap C_\infty(\mathbb{X})$ the solution to the MPC problem results in $x_1^* \in \mathbb{X}_N \cap C_\infty(\mathbb{X})$.*

If an MPC is feasible for all time for every feasible initial condition, then we say that it is *recursively feasible*. The following results are from Kerrigan and Maciejowski (2000a):

Definition 2.6.3 (Strongly (or recursively) feasible MPC) *The MPC problem is strongly feasible if and only if, for all $x(0) \in \mathbb{X}_N$ the MPC problem is feasible for all time.*

Theorem 2.6.3 (Kerrigan and Maciejowski 2000a)

1. *The MPC problem is strongly feasible if and only if the feasible set is a positively invariant set for the (nominal) closed-loop system $x(k+1) = \tilde{f}(x(k), K_N(x(k)))$.*
2. *The MPC problem is strongly feasible only if the feasible set is a control invariant set for the system $x(k+1) = \tilde{f}(x(k), u(k))$.*

Thanks to Theorem 2.6.3 the conditions for recursive feasibility are translated into set-theoretic control conditions. The next section deals with common MPC assumptions guaranteeing asymptotic stability and, indirectly, the invariance of $\mathbb{X}_N$ and recursive feasibility of the MPC problem when $w(k) = 0, \forall k$.

Remark 2.6.3 In the perturbed case ($w(k) \neq 0$ for some k), the notion of *robustly feasible MPC* is introduced, which means that the MPC is strongly feasible $\forall w \in \mathbb{W}$. This (strong) condition is verified if $\mathbb{X}_N$ is robustly control invariant (Kerrigan and Maciejowski 2000a).

2.6.4 Sufficient Conditions for Asymptotic Stability

A standard set of sufficient assumptions for MPC stability is the following (Rawlings and Mayne 2009):

Assumption 2.6.3 There exist two $\mathcal{K}_\infty$ -functions, α_1, α_2 , such that

- (H5) $\alpha_1(\|x\|) \leq \ell(x, u), \forall x \in \mathbb{X}_N, \forall u \in \mathbb{U}$,
- (H6) $F(x) \leq \alpha_2(\|x\|), \forall x \in \mathbb{X}_f$,
- (H7) $\min_{u \in \mathbb{U}} \{F \circ f(x, u) + \ell(x, u) \mid f(x, u) \in \mathbb{X}_f\} \leq F(x), \forall x \in \mathbb{X}_f$.

It can be noticed that (H7) implicitly assumes the control invariance of $\mathbb{X}_f$.

Theorem 2.6.4 (Rawlings and Mayne 2009) *Assume Assumptions 2.6.1–2.6.3. Then, the origin of (2.5.8) is AS in $\mathbb{X}_N$ under the MPC controller, $\kappa(x) = K_N(x)$, with prediction model $\hat{f} \equiv \bar{f}$.*

Proof (Sketch) The result can be obtained, for instance, by means of the *direct method* (Rawlings and Mayne 2009). The proof rationale is reported. At time $k + 1$, take the previous state and input prediction sequences without their first step. This sequences are still feasible and, since the terminal prediction at time k was in $\mathbb{X}_f$, one can append to the control sequence the move satisfying (H7), obtaining a feasible suboptimal control solution, $\tilde{u}$, with the new terminal state prediction being also in $\mathbb{X}_f$. Thanks to (H7), the MPC cost under $\tilde{u}$ is smaller than its value at time k , in particular, $V_N(x(k + 1), \tilde{u}) \leq V_N^o(x(k)) - \ell(x(k), K_N(x(k)))$. By optimality, $V_N^o(x(k + 1)) \leq V_N(x(k + 1), \tilde{u})$, for any feasible $\tilde{u}$. Hence, under (H5)–(H7), the optimal cost $V_N^o(x)$ is a Lyapunov function in $\mathbb{X}_N$ for the nominal system, with $V_N^o(x(k + 1)) - V_N^o(x(k)) \leq -\alpha_1(\|x(k)\|)$. Finally, by Theorem 2.5.1, the system origin is AS with domain of attraction $\mathbb{X}_N$. ■

2.6.5 Terminal Ingredients

In most common formulations, (H5) is satisfied by taking $Q \succ 0$. Then, (H7) and, consequently, (H5) are satisfied by means of a candidate state feedback *terminal controller*.⁴ Given $K \in \mathbb{R}^{m \times n}$, define the $i + 1$ -step evolution of (2.5.6) under $u = Kx$ as

$$\bar{f}_K^{i+1}(x) = \bar{f}_K^1 \circ \bar{f}_K^i(x), \quad \forall i \in \mathbb{I}_{\geq 2}, \quad \bar{f}_K^1(x) = \bar{f}(x, Kx), \quad \bar{f}_K^0(x) = x. \quad (2.6.5)$$

The following assumption is the most common in the MPC literature:

Assumption 2.6.4 There exists a matrix K and a C-set $\mathbb{X}_f$, containing the origin, such that

$$F(\bar{f}_K^i(x)) \leq F(\bar{f}_K^{i-1}(x)) - \ell(\bar{f}_K^{i-1}(x), K \bar{f}_K^{i-1}(x)) \quad \forall x \in \mathbb{X}_f, \quad \forall i \in \mathbb{I}_{\geq 0}. \quad (2.6.6)$$

It is common practice, for differentiable systems and a single set point, to linearise the model around the desired equilibrium, then to compute a K that stabilises the linearisation (if stabilisable). Lyapunov theory is then used to determine an admissible invariant set (Chen and Allgöwer 1998). This, however, can be very conservative. A review of common approaches follows.

⁴The terminal controller is never applied to the plant.

2.6.5.1 Linear Quadratic MPC

For LTI systems and quadratic cost, it is commonly assumed that (Rawlings and Mayne 2009)

Assumption 2.6.5 $Q \succeq 0$, $R \succ 0$, (A, B) is stabilisable and $(Q^{1/2}, A)$ detectable.

In order to satisfy Assumption 2.6.4, assume P and K to solve the following DARE

$$(A + BK)^T P (A + BK) - P = -(Q + K^T R K + K^T T + T^T K), \quad (2.6.7)$$

that characterises the discrete-time infinite-horizon LQR (Astrom 1970). Then, the terminal constraint set can be computed by defining the *input admissible* set, $\bar{X}$, and the *maximal output admissible set*, $\mathcal{O}_\infty$, as (Chmielewski and Manousiouthakis 1996; Kerrigan and Maciejowski 2000a)

$$\bar{X} = \{x \in \mathbb{X} \mid Kx \in \mathbb{U}\}, \quad \mathcal{O}_\infty = \{x \mid (A + BK)^k x \in \bar{X}, \forall k \geq 0\}. \quad (2.6.8)$$

Then, one can take either

$$\mathbb{X}_f = \mathcal{O}_\infty, \quad (2.6.9)$$

or (Bemporad et al. 2002)

$$\mathbb{X}_f = \{x \mid F(x) \leq c\}, \quad \text{where } 0 < c < c_m \triangleq \inf_{x \notin \bar{X}} \{x^T P_\infty x\}. \quad (2.6.10)$$

Using (2.6.9) causes the MPC problem to be a convex QP, while using (2.6.10) gives a more demanding QCQP. Note that both sets from (2.6.9) and (2.6.10) are PI under $u = Kx$, however, the $\mathcal{O}_\infty$ set in (2.6.9) is by definition the maximal of all invariant sets. Therefore, the choice of $\mathcal{O}_\infty$ is always less conservative than any other control invariant set, for the same K . Note that, if $(A + BK)$ has eigenvalues strictly inside the unit circle (for instance, in the LQR case), or at particular locations on the circle, then $\mathcal{O}_\infty$ is a polytope, and it can be computed in finite time by intersecting a finite number of linear inequalities (Chmielewski and Manousiouthakis 1996; Kerrigan 2000). The locations on the unit circle are characterised in Theorem 4.47 of Blanchini and Miani (2008).

2.6.5.2 Nonlinear Quadratic MPC and Differential Inclusion

Computation of K and $\mathbb{X}_f$ for Assumption 2.6.4, is generally non-trivial. Taking $\mathbb{X}_f = \{0\}$ works theoretically, but it is not recommended in practice, as this could cause numerical difficulties and a small feasible region. Recall that this thesis considers differentiable systems with the origin as equilibrium and take $(\bar{A}, \bar{B})$ from the linearisation of (2.5.6) at the origin. Then, a possible solution is the following (a novel discrete-time version of Lemma 1 of Chen and Allgöwer 1998):

Lemma 2.6.5 Assume $(\bar{A}, \bar{B})$ is stabilisable, $T = 0$. Then there exists a $K \in \mathbb{R}^{m \times n}$, $P \succ 0$, and a scalar, $\alpha > 0$, such that if $\mathbb{X}_f = \{x \in \mathbb{X} \mid F(x) \leq \alpha\}$ then Assumption 2.6.4 is satisfied.

Proof (Lemma 2.6.5) Define $v(x) = \bar{f}(x, Kx) - (\bar{A} + \bar{B}K)x$. Then, (2.5.6) is equivalent, in closed loop with $u = Kx$, to

$$x(k+1) = \tilde{f}(x) \equiv (\bar{A} + \bar{B}K)x + v(x). \quad (2.6.11)$$

From Lazar et al. (2009) an appropriate matrix gain K can be designed, by means of an LMI, such that

$$\tilde{f}(x)^T P \tilde{f}(x) - x^T P x \leq -x^T \tilde{Q} x + \tau v(x)^T v(x), \quad (2.6.12)$$

for some $P \succ 0$, given $\tilde{Q} \succ 0$, $\tau > 0$. Such (K, P) always exists, since $(\bar{A}, \bar{B})$ is stabilisable. Define $\tilde{Q} = Q + K^T R K + \kappa I$, $\kappa > 0$. Define also $\mathcal{B}_{P,\alpha} = \{x \mid x^T P x \leq \alpha\}$, and take α_1 as the maximum constant for which Kx is admissible in $\mathcal{B}_{P,\alpha_1}$. Similarly to Chen and Allgöwer (1998), taking $\alpha \in (0, \alpha_1]$ such that

$$\max_{x \in \mathcal{B}_{P,\alpha}} \frac{\|v(x)\|_2^2}{\|x\|_2^2} \leq \kappa/\tau, \quad (2.6.13)$$

satisfies (2.6.6). ■

Remark 2.6.4 In the proof of Lemma 2.6.5, the gain κ/τ can be seen as a design parameter that affects the size of $\mathbb{X}_f$.

Remark 2.6.5 In Lemma 2.6.5, continuity and differentiability of $\bar{f}$ are required only at the origin.

Alternative methods are available, when $\bar{f}$ is differentiable in a convex set, that potentially enlarge $\mathbb{X}_f$. These methods however *might require a significant increase of offline computation*. In Chap. 5, the terminal ingredients are computed using Linear Differential Inclusion (LDI) (Blanchini and Miani 2008; Boyd et al. 1994). In particular, the following results are considered:

Lemma 2.6.6 (LPV embedding or model absorption) Assume $\bar{f}(x, u)$ is differentiable with $\bar{f}(0, 0) = 0$ and consider the artificial Linear Parameter-Varying (LPV) system:

$$\begin{aligned} x(k+1) &= A(r)x(k) + B(r)u(k), \\ A(r) &= \sum_i^{n_r} A_i r_i, \quad B(r) = \sum_i^{n_r} B_i r_i, \\ r &\in \mathbb{R}_{\geq 0}^{n_r}, \quad \sum_i^{n_r} r_i = 1. \end{aligned} \quad (2.6.14)$$

If there exists a set $\mathcal{S}$ such that

$$\frac{\partial \bar{f}(x, u)}{\partial x, u} \in \mathbf{conv}([A_i, B_i], i \in \mathbb{I}_{[1, n_r]}), \quad \forall x \in \mathcal{S}, \quad \forall u \in \mathbb{U}, \quad (2.6.15)$$

then $\forall x \in \mathcal{S}, \forall u \in \mathbb{U}$, there exists $\bar{r} = r(x, u) \in \mathbb{R}_{\geq 0}^{n_r}$ with $\sum_i \bar{r}_i = 1$ such that

$$\bar{f}(x, u) = A(\bar{r})x + B(\bar{r})u. \quad (2.6.16)$$

Then, $\forall x \in \mathcal{S}$, every trajectory of the original system (2.5.6) is also a trajectory of (2.6.14), for an appropriate law $r(k)$. We say that (2.6.14) embeds (or absorbs) (2.5.6) in $\mathcal{S}$.

Proof Lemma 2.6.6 follows from the arguments in Boyd et al. (1994), p. 55, Sect. 4.3, which make use of the mean value theorem (Trench 2013) Theorem 2.3.10, p. 83. ■

Theorem 2.6.7 (Gain scheduling control) *Assume that Lemma 2.6.6 holds. Assume also that, at each time k , $\bar{r} = r(k)$ is available so that (2.6.16) is satisfied. If there exists a controller $\kappa(x, \bar{r})$ and a C -set $\mathcal{X} \subseteq \mathcal{S}$, positively invariant for (2.6.14) in closed loop with $\kappa(x, \bar{r})$, then $\mathcal{X}$ is positively invariant also for (2.5.6) in closed loop with $\kappa(x, \bar{r})$.*

Proof If $\kappa(x, r)$ keeps any trajectory of (2.6.14) inside $\mathcal{X} \subseteq \mathcal{S}$, given x and r , then from Lemma 2.6.6 there exists an appropriate law $\bar{r} = r(k)$ for which every trajectory of the original system (2.5.6) is also a trajectory of system (2.6.14) under $\kappa(x, \bar{r})$. Therefore, every trajectory of (2.5.6) is contained in $\mathcal{X}$, which is then positively invariant. ■

If Lemma 2.6.6 holds, then one can search for a control gain K , or for a more general gain scheduling terminal controller $K(r) = \sum_i^{n_r} K_i r_i$, that provides a control invariant set, $\mathbb{X}_f$, for (2.6.14) such that $\mathbb{X}_f \subseteq \mathcal{S}$. Then, from Lemma 2.6.6 and Theorem 2.6.7, the set $\mathbb{X}_f$ is also invariant for (2.5.6). Note that, for what concerns the MPC design, the parameter $\bar{r}$ never needs to be computed.

The matrices $(P, K(r))$ can be computed from the solution of the following set of matrix inequalities (Boyd et al. 1994; Kothare et al. 1996) at the parameter space vertices (here assuming $T = 0$):

$$(A_i + B_i K_i)^T P (A_i + B_i K_i) - P \leq -Q - K_i^T R K_i, \quad \forall i \in \mathbb{I}_{[0, n_r]}. \quad (2.6.17)$$

If (2.6.17) admits a solution, for some $\mathcal{S}$, then $\mathbb{X}_f$ can be taken as a level set of $x^T P x$, or as any admissible robust invariant set inside $\mathcal{S}$. This will be proven in Chap. 5. A drawback of LDI methods is that, depending on the system, a solution for (2.6.17) might not exist, for various candidate $\mathcal{S}$. Computation is also intense, and it grows dramatically with the number of parameters.

In Chap. 7 of Blanchini and Miani (2008), it is pointed out that, even when (2.6.17) is not solvable (a quadratic CLF cannot be found), the system (2.6.14) could still be robustly stabilisable, for instance, with a more general PWA controller (e.g. an

MPC). Constraints and dynamics are assumed to be polytopic, causing the largest robustly invariant set for (2.6.14) to be also a polytope. This polytope must satisfy the conditions of Theorem 2.5.9, for all A_i and B_i (see Blanchini and Miani 2008, Chap. 7.3). A technique for computing the maximal RPI set for (2.6.14) has been proposed in Blanchini (1999). In particular, λ -contractive sets are constructed by means of polytope projections (Kerrigan 2000). This is basically a dynamic programming approach. Then, the resulting set is also λ -contractive for the original system (2.5.6). Polytope projections, required for the technique in Blanchini (1999), can be quite demanding. In particular, they can be performed only for relatively small state dimensions. A fundamental step for using the difference inclusion is to determine the set $\mathcal{S}$. A search is required for (relatively large) $\mathcal{S}$, for which the resulting system (2.6.14) is robustly stabilisable, under constraints. This idea is used, for LTI systems, in Grammatico and Pannocchia (2013), where $F(x)$ is proportional to the Minkowski function of $\mathbb{X}_f$.

The use of projections can be avoided (at the expense of a smaller terminal set) if, for instance, a stabilising LPV controller is known for system (2.6.14). Then, a λ -contractive set is computed for this closed-loop system, as done for $\mathcal{O}_\infty$. Alternatively, one can use a matrix of candidate vertices, X , and then impose the conditions in Theorem 2.5.9, for all A_i and B_i . This is done in Chap. 5, by solving an LP. In the same Chapter a Minkowski terminal cost is proposed, similarly to Grammatico and Pannocchia (2013), the difference being the method for computing $\mathbb{X}_f$. In Chap. 5, as well as in Gallieri and Maciejowski (2013a, b), Grammatico and Pannocchia (2013), the terminal set $\mathbb{X}_f$ is computed *a priori* and it is independent from the cost function. Then, the terminal set is scaled, according to the stage cost. This facilitates fast online tuning, as computing $\mathbb{X}_f$ is effectively the main bottleneck in MPC design. Further alternatives exist, among which are Bravo et al. (2005), Fiacchini et al. (2012).

2.6.6 Domain of Attraction

Recall Definitions 2.5.1, 2.5.19, 2.5.20 and 2.5.21. The following is a standard result of MPC theory:

Theorem 2.6.8 *Assume Assumptions 2.6.1–2.6.3. Then, the feasible region of problem (2.6.1) is equivalent to the domain of attraction (DOA) of the closed-loop system (2.5.8), and it is given by*

$$\mathbb{X}_N = \{x \in \mathbb{X} : \exists \underline{u}_{[N-1]} \in \mathbb{U}^N | \bar{\phi}(i, x, \underline{u}_{[i-1]}) \in \mathbb{X} \forall i \in \mathbb{I}[0, N], \bar{\phi}(N, x, \underline{u}_{[N-1]}) \in \mathcal{O}_\infty\}. \quad (2.6.18)$$

or, equivalently,

$$\mathbb{X}_N = \mathcal{K}_N(\mathbb{X}, \mathbb{X}_f) = \mathcal{S}_N(\mathbb{X}, \mathbb{X}_f), \text{ and } \mathbb{X}_N \supseteq \mathbb{X}_f. \quad (2.6.19)$$

The above results are connected to these in Sect. 2.6.3. In particular, recursive feasibility of the MPC problem is used to prove stability, and is obtained by means of a control invariant terminal set $\mathbb{X}_f$. The presence of this set, on the other hand, can limit the size of the feasible region and the system operations. The set-theoretic arguments in Kerrigan and Maciejowski (2000a) can be used to show how the choice of $\mathbb{X}_f = \mathcal{O}_\infty$ provides a DOA that is at least as large as when a level set of the terminal cost is used instead. For LTI systems the maximal DOA can be obtained if, for instance, the approach of Grammatico and Pannocchia (2013) is used. This requires an increase of computation as discussed in Chap. 5.

Remark 2.6.6 The case of $c = 0$ is the trivial completion constraint, $x_N = 0$, which automatically satisfies the conditions for asymptotic stability of the origin for the considered nonlinear systems (Rawlings and Mayne 2009). However, for such a constraint one can expect the feasible region (the DOA) to be smaller than for a terminal constraint set, thus restricting the plant operating points. Moreover, since the predictions will reach the origin in finite time such a controller might present an unnecessarily high gain.

2.6.7 Sufficient Conditions for Intrinsic Robustness

It is desirable for a feedback controller to possess a certain degree of robustness to bounded uncertainties. In the nonlinear case, this cannot be guaranteed a priori by a nominally stabilising controller, without performing a robust design (Grimm et al. 2004).

In this thesis, robustness is characterised in terms of robust invariance and IS(p)S. To introduce the discussion, consider the following (Limon et al. 2009):

Theorem 2.6.9 *Assume that Theorem 2.6.4 holds. Moreover, assume that either,*

1. *$f(x, K_N(x), w)$ is continuous $\forall x \in \mathbb{X}_N, \forall w \in \mathbb{W}$,*
2. *The MPC optimal cost $V_N^o(x)$ is continuous in $\mathbb{X}_f$.*

Then, there exists an $r > 0$ such that, if $\mathbb{W} \subseteq \mathcal{B}_{\ell_p, r}(0)$, then the system (2.6.4) admits an RPI set $\mathbb{X}_N^r \subseteq \mathbb{X}_N$, $0 \in \text{int}\mathbb{X}_N^r$. Moreover, the system is ISS in $\mathbb{X}_N^r$.

Proof By the Heine–Cantor Theorem 2.3.3, the uniform continuity assumed in Theorem 4 of Limon et al. (2009) can be relaxed to simple continuity. Then, similarly to Theorem 2.5.6, the existence of a uniformly continuous Lyapunov function $V(x)$ provides the satisfaction of the ISS property $V(x(k+1)) \leq V(x(k)) - \alpha(\|x(k)\|) + \sigma(\|w\|)$, with $\alpha \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$. As shown in the proof of Theorem 4 of Limon et al. (2009), uniform continuity of the model (the first assumption) provides the existence of such $V(x)$. Under the second assumption, $V(x)$ can be taken as the MPC optimal cost $V_N^o(x)$. The existence of the RPI set $\mathbb{X}_N^r$, given a sufficiently small $r > 0$, is shown by Limon et al. (2009) to be a result of the ISS property of $V(x)$, being $\mathbb{X}_N^r$ one of its level sets for which $x \in \mathbb{X}_N^r \Rightarrow \alpha(\|x\|) \geq \sigma(r)$. ■

Remark 2.6.7 Recall that in this thesis the considered systems models are assumed to be continuous, the cost function is continuous and convex. If the system is also LTI, then $K_N(x)$ is also continuous and the intrinsic robustness result of Theorem 2.6.9 applies to an MPC with nominal stability. The nonlinear case is investigated, for instance, in Pannocchia et al. (2011).

The ISS gain of the system is a measure of robust stability and performance. In MPC, this gain depends mainly on the terminal cost, $F(x)$, as will be further discussed in Chap. 7.

2.6.8 Explicit Solution of Linear Quadratic MPC

The term *explicit MPC* (Alessio and Bemporad 2009; Bemporad et al. 2002) stands for the closed-form representation of an MPC control law, computed offline. For LTI systems this can be done by means of multi-parametric linear or quadratic programming (respectively, mpLP and mpQP). This representation allows for the MPC law to be evaluated without online optimisation, at the cost of additional memory usage (Borrelli et al. 2008, 2009). Explicit MPC is generally used for applications with fast sampling, including, for instance, automotive (Di Cairano et al. 2013) and power electronics (Mariéthoz et al. 2012). This form of MPC can also be useful for post-design analysis, as it provides further insight on the control law and the optimal cost. The material presented in this section will be used, in Chap. 4, to describe the solution of ℓ_{asso} -MPC, as well as in Chap. 6 for the synthesis of a particular type of ℓ_{asso} -MPC.

Procedures for computing the explicit MPC solution are presented, for instance, in Alessio and Bemporad (2009), Bemporad et al. (2002), Tondel et al. (2003). The technique is quite consolidated for quadratic, 1- and ∞ -norm costs and for systems with piece-wise affine dynamics (Alessio and Bemporad 2009). The basic idea is to partition the feasible region $\mathbb{X}_N$ in a set of N_r *critical regions*, $\mathcal{P}_i$, each one associated with an *active set of constraints* $\mathcal{A}_i$. Complexity of the resulting controller is expressed in terms of N_r and depends (in the worst case exponentially) on the number of variables and constraints, as well as on the cost function parameters (Alessio and Bemporad 2009; Borrelli et al. 2009). Several strategies have been proposed for the exploration of the feasible space, required to determine the critical regions, as well as for reducing N_r (Bemporad et al. 2002; Tondel et al. 2003). Computational complexity can become an issue for systems with, for instance, more than $n \geq 5$ states and horizon length $N \geq 5$.

Consider the Quadratic MPC defined by (2.6.1)–(2.6.3) with an LTI prediction model

$$\hat{f}(x, u) = Ax + Bu,$$

and closed polytopic constraints. The following matrices are widely used throughout the thesis

$$\Psi = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \Theta = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}, \quad \begin{aligned} \mathbf{Q} &= \text{BlockDiag}(I_{N-1} \otimes Q, P), \\ \mathbf{R} &= I_N \otimes R, \\ H &= (\Theta^T \mathbf{Q} \Theta + \mathbf{R}), \\ \Gamma &= \Theta^T \mathbf{Q} \Psi. \end{aligned} \quad (2.6.20)$$

The Quadratic MPC problem (2.6.1)–(2.6.3) can be formulated as the following mpQP:

$$\begin{aligned} V_N^o(x) &= x^T Y x + \min_{\underline{u}} (\underline{u}^T H \underline{u} + 2\underline{u}^T \Gamma x) \\ \text{s.t. } \Omega \underline{u} &\leq \underline{1} - Mx = \underline{b}(x), \end{aligned} \quad (2.6.21)$$

where $Y = \Psi^T \mathbf{Q} \Psi + Q$, and the term $x^T Y x$ does not affect the optimal solution. With $R \succ 0$ and polyhedral constraints the following holds:

Theorem 2.6.10 (Alessio and Bemporad 2009; Bemporad et al. 2002) *Assume $H \succ 0$, $\begin{bmatrix} H, \Gamma \\ \Gamma^T, Y \end{bmatrix} \succeq 0$. Then $\mathbb{X}_N$ is a polyhedral set, the value function $V_N^o : \mathbb{X}_N \rightarrow \mathbb{R}$ is continuous, convex and piece-wise quadratic in x . The optimiser $\underline{u}^* : \mathbb{X}_N \rightarrow \mathbb{U}^N$ is unique, continuous and piece-wise affine in x .*

Remark 2.6.8 Theorem 2.6.10 is not only useful to compute an explicit MPC solution. It provides a theoretical characterisation for $K_N(x)$ and $V_N^o(x)$.

Theorem 2.6.10 is proved by means of the KKT conditions for (2.6.21)

$$\begin{aligned} 2H \underline{u}^*(x) + 2\Gamma x + \Omega^T \nu^*(x) &= 0, \\ -\Omega \underline{u}^*(x) - s^*(x) &= -\underline{b}(x), \\ (s^*(x))^T \nu^*(x) &= 0, \\ s^*(x) &\geq 0, \\ \nu^*(x) &\geq 0. \end{aligned} \quad (2.6.22)$$

An illustrative proof sketch of Theorem 2.6.10 follows, based upon Alessio and Bemporad (2009), Bemporad et al. (2002), Kerrigan and Maciejowski (2000b). Solving (2.6.22) for $\underline{u}^*$ provides (since H is invertible)

$$\underline{u}^*(x) = -H^{-1} \left(\Gamma x + \frac{1}{2} \Omega^T \nu^*(x) \right). \quad (2.6.23)$$

If constraints are not active at a particular x , the optimal solution (2.6.23) is linear in x and the LQ MPC will act as a linear controller. Given an active set of constraints, $\mathcal{A}(x)$, denote $\tilde{\nu}$ as the corresponding Lagrange multipliers (the remaining ones being

equal to zero). The set of x with the same active set is called a *critical region*. Assume without loss of generality that (2.6.21) is not primal degenerate, namely that Ω has full row rank.⁵ Then, substituting (2.6.23) into (2.6.22) provides

$$\tilde{v}^*(x) = -2(\tilde{\Omega}H^{-1}\tilde{\Omega}^T)^{-1} \left(\underline{1} + (\tilde{\Omega}H^{-1}\Gamma - \tilde{M})x \right). \quad (2.6.24)$$

Therefore, the optimal Lagrange multipliers are PWA in x . By substituting this back into (2.6.23) it is also possible to show the optimiser to be PWA in $x(k)$. In particular

$$\underline{u}^*(x) = -H^{-1}\Gamma x + \tilde{\Omega}^T (\tilde{\Omega}H^{-1}\tilde{\Omega}^T)^{-1} \left(\underline{1} + (\tilde{\Omega}H^{-1}\Gamma - \tilde{M})x \right). \quad (2.6.25)$$

Continuity of the solution is shown in Bemporad et al. (2002) by means of the adjacency of critical regions. These regions form in fact a partition of the feasible set. By similar arguments (Bemporad et al. 2002) has shown that the value function is continuous, convex (recall Lemma 2.3.7) and PWQ in $x(k)$.

The techniques used for explicit MPC aim to determine the active sets for all $x \in \mathbb{X}_N$ by means of ad hoc exploration strategies (Alessio and Bemporad 2009; Bemporad et al. 2002; Kerrigan and Maciejowski 2000b). The parameters describing the first m elements of (2.6.25) are then stored in a lookup table. If numerical precision is an issue, for instance, when constraints are close to degeneracy, the coefficients of (2.6.24) and (2.6.25) can also be computed with different methods, for instance, using QR decomposition (Borrelli et al. 2009; Fletcher 1987).

The required computation for online evaluation of the control law, as well as the required storage, has been investigated, for instance, by Borrelli et al. (2008, 2009). Define N_c^i the number of halfspaces describing $\mathcal{P}_i$, and $N_C = \sum_{i=0}^{N_r} N_c^i$. The simplest possible implementation, reviewed in Borrelli et al. (2008) and used in Chap. 6 of this thesis, requires in the worst case $2nN_C$ floating-point operations (flops) and a storage of $(n+1)N_C$ real numbers (including the inequalities for $\mathcal{P}_i$). On the other hand, the algorithm proposed in Borrelli et al. (2008) (not investigated here) requires, for quadratic cost, $(2n-1)N_r + N_C$ flops and a storage of $(2n-1)N_r + N_C$ reals in the worst case. The control law can also be approximated for further complexity reduction as proposed in Johansen and Grancharova (2003), Summers et al. (2011), where stability is guaranteed for a certain level of suboptimality.

Remark 2.6.9 Theorem 2.6.10 can also be used to compute a lower bound on the penalty weight for an exact penalty function (Kerrigan and Maciejowski 2000b). This can be used for instance to replace some state constraints with penalties (or *soft constraints*) (Kerrigan and Maciejowski 2000b), and will be further discussed in Chaps. 6 and 7.

⁵Redundant inequalities can be removed at a preliminary stage.

2.6.9 Robust MPC and Soft Constraints

In real-world MPC applications the presence of constraints is twofold. On one hand, control performance can be notably enhanced, as mentioned above. On the other hand, handling constraints can become problematic when recursive feasibility is lost due to excessive disturbances or plant/model mismatch. Uncertainty, feedback control's *raison d'être* (Vinnicombe 2000) has been the object of study within the MPC community, through several approaches. For the purpose of this thesis, the focus is on the more classic deterministic *robust MPC* strategies. A possible choice is the restriction of constraints based on a set-valued uncertainty description. This is done in, among others, (Blanchini and Miani 2008; Chisci et al. 2001; Goulart et al. 2006; Kuwata et al. 2007; Richards and How 2006; Shekhar and Maciejowski 2012) to allow for the existence (or application) of an additional controller, such that the closed-loop trajectory is bounded inside the original constraints. This offers neat but conservative results at the expense of the nominal feasible region.

An indirect way to deal with uncertainty is to use soft-constrained MPC, where slack variables and penalty functions are used instead of the standard “hard” constraints (Kerrigan and Maciejowski 2000a; Maciejowski 2002; Zeilinger et al. 2010). This is generally done to the state constraints. A special case of this is the relaxation of the terminal constraint (Grüne 2011; Grüne and Pannek 2011; Limon et al. 2006) that can allow for a larger feasible region. On the other hand, softening the terminal constraint does not always guarantee closed-loop stability from every point of feasible region. The arguments in Limon et al. (2006) apply to ℓ_{asso} -MPC, but are not addressed here. A further indirect way to deal with uncertainty is to have intermediate targets, as in the recently introduced MPC for tracking (Ferramosca et al. 2008, 2009b; Limon et al. 2008; Zeilinger et al. 2010). Chapter 7 formulates a synthesis of these ideas, with a novel robust uncertainty bound for soft-constrained MPC for tracking.

2.6.10 MPC for Tracking

In the above sections, the use of a terminal constraint has been assumed. This constraint generally depends on the particular set point we want to regulate to. For changing set points this means changing the terminal constraint, which is computationally demanding and could also make the problem infeasible. The MPC for tracking, introduced by Ferramosca et al. (2008, 2009b), Limon et al. (2008) for LTI systems and extended to nonlinear systems in Ferramosca et al. (2009a), overcomes the limitations of MPC for regularisation by means of an invariant set that is common to a range of feasible steady states. This tracking approach can also benefit from a larger feasible region than standard MPC for regulation. In this thesis MPC for tracking is investigated for LTI systems in Chap. 7. Considering system (2.5.7), the steady states can be characterised by the following:

Lemma 2.6.11 (based on Ferramosca et al. 2008) *If (A, B) is stabilisable, $\exists M_\theta \in \mathbb{R}^{n+m \times n_\theta}$, $n_\theta \leq n + m$ such that the steady states of the nominal system satisfy $(x_s, u_s) = M_\theta \theta_s$, with $M_\theta^T = [M_x^T \ M_u^T]$, and with $\theta_s = M_x^T x_s + M_u^T u_s$. In particular,*

$$M_\theta = \text{null}([A - I, \ B]). \quad (2.6.26)$$

In Ferramosca et al. (2009b) the case of tracking a piece-wise constant signal, $y_t(k) = Cx_t(k) + Du_t(k)$ is considered. In particular, a virtual set point is introduced so that the standard MPC cost penalises deviations from this artificial variable. Then a (possibly non-smooth) function is added to the MPC cost, penalising the virtual and the actual set point mismatch. In Ferramosca et al. (2009b), the MPC problem (2.6.1) is modified as follows:

1. Add an extra decision variable, θ , and an external reference signal $y_t(k)$,
2. Penalise $\delta \hat{x}_j = \hat{x}_j - M_x \theta$ and $\delta \hat{u}_j = \hat{u}_j - M_u \theta$ instead of $\hat{x}_j$, $\hat{u}_j$,
3. Define a constrained system, Σ_θ , with states $(x(k), \theta(k))$ (where $\theta(k)$ is simply a constant), in closed loop with a static terminal controller, $u(k) = K(x(k) - M_x \theta(k)) + M_u \theta(k)$,
4. Impose a different terminal constraint set, $(\hat{x}_N, \theta) \in \mathbb{X}'_f$, with $\mathbb{X}'_f$ positively invariant for the closed-loop system Σ_θ ,
5. Add to the cost function a sub-differentiable penalty, $V_O([C, \ D]M_\theta \theta - y_t(k)) > 0$.

2.6.10.1 Stability and Convergence

Sufficient conditions for recursive feasibility and asymptotic convergence to a feasible steady state are given in (Ferramosca et al. 2009b), where it is also shown that convergence occurs to a minimiser of the offset cost V_O if an asymptotically constant $y_t(k)$ is assumed. Results are only sketched here, as they will be further revised in Chap. 7. In particular, in (Ferramosca et al. 2009b) stability of a steady state is proven by the same argument as for Theorem 2.6.4, posed in the coordinates δx , δu . This is obtained through Assumption 2.6.4, assumed to hold for Σ_θ in the invariant set $\mathbb{X}'_f$, with K solving (2.6.7). Then convergence to a minimiser of V_O is proven by contradiction, by showing the sub-differential of the value function (V_N^o) to be non-negative for any feasible steady state that is not a minimiser of V_O .

2.6.10.2 Robustness

The recent development of (Zeilinger et al. 2010) combined the approach of (Ferramosca et al. 2008) with the concept of soft constraints (Kerrigan and Maciejowski 2000b; Maciejowski 2002). In particular, the approach of (Zeilinger et al. 2010) results in an even larger feasible region than the one of (Ferramosca et al. 2008), and provides a level of intrinsic robustness to bounded uncertainties in the form of

Input-to-State Stability (ISS) (Grimm et al. 2004; Jiang and Wang 2001; Lazar 2006; Limon et al. 2009).

2.6.10.3 Proposed Contribution

In Chap. 7 a novel set of conditions is given for robust feasibility under bounded additive uncertainties. These conditions require minor modification of the terminal constraint of Ferramosca et al. (2008) and, if satisfied, they provide an uncertainty bound for which the MPC is always feasible and the feasible region is an RPI set. This region is allowed to be greater than the state constraints and, in most cases, is greater than the feasible region of standard robust MPC. Finally, a local ISS gain, which depends on the system and the cost parameters, is computed.⁶

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⁶This includes a corrigendum of the ISS gain provided in Theorem III.7 of Gallieri and Maciejowski (2013a).

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Lasso-MPC – Predictive Control with ℓ_1 -Regularised
Least Squares

Gallieri, M.

2016, XXX, 187 p. 64 illus., 54 illus. in color., Hardcover

ISBN: 978-3-319-27961-9