

## Chapter 2

# Brouwer in Dimension Two

### THE BROUWER FIXED-POINT THEOREM VIA SPERNER'S LEMMA

**Overview.** In dimension two the Brouwer Fixed-Point Theorem states that every continuous mapping taking a closed disc into itself has a fixed point. In this chapter we'll give a proof of this special case of Brouwer's result, but for triangles rather than discs; closed triangles are homeomorphic to closed discs (Exercise 2.2 below) so our result will be equivalent to Brouwer's. We'll base our proof on an apparently unrelated combinatorial lemma due to Emanuel Sperner, which—in dimension two—concerns a certain method of labeling the vertices of “regular” decompositions of triangles into subtriangles. We'll give two proofs of this special case of Sperner's Lemma, one of which has come to serve as a basis for algorithms designed to approximate Brouwer fixed points.

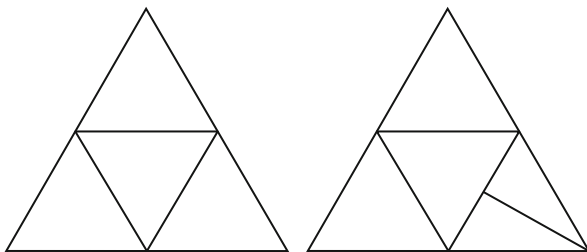
**Prerequisites.** Undergraduate real analysis: compactness and continuity in the context of  $\mathbb{R}^2$ .

## 2.1 Sperner's Lemma

Throughout this discussion, “triangle” means “closed triangle,” i.e., the convex hull of three points in Euclidean space that don't all lie on the same straight line. A “regular decomposition” of a triangle is a collection of subtriangles whose union is the original triangle and for which the intersection of two distinct subtriangles is either a vertex or a complete common edge. Figure 2.1 below illustrates both a regular and an irregular decomposition of a triangle into subtriangles.

A “Sperner Labeling” of the subvertices (the vertices of the subtriangles) in a regular decomposition is an assignment of labels “1,” “2,” or “3” to each subvertex in such a way that:

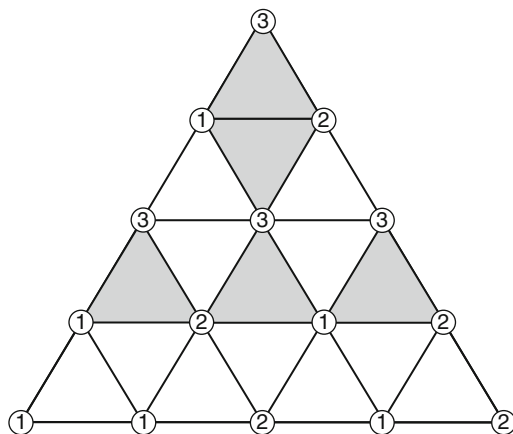
- (a) No two vertices of the original triangle get the same label (i.e., all three labels get used for the original vertices),



**Fig. 2.1** Regular (*left*) and irregular (*right*) decomposition of a triangle into subtriangles

- (b) Each subvertex lying on an edge of the original triangle gets labels drawn only from the labels of that edge, e.g., subvertices on the original edge labeled “1” and “2” (henceforth: a “ $\{1, 2\}$  edge”) get only the labels “1” or “2,” but with no further restriction.
- (c) Subvertices lying in the interior of the original triangle can be labeled without restriction.

We’ll call a subtriangle whose vertices have labels “1,” “2,” and “3” a *completely labeled subtriangle*. Figure 2.2 shows a regular decomposition of a triangle into Sperner-labeled subtriangles, five of which (the shaded ones) are completely labeled.



**Fig. 2.2** A Sperner-labeled regular decomposition into subtriangles

**Theorem 2.1** (Sperner’s Lemma for Dimension Two). *Every Sperner-labeled regular decomposition of a triangle has an odd number of completely labeled subtriangles; in particular there is at least one.*

**The One dimensional case.** Here, instead of triangles split “regularly” into subtriangles, we just have a closed finite line segment split into finitely many closed subsegments which can intersect in at most a common endpoint. One end of the original segment is labeled “1” and the other is labeled “2.” The remaining subsegment endpoints get these labels without restriction.

Sperner's Lemma for this situation asserts that: *There is an odd number of subsegments (in particular, at least one!) whose endpoints get different labels.*

To prove this let's imagine moving from the one-labeled endpoint of our initial interval toward the two-labeled one. If there are no subintervals, we're done. Otherwise there has to be a first subinterval endpoint whose label switches from “1” to “2,” thus yielding a completely labeled subinterval with final endpoint “2.” At the next switch, if there is one, the initial endpoint is “2” and the final endpoint is “1,” thus yielding another completely labeled subinterval which must, somewhere further on the line, have an oppositely labeled companion (else we'd never be able to end up with the final subinterval labeled “2”). Thus there must be an odd number of completely labeled subintervals.  $\square$

**The Two dimensional case.** We start with a triangle  $\Delta$  regularly decomposed into a finite collection of subtriangles  $\{\Delta_j\}$ . Let  $v(\Delta_j)$  denote the number of “ $\{1,2\}$ -labeled edges” belonging to the boundary of  $\Delta_j$ , and set  $S = \sum_j v(\Delta_j)$ . We'll compute  $S$  in two different ways:

*By counting edges.* If a  $\{1,2\}$ -labeled edge of  $\Delta_j$  does not belong to the boundary of  $\Delta$  then it belongs to exactly one other subtriangle. If a  $\{1,2\}$ -labeled edge of  $\Delta_j$  lies on the boundary of  $\Delta$ , then that edge belongs to no other subtriangle. Thus  $S$  is twice the number of “non-boundary”  $\{1,2\}$ -labeled edges plus the number of “boundary”  $\{1,2\}$ -labeled edges. But by the one dimensional Sperner Lemma, the number of boundary  $\{1,2\}$ -labeled edges is odd. Thus  $S$  is odd.

*By counting subtriangles.* Each completely labeled subtriangle has exactly one  $\{1,2\}$ -labeled edge. All the others have either zero or two such edges. Thus the odd number  $S$  is the number of completely labeled subtriangles plus twice the number of subtriangles with  $\{1,2\}$  edges, hence our Sperner-labeled regular decomposition of  $\Delta$  has an odd number of completely labeled subtriangles.

## 2.2 Proof of Brouwer's Theorem for a Triangle

We may assume, without loss of generality (see the exercise below), that our triangle  $\Delta$  is the standard simplex  $\Pi_3$  of  $\mathbb{R}^3$  (see Definition 1.7). Fix a continuous self-map  $f$  of  $\Delta$ ; for each  $x \in \Delta$  write  $f(x) = (f_1(x), f_2(x), f_3(x))$ . Thus for each index  $j = 1, 2, 3$  we have a continuous “coordinate function”  $f_j: \Delta \rightarrow [0, 1]$  with  $f_1(x) + f_2(x) + f_3(x) = 1$  for each  $x \in \Delta$ .

*A Sperner labeling induced by  $f$ .* Consider a regular decomposition of  $\Delta$  into subtriangles and suppose  $f$  fixes no subvertex (if  $f$  fixes a subvertex, we are done). Then  $f$  determines a Sperner labeling of subvertices in the following manner. Fix a subtriangle vertex  $p$ . Since  $f(p) \neq p$ , and since both  $p$  and  $f(p)$  have non-negative coordinates that sum to 1, at least one coordinate of  $f(p)$  is strictly less than the corresponding coordinate of  $p$ . Choose such a coordinate and use its index to label the subvertex  $p$ .

In this way the three original vertices  $e_1 = (1,0,0)$ ,  $e_2 = (0,1,0)$ , and  $e_3 = (0,0,1)$ , get the labels “1,” “2,” and “3,” respectively. For example,  $f(e_1) \neq e_1$ , so the first coordinate of  $f(e_1)$  must be strictly less than 1, and similarly for the other two vertices of  $\Delta$ . Each vertex on the  $\{1,2\}$  edge of  $\partial\Delta$  (the line segment joining  $e_1$  to  $e_2$ ) has third coordinate zero, so this coordinate cannot strictly decrease when that vertex is acted upon by  $f$ . Thus (since that vertex is not fixed by  $f$ ) at least one of the other coordinates must strictly decrease, so each vertex on the  $\{1,2\}$ -edge gets only the labels “1” or “2,” as required by Sperner labeling. Similarly for the other edges of  $\partial\Delta$ ; the vertices on the  $(2,3)$ -edge get only labels “2” and “3,” and the vertices on the  $(1,3)$ -edge get only labels “1” and “3.” No further checking is required for the labels induced by  $f$  on the interior vertices; Sperner labeling places no special restrictions here. In this way  $f$  determines, for each regular subdivision of  $\Delta$ , a Sperner labeling of the subvertices (note that the continuity assumed for  $f$  has not yet been used).

*Approximate fixed points for  $f$ .* Let  $\varepsilon > 0$  be given. We’re going to show that our continuous self-map  $f$  of  $\Delta$  has an  $\varepsilon$ -approximate fixed point, i.e., a point  $p \in \Delta$  such that  $\|f(p) - p\|_1 \leq \varepsilon$ . Here  $\|x\|_1$  is the “one-norm” of  $x \in \mathbb{R}^3$ , as defined by Eq. (1.5) (page 11). Being continuous on the compact set  $\Delta$ , the mapping  $f$  is *uniformly* continuous there, so there exists  $\delta > 0$  such that  $x, y \in \Delta$  with  $\|x - y\|_1 < \delta$  implies  $\|f(x) - f(y)\|_1 < \varepsilon/8$ . Upon decreasing  $\delta$  if necessary we may assume that  $\delta < \varepsilon/8$ . Now suppose  $\Delta$  is regularly decomposed into subtriangles of  $\|\cdot\|_1$ -diameter  $< \delta$ . If some subvertex of this decomposition is a fixed point of  $f$ , we’re done. Suppose otherwise. Thus  $f$  creates a Sperner labeling of the subvertices of this decomposition. Let  $\Delta_\varepsilon$  be a completely labeled subtriangle, as promised by Sperner’s Lemma.

*Claim.*  $\Delta_\varepsilon$  contains an  $\varepsilon$ -approximate fixed point.

*Proof of Claim.* Let  $p$ ,  $q$ , and  $r$  be the vertices of  $\Delta_\varepsilon$ , carrying the labels “1,” “2,” and “3,” respectively, so that  $f_1(p) < p_1$ ,  $f_2(q) < q_2$ , and  $f_3(r) < r_3$ . Thus:

$$\begin{aligned} \|p - f(p)\|_1 &= \underbrace{p_1 - f_1(p)}_{>0} + |p_2 - f_2(p)| + |p_3 - f_3(p)| \\ &= p_1 - f_1(p) + |q_2 - f_2(q) + p_2 - q_2 + f_2(q) - f_2(p)| \\ &\quad + |r_3 - f_3(r) + p_3 - r_3 + f_3(r) - f_3(p)| \end{aligned}$$

$$\begin{aligned}
&\leq \underbrace{p_1 - f_1(p)}_{>0} + \underbrace{q_2 - f_2(q_2)}_{>0} + \underbrace{r_3 - f_3(r_3)}_{>0} \\
&\quad + |p_2 - q_2| + |f_2(q) - f_2(p)| \\
&\quad + |p_3 - r_3| + |f_3(r) - f_3(p)|,
\end{aligned}$$

so  $\|p - f(p)\| \leq A + B$ , where

$$A = [p_1 - f_1(p)] + [q_2 - f_2(q_2)] + [r_3 - f_3(r_3)]$$

which is  $> 0$  since this is true of each bracketed term, and

$$B = |p_2 - q_2| + |f_2(q) - f_2(p)| + |p_3 - r_3| + |f_3(r) - f_3(p)|. \quad (2.1)$$

Now each summand on the right-hand side of (2.1) is  $< \varepsilon/8$ , hence  $B < \varepsilon/2$ . As for  $A$ , the same “adding-zero trick” we used above yields

$$\begin{aligned}
A &= \underbrace{p_1 + p_2 + p_3}_{=1} - \underbrace{f_1(p) + f_2(p) + f_3(p)}_{=1} \\
&\quad + [q_2 - p_2] + [f_2(p) - f_2(q)] \\
&\quad + [r_3 - p_3] + [f_3(p) - f_3(r)].
\end{aligned}$$

On the right-hand side of this equation, the top line equals zero and each bracketed term has absolute value  $< \varepsilon/8$ , so by the triangle inequality,  $A < \varepsilon/2$ . These estimates on  $A$  and  $B$  yield  $\|p - f(p)\|_1 < \varepsilon$ , the vertex  $p$  of  $\Delta_\varepsilon$  is an  $\varepsilon$ -approximate fixed point of  $f$ .  $\square$

The same argument shows that the other two vertices of  $\Delta_\varepsilon$  are also  $\varepsilon$ -approximate fixed points of  $f$ ; the triangle inequality shows that *every* point of  $\Delta_\varepsilon$  is a  $\frac{5}{4}\varepsilon$ -approximate fixed point.

*A fixed point for  $f$ .* So far we know that our self-map  $f$  of  $\Delta$  has an  $\varepsilon$ -approximate fixed point for every  $\varepsilon > 0$ . In particular, for each positive integer  $n$  there is a  $1/n$ -approximate fixed point  $x_n$ . Since  $\Delta$  is compact there is a subsequence  $(x_{n_k})$  convergent to some point  $x \in \Delta$ . By the triangle inequality for the norm  $\|\cdot\|_1$ :

$$\|x - f(x)\|_1 \leq \|x - x_{n_k}\|_1 + \|x_{n_k} - f(x_{n_k})\|_1 + \|f(x_{n_k}) - f(x)\|_1$$

On the right-hand side of this inequality, as  $k \rightarrow \infty$ :

The first summand  $\rightarrow 0$  (since  $x_{n_k} \rightarrow x$ ).

Therefore the third summand  $\rightarrow 0$  by the continuity of  $f$ .

The second summand  $\rightarrow 0$  because it's  $< 1/n_k$ .

*Conclusion:*  $\|x - f(x)\|_1 = 0$ , hence  $f(x) = x$ , as desired.  $\square$

The argument above works much more generally to prove:

**Lemma 2.2** (The Approximate-Fixed-Point Lemma). *Suppose  $(X, d)$  is a compact metric space and  $f: X \rightarrow X$  is a continuous map. Suppose that for every  $\varepsilon > 0$  there exists a point  $x_\varepsilon \in X$  with  $d(f(x_\varepsilon), x_\varepsilon) \leq \varepsilon$ . Then  $f$  has a fixed point.*

*Proof.* Exercise: generalize the proof given above for the metric induced by the one-norm to arbitrary metrics.  $\square$

*Exercise 2.1.* Here's another way to produce fixed points from completely labeled subtriangles. Make a regular decomposition of  $\Delta$  into subtriangles of diameter  $< 1/n$ . For this decomposition of  $\Delta$ , use  $f$  to Sperner-label the subvertices, and let  $\Delta_n$  be a resulting completely labeled subtriangle. Denote the vertices of  $\Delta_n$  by  $p^{(n)}$ ,  $q^{(n)}$ , and  $r^{(n)}$ , using the previous numbering scheme so that  $f_1(p^{(n)}) \leq p_1^{(n)}$ , etc. Show that it's possible to choose a subsequence of integers  $n_k \nearrow \infty$  such that the corresponding subsequences of  $p$ 's,  $q$ 's, and  $r$ 's all converge. Show that these three subsequences all converge to the same point of  $\Delta$ , and that this point is a fixed point of  $f$ .

*Exercise 2.2.* Show that every triangle is homeomorphic to a closed disc.

*Suggestion:* First argue that without loss of generality we can suppose that our triangle  $T$  lies in  $\mathbb{R}^2$ , contains the origin in its interior, and is contained in the closed disc  $D$  of radius 1 centered at the origin. Then each point  $z \in T \setminus \{0\}$  is uniquely represented as  $z = r\zeta$  for  $\zeta \in \partial D$  and  $r > 0$ . Let  $w = \rho\zeta$  be the point at which the line through the origin and  $z$  intersects  $\partial T$ . Show that the map that fixes the origin and takes  $z \neq 0$  to  $(r/\rho)\zeta$  is a homeomorphism of  $T$  onto  $D$ .

## 2.3 Finding Fixed Points by “Walking Through Rooms”

Finding fixed points “computationally” amounts to finding an algorithm that produces sufficiently accurate approximate fixed points. Thanks to the work just done in Sect. 2.2, an algorithm for finding a completely labeled subtriangles will do the trick. Here's an alternate proof of Sperner's Lemma that speaks to this issue.

Imagine our triangle  $\Delta$  to be a house, and that the subtriangles of a regular subdivision are its rooms. Given a Sperner labeling of the subvertices that arise from this decomposition, think of each  $\{1, 2\}$ -labeled segment of a subtriangle boundary as a door; these are the only doors. For example, a  $\{1, 2, 2\}$ -labeled subtriangle has two doors, some rooms have no doors (e.g., those with no subvertex labeled “2”); the completely labeled subtriangles are those rooms with exactly one door.

Now imagine that you are outside the house. *There is a door to the inside*; the Sperner labeling of the subvertices induces on the original  $\{1, 2\}$  edge a one dimensional Sperner labeling which, by the  $N = 1$  case of Sperner's Lemma, must produce a  $\{1, 2\}$ -labeled subinterval. Go through this door. Once inside either the room you're in has no further door, in which case you're in a completely labeled subtriangle, or there is another door to walk through. Keep walking, subject to the rule that you can't pass through a door more than once (i.e., the doors are “trap-doors”). There are two possibilities. Either your walk terminates in a completely

labeled room, in which case you’re done, or it doesn’t, in which case you find yourself back outside the house. In that case, you’ve used up two doors on the  $\{1, 2\}$  edge of  $\Delta$ : one to go into the house, and the other to come back out. But according to the one dimensional Sperner Lemma, there are an odd number of such boundary doors, so there’s one you haven’t used. Re-enter the house. Continue. In a finite number of steps you must encounter a room with just one door: a completely labeled one.  $\square$

Figure 2.3 below illustrates this process. Starting at point A one travels through three rooms, arriving outside at point B. The process starts again at B, this time terminating at C, inside a completely labeled subtriangle.

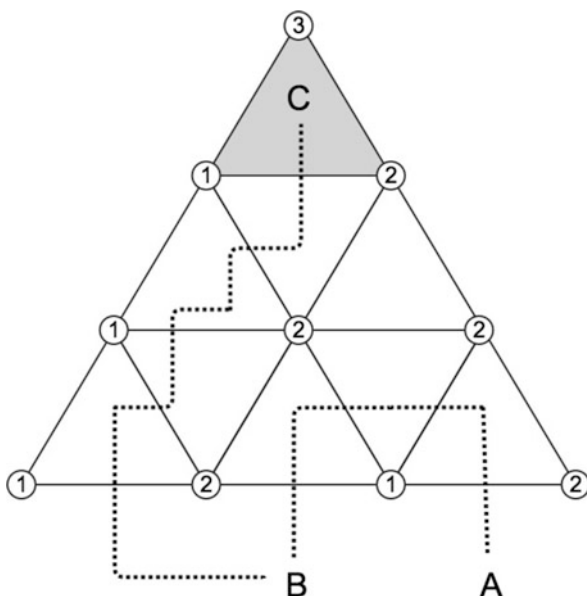


Fig. 2.3 Finding a completely labeled subtriangle by walking through rooms

## Notes

*Sperner’s Lemma, higher dimensions.* This result for all finite dimensions appears in Sperner’s 1928 doctoral dissertation [111]. In dimensions  $> 2$  the analogue of a triangle is an “ $N$ -simplex” in  $\mathbb{R}^N$ ; the convex hull of  $N + 1$  points of  $\mathbb{R}^N$  in “general position,” i.e., no point belongs to the convex hull of the others. The analogue of our regular decomposition of a triangle is a “triangulation” of an  $N$ -simplex into “elementary sub-simplices,” each of which is itself an  $N$ -simplex.

Nice descriptions of this generalization occur in [40, Chap. 3, Sect. 4], and in E.F. Su’s expository article [113], which also provides a proof of the general Brouwer theorem based on “walking through rooms.” Su’s article also contains interesting applications of Sperner’s Lemma to problems of “fair division.”

*Walking through rooms.* In [113] Su attributes this argument to Kuhn [64] and Cohen [26]. According to Scarf [106], however, the argument has its origin in Lemke's 1965 paper [68]. This technique has been greatly refined to produce useful algorithms for finding approximate fixed points, especially by Scarf, whose survey [106], in addition to providing a nice introduction to the legacy of Sperner and Lemke in the algorithmic search for fixed points, also introduces the reader to the way in which economists view Brouwer's theorem.





<http://www.springer.com/978-3-319-27976-3>

A Fixed-Point Farrago

Shapiro, J.H.

2016, XIV, 221 p. 8 illus., Hardcover

ISBN: 978-3-319-27976-3