Problem-Posing and Questioning: Two Tools to Help Solve Problems

José Carrillo and Jorge Cruz

Abstract This paper analyses the solutions put forward by two secondary school pupils to two mathematical problems. The task of working out the solutions was framed by two questionnaires aimed at encouraging self-reflection (completed before and after the activity). The pupils were also asked to pose a new problem with a similar structure to each of the original problems. The results from the different data collection instruments are mutually congruent, from which we can conclude that the methodology is suitable for the design, implementation and evaluation of problem-posing and problem-solving. This methodology can be useful in terms of both research and teaching itself.

Keywords Problem-solving • Problem-posing • Metacognitive questioning

Introduction

Problem-solving has for some time occupied a prominent position in mathematics education research and the mathematics curriculum (Törner, Schoenfeld & Reiss, 2007).

In Portugal, where the research presented here was undertaken, problem-solving continues to receive significant attention throughout the education system as can be seen in syllabus content and teacher training, as well as in systems of evaluation, whether via standardised tests or continuous assessment.

By contrast, problem-posing has received less attention in the syllabus than problem-solving, as it is a more recent approach (Brown & Walter, 1983; Kilpatrick, 1987, cited in Cai & Hwang, 2002) and represents an emergent area of research as well as a significant new tool for teaching (Singer, Ellerton & Cai, 2013).

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Cruz (2003) found that in a study of students aged 12 and 14, the latter were not always the best able to mobilise resources or to apply heuristic and checking techniques. On the other hand, those with the best grades in mathematics (irrespective of age) showed better results in problem-solving in terms of the categories considered in the study (Cruz & Carrillo, 2004). The awarding of grades followed the *Principles and Standards for School Mathematics* (NCTM, 2000), which are consistent with the Portuguese assessment guidelines, both focusing on problem-solving, reasoning and proof, making connections, oral and written communication and uses of mathematical representation.

In the light of this, further studies into how students tackle problem-solving activities would be valuable, in particular through observation of the behaviour of those who habitually achieve the best grades. The decision to select students with good school grades draws on studies to be found in Lesh and Zawojewski (2007). These studies show that such students have at their disposal better structuring of ideas, a wider range of strategies and more representations and are more adept in creating an image of the problem. Consideration of the problems formulated by such students could provide insights into the links between problem-solving and problem-posing and suggest directions that could be followed in teaching in terms of characterising what makes a good problem-solver.

Problem-posing, included in the *Standards* since 1989 as an activity of value to mathematics teaching (NCTM, 1998, p. 163), can be regarded as a task which, by virtue of its design, allows student solutions to be evaluated in terms of their quality (Kilpatrick, 1987, cited in Cai & Hwang, 2003; Goldenberg & Walter, 2006). Cai and Nie (2007) consider the teaching of problem-solving in China and describe three types of tasks, one of which requires students to pose new problems modelled on an original (cf. also Cai et al.'s contribution in this book). The researchers found that such tasks helped students to make connections and make sense of mathematics. Kontorovich and Koichu (2009) suggest a framework for characterising problem-posing [PP]. Amongst the four aspects that these researchers consider are resources, in which the stimulus for PP is regarded as essential. One means of furnishing this stimulus is through an original problem which serves as the basis for formulating a new problem, and it is this option which is followed in this study. In this regard, we consider whether (metacognitive) reflection and problem-posing help to bring students to a clearer understanding of the problem to be solved. This question can be broken down into two related questions:

(a) Does reflection enhance pupils' awareness of the structure of the problem to be solved? (The structure of the problem can be understood as the configuration of relationships, concepts, procedures and degree of difficulty. Several structures could be linked to the same problem, corresponding to different solutions or approaches to the problem.)
(b) Do the problems devised by the pupils benefit from the prior solution of similarly structured problems and vice versa?

Method

Participants

The study focussed on two subjects, both 14 years old, chosen from a group of 27 pupils according to two criteria: good academic grades in mathematics and a positive attitude towards mathematics and problem-solving.

Design

A questionnaire for identifying mathematical beliefs and attitudes to problem-solving (Villa, 2001) was completed by the group of 27 students. The aim of the questionnaire was to identify students who were both academically successful in mathematics and had a positive attitude to problem-solving. Three students obtained scores (academic results and mathematical beliefs) significantly above their classmates, suggesting a favourable disposition towards problem-solving. These students were doing exceptionally well in mathematics and so were ideally suited to participating in the study. This is consistent with Schoenfeld's (1985, 1992) model, in which a good problem-solving profile includes appropriate resources, strategies, control and a favourable system of beliefs and affects. Of these four dimensions, the latter was the least likely to be guaranteed by purely good academic results, for which reason the questionnaire for identifying mathematical beliefs was employed.

Before and after the solution of each problem, the students completed a questionnaire (pre-PS and post-PS) specifically designed to gather data on the students' understanding of the problems. The questionnaires played a significant role in encouraging students to question their own reasoning and procedures, a process which according to Flavell (1976) can be described as underpinning the capacity for metacognitive reflection. Nevertheless, it should be borne in mind that the study concerned students with good academic grades, in the expectation of finding indicators within the data of their ability to achieve the structure of the problems (considered beneficial to finding solutions).

After solving each problem, the students were then invited to pose a new problem which could be solved using the same method. The analysis of these posed problems drew on two instruments (Cai & Hwang, 2003 and Leung, 1997), whilst the analysis of the solutions themselves followed Carrillo (1998).

The present article is concerned with the information provided by the students via protocol sheets completed during the process of solving the two problems. The students also answered pre-PS and post-PS questionnaires and attempted to pose new problems with the same structure as the original.

Instruments

A Set of 12 Problems, of Which This Paper Deals with Just Two

The first problem consisted of two questions, the first not too difficult and the second somewhat more demanding. In order to answer the questions, it was not necessary to use an algebraic model of the two contracts (see below), although the recognition that such a structure was implicit in the problem was important for posing a new problem along the same lines.

P2

Based on the advertisement, which type of contract would be preferable:

(a) For 2 years' employment
(b) For 9 years' employment

Propose a problem which could be solved using a similar method. (It is not necessary to provide a solution.)

International company seeks engineer
REQUIREMENTS:
- Degree in Chemical Engineering
- Age up to 35
- Good knowledge of English

CONDITIONS:
Contract A
- Annual salary in 1st year of € 25,000.
- Annual salary increment of € 3,000.

Contract B
- half-yearly salary in 1st 6 month period of € 10,000.
- half-yearly increment of € 1,000.

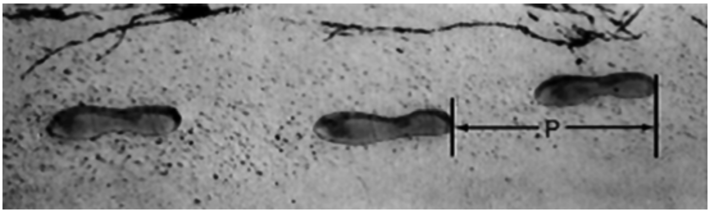
Send *CV* to situations vacant n° 1251 in this magazine.

The reason for choosing this problem was its algebraic structure, which the students had to demonstrate they understood if they were to propose a similar problem.

P8 (selected from GAVE, 2004)

Table 1 Questionnaire prior to problem solving

QUESTIONNAIRE PRIOR TO PROBLEM SOLVING:
After carefully reading the first problem, answer the questions below:
1. How far do you think you have understood this problem: (Tick the appropriate box)
Fully ☐ The main points ☐ A little ☐ Not at all ☐
2. How do you think you can solve this problem?
3. Do you think you are lacking anything at the moment to solve the problem? What?
4. Where do you think the difficulty of this problem lies?
Now try to solve the problem.



The figure on the right shows a man's footprints. The length of the stride (pace-length), P, is the distance between the heel marks of two consecutive footprints.

For men, the formula $\frac{n}{P} = 140$ establishes an approximate relation between n and P, in which

n = the number of steps per minute and
P = length of stride in metres.

– If this formula is applied to Pedro's stride and he takes 70 steps per minute, what is Pedro's stride length?
– Bernardo knows that the length of his stride is 0.80 m. Apply the formula to Bernardo's walking. Calculate the speed he walks in metres per minute and kilometres per hour.
– After you have solved this problem, formulate a problem which can be solved in the same way.

A problem from the PISA test was selected in order to enable comparison between PISA and the Portuguese syllabus in terms of problem-solving. The larger study, from which this paper is drawn, is currently scrutinising results for possible interpretations with regard to PISA performance. This problem was of particular interest for its algebraic structure, which the students had to demonstrate they understood if they were to propose a similar problem (Table 1-3).

Table 2 Questionnaire subsequent to problem solving

QUESTIONNAIRE SUBSEQUENT TO PROBLEM SOLVING:
1.How far do you think you managed to solve the problem: (Tick the appropriate box)
Completely ☐ Not very well ☐
The main elements ☐ Didn't solve or the answer was unsatisfactory ☐
2.Did you solve the problem as you initially expected to? If not, what changed your expectations?
3.What main difficulties did you find while solving the problem?
Thank you for participating.

Pre-PS and Post-PS Questionnaires About the Processes of Problem-Solving and Problem-Posing

Instruments for Processing/Analysing the Information

The data provided by the pre- and post-questionnaires was processed using an adapted version of Efklides' (2006) analysis (adapted) for classifying metacognitive knowledge and experiences. Efklides' concept of metacognition, based on Flavell (1979), sees it as knowledge acting upon an objective world (the task) at a metalevel through monitoring and checking. She proposes, in a summary table, three characterising features (metacognitive knowledge, metacognitive experience and metacognitive competencies/skills), along with their manifestations, grouped under monitoring and checking, although she recognises certain difficulties in distinguishing these (Efklides, 2006, p. 4). This instrument, however, was not applied across the full range of features, as the design of the pre- and post-PS questionnaires did not intend to supply such comprehensive data. Rather, the questionnaires aimed to capture the structure of the problem which each student managed to construct after carefully reading the rubric. It was to meet this requirement that the instrument needed to be adapted (Table 2).

This adaptation consisted in omitting the category of *metacognitive competencies*, in which Efklides included procedural knowledge, that is, actions for controlling cognition. As these take place in action, specifically the context of solving the proposed task, it was possible to analyse them separately via the students' solution protocols using other instruments. Finally, it should be noted that the post-PS questionnaire was not designed to evaluate how the task has been carried out but simply to measure the degree to which the original ideas about the problem had been carried through. The classification instrument—omitting metacognitive competencies—is set out in the Appendix (Table 4).

The solutions themselves were analysed using a slightly adapted version of Carrillo's (1998) scheme, omitting the category concerning the personal characteristics of the solver and giving prominence instead to the tactical features of the process. This scheme, set out below, consists of five analytical dimensions, each with five levels of acquisition (for an example of these levels, see Table 5 in the Appendix).

For dealing with the reformulation of problems, we drew on the instruments for classifying *problem-posing* offered by Cai and Hwang (2003) and Leung (1997). Cai and Hwang classify the formulation of the new problem according to its similarity with the original problem and its structure. Leung, on the other hand, focuses on the plausibility of the new problem in terms of the quality of information included in the reformulation.

According to Cai and Hwang (2003), a newly formulated problem can be classified as 'extensive', 'not extensive' or 'other'. If extensive (E), it follows the structure of the original problem but is more demanding in terms of the mathematical work required to solve it. If it is not extensive (NE), it fully patterns the structure of the original problem and maintains the same level of difficulty, and if other (O), it fails to follow the structure of the original problem. Leung, on the other hand, classifies the problems as follows: not a problem, that is, the suggested situation is descriptive only and fails to ask a question that can be answered; non-mathematical problem, in which the question posed falls outside the scope of mathematics; implausible mathematical problem, according to which the problem falls within the scope of mathematics but the data involved or the solution do not make sense in the context; insufficient plausible mathematical problem, whereby the problem can receive a mathematical treatment but the data involved are insufficient to arrive at a solution; and sufficient plausible mathematical problem, in other words, a well-formulated problem that can be solved.

Employing these codes, we propose the following categorisation:

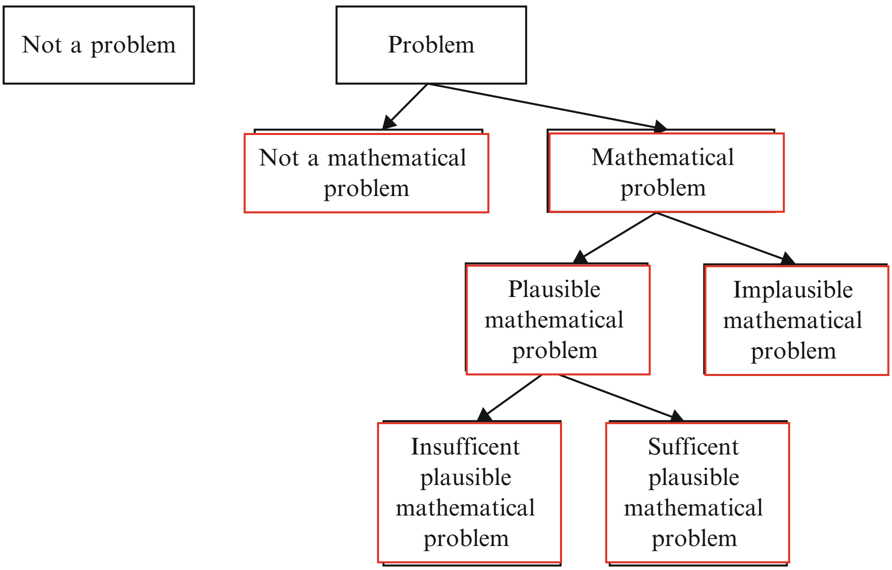


Table 3 Tactical characteristics of the process (efficacy of the action)

2.	Tactical characteristics of the process (efficacy of the action)
2.1.	Obtaining a meaningful representation (including obtaining the structure of the problem)
2.2.	Efficacy and adequacy of the planning (with data possibly provided by the use of intuition)
2.3.	Efficacy and adequacy of the execution (with data possibly provided by the use of generalisation of objects, relations and calculations and by reasoning with mathematical symbols and spatial relations)
2.4.	Efficacy in the use of revision
2.5.	Quality of the final version of the solution (including clarity, simplicity and economy, as well as reasonableness of the solution and rationale for the process)

To summarise, the students' answers to the pre- and post-questionnaires were considered and coded according to Efklides' (2006) system of analysis (Table 4, Appendix). Various relevant information units (student responses) have been included in order to substantiate the commentaries and coding (U for unit and P for problem). The students' progress in solving the problems were analysed using Carrillo's (1998) instrument (Table 3), again supported with illustrative samples of the students' solutions, accompanied by appropriate excerpts from the descriptors of the five analytical dimensions. Finally, the analysis of the new problems posed by the students was effected using the instruments developed by Cai and Hwang (2003) and Leung (1997). Some of the rubric from the problems has been included.

Analysis and Results

This section presents the analysis of the data provided by the students Clara and Rafael.

Clara

Her level of achievement is reflected in the pre-PS questionnaire, where she shows knowledge of the type of problem (task) and the demands of the task (explanation of the abbreviations see above):

> I think I can solve the problem calculating the amount I would receive in the four situations. (U1P2)

She also knows which strategy to use (strategy) and seems to have the appropriate mathematical knowledge (knowledge) to find a solution:

> calculating the values of the right unknown, substituting certain values in the formula (U1P8)

and estimates where the greatest difficulty might lie (difficulty):

> but the fact that the calculations don't involve any unknowns, just specific numbers, means it shouldn't cause any difficulties. (U2P2)

Likewise, in the post-PS questionnaire, she expresses recognition of some difficulty in the task (difficulty):

> the main difficulty in the solution… was the number of values and their size. (U3P2)

Her solutions to the two problems are almost uniformly situated at the highest level descriptors of the instrument used.

In problem 2, she shows a good ability to achieve a meaningful representation, so that the descriptor corresponding top level 5 ('the solver understands the structure of the problem perfectly and usually retrieves the mathematics underlying the data in the problem statement') best describes the work she produces:

'Contract A: $25000+(25000+3000)=53000$'.

In problem 8, the implementation of her solution is effective and appropriate, corresponding to level 5 of the descriptor ('execution is consistent with planning and is effective in contributing key results towards the overall solution') best describes the work she produces, in particular, when she converts 89.6 m/min to 5.376 km/h:

$$\text{“}112\times 0.80 = 89.6\ \text{m}/\text{min}$$
$$(112\times 60)\times\left(\frac{0.80}{1000}\right)=\ldots\text{”}.$$

The new problems she poses represent sufficient plausible mathematical problems, with an identical structure to the original problems, as required.

Her awareness of the difficulties shown in the questionnaires can be seen in the problem she poses based on problem 2, as is her confidence in the solution. On the one hand, she reduces the amounts involved, but, on the other hand, because she perfectly understands the structure of problem 2, she is sufficiently confident to change the amount to be added in the original to an amount to be subtracted in the new problem. This is the problem she poses:

'Manuel decided to buy a car on special offer, making payments of €350/month in the first year, after which, every year the payments are reduced by €25. How many years would it take to pay for a €20,000 car, assuming that the price includes interest?'

In the case of problem 8, she added a question, which represents an extension of what was originally required.

In summary, the data from the pre-PS and post-PS questionnaires coincided in revealing a high level of metacognitive knowledge in terms of the indicators for (mathematical) knowledge and (estimate of) difficulty. The task is tackled coherently throughout, without any difficulties arising, and correct solutions arrived at. The posed problem is completely consistent with the original, having an identical structure and even adding an extension.

Rafael

In the pre-PS questionnaire, he shows he understands the task (task):

> I put in what I earn by the end of the year with contract A (already done) and then I put in what I earn by the end of the year with contract B. (U1P2)
> Substituting the letters with numbers. (U1P8)

He shows uncertainty as to how to proceed during the task (strategy):

> I think what can make this problem difficult is finding a way to solve it. (U2P2)

and insecurity as to his chances of success (confidence).

In the post-PS questionnaire, he shows himself uncertain of his solution (validity):

> Check that the money was always added to what he had earned. (U3P2)

The solution to problem 2 reveals a good mental representation, and consistent planning and execution, but also reveals difficulties when it comes to reviewing his work, in that he fails to notice an error in calculation which accumulates over the course of working out the final total. As he adds up the half-yearly salary increments in contract B, he misses out the 16th payment and so miscalculates the total, which should be 333,000€ instead of 306,000€:

“…	13º	14º	15º	17º	18º		”
	22000	23000	24000	25000	26000	= 306000	

His work would benefit with greater attention given to the final two items of the instrument (2.4. Efficacy in the use of review strategies and 2.5. Quality of the final version of the solution), given that, as the questionnaires show, he felt little confidence in the validity of his work.

His solution to problem 8 also shows difficulties in the last two levels of action described by the instrument (2.4 and 2.5). He again fails to spot an error and does not attempt to find a solution through other means, despite having also given signs of a lack of confidence and uncertainty in his answers to the questionnaires.

With respect to problem 2, he is unable to reformulate it and presents an insufficient plausible mathematical problem, the structure of which fails to mirror that of the original. He proposes the following problem:

‘Make a formula for each contract’.

This formulation, which goes little way to meet the requirements of the task, clearly shows that the difficulties revealed in the questionnaires and in solving this problem were real. The student is unable to extract the mathematical structure of the problem, which would enable him to pose another with a comparable structure, although he is aware that there is one.

In reformulating problem 8, he presents a sufficient plausible mathematical problem, but fails to preserve the mathematical structure of the original. The original problem concerns a direct proportional relation, whilst the reformulation is based on inverse proportionality.

In summary, the pre-PS and post-PS questionnaires reveal metacognitive knowledge in terms of the indicators for task, strategy, confidence and validity. The student's working through of the task is consistent with the data supplied by the questionnaires, as revealed by his inability to overcome perceived difficulties, and the solutions to the problems themselves. The posed problems provide further confirmation, as they lack any content or mathematical structure (in the first instance) and any mathematical structure (in the second).

Conclusions

The use of a questionnaire prior to attempting the problem strengthens the students' questioning of aspects of the problem and the resources and strategies to be employed. It also provides data on their self-awareness and decision-making capacity. Unfortunately, although in the global study questionnaires were used with some problems and not with others, it was not possible to draw a comparison of the same problem with and without a questionnaire, which could have provided further information of interest.

The data revealed by the pre-PS questionnaire, with regard to awareness of task typology, strategies, type of knowledge involved and identification of difficulties (or not), are consistent with the students' actual solutions to the problems and with the new problems they posed based on the original tasks. We acknowledge that this correlation represents what was to be expected, but it is worth noting as it indicates the methodological consistency of the following scheme for fieldwork we would like to propose: (a) pre- and post-PS questionnaires, (b) task of solving a problem and (c) posing a problem with the same structure as the original. It also indicates that the instruments used in the analysis produced consistent data.

In the case where the problem-solving task was not fully completed, the post-PS questionnaire indicated less knowledge regarding the student's own capabilities, little confidence in the validity of the revealed knowledge and a sense of difficulty (which might vary during the task). In the case where the problem-posing task was not successfully accomplished, the proposed problems fell short of what was desired, either because the data were insufficient (Rafael's reformulation of problem 2) or because its structure did not pattern the structure of the original problem (Rafael's reformulation of problems 2 and 8).

The case of Clara shows an evident relation between the results of the questionnaires (knowledge of task, strategy, mathematical content) and her grasp of the tactical aspects of the process of problem-solving. The problems she posed respected the mathematical structure of the original problems and even introduced small changes and extensions which indicated a total understanding of the structure of the problems which were given to her to solve and reformulate. In this case problem-posing represents a good indicator of the PS process. At the same time, it promotes an amplification in the understanding of this process.

If we consider the problems posed by the students, it becomes evident that, when they respected the structure of the original problems (implying the creation of a sufficient plausible mathematical problem), this meant an understanding of the structure of the original problem, which had a fairly complete solution in the tactical aspects of the process (see the case of Clara). The opposite of this affirmation is also true (see the case of Rafael).

The results of this study lead us to conjecture that:

– The use of pre- and post-PS questionnaires improves problem-solving (presumably because it urges questioning), and problem-solving, in its turn, benefits problem-posing.
– The formulation of problems by the students (with appropriate data and structure) indicates the use of more meaningful representations, gaining access to the structure of the problem.

Further studies into how this method could improve the problem-solving abilities of lower achievers would also be valuable.

Appendix

Table 4

Efklides (2006)	Meaning as understood in this study
1. Metacognitive knowledge	Declared knowledge of cognition, deriving from long-term memory
Ideas, beliefs, 'theories'	May be explicit or implicit
1.1. Person/self	Concerns oneself and one's possibilities (related to 1.2, 1.3 and 1.4)
1.2. Task	Concerns categories (classes) of tasks and the means of solving them
1.3. Strategies	Concerns general ways (modes) for acting (may be heuristic)
1.4. Goals	Concerns objectives or the type of solution
1.5. Cognitive functions	Concerning memory or thinking (what they are and how they act), attention
1.6. Validity of knowledge	Concerns epistemological knowledge, quality of knowledge
2. Metacognitive experience	Involves aspects of the mobilisation of knowledge
2.1. Feelings	Considered as products of monitoring (good functioning)
2.1.1. Familiarity	Denotes previous occurrence of a stimulus (frequency) and fluency in the mode of action
2.1.2. Difficulty	Results from complexity of task, context, personal characteristics (cognitive), self-image, affective factors, extrinsic feedback such as positive or negative sensations. May vary during task, may be illusory
2.1.3. Knowing	Concerns appropriate mathematical knowledge for task
2.1.4. Confidence	Derives from previous experience, hesitation vs. overconfidence
2.1.5. Satisfaction	Monitors the personal criteria and standards by which the quality of response is judged
2.2. Judgements/estimates	
2.2.1. Estimate of effort	Monitors work to be carried out, related to difficulty
2.2.2. Estimate of time	Monitors time taken, related to difficulty

Table 5

2.1. Obtaining a meaningful representation
1. The solver never obtains a representation for the problem and does not understand the given situation, which is totally unfamiliar to him or her. Because the solver does not understand the structure of the problem, he or she is unable to articulate the reasoning by which the problem is introduced (sometimes the solver does not even state it)
2. The structure of the problem is occasionally understood, usually imperfectly; that is to say, he or she is able to express in his or her own words some, but not all, elements of the problem; abstract reasoning is not used
3. Understanding of the problem extends to all or most elements, though not in depth. The basic structure of the problem is usually understood, though sometimes imperfectly; both concrete and abstract reasoning are used
4. Largely understands the problem, although there may be an element which is not understood. The structure of the problem may be understood, but the posing of a new, similar problem causes difficulty
5. The solver obtains a highly meaningful representation of the situation, allowing a successful planning process to begin, after formulating the problem in his or her own terms. Therefore, the solver may understand the structure of the problem perfectly and usually retrieves the mathematics underlying the data in the problem statement. The solver abstracts, starting with concrete relations and moving towards formal structures

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