Chapter 2
Basic Theory

The present chapter introduces the basic theory for the qualitative investigations of impulsive mathematical models.

Section 2.1 offers the main classes of impulsive differential equations which will be investigated in the book. The problems of existence, uniqueness, and continuability of the solutions are discussed.

Section 2.2 deals with almost periodic sequences and almost periodic piecewise continuous functions. The main definitions and properties are considered.

Section 2.3 is devoted to the main definitions of stability and boundedness properties of solutions of impulsive models.

In Sect. 2.4 different classes of Lyapunov functions are introduced. Lyapunov functionals, which are used in the stability and boundedness of the solutions of impulsive models with delays, are defined.

In Sect. 2.5, some main impulsive differential inequalities are considered.

Finally, in Sect. 2.6 the main lemmas from coincidence degree theory for the impulsive case are given.

2.1 Impulsive Differential Equations

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $||.||$, $\Omega \subseteq \mathbb{R}^n$, $\Omega \neq \emptyset$, and let $\mathbb{R}_+ = [0, \infty)$.

First, we shall give a brief description of the main classes of impulsive differential equations that will be used for mathematical modeling within the book.

A. Impulsive ordinary differential equations. Consider the system of impulsive ordinary differential equations

$$\begin{cases} \dot{x}(t) = f(t,x), \ t \neq \tau_k(x(t)), \\ \Delta x(t) = I_k(x(t)), \ t = \tau_k(x(t)), \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.1}$$

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where $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$, $\tau_k : \Omega \to \mathbb{R}$, $I_k : \Omega \to \mathbb{R}^n$, $\Delta x(t) = x(t^+) - x(t^-)$.

Let $t_0 \in \mathbb{R}$ and $x_0 \in \Omega$. Denote by $x(t) = x(t; t_0, x_0)$ the solution of system (2.1) satisfying the initial condition

$$x(t_0^+; t_0, x_0) = x_0. \tag{2.2}$$

We note that, instead of the initial condition $x(t_0) = x_0$, we have imposed the limiting condition $x(t_0^+) = x_0$ which, in general, is natural for Eq. (2.1) since (t_0, x_0) may be such that $t_0 = \tau_k(x_0)$ for some k. Whenever $t_0 \neq \tau_k(x_0)$, for all k, we shall understand the initial condition $x(t_0^+) = x_0$ in the usual sense, that is, $x(t_0) = x_0$.

The solutions $x(t; t_0, x_0)$ of system (2.1) are, in general, piecewise continuous functions with points of discontinuity of the first type at which they are left continuous, that is, at the moments t_k, $k = \pm 1, \pm 2, \ldots$, when the integral curve of a solution $x(t)$ meets the hypersurfaces

$$\sigma_k = \Big\{(t, x) : \ t = \tau_k(x), \ x \in \Omega\Big\},$$

the following relations are satisfied:

$$x(t_k^-) = x(t_k) \quad \text{and} \quad x(t_k^+) = x(t_k) + I_k(x(t_k)).$$

We shall assume that for each $x \in \Omega$ and $k = \pm 1, \pm 2, \ldots,$

$$\tau_k(x) < \tau_{k+1}(x)$$

and

$$\tau_k(x) \to \infty \quad \text{as} \quad k \to \infty \ (\tau_k(x) \to -\infty \quad \text{as} \quad k \to -\infty), \ \text{ uniformly on } \ x \in \Omega,$$

and the integral curve of each solution of the system (2.1) meets each of the hypersurfaces $\{\sigma_k\}$ at most once.

The above condition means the absence of the phenomenon of "beating" of the solutions to the system (2.1), i.e. the phenomenon where a given integral curve meets the same hypersurface $\{\sigma_k\}$ more than once (possibly infinitely many times) [34, 178, 256]. Some of the difficulties in the investigation of systems with variable impulsive perturbations are related to the possibilities of "merging" different integral curves after a given moment, loss of the property of autonomy, etc.

Note that the phenomenon of "beating" is not present in the case when $\tau_k(x) \equiv t_k$, $k = \pm 1, \pm 2, \ldots, x \in \Omega$, i.e. when the impulses are realized at fixed moments $t = t_k$, $k = \pm 1, \pm 2, \ldots$. Then the system (2.1) reduces to

$$\begin{cases} \dot{x}(t) = f(t, x), \ t \neq t_k, \\ \Delta x = I_k(x), \ t = t_k, \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.3}$$

where the moments of impulsive effects are such that $t_k < t_{k+1}$, $k = \pm 1, \pm 2, \ldots$, and $\lim_{k \to \pm\infty} t_k = \pm\infty$.

It is clear that systems of impulsive differential equations with fixed moments of impulse effect (2.3) can be considered as a particular case of systems with variable impulsive perturbations (2.1), and they will be one of the main objects of investigation in our book.

In the following, some of the main results on the fundamental theory of impulsive ordinary differential equations with fixed moments of impulsive effect (2.3) are presented.

Let $J_1 = [t_0, \omega)$, $J_2 = [t_0, \tilde{\omega})$, and $J_1 \subseteq J_2$.

Definition 2.1. If:

1. $x(t) = x(t;\, t_0, x_0)$ and $y(t) = y(t;\, t_0, x_0)$ are two solutions of the system (2.3) on the intervals J_1 and J_2, respectively;
2. $x(t) = y(t)$ for $t \in J_1$,

then $y(t)$ is said to be a *continuation* of $x(t)$ on the interval J_2 (*continuation to the right*).

The solution $x(t) = x(t;\, t_0, x_0)$ is said to be *continuable* on the interval J_2, if there exists a continuation $y(t)$ of $x(t)$ on J_2. Otherwise $x(t) = x(t;\, t_0, x_0)$ is said to be *noncontinuable* and the interval J_1 is called a *maximal interval of existence* of $x(t)$.

Definition 2.2. The solution $x(t) = x(t;\, t_0, x_0)$ of system (2.3) is said to be *unique* if, given any other solution $y(t) = y(t;\, t_0, x_0)$ of the system, $x(t) = y(t)$ on their common interval of existence.

Theorem 2.1 ([36]). *Let the following conditions hold.*

1. *The function $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is continuous on the sets $(t_k, t_{k+1}] \times \Omega$, $k = \pm 1, \pm 2, \ldots$.*
2. *For each $k = \pm 1, \pm 2, \ldots$ and $x \in \Omega$ the finite limit of $f(t, y)$ as $(t, y) \to (t_k, x)$, $t > t_k$, exists.*

Then for each $(t_0, x_0) \in \mathbb{R} \times \Omega$ there exist $\omega > t_0$ and a solution $x : [t_0, \omega) \to \mathbb{R}^n$ of the initial value problem (2.3) and (2.2).

If, moreover, the function $f(t, x)$ is locally Lipschitz continuous with respect to $x \in \Omega$, then this solution is unique.

Let us consider the problem of the continuability to the right of a given solution $\varphi(t)$ of system (2.3).

Theorem 2.2 ([36]). *Let the following conditions hold.*

1. *The function $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is continuous on the sets $(t_k, t_{k+1}] \times \Omega$, $k = \pm 1, \pm 2, \ldots$.*
2. *For each $k = \pm 1, \pm 2, \ldots$ and $x \in \Omega$ the finite limit of $f(t, y)$ as $(t, y) \to (t_k, x)$, $t > t_k$, exists.*

3. *The function* $\varphi : (\alpha, \beta) \to \mathbb{R}^n$ *is a solution of* (2.3).

Then the solution $\varphi(t)$ *is continuable to the right of* β *if and only if the limit*

$$\lim_{t \to \beta^-} \varphi(t) = \eta$$

exists and one of the following conditions hold:

(a) $\beta \neq t_k$ *for each* $k = \pm 1, \pm 2, \ldots$ *and* $\eta \in \Omega$*;*
(b) $\beta = t_k$ *for some* $k = \pm 1, \pm 2, \ldots$ *and* $\eta + I_k(\eta) \in \Omega$.

Theorem 2.3 ([36]). *Let the following conditions hold.*

1. *Conditions* 1 *and* 2 *of Theorem* 2.2 *hold.*
2. *The function* f *is locally Lipschitz continuous with respect to* $x \in \Omega$.
3. $\eta + I_k(\eta) \in \Omega$ *for each* $k = \pm 1, \pm 2, \ldots$ *and* $\eta \in \Omega$.

Then for any $(t_0, x_0) \in \mathbb{R} \times \Omega$ *there exists a unique solution of the initial value problem* (2.3), (2.2) *which is defined on an interval of the form* $[t_0, \omega)$ *and is not continuable to the right of* ω.

Let the conditions of Theorem 2.3 be satisfied and let $(t_0, x_0) \in \mathbb{R} \times \Omega$. Denote by $J^+ = J^+(t_0, x_0)$ the maximal interval of the form $[t_0, \omega)$ in which the solution $x(t; t_0, x_0)$ is defined.

Theorem 2.4 ([36]). *Let the following conditions hold.*

1. *The conditions of Theorem* 2.3 *are met.*
2. $\varphi(t)$ *is a solution of the initial value problem* (2.3), (2.2).
3. *There exists a compact* $Q \subset \Omega$ *such that* $\varphi(t) \in Q$ *for* $t \in J^+(t_0, x_0)$.

Then $J^+(t_0, x_0) = (t_0, \infty)$.

Let $\varphi(t) : (\alpha, \omega) \to \mathbb{R}^n$ be a solution of system (2.3) and consider the question of the continuability of this solution to the left of α.

If $\alpha \neq t_k,\ k = \pm 1, \pm 2, \ldots$, then the problem of continuability to the left of α is solved in the same way as for ordinary differential equations without impulses [133]. In this case, such an extension is possible if and only if the limit

$$\lim_{t \to \sigma^+} \varphi(t) = \eta \tag{2.4}$$

exists and $\eta \in \Omega$.

If $\alpha = t_k$ for some $k = \pm 1, \pm 2, \ldots$, then the solution $\varphi(t)$ will be continuable to the left of t_k when the limit (2.4), $\eta \in \Omega$, exists and the equation $x + I_k(x) = \eta$ has a unique solution $x_k \in \Omega$. In this case, the extension $\psi(t)$ of $\varphi(t)$ for $t \in (t_{k-1}, t_k]$ coincides with the solution of the initial value problem

$$\begin{cases} \dot{\psi}(t) = f(t, \psi(t)), & t_{k-1} < t \leq t_k, \\ \psi(t_k) = x_k. \end{cases}$$

If the solution $\varphi(t)$ can be continued up to t_{k-1}, then the above procedure is repeated, and so on. Under the conditions of Theorem 2.3 for each $(t_0, x_0) \in \mathbb{R} \times \Omega$ there exists a unique solution $x(t; t_0, x_0)$ of the initial value problem (2.3), (2.2) which is defined in an interval of the form (α, ω) and is not continuable to the right of ω or to the left of α. Denote by $J(t_0, x_0)$ this maximal interval of existence of the solution $x(t; t_0, x_0)$ and set $J^- = J^-(t_0, x_0) = (\alpha, t_0]$. A straightforward verification shows that the solution $x(t) = x(t; t_0, x_0)$ of the initial value problem (2.3), (2.2) satisfies the following integro-summary equation

$$x(t) = \begin{cases} x_0 + \displaystyle\int_{t_0}^{t} f(s, x(s))ds + \sum_{t_0 < t_k < t} I_k(x(t_k)), & \text{for t} \in \mathrm{J}^+, \\ x_0 + \displaystyle\int_{t_0}^{t} f(s, x(s))ds - \sum_{t < t_k < t_0} I_k(x(t_k)), & \text{for t} \in \mathrm{J}^-. \end{cases} \tag{2.5}$$

Now we consider the linear system of impulsive equations

$$\begin{cases} \dot{x}(t) = A(t)x(t), & t \neq t_k, \\ \Delta x(t_k) = B_k x(t_k), & k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.6}$$

under the assumption that the following conditions hold:

H2.1. $t_k < t_{k+1}$, $k = \pm 1, \pm 2, \ldots$, and $\lim\limits_{k \to \pm\infty} t_k = \pm\infty$.

H2.2. $A \in C[\mathbb{R}, \mathbb{R}^{n \times n}]$, $B_k \in \mathbb{R}^{n \times n}$, $k = \pm 1, \pm 2, \ldots$.

Theorem 2.5 ([36]). *Let conditions* H2.1 *and* H2.2 *hold. Then for any* $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ *there exists a unique solution* $x(t)$ *of system* (2.6) *with* $x(t_0^+) = x_0$ *and this solution is defined for* $t \geq t_0$.

If, moreover, $\det(E + B_k) \neq 0$, $k = \pm 1, \pm 2, \ldots$, *where E is the* $(n \times n)$ *identity matrix, then this solution is defined for all* $t \in \mathbb{R}$.

Let $U_k(t, s)$ $(t, s \in (t_{k-1}, t_k])$ be the Cauchy matrix [133] for the linear equation

$$\dot{x}(t) = A(t)x(t), \; t_{k-1} < t \leq t_k, \; k = \pm 1, \pm 2, \ldots.$$

Then, by virtue of Theorem 2.5, the solution of the initial problem (2.6), (2.2) can be decomposed as:

$$x(t; t_0, x_0) = W(t, t_0^+)x_0, \tag{2.7}$$

where

$$W(t,s)=\begin{cases} U_k(t,s) \ \text{ as } \ t,s\in(t_{k-1},t_k],\\ U_{k+1}(t,t_k^+)(E+B_k)U_k(t_k,s) \text{ as } \ t_{k-1}<s\le t_k<t\le t_{k+1},\\ U_k(t,t_k)(E+B_k)^{-1}U_{k+1}(t_k^+,s) \text{ as } \ t_{k-1}<t\le t_k<s\le t_{k+1},\\ U_{k+1}(t,t_k^+)\displaystyle\prod_{j=k}^{i+1}(E+B_j)U_j(t_j,t_{j-1}^+)(E+B_i)U_i(t_i,s)\\ \quad \text{as } \ t_{i-1}<s\le t_i<t_k<t\le t_{k+1},\\ U_i(t,t_i)\displaystyle\prod_{j=i}^{k-1}(E+B_j)^{-1}U_{j+1}(t_j^+,t_{j+1})(E+B_k)^{-1}U_{k+1}(t_k^+,s)\\ \quad \text{as } \ t_{i-1}<t\le t_i<t_k<s\le t_{k+1}, \end{cases}$$

is the solving operator of the Eq. (2.6).

B. Impulsive functional differential equations. Let $J\subseteq\mathbb{R}$. Define the following classes of functions:

$PC[J,\Omega]=\{\sigma:J\to\Omega:\ \sigma(t)$ is a piecewise continuous function with points of discontinuity $\tilde t\in J$ at which $\sigma(\tilde t^-)$ and $\sigma(\tilde t^+)$ exist and $\sigma(\tilde t^-)=\sigma(\tilde t)\}$;

$PC^1[J,\Omega]=\{\sigma\in PC[J,\Omega]:\ \sigma(t)$ is continuously differentiable everywhere except the points $\tilde t\in J$ at which $\dot\sigma(\tilde t^-)$ and $\dot\sigma(\tilde t^+)$ exist and $\dot\sigma(\tilde t^-)=\dot\sigma(\tilde t)\}$.

Let $r>0$. For any $t\in\mathbb{R}$, we denote by x_t an element of $PC[[-r,0],\Omega]$ defined by $x_t(s)=x(t+s),\quad -r\le s\le 0$.

In the present book we shall consider systems of impulsive functional differential equations with fixed moments of impulse effect. The systems of this class are written as follows

$$\begin{cases}\dot x(t)=f(t,x_t),\ t\ne t_k,\\ \Delta x(t)=I_k(x(t)),\ t=t_k,\ k=\pm1,\pm2,\ldots,\end{cases}\tag{2.8}$$

where $f:\mathbb{R}\times PC[[-r,0],\Omega]\to\mathbb{R}^n$, $I_k:\Omega\to\mathbb{R}^n$, $t_k<t_{k+1}$, $k=\pm1,\pm2,\ldots$, and $\lim\limits_{k\to\pm\infty}t_k=\pm\infty$.

Instead of an initial point value for an impulsive ordinary differential equation (2.3), an initial function is required for the type (2.8) systems, which is defined over the range of time J delimited by the delay. Let $t_0\in\mathbb{R}$, $\varphi_0\in PC[[-r,0],\Omega]$. Denote by $x(t)=x(t;t_0,\varphi_0)$ the solution of system (2.8) satisfying the initial conditions

$$\begin{cases}x(t)=\varphi_0(t-t_0),\ t_0-r\le t\le t_0,\\ x(t_0^+)=\varphi_0(0),\end{cases}\tag{2.9}$$

and by $J^+(t_0,\varphi_0)$ the maximal interval of type $[t_0,\beta)$ in which the solution $x(t;t_0,\varphi_0)$ is defined.

Without loss of generality we can assume that $\dots t_1 < t_0 < t_1 < t_2 < \dots < t_k < t_{k+1} < \dots$. Then the solution $x(t) = x(t; t_0, \varphi_0)$ of the initial value problem (2.8), (2.9) is characterized by the following:

1. For $t_0 - r \leq t \leq t_0$ the solution $x(t)$ satisfies the initial conditions (2.9).
2. For $t_0 < t \leq t_1$, $x(t)$ coincides with the solution of the problem

$$\begin{aligned} &\dot{x}(t) = f(t, x_t),\ t > t_0, \\ &x_{t_0} = \varphi_0(s),\ -r \leq s \leq 0, \\ &x(t_0^+) = \varphi_0(0). \end{aligned}$$

   At the moment $t = t_1$ the mapping point $(t, x(t; t_0, \varphi_0))$ of the extended phase space jumps momentarily from the position $(t_1, x(t_1; t_0, \varphi_0))$ to the position $(t_1, x(t_1; t_0, \varphi_0) + I_1(x(t_1; t_0, \varphi_0)))$.
3. For $t_1 < t \leq t_2$ the solution $x(t)$ coincides with the solution of

$$\begin{cases} \dot{y}(t) = f(t, y_t),\ t > t_1, \\ y_{t_1} = \varphi_1,\ \varphi_1 \in PC[[-r, 0], \Omega], \end{cases}$$

   where

$$\varphi_1(t - t_1) = \begin{cases} \varphi_0(t - t_1),\ t \in [t_0 - r, t_0] \cap [t_1 - r, t_1], \\ x(t; t_0, \varphi_0),\ t \in (t_0, t_1) \cap [t_1 - r, t_1], \\ x(t; t_0, \varphi_0) + I_1(x(t; t_0, \varphi_0)),\ t = t_1. \end{cases}$$

At the moment $t = t_2$ the mapping point $(t, x(t))$ jumps momentarily, etc.

Thus the solution $x(t; t_0, \varphi_0)$ of problem (2.8), (2.9) is a piecewise continuous function with points of discontinuity of the first kind $t = t_k,\ k = \pm 1, \pm 2, \dots$, at which it is continuous from the left, i.e. the following relations are satisfied

$$x(t_k^-) = x(t_k),\ k = \pm 1, \pm 2, \dots,$$

$$x(t_k^+) = x(t_k) + I_k(x(t_k)),\ k = \pm 1, \pm 2, \dots.$$

The system (2.8) is a universal type of impulsive functional differential system. In the particular case $r = 0$, it contains a system of impulsive ordinary differential equations. The systems of the type (2.8) also include the following systems of impulsive functional differential equations:

– systems of impulsive differential-difference equations of the type

$$\begin{cases} \dot{x}(t) = F(t, x(t), x(t - r_1(t)), x(t - r_2(t)), \dots, x(t - r_p(t))),\ t \neq t_k, \\ \Delta x(t) = I_k(x(t)),\ t = t_k,\ k = \pm 1, \pm 2, \dots, \end{cases}$$

  where $0 \leq r_j(t) \leq r, j = 1, 2, \dots, p$;

– systems of impulsive integro-differential equations of the type

$$\begin{cases} \dot{x}(t) = \int_{-r}^{0} g(t, s, x(t+s))ds, \ t \neq t_k, \\ \Delta x(t) = I_k(x(t)), \ t = t_k, \ k = \pm 1, \pm 2, \dots; \end{cases}$$

– systems of impulsive integro-differential equations with infinite delays

$$\begin{cases} \dot{x}(t) = \int_{-\infty}^{t} k(t, s) f(t, x(s))\, ds, \ t \neq t_k, \\ \Delta x(t) = I_k(x(t)), \ t = t_k, \ k = \pm 1, \pm 2, \dots, \end{cases}$$

where $k : \mathbb{R}^2 \to \mathbb{R}_+$ is continuous;

– systems of impulsive differential equations with supremums of the type

$$\begin{cases} \dot{x}(t) = f(t, \sup_{s \in [-r,0]} y(t+s)), \ t \neq t_k, \\ \Delta x(t) = I_k(x(t)), \ t = t_k, \ \pm 1, \pm 2, \dots, \end{cases}$$

where

$$\sup_{s \in [-r,0]} x(t+s) = \Big(\sup_{s \in [-r,0]} x_1(t+s), \sup_{s \in [-r,0]} x_2(t+s), \dots, \sup_{s \in [-r,0]} x_n(t+s) \Big).$$

Some systems of more general types are also included in the type (2.8) system.

The system (2.8) is *linear* when $f(t, x_t) = L(t, x_t) + h(t)$, where $L(t, x_t)$ is linear with respect to x_t.

One special class of impulsive functional differential systems are the systems of impulsive functional differential equations of neutral type

$$\begin{cases} \frac{d}{dt} h(t, x_t) = f(t, x_t), \ t \neq t_k, \\ \Delta x(t) = I_k(x(t)), \ t = t_k, \ k = \pm 1, \pm 2, \dots, \end{cases} \tag{2.10}$$

where $h : \mathbb{R} \times PC[[-r, 0], \Omega] \to \mathbb{R}^n$.

For systems (2.10) the initial function $\varphi_0 \in PC^1[[-r, 0], \Omega]$. In particular, it is possible for φ_0 to be a continuous and continuously differentiable function, i.e. $\varphi_0 \in C^1[[-r, 0], \Omega]$.

For $t_0 - r \leq t \leq t_0$ the solution $x(t) = x(t; t_0, \varphi_0)$ of the initial value problem (2.10), (2.9) with $\varphi_0 \in PC^1[[-r, 0], \Omega]$ coincides with the function $\varphi_0(t)$ (and $\dot{x} = \dot{\varphi}_0$ by definition everywhere in this interval).

The solution $x(t; t_0, \varphi_0)$ of problem (2.10), (2.9) is a piecewise continuous function for $t > t_0$ with points of discontinuity of the first kind $t = t_k$, $k = \pm 1, \pm 2, \dots$, at which it is continuous from the left and

$$x(t_k^+) = x(t_k) + I_k(x(t_k)), \ k = \pm 1, \pm 2, \dots.$$

Its derivative $\dot{x}(t)$ may have discontinuities of the first kind at some points of the form $t_i - js$, where $-r \leq s \leq 0$, $i < k$, $j \in \mathbb{N}$, which are situated in the interval $[t_k, t_{k+1})$ (see [28, 37]), and there is a finite number of these points.

The proof of the following theorem is analogous to the proof of Theorem 2.17 in [298]. We omit the details here.

Theorem 2.6. *Let condition* H2.1 *hold. Assume that:*

1. *The function f is continuous and bounded in* $\mathbb{R} \times PC[[-r, 0], \Omega]$.
2. $I_k \in C[\Omega, \Omega]$*, and* $(E + I_k) : \Omega \to \Omega,\ k = \pm 1, \pm 2, \ldots$.

Then for each $(t_0, \varphi_0) \in \mathbb{R} \times PC[[-r, 0], \Omega]$*:*

1. *There exists a solution* $x(t) = x(t; t_0, \varphi_0)$ *of the initial value problem* (2.8)*,* (2.9) *defined on* $J^+(t_0, \varphi_0)$.
2. $J^+(t_0, \varphi_0) = [t_0, \infty)$.
3. *If, moreover, the function f is locally Lipschitz continuous with respect to its second argument in* $[t_0, \infty) \times PC[[-r, 0], \Omega]$*, then the solution* $x(t; t_0, \varphi_0)$ *is unique.*

Remark 2.1. Since we shall consider solutions $x(t)$ of real-world systems with non-negative components, the set Ω will usually be $\mathbb{R}^n_+$.

More existence and uniqueness criteria are given in [298], so we have not included them here.

2.2 Almost Periodic Sequences and Almost Periodic Functions

One objective of this book is to investigate the existence, uniqueness and qualitative properties of almost periodic solutions of the impulsive models under consideration. In this section, the main definitions and properties of almost periodic sequences and almost periodic functions are introduced. These notions are used throughout the book.

2.2.1 Almost Periodic Sequences

In this part, we shall follow [256] and [284], and consider the main definitions and properties of almost periodic sequences.

We shall consider the sequence $\{x_k\},\ x_k \in \mathbb{R}^n,\ k = \pm 1, \pm 2, \ldots$, and let $\varepsilon > 0$.

Definition 2.3. An integer p is said to be an *ε-almost period* of $\{x_k\}$ if for each $k = \pm 1, \pm 2, \ldots$,

$$||x_{k+p} - x_k|| < \varepsilon. \tag{2.11}$$

It is easy to see that if p and q are ε-almost periods of $\{x_k\}$, then $p+q,\ p-q$ are 2ε-almost periods of the sequence $\{x_k\}$.

Definition 2.4. The sequence $\{x_k\},\ x_k \in \mathbb{R}^n,\ k = \pm 1, \pm 2, \ldots$, is said to be *almost periodic* if, for an arbitrary $\varepsilon > 0$, there exists a relatively dense set of its ε-almost periods, i.e. there exists a natural number $N = N(\varepsilon)$ such that for an arbitrary integer k, there exists at least one integer p in the interval $[k, k+N]$ for which the inequality (2.11) holds.

Let $B_\alpha = \{x \in \mathbb{R}^n : ||x|| < \alpha\},\ \alpha > 0$.

Theorem 2.7 ([284]). *Let the following conditions hold.*

1. *The sequence $\{x_k\} \subset B_\alpha$, $k = \pm 1, \pm 2, \ldots$, is almost periodic.*
2. *The function $y = f(x)$ is uniformly continuous in B_α.*

Then:

1. *The sequence $\{x_k\}$, $k = \pm 1, \pm 2, \ldots$, is bounded.*
2. *The sequence $\{y_k\},\ y_k = f(x_k),\ k = \pm 1, \pm 2, \ldots$, is almost periodic.*

Theorem 2.8 ([256]). *Let the following conditions hold.*

1. *For each $m = 1, 2, \ldots$ the sequence $\{x_k^m\},\ k = \pm 1, \pm 2, \ldots,$ is almost periodic.*
2. *There exists a limit sequence $\{y_k\},\ k = \pm 1, \pm 2, \ldots$, of the sequence $\{x_k^m\},\ k = \pm 1, \pm 2, \ldots$, as $m \to \infty$.*

Then the limit sequence $\{y_k\},\ k = \pm 1, \pm 2, \ldots$, is almost periodic.

Theorem 2.9 ([284]). *The sequence $\{x_k\},\ k = \pm 1, \pm 2, \ldots$, is almost periodic if and only if for any sequence of integers $\{m_i\},\ i = \pm 1, \pm 2, \ldots$, there exists a subsequence $\{m_{i_j}\}$ such that $\{x_{k+m_{i_j}}\}$ is convergent for $j \to \infty$ uniformly on $k = \pm 1, \pm 2, \ldots$.*

From this theorem, we get the next corollary.

Corollary 2.1. *Let the sequences $\{x_k\},\ \{y_k\},\ x_k, y_k \in \mathbb{R}^n$, and the sequence $\{\alpha_k\}$, $k = \pm 1, \pm 2, \ldots$, of real numbers be almost periodic.*

Then the sequences $\{x_k + y_k\}$ and $\{\alpha_k x_k\}$, $k = \pm 1, \pm 2, \ldots$, are almost periodic.

From Theorem 2.9 and Corollary 2.1 it follows that the set of all almost periodic sequences $\{x_k\},\ k = \pm 1, \pm 2, \ldots,\ x_k \in \mathbb{R}^n$, is a linear space, and equipped with the norm $|x_k|_\infty = \sup\limits_{k=\pm 1, \pm 2, \ldots} ||x_k||$ is a Banach space.

Theorem 2.10 ([284]). *Let the sequences $\{x_k\},\ \{y_k\},\ k = \pm 1, \pm 2, \ldots,\ x_k, y_k \in \mathbb{R}^n$ be almost periodic.*

Then for any $\varepsilon > 0$ there exists a relatively dense set of their common ε-almost periods.

Theorem 2.11 ([256]). *For any almost periodic sequence* $\{x_k\}$, $k = \pm 1, \pm 2, \ldots$, $x_k \in \mathbb{R}^n$, *the following average value exists (uniformly on k)*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=k}^{k+n-1} x_k = M(x_k) < \infty.$$

Now we shall consider the set

$$\mathcal{B} = \left\{\{t_k\},\ t_k \in \mathbb{R},\ t_k < t_{k+1},\ k = \pm 1, \pm 2, \ldots,\ \lim_{k\to\pm\infty} t_k = \pm\infty\right\}$$

of all unbounded increasing sequences of real numbers, and let $i(t, t+A)$ be the number of the points t_k in the interval $(t, t+A]$.

Lemma 2.1 ([284]). *Let* $\{t_k\} \in \mathcal{B}$ *be such that the sequence* $\{t_k^1\}$, $t_k^1 = t_{k+1} - t_k$, $k = \pm 1, \pm 2, \ldots$, *is almost periodic.*

Then, uniformly on $t \in \mathbb{R}$, *the following limit exists*

$$\lim_{A\to\infty} \frac{i(t, t+A)}{A} = p < \infty. \tag{2.12}$$

We shall consider the sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k, j = \pm 1, \pm 2, \ldots$. It is easy to see that

$$t_{k+i}^j - t_k^j = t_{k+j}^i - t_k^i,\ \ t_k^j - t_k^i = t_{k+i}^{j-i},\ i, j, k = \pm 1, \pm 2, \ldots. \tag{2.13}$$

Definition 2.5. The set of sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k, j = \pm 1, \pm 2, \ldots$, is said to be *uniformly almost periodic* if, for an arbitrary $\varepsilon > 0$, there exists a relatively dense set of ε-almost periods, common for all sequences $\{t_k^j\}$.

Example 2.1 ([256]). Let $\{\alpha_k\}$, $\alpha_k \in \mathbb{R}$, $k = \pm 1, \pm 2, \ldots$, be an almost periodic sequence such that

$$\sup_{k=\pm 1, \pm 2, \ldots} |\alpha_k| = \alpha < \frac{A}{2},\ A > 0,$$

and let $t_k = kA + \alpha_k$, $k = \pm 1, \pm 2, \ldots$.

Then

$$t_{k+1} - t_k \geq A - 2\alpha > 0,$$

and $\lim_{k\to\pm\infty} t_k = \pm\infty$.

Let $\varepsilon > 0$ and p be an $\frac{\varepsilon}{2}$-almost period of the sequence $\{\alpha_k\}$. Then, for all integers k and j it follows that

$$\left|t^j_{k+p} - t^j_k\right| = \left|\alpha_{k+j+p} - \alpha_{k+j}\right| + \left|\alpha_{k+p} - \alpha_k\right| < \varepsilon.$$

The last inequality shows that the set of sequences $\{t^j_k\}$ is uniformly almost periodic.

Example 2.2 ([137]). Let $t_k = k + \alpha_k$, where

$$\alpha_k = \frac{1}{4}|\cos k - \cos k\sqrt{2}|,\ k = \pm 1, \pm 2, \ldots.$$

The sequence $\{t_k\}$ is strictly increasing, since we have

$$t_{k+1} - t_k = 1 + \frac{1}{4}|\cos(k+1) - \cos(k+1)\sqrt{2}| - \frac{1}{4}|\cos k - \cos k\sqrt{2}| \geq \frac{1}{2},$$

and it is easy to see that $\lim\limits_{k \to \pm\infty} t_k = \pm\infty$.

We shall prove that the set of sequences $\{t^j_k\}$ is uniformly almost periodic. Let $\varepsilon > 0$ and p be an $\frac{\varepsilon}{2}$-almost period of the sequence $\{\alpha_k\}$. Then for all integers k and j, we have

$$\begin{aligned}\left|t^j_{k+p} - t^j_k\right| &= \left|t_{k+p+j} - t_{k+p} - t_{k+j} + t_p\right| \\ &\leq \left|\alpha_{k+p+j} - \alpha_{k+j}\right| + \left|\alpha_{k+p} - \alpha_k\right| < \varepsilon,\end{aligned}$$

and from Definition 2.5 it follows that the set of sequences $\{t^j_k\}$ is uniformly almost periodic.

We shall use the following properties of uniformly almost periodic sequences.

Lemma 2.2 ([256]). *Let the set of sequences $\{t^j_k\}$, $t^j_k = t_{k+j} - t_k$, $k, j = \pm 1, \pm 2, \ldots$, be uniformly almost periodic. Then for each $p > 0$ there exists a positive integer N such that on each interval of length p, there exist no more than N elements of the sequence $\{t_k\}$ and*

$$i(s,t) \leq N(t-s) + N. \tag{2.14}$$

Lemma 2.3 ([256]). *Let the set of sequences $\{t^j_k\}$, $t^j_k = t_{k+j} - t_k$, $k, j = \pm 1, \pm 2, \ldots$, be uniformly almost periodic. Then for each $\varepsilon > 0$ there exists a positive number $l = l(\epsilon)$ such that for each interval A of length l, there exist a subinterval $I \subset A$ of length ε and an integer q such that*

$$\left|t^q_k - r\right| < \varepsilon,\ k = \pm 1, \pm 2, \ldots,\ r \in I. \tag{2.15}$$

Lemma 2.4 ([129]). *Let the set of sequences* $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k,j = \pm 1, \pm 2, \ldots$, *be uniformly almost periodic, and let the function* $\Phi(t)$ *be almost periodic in the sense of Bohr. Then, for each* $\varepsilon > 0$ *there exists a positive* $l = l(\epsilon)$ *such that for each interval* A *of length* l, *there exists an* $r \in A$ *and an integer* q *such that*

$$\left|t_k^q - r\right| < \varepsilon, \quad \left|\Phi(t+r) - \Phi(t)\right| < \varepsilon,$$

for all $k = \pm 1, \pm 2, \ldots$, *and* $t \in \mathbb{R}$.

Lemma 2.5 ([129]). *Let the set of sequences* $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k,j = \pm 1, \pm 2, \ldots$, *be uniformly almost periodic, and let the function* $\Phi(t)$ *be almost periodic in the sense of Bohr. Then the sequence* $\{\Phi(t_k)\}$ *is almost periodic.*

Definition 2.6 ([257]). The set $T \in \mathcal{B}$ *is almost periodic* if for every sequence of real numbers $\{s'_m\}$ there exists a subsequence $\{s_n\}$, $s_n = s'_{m_n}$, such that $T - s_n = \{t_k - s_n\}$ is uniformly convergent as $n \to \infty$ to a set $T_1 \in \mathcal{B}$.

Lemma 2.6. *The set of sequences* $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k,j = \pm 1, \pm 2, \ldots$, *is uniformly almost periodic if and only if for every sequence of real numbers* $\{s'_m\}$ *there exists a subsequence* $\{s_n\}$, $s_n = s'_{m_n}$, *such that* $T - s_n = \{t_k - s_n\}$ *is uniformly convergent for* $n \to \infty$ *on* $\mathcal{B}$.

Proof. The proof follows directly from Theorem 1 in [257].

In the investigation of the existence of almost periodic solutions of impulsive models, the question of the separation from the origin of the sequences $\{t_k\} \in \mathcal{B}$ is very important. Hence, we will always assume that the following inequality

$$\inf_{k=\pm 1, \pm 2, \ldots} t_k^1 = \theta > 0$$

holds.

We shall also use the set *UAPS*, $UAPS \subset \mathcal{B}$, for which the sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k,j = \pm 1, \pm 2, \ldots$, form a uniformly almost periodic set and $\inf\limits_{k=\pm 1, \pm 2, \ldots} t_k^1 = \theta > 0$.

2.2.2 Almost Periodic Functions

In this part, we shall consider the main definitions and properties of almost periodic piecewise continuous functions.

Definition 2.7. A function $\varphi \in PC[\mathbb{R}, \mathbb{R}^n]$ is said to be *almost periodic* if the following holds:

(a) $\{t_k\} \in UAPS$.
(b) For any $\varepsilon > 0$ there exists a real number $\delta = \delta(\varepsilon) > 0$ such that, if the points t' and t'' belong to one and the same interval of continuity of $\varphi(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $||\varphi(t') - \varphi(t'')|| < \varepsilon$.
(c) For any $\varepsilon > 0$ there exists a relatively dense set $\overline{T}$ such that, if $\tau \in \overline{T}$, then $||\varphi(t + \tau) - \varphi(t)|| < \varepsilon$ for all $t \in \mathbb{R}$ satisfying the condition $|t - t_k| > \varepsilon$, $k = \pm 1, \pm 2, \ldots$.

The elements of $\overline{T}$ are called *ε-almost periods*.

Example 2.3 ([137]). Let $\{\mu_k\},\ \mu_k \in \mathbb{R},\ k = \pm 1, \pm 2, \ldots$, be an almost periodic sequence and $\{t_k\} \in UAPS$ be uniformly almost periodic. Then the function $\varphi(t) = \mu_k,\ t_k \le t < t_{k+1}$, is almost periodic.

Now we shall consider some properties of almost periodic functions.

Theorem 2.12 ([284]). *Every almost periodic function is bounded on the real line.*

Theorem 2.13 ([284]). *If $\varphi \in PC[\mathbb{R}, \mathbb{R}^n]$ is an almost periodic function, then for any $\varepsilon > 0$ there exists a relative dense set of intervals with a fixed length γ, $0 < \gamma < \varepsilon$, which contains ε-almost periods of the function $\varphi(t)$.*

Theorem 2.14. *Let $\varphi \in PC[\mathbb{R}, \mathbb{R}^n]$ be an almost periodic function with range $Y \subset \mathbb{R}^n$. If the function $F(y)$ is uniformly continuous with domain Y, then the function $F(\varphi(t))$ is almost periodic.*

Proof. The proof is trivial, so we omit the details.

Theorem 2.15 ([284]). *For every pair of almost periodic functions with points of discontinuity from the sequence $\{t_k\} \in UAPS$ and for arbitrary $\varepsilon > 0$, there exists a relatively dense set of their common ε-almost periods.*

Theorem 2.16. *The sum of two almost periodic functions with points of discontinuity $t_k,\ k = \pm 1, \pm 2, \ldots,\ \{t_k\} \in UAPS$, is an almost periodic function.*

Theorem 2.17. *The quotient $\dfrac{\varphi(t)}{\psi(t)}$ of two almost periodic functions with points of discontinuity $t_k,\ k = \pm 1, \pm 2, \ldots,\ \{t_k\} \in UAPS$, is an almost periodic function if*

$$\inf_{t \in \mathbb{R}} ||\psi(t)|| > 0.$$

Now, let us consider the following system of impulsive differential equations

$$\begin{cases} \dot{x} = A(t)x + f(t),\ t \neq t_k, \\ \Delta x(t_k) = B_k x(t_k) + I_k(x(t_k)),\ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.16}$$

where the $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ is an almost periodic matrix in the sense of Bohr.

Lemma 2.7 ([284]). *Let the following conditions hold.*

1. *$U(t,s)$ is the fundamental matrix of the linear part of* (2.16).
2. *$f(t)$ is an almost periodic function.*
3. *The sequence of functions $\{I_k\}$ and the sequence of matrices $\{B_k\}$ are almost periodic.*
4. *The set of sequences $\{t_k^j\}$ is uniformly almost periodic.*

Then for every $\varepsilon > 0$ and every $\theta > 0$, there exist $\varepsilon_1,\ 0 < \varepsilon_1 < \varepsilon$, a relatively dense set $\overline{T}$ of real numbers and a set P of integers, such that the following relations hold:

(a) $||U(t+\tau, s+\tau) - U(t,s)|| < \varepsilon,\ t \in \mathbb{R},\ \tau \in \overline{T},\ 0 \le t-s \le \theta;$
(b) $||f(t+\tau) - f(t)|| < \varepsilon,\ t \in \mathbb{R},\ \tau \in \overline{T},\ |t-t_k| > \varepsilon,\ k = \pm 1, \pm 2, \ldots;$
(c) $||B_{k+q} - B_k|| < \varepsilon,\ q \in P,\ k = \pm 1, \pm 2, \ldots;$
(d) $||I_{k+q} - I_k|| < \varepsilon,\ q \in P,\ k = \pm 1, \pm 2, \ldots;$
(e) $|t_k^q - \tau| < \varepsilon_1,\ q \in P,\ \tau \in \overline{T},\ k = \pm 1, \pm 2, \ldots.$

We shall also consider the following definition for almost periodic piecewise continuous functions.

Let $T, P \in \mathcal{B}$, and let $s(T \cup P)\ :\ \mathcal{B} \to \mathcal{B}$ be a map such that the set $s(T \cup P)$ forms a strictly increasing sequence. For $D \subset \mathbb{R}$ and $\varepsilon > 0$, we introduce the notations $\theta_\varepsilon(D) = \{t + \varepsilon,\ t \in D\}$ and $F_\varepsilon(D) = \cap\{\theta_\varepsilon(D)\}$.

By $\phi = (\varphi(t), T)$ we denote an element from the space $PC[\mathbb{R}, \mathbb{R}^n] \times \mathcal{B}$ and for every sequence of real numbers $\{s_n\}, n = 1, 2, \ldots$, by $\theta_{s_n}\phi$, we shall mean the set $\{\varphi(t+s_n), T - s_n\} \subset PC \times \mathcal{B}$, where

$$T - s_n = \{t_k - s_n,\ k = \pm 1, \pm 2, \ldots,\ n = 1, 2, \ldots\}.$$

Definition 2.8. The sequence $\{\phi_n\}$, $\phi_n = (\varphi_n(t), T_n) \in PC[\mathbb{R}, \mathbb{R}^n] \times \mathcal{B}$, converges to $\phi,\ \phi = (\varphi(t), T),\ (\varphi(t), T) \in PC[\mathbb{R}, \mathbb{R}^n] \times \mathcal{B}$, if and only if for any $\varepsilon > 0$ there exists an $n_0 > 0$ such that $n \ge n_0$ implies

$$\rho(T, T_n) < \varepsilon,\ ||\varphi_n(t) - \varphi(t)|| < \varepsilon$$

uniformly for $t \in \mathbb{R} \setminus F_\varepsilon(s(T_n \cup T))$, where $\rho(.,.)$ is an arbitrary distance in $\mathcal{B}$.

Definition 2.9. The function $\varphi \in PC[\mathbb{R}, \mathbb{R}^n]$ is said to be an *almost periodic piecewise continuous function* with points of discontinuity of the first kind from the set $T \in \mathcal{B}$ if for every sequence of real numbers $\{s'_m\}$ there exists a subsequence $\{s_n\},\ s_n = s'_{m_n}$, such that $\theta_{s_n}\phi$ is uniformly convergent on $PC[\mathbb{R}, \mathbb{R}^n] \times \mathcal{B}$.

Now, let $\Omega \subseteq \mathbb{R}^n$ and consider the impulsive differential system (2.3).

We introduce the following conditions:

H2.3. The function $f(t,x)$ is almost periodic in t uniformly with respect to $x \in \Omega$.
H2.4. The sequence $\{I_k(x)\}$, $k = \pm 1, \pm 2, \ldots$, is almost periodic uniformly with respect to $x \in \Omega$.
H2.5. The set of sequences $\{t_k\} \in UAPS$.

Let the assumptions H2.3–H2.5 hold, and let $\{s'_m\}$ be an arbitrary sequence of real numbers. Then there exists a subsequence $\{s_n\}$, $s_n = s'_{m_n}$, such that the sequence $\{f(t+s_n,x)\}$ converges uniformly to the function $f^s(t,x)$, and from Lemma 2.6 it follows that the set of sequences $\{t_k - s_n\}$, $k = \pm 1, \pm 2, \ldots$, is convergent to the sequence $\{t_k^s\}$ uniformly with respect to $k = \pm 1, \pm 2, \ldots$ as $n \to \infty$.

By $\{k_{n_i}\}$ we denote the sequence of integers such that the subsequence $\{t_{k_{n_i}}\}$ is convergent to the sequence $\{t_k^s\}$ uniformly with respect to $k = \pm 1, \pm 2, \ldots$ as $i \to \infty$.

Then, for every sequence $\{s'_m\}$, the system (2.3) transforms to the system

$$\begin{cases} \dot{x}(t) = f^s(t,x), \ t \neq t_k^s, \\ \Delta x(t_k^s) = I_k^s(x(t_k^s)), \ k = \pm 1, \pm 2, \ldots. \end{cases} \tag{2.17}$$

Remark 2.2. In many papers, the limiting system (2.17) is called the *Hull* of the system (2.3), and is denoted by $H(f, I_k, t_k)$.

2.3 Stability and Boundedness Definitions

We shall use the following stability definitions for systems of the form (2.3).

Let $\psi(t) = \psi(t; t_0, \psi_0)$, $\psi(t_0^+) = \psi_0 \in \Omega$, be a solution of system (2.3).

Definition 2.10. The solution $\psi(t)$ is said to be:

(a) *stable* if

$$(\forall t_0 \in \mathbb{R})(\forall \varepsilon > 0)(\exists \delta = \delta(t_0, \varepsilon) > 0)$$
$$(\forall x_0 \in \Omega : \ ||x_0 - \psi(t_0^+)|| < \delta)(\forall t \geq t_0) :$$
$$||x(t; t_0, x_0) - \psi(t)|| < \varepsilon;$$

(b) *uniformly stable* if the number δ in (a) is independent of $t_0 \in \mathbb{R}$;

(c) *attractive* if

$$(\forall t_0 \in \mathbb{R})(\exists \lambda = \lambda(t_0) > 0)(\forall x_0 \in \Omega : \ ||x_0 - \psi(t_0^+)|| < \lambda)$$
$$\lim_{t \to \infty} ||x(t; t_0, x_0) - \psi(t)|| = 0;$$

(d) *equi-attractive* if

$$(\forall t_0 \in \mathbb{R})(\exists \lambda = \lambda(t_0) > 0)(\forall \varepsilon > 0)(\exists T = T(t_0, \varepsilon) > 0)$$
$$(\forall x_0 \in \Omega : \ ||x_0 - \psi(t_0^+)|| < \lambda)(\forall t \geq t_0 + T) : ||x(t; t_0, x_0) - \psi(t)|| < \varepsilon;$$

(e) *uniformly attractive* if the numbers λ and T in (d) are independent of $t_0 \in \mathbb{R}$;

(f) *asymptotically stable* if it is stable and attractive;

(g) *uniformly asymptotically stable* if it is uniformly stable and uniformly attractive;
(h) *exponentially stable* if

$$(\exists\lambda > 0)(\forall\alpha > 0)(\exists\gamma = \gamma(\alpha) > 0)(\forall t_0 \in \mathbb{R})$$
$$(\forall x_0 \in \Omega: \ ||x_0 - \psi(t_0^+)|| < \alpha)(\forall t \geq t_0):$$
$$||x(t; t_0, x_0) - \psi(t)|| < \gamma(\alpha)||x_0 - \psi(t_0^+)|| \exp\{-\lambda(t - t_0)\}.$$

In the case when $\psi(t) \equiv 0$, we shall use the following definition.

Definition 2.11. The zero solution $\psi(t) \equiv 0$ of system (2.3) is said to be:

(a) *stable* if

$$(\forall t_0 \in \mathbb{R})(\forall\varepsilon > 0)(\exists\delta = \delta(t_0, \varepsilon) > 0)$$
$$(\forall x_0 \in B_\delta)(\forall t \geq t_0): ||x(t; t_0, x_0)|| < \varepsilon;$$

(b) *uniformly stable* if the number δ in (a) is independent of $t_0 \in \mathbb{R}$;
(c) *attractive* if

$$(\forall t_0 \in \mathbb{R})(\exists\lambda = \lambda(t_0) > 0)(\forall x_0 \in B_\lambda): \lim_{t\to\infty} ||x(t; t_0, x_0)|| = 0;$$

(d) *equi-attractive* if

$$(\forall t_0 \in \mathbb{R})(\exists\lambda = \lambda(t_0) > 0)(\forall\varepsilon > 0)(\exists T = T(t_0, \varepsilon) > 0)$$
$$(\forall x_0 \in B_\lambda)(\forall t \geq t_0 + T): \ ||x(t; t_0, x_0)|| < \varepsilon;$$

(e) *uniformly attractive* if the numbers λ and T in (d) are independent of $t_0 \in \mathbb{R}$;
(f) *asymptotically stable* if it is stable and attractive;
(g) *uniformly asymptotically stable* if it is uniformly stable and uniformly attractive.

We introduce the following notation:

$$||\varphi||_r = \sup_{t\in[t_0-r,t_0]} ||\varphi(t - t_0)|| \text{ is the norm of the function } \varphi \in PC[[-r, 0], \Omega].$$

In the case $r = \infty$ we have $||\varphi||_r = ||\varphi||_\infty = \sup_{t\in(-\infty,t_0]} ||\varphi(t - t_0)||$.

Let $\varphi_1 \in PC[[-r, 0], \Omega]$. Denote by $x_1(t) = x_1(t; t_0, \varphi_1)$ the solution of system (2.8) satisfying the initial conditions

$$\begin{cases} x_1(t; t_0, \varphi_1) = \varphi_1(t - t_0), \ t_0 - r \leq t \leq t_0, \\ x_1(t_0^+; t_0, \varphi_1) = \varphi_1(0). \end{cases}$$

Definition 2.12. The solution $x_1(t)$ of system (2.8) is said to be:

(a) *stable* if

$$(\forall t_0 \in \mathbb{R})(\forall \varepsilon > 0)(\exists \delta = \delta(t_0, \varepsilon) > 0)$$
$$(\forall \varphi_0 \in PC[[-r, 0], \Omega] : \ ||\varphi_0 - \varphi_1||_r < \delta)$$
$$(\forall t \geq t_0) : \ ||x(t; t_0, \varphi_0) - x_1(t; t_0, \varphi_1)|| < \varepsilon;$$

(b) *uniformly stable* if the number δ in (a) is independent of $t_0 \in \mathbb{R}$;
(c) *attractive* if

$$(\forall t_0 \in \mathbb{R})(\exists \lambda = \lambda(t_0) > 0)$$
$$(\forall \varphi_0 \in PC[[-r, 0], \Omega] : \ ||\varphi_0 - \varphi_1||_r < \lambda)$$
$$\lim_{t\to\infty} ||x(t; t_0, \varphi_0) - x_1(t; t_0, \varphi_1)|| = 0;$$

(d) *equi-attractive* if

$$(\forall t_0 \in \mathbb{R})(\exists \lambda = \lambda(t_0) > 0)(\forall \varepsilon > 0)(\exists T = T(t_0, \varepsilon) > 0)$$
$$(\forall \varphi_0 \in PC[[-r, 0], \Omega] : \ ||\varphi_0 - \varphi_1||_r < \lambda)$$
$$(\forall t \geq t_0 + T) : \ ||x(t; t_0, \varphi_0) - x_1(t; t_0, \varphi_1)|| < \varepsilon;$$

(e) *uniformly attractive* if the numbers λ and T in (d) are independent of $t_0 \in \mathbb{R}$;
(f) *asymptotically stable* if it is stable and attractive;
(g) *uniformly asymptotically stable* if it is uniformly stable and uniformly attractive;
(h) *unstable* if (a) does not hold.

We shall use the following definitions of Lyapunov-like stability of the zero solution of (2.8).

Definition 2.13. The zero solution $x_1(t) \equiv 0$ of system (2.8) is said to be:

(a) *stable* if

$$(\forall t_0 \in \mathbb{R})(\forall \varepsilon > 0)(\exists \delta = \delta(t_0, \varepsilon) > 0)$$
$$(\forall \varphi_0 \in PC[[-r, 0], \Omega] : \ ||\varphi_0||_r < \delta)(\forall t \geq t_0) : \ ||x(t; t_0, \varphi_0)|| < \varepsilon;$$

(b) *uniformly stable* if the number δ in (a) is independent of $t_0 \in \mathbb{R}$;
(c) *attractive* if

$$(\forall t_0 \in \mathbb{R})(\exists \lambda = \lambda(t_0) > 0)(\forall \varphi_0 \in PC[[-r, 0], \Omega] : \ ||\varphi_0||_r < \lambda)$$
$$\lim_{t\to\infty} ||x(t; t_0, \varphi_0)|| = 0;$$

(d) *equi-attractive* if

$$(\forall t_0 \in \mathbb{R})(\exists \lambda = \lambda(t_0) > 0)(\forall \varepsilon > 0)(\exists T = T(t_0, \varepsilon) > 0)$$
$$(\forall \varphi_0 \in PC[[-r, 0], \Omega] : \ ||\varphi_0||_r < \lambda)(\forall t \geq t_0 + T) : \ ||x(t; t_0, \varphi_0)|| < \varepsilon;$$

(e) *uniformly attractive* if the numbers λ and T in (d) are independent of $t_0 \in \mathbb{R}$;
(f) *asymptotically stable* if it is stable and attractive;
(g) *uniformly asymptotically stable* if it is uniformly stable and uniformly attractive;
(h) *unstable* if

$$(\exists t_0 \in \mathbb{R})(\exists \varepsilon > 0)(\forall \delta > 0)(\exists \varphi_0 \in PC[[-r, 0], \Omega] : \ ||\varphi_0||_r < \delta)$$
$$(\exists t \geq t_0) : \ ||x(t; t_0, \varphi_0)|| \geq \varepsilon.$$

In this book, we shall apply Lyapunov's second method for investigating the boundedness of solutions of system (2.8) for $\Omega = \mathbb{R}^n$, i.e. we shall consider the following system

$$\begin{cases} \dot{x}(t) = f(t, x_t), \ t \neq t_k, \\ \Delta x(t) = I_k(x(t)), \ t = t_k, \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.18}$$

where $f : \mathbb{R} \times PC[[-r, 0], \mathbb{R}^n] \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = \pm 1, \pm 2, \ldots$, $t_k < t_{k+1} < \ldots$ and $\lim_{k \to \pm\infty} t_k = \pm\infty$.

Let $\varphi_0 \in PC[[-r, 0], \mathbb{R}^n]$. Denote by $x(t) = x(t; t_0, \varphi_0)$ the solution of system (2.18) satisfying the initial conditions

$$\begin{cases} x(t; t_0, \varphi_0) = \varphi_0(t - t_0), \ t_0 - r \leq t \leq t_0, \\ x(t_0^+; t_0, \varphi_0) = \varphi_0(0), \end{cases} \tag{2.19}$$

and by $J^+(t_0, \varphi_0)$ the maximal interval of type $[t_0, \beta)$, in which the solution $x(t; t_0, \varphi_0)$ is defined.

Definition 2.14. We say that the solutions of system (2.18) are:

(a) *equi-bounded* if

$$(\forall t_0 \in \mathbb{R})(\forall \alpha > 0)(\exists \beta = \beta(t_0, \alpha) > 0)$$
$$(\forall \varphi_0 \in PC[[-r, 0], \mathbb{R}^n] : \ \|\varphi_0\|_r < \alpha)(\forall t \geq t_0) : \quad ||x(t; t_0, \varphi_0)|| < \beta;$$

(b) *uniformly bounded* if the number β in (a) is independent of $t_0 \in \mathbb{R}$;
(c) *quasi-uniformly ultimately bounded* if

$$(\exists B > 0)(\forall \alpha > 0)(\exists T = T(\alpha) > 0)(\forall t_0 \in \mathbb{R})$$
$$(\forall \varphi_0 \in PC[[-r, 0], \mathbb{R}^n] : \ \|\varphi_0\|_r < \alpha)(\forall t \geq t_0 + T) : \quad ||x(t; t_0, \varphi_0)|| < B;$$

(d) *uniformly ultimately bounded* if (b) and (c) hold together.

We shall use the following definitions of global stability of the zero solution of (2.18).

Definition 2.15. The zero solution $x(t) \equiv 0$ of system (2.18) is said to be:

(a) *stable* if

$$(\forall t_0 \in \mathbb{R})(\forall \varepsilon > 0)(\exists \delta = \delta(t_0, \varepsilon) > 0)$$
$$(\forall \varphi_0 \in PC[[-r, 0], \mathbb{R}^n] : \, ||\varphi_0||_r < \delta)$$
$$(\forall t \geq t_0) : \, ||x(t; t_0, \varphi_0)|| < \varepsilon;$$

(b) *uniformly stable* if the number δ in (a) is independent of $t_0 \in \mathbb{R}$;
(c) *globally equi-attractive* if

$$(\forall t_0 \in \mathbb{R})(\forall \alpha > 0)(\forall \varepsilon > 0)(\exists \gamma = \gamma(t_0, \alpha, \varepsilon) > 0)$$
$$(\forall \varphi_0 \in PC[[-r, 0], \mathbb{R}^n] : \, ||\varphi_0||_r < \alpha)(\forall t \geq t_0 + \gamma) : \, ||x(t; t_0, \varphi_0)|| < \varepsilon;$$

(d) *uniformly globally attractive* if the number γ in (c) is independent of $t_0 \in \mathbb{R}$;
(e) *globally equi-asymptotically stable* if it is stable and globally equi-attractive;
(f) *uniformly globally asymptotically stable* if it is uniformly stable, uniformly globally attractive and the solutions of system (2.18) are uniformly bounded;
(g) *globally exponentially stable* if

$$(\exists c > 0)(\forall \alpha > 0)(\exists \gamma = \gamma(\alpha) > 0)(\forall t_0 \in \mathbb{R})$$
$$(\forall \varphi_0 \in PC[[-r, 0], \mathbb{R}^n] : \, ||\varphi_0||_r < \alpha)(\forall t \geq t_0) :$$
$$||x(t; t_0, \varphi_0)|| \leq \gamma(\alpha)||\varphi_0||_r \exp[-c(t - t_0)].$$

2.4 Piecewise Continuous Lyapunov Functions and Lyapunov Functionals

An interesting and fruitful technique that has gained increasing significance and has given decisive impetus to the modern development of the stability theory of impulsive functional differential equations is Lyapunov's second method [214]. A manifest advantage of this method is that it does not require the knowledge of solutions and therefore has great power in applications.

Different aspects of applications of Lyapunov's second method to differential equations are given in [44, 56, 81, 93, 96, 130, 131, 151, 165, 171, 180–183, 217, 249, 251, 298, 351]. There has been a gradual expansion both in the class of objects studied and in the mathematical problems investigated by means of the method.

Gurgulla and Perestyuk were the first to apply the Lyapunov direct method for impulsive systems. In the work [128] they used classical (continuous) Lyapunov functions. The application of continuous Lyapunov functions to the investigation of impulsive systems restricts the possibilities of Lyapunov's second method. The fact that the solutions of impulsive systems are piecewise continuous functions requires the introduction of an analogue of the classical Lyapunov functions which have discontinuities of the first kind [34]. By means of such functions it becomes possible to solve basic problems related to the application of Lyapunov's second method to impulsive systems.

Let $\tau_0(x) \equiv t_0$ for $x \in \Omega$ and introduce the sets

$$G_k = \Big\{(t,x) \in \mathbb{R} \times \Omega : \ \tau_{k-1}(x) < t < \tau_k(x)\Big\}, \ G = \bigcup_{k=\pm 1,\pm 2,\dots} G_k.$$

Definition 2.16. A function $V : \ \mathbb{R} \times \Omega \to \mathbb{R}_+$ *belongs to the class* V_0 if:

1. $V(t,x)$ is continuous on G and locally Lipschitz continuous with respect to its second argument on each of the sets G_k, $k = \pm 1, \pm 2, \dots$.
2. For each $k = \pm 1, \pm 2, \dots$ and $(t_0^*, x_0^*) \in \sigma_k$ the following finite limits exist

$$V(t_0^{*-}, x_0^*) = \lim_{\substack{(t,x)\to(t_0^*,x_0^*)\\(t,x)\in G_k}} V(t,x), \ \ V(t_0^{*+}, x_0^*) = \lim_{\substack{(t,x)\to(t_0^*,x_0^*)\\(t,x)\in G_{k+1}}} V(t,x)$$

and the equality $V(t_0^{*-}, x_0^*) = V(t_0^*, x_0^*)$ holds.

Let the function $V \in V_0$ and $(t,x) \in G$. We define the derivative

$$\dot V_{(2.1)}(t,x) = \lim_{\delta\to 0+} \sup \frac{1}{\delta}\big[V(t+\delta, x+\delta f(t,x)) - V(t,x)\big].$$

Note that if $x = x(t)$ is a solution of system (2.1), then for $t \neq \tau_k(x(t))$, $k = \pm 1, \pm 2, \dots$, we have $\dot V_{(2.1)}(t,x) = D^+_{(2.1)} V(t, x(t))$, where

$$D^+_{(2.1)} V(t,x(t)) = \lim_{\delta\to 0+} \sup \frac{1}{\delta}\big[V(t+\delta, x(t+\delta)) - V(t,x(t))\big] \tag{2.20}$$

is *the upper right-hand Dini derivative* of $V \in V_0$ (with respect to the system (2.1)).

The class of functions V_0 is also used to investigate the qualitative properties of solutions of impulsive models with fixed moments of impulse effect (2.3). In this case, $\tau_k(x) \equiv t_k, \ k = \pm 1, \pm 2, \dots, \sigma_k$, are hyperplanes in $\mathbb{R}^{n+1}$, the sets G_k are

$$G_k = \{(t,x) \in \mathbb{R} \times \Omega : \quad t_{k-1} < t < t_k\},$$

and condition 2 of Definition 2.16 is substituted by the condition:

2'. For each $k = \pm 1, \pm 2, \dots$ and $x \in \Omega$, the finite limits

$$V(t_k^-, x) = \lim_{\substack{t\to t_k\\ t<t_k}} V(t,x), \qquad V(t_k^+, x) = \lim_{\substack{t\to t_k\\ t>t_k}} V(t,x),$$

exist and the following equalities hold

$$V(t_k^-, x) = V(t_k, x).$$

For $t \neq t_k,\ k = \pm 1, \pm 2, \ldots$, the upper right-hand derivative of Lyapunov's function $V \in V_0$ with respect to system (2.3) is

$$D^+_{(2.3)} V(t, x(t)) = \lim_{\delta \to 0^+} \sup \frac{1}{\delta}\big[V(t+\delta, x(t+\delta)) - V(t, x)\big].$$

In subsequent chapters we shall also use the following classes of piecewise continuous Lyapunov functions

$$V_1 = \Big\{V : \mathbb{R} \times \Omega \times \Omega \to \mathbb{R}_+,\ V \text{ is continuous in } (t_{k-1}, t_k] \times \Omega \times \Omega,$$
$$V(t_k, x, y) = V(t_k^-, x, y) \text{ and } \lim_{t > t_k} V(t, x, y) = V(t_k^+, x, y),\ x, y \in \Omega\Big\}.$$

Definition 2.17. A function $V \in V_1$ *belongs to the class* V_2 if:

1. $V(t, 0, 0) = 0,\ t \in \mathbb{R}$.
2. The function $V(t, x, y)$ is locally Lipschitz continuous with respect to its second and third arguments with a Lipschitz constant $H_1 > 0$, i.e. for $x_1, x_2 \in \Omega,\ y_1, y_2 \in \Omega$ and for $t \in \mathbb{R}$ it follows that

$$|V(t, x_1, y_1) - V(t, x_2, y_2)| \leq H_1\big(||x_1 - x_2|| + ||y_1 - y_2||\big),$$

$t \neq t_k, k = \pm 1, \pm 2, \ldots$.

Let $V \in V_2,\ t \neq t_k,\ k = \pm 1, \pm 2, \ldots,\ x \in PC[\mathbb{R}, \Omega],\ y \in PC[\mathbb{R}, \Omega]$.
We introduce

$$D^+_{(2.3)} V(t, x(t), y(t))$$
$$= \lim_{\delta \to 0^+} \sup \frac{1}{\delta}\big[V(t+\delta, x(t) + \delta f(t, x(t)), y(t) + \delta f(t, y(t))) - V(t, x(t), y(t))\big].$$

When applying Lyapunov's second method, there are two main approaches to investigating the qualitative properties of solutions of functional differential equations. Krasovskii [171] adopted a functional analysis approach. He replaced the Lyapunov function with a Lyapunov functional. The method of *Lyapunov–Krasovskii functionals* has been used by many researchers to investigate the stability theory of functional differential equations and their applications [2, 18, 19, 48, 64, 65, 71, 79, 121, 130, 131, 159, 164, 165, 206].

The presence of impulses as well as delays in impulsive functional differential equations requires the use of piecewise continuous Lyapunov functionals or a combination of the methods of piecewise continuous Lyapunov functions and the Razumikhin technique. By means of such approaches, many interesting results in the qualitative theory of these equations have been obtained [37–42, 130, 180, 182, 184, 213, 220, 260, 293–299, 301, 302, 305, 312–314].

Definition 2.18. A functional $V : \mathbb{R} \times PC[[-r,0],\Omega] \to \mathbb{R}_+$ *belongs to the class* $V_0(.)$ if:

1. $V(t,\varphi)$ is continuous on each of the sets $(t_{k-1},t_k) \times PC[[-r,0],\Omega]$, $k = \pm 1, \pm 2, \ldots$, and locally Lipschitz in φ on each compact set in $PC[[-r,0],\Omega]$.
2. For each $k = \pm 1, \pm 2, \ldots$ and $\varphi \in PC[[-r,0],\Omega]$ the finite limits

$$V(t_k^-,\varphi) = \lim_{\substack{t\to t_k\\ t<t_k}} V(t,\varphi), \qquad V(t_k^+,\varphi) = \lim_{\substack{t\to t_k\\ t>t_k}} V(t,\varphi),$$

exist and the following equalities hold

$$V(t_k^-,\varphi) = V(t_k,\varphi).$$

Let $V \in V_0(.)$ and $(t,\varphi) \in \mathbb{R} \times PC[[-r,0],\Omega]$. We define

$$D^+_{(2.8)}V(t,\varphi) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h,x_{t+h}(t,\varphi)) - V(t,\varphi)]. \tag{2.21}$$

The functional $D^+_{(2.8)}V(t,\varphi)$, defined by (2.21), is *the upper right-hand Dini derivative* of $V \in V_0(.)$ with respect to system (2.8).

When using the method of Lyapunov functions for functional differential equations, the direct transfer of the Lyapunov theorems leads to significant difficulties when the sign of the derivative of the Lyapunov function with respect to the system has to be determined.

We shall employ Lyapunov functions from the class V_0 and develop the corresponding stability theory for the system (2.8).

Definition 2.19. Given a function $V \in V_0$. For $(t_0,\varphi) \in \mathbb{R} \times PC[[-r,0],\Omega]$ the *upper right-hand derivative of V with respect to system* (2.8) is defined by

$$D^+_{(2.8)}V(t,\varphi(0)) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h,x(t+h;t_0,\varphi)) - V(t,\varphi(0))], \tag{2.22}$$

where $x(t;t_0,\varphi)$ is a solution of (2.8) with an initial function $\varphi \in PC[[-r,0],\Omega]$.

Note that in Definition 2.19, $D^+_{(2.8)}V(t,\varphi(0))$ is a functional whereas V is a function. This special feature was a source of difficulty when applying Lyapunov's second method to functional differential equations. In order to find a positive definite function V such that $D^+_{(2.8)}V(t,\varphi(0)) \le 0$, the point $\varphi(0)$ has a dominant role. Using simple considerations, Razumikhin [249] proved that the derivative $D^+_{(2.8)}V(t,\varphi(0))$ should be estimated only by the elements of minimal subsets of the integral curves of the investigated system when the following condition

$$V(t+s,\varphi(s)) \le V(t,\varphi(0)),\ s \in [-r,0] \tag{2.23}$$

holds. The condition (2.23) is called the *Razumikhin condition*, and the corresponding technique is known as the *Razumikhin technique*.

We shall use the next class of piecewise Lyapunov functions, which are connected with the system (2.17).

Definition 2.20. A function $W: \mathbb{R} \times \Omega \to \mathbb{R}_+$ *belongs to the class* W_0 if:

1. The function $W(t,x)$ is continuous on $(t,x) \in \mathbb{R} \times \Omega$, $t \neq t_k^s$, $k = \pm 1, \pm 2, \ldots$, and $W(t,0) = 0$, $t \in \mathbb{R}$.
2. The function $W(t,x)$ is locally Lipschitz continuous with respect to its second argument.
3. For each $k = \pm 1, \pm 2, \ldots$ and $x \in \Omega$ the finite limits

$$W(t_k^{s-}, x) = \lim_{\substack{t \to t_k^s \\ t < t_k^s}} W(t,x), \quad W(t_k^{s+}, x) = \lim_{\substack{t \to t_k^s \\ t > t_k^s}} W(t,x)$$

exist and the equality $W(t_k^{s-}, x) = W(t_k^s, x)$ holds.

Let the function $W \in W_0$ and $x \in PC[\mathbb{R}, \Omega]$. The upper right-hand Dini derivative of W with respect to (2.17) is defined by

$$D^+_{(2.17)} W(t, x(t)) = \lim_{\delta \to 0+} \sup \frac{1}{\delta} \big[W(t+\delta, x(t) + \delta f^s(t, x(t))) - W(t, x(t)) \big].$$

In the investigation of the qualitative properties of solutions of differential equations, it is well known that employing several Lyapunov functions is more useful than employing a single one since each function can satisfy less rigid requirements. Hence, the corresponding theory, known as the *method of vector Lyapunov functions*, offers a very flexible mechanism (see [183] and the references therein).

Moreover, by means of the method of vector Lyapunov functions we can prove the results in some cases where using scalar Lyapunov functions is impossible.

In the present book we shall use vector Lyapunov functions $V: \mathbb{R} \times \Omega \to \mathbb{R}^m_+$, $V = \mathrm{col}(V_1, V_2, \ldots, V_m)$ such that $V_j \in V_0, j = 1, 2, \ldots, m.$

In the presence of delays, we shall use the corresponding modifications and generalizations.

2.5 Impulsive Differential Inequalities

In this section we present the main comparison results and integral inequalities we will use. The essence of the comparison method is in studying the relations between the given system and a comparison system so that some properties of the solutions of the comparison system should imply the corresponding properties of the solutions of the system under consideration. These relations are obtained by employing differential inequalities. The comparison system is usually of lower order and its right-hand side possesses a certain type of monotonicity, which considerably simplifies the study of its solutions.

Consider the system of impulsive differential equations (2.3). Together with system (2.3) we shall consider the comparison system

$$\begin{cases} \dot{u}(t) = F(t, u(t)), \ t \neq t_k, \\ \Delta u(t_k) = u(t_k^+) - u(t_k) = J_k(u(t_k)), \ t_k > t_0, \end{cases} \tag{2.24}$$

where $F : \mathbb{R} \times \mathbb{R}_+^m \to \mathbb{R}^m$; $J_k : \mathbb{R}_+^m \to \mathbb{R}^m$, $k = 1, 2, \ldots$.

Let $u_0 \in \mathbb{R}_+^m$. Denote by $u(t) = u(t; t_0, u_0)$ the solution of system (2.24) satisfying the initial condition $u(t_0^+) = u(t_0) = u_0$ and by $J^+(t_0, u_0)$ the maximal interval of type $[t_0, \beta)$ in which the solution $u(t; t_0, u_0)$ is defined.

We introduce the following partial ordering on $\mathbb{R}^m$: for the vectors $u, \ v \in \mathbb{R}^m$ we shall say that $u \geq v$ if $u_j \geq v_j$ for each $j = 1, 2, \ldots, m$ and $u > v$ if $u_j > v_j$ for each $j = 1, 2, \ldots, m$.

Definition 2.21. A solution $u^+ : \ J^+(t_0, u_0) \to \mathbb{R}_+^m$ of the system (2.24) for which $u^+(t_0; t_0, u_0) = u_0$ is said to be a *maximal solution* if any other solution $u : [t_0, \tilde{\omega}) \to \mathbb{R}_+^m$ for which $u(t_0) = u_0$ satisfies the inequality $u^+(t) \geq u(t)$ for $t \in J^+(t_0, u_0) \cap [t_0, \tilde{\omega})$.

Analogously, a *minimal solution* of system (2.24) is defined as follows.

Definition 2.22. The function $\psi : \mathbb{R}_+^m \to \mathbb{R}^m$ is said to be:

(a) *non-decreasing* in $\mathbb{R}_+^m$ if $\psi(u) \geq \psi(v)$ for $u \geq v, u, v \in \mathbb{R}_+^m$.
(b) *monotone increasing* in $\mathbb{R}_+^m$ if $\psi(u) > \psi(v)$ for $u > v$ and $\psi(u) \geq \psi(v)$ for $u \geq v, u, v \in \mathbb{R}_+^m$.

Definition 2.23. The function $F : \mathbb{R} \times \mathbb{R}_+^m \to \mathbb{R}^m$ is said to be *quasi-monotone increasing* in $\mathbb{R} \times \mathbb{R}_+^m$ if for each pair of points (t, u) and (t, v) from $\mathbb{R} \times \mathbb{R}_+^m$ and for $i \in \{1, 2, \ldots, m\}$ the inequality $F_i(t, u) \geq F_i(t, v)$ holds whenever $u_i = v_i$ and $u_j \geq v_j$ for $j = 1, 2, \ldots, m, \ \ i \neq j$, i.e. for any fixed $t \in \mathbb{R}$ and any $i \in \{1, 2, \ldots, m\}$ the function $F_i(t, u)$ is non-decreasing with respect to $(u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_m)$.

In the case when the function $F : \mathbb{R} \times \mathbb{R}_+^m \to \mathbb{R}^m$ is continuous and quasi-monotone increasing, all solutions of problem (2.24) starting from the point $(\overline{t_0}, u_0) \in [t_0, \infty) \times \mathbb{R}_+^m$ lie between two singular solutions – the maximal and the minimal ones.

We need the following known result for our discussion, whose proof may be found in [178] and [183].

Theorem 2.18. *Assume that:*

1. *The conditions of Theorem 2.4 hold for* $k = 1, 2, \ldots$.
2. *The function F is quasi-monotone increasing, continuous on the sets* $(t_k, t_{k+1}] \times \mathbb{R}_+^m, \ \ k \in \mathbb{N} \cup \{0\}$ *and for* $k = 1, 2, \ldots$ *and* $v \in \mathbb{R}_+^m$ *the following finite limit exists*

$$\lim_{\substack{(t,u) \to (t,v) \\ t > t_k}} F(t, u).$$

3. *The maximal solution* $u^+ : J^+(t_0, u_0) \to \mathbb{R}^m_+$ *of the system* (2.24) *is defined for* $t \geq t_0$.
4. *The functions* $\psi_k : \mathbb{R}^m_+ \to \mathbb{R}^m_+,\ \ \psi_k(u) = u + J_k(u),\ \ k = 1, 2, \ldots$, *are non-decreasing on* $\mathbb{R}^m_+$.
5. *The function* $V : [t_0, \infty) \times \Omega \to \mathbb{R}^m_+$, $V = \mathrm{col}(V_1, V_2, \ldots, V_m)$, $V_j \in V_0$, $j = 1, 2, \ldots, m$ *is such that*

$$V(t_0^+, x_0) \leq u_0,$$

$$V(t^+, x + I_k(x)) \leq \psi_k(V(t, x)),\quad x \in \Omega,\ \ t = t_k,\ \ k = 1, 2, \ldots,$$

and the inequality

$$D^+_{(2.3)} V(t, x(t)) \leq F(t, V(t, x(t))),\ t \neq t_k,\ k = 1, 2, \ldots$$

holds for $t \in [t_0, \infty)$.

Then

$$V(t, x(t; t_0, x_0)) \leq u^+(t; t_0, u_0)\ \textit{for}\ t \in [t_0, \infty).$$

For the scalar case $m = 1$, we shall consider the comparison equation

$$\begin{cases} \dot{u}(t) = g(t, u(t)),\ t \neq t_k, \\ \Delta u(t_k) = B_k(u(t_k)),\ t_k > t_0, \end{cases} \tag{2.25}$$

where $g : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, $B_k : \mathbb{R}_+ \to \mathbb{R}_+$, $k = \pm 1, \pm 2, \ldots$.

Let $u_0 \in \mathbb{R}_+$. We denote again by $u^+(t) = u^+(t; t_0, u_0)$ the maximal solution of Eq. (2.25) which satisfies the initial condition

$$u^+(t_0; t_0, u_0) = u_0.$$

The next result follows directly from the similar results in [178].

Theorem 2.19. *Assume that:*

1. *The conditions of Theorem* 2.4 *hold.*
2. *The function* $g : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ *is continuous on each of the sets* $(t_{k-1}, t_k] \times \mathbb{R}_+$, $t_k > t_0$.
3. $B_k \in C[\mathbb{R}_+, \mathbb{R}_+]$ *and* $\tilde{\psi}_k(u) = u + B_k(u) \geq 0$, $k = \pm 1, \pm 2, \ldots$, *are non-decreasing with respect to u.*
4. *The maximal solution* $u^+ : J^+(t_0, u_0) \to \mathbb{R}_+$ *of* (2.25), $u^+(t_0^+; t_0, u_0) = u_0$, $t_0 \in \mathbb{R}$, *is defined on* $[t_0, \infty)$.
5. *The function* $V \in V_0$ *is such that* $V(t_0^+, x_0) \leq u_0$,

$$V(t^+, x + I_k(x)) \leq \tilde{\psi}_k(V(t, x)),\quad x \in \Omega,\ \ t = t_k,\ \ t_k > t_0,$$

$$D^+_{(2.3)} V(t, x(t)) \leq g(t, V(t, x(t))),\ \ t \neq t_k,\ \ t \in [t_0, \infty).$$

Then

$$V(t, x(t; t_0, x_0)) \le u^+(t; t_0, u_0),\ t \in [t_0, \infty).$$

In the case when $g(t, u) = 0$ for $(t, u) \in \mathbb{R} \times \mathbb{R}_+$ and $\tilde{\psi}_k(u) = u$ for $u \in \mathbb{R}_+,\ k = \pm 1, \pm 2, \ldots$, the following corollary holds.

Corollary 2.2. *Assume that:*

1. *The conditions of Theorem 2.4 hold.*
2. *The function $V \in V_0$ is such that*

$$V(t^+, x + I_k(x)) \le V(t, x), \quad x \in \Omega,\ t = t_k,\ t_k > t_0,$$
$$D^+_{(2.3)} V(t, x(t)) \le 0,\ t \ne t_k,\ t \in [t_0, \infty).$$

Then

$$V(t, x(t; t_0, x_0)) \le V(t_0^+, x_0),\ t \in [t_0, \infty).$$

For impulsive delay differential systems (2.8) we need the following results, whose proof is similar to the proof of the comparison lemma in [298].

Theorem 2.20. *Let the following conditions hold.*

1. *The conditions of Theorem 2.6 hold for $k = 1, 2, \ldots$.*
2. *Conditions 2–4 of Theorem 2.18 are met.*
3. *The function $V : [t_0, \infty) \times \Omega \to \mathbb{R}^m_+$, $V = \mathrm{col}(V_1, V_2, \ldots, V_m)$, $V_j \in V_0$, $j = 1, 2, \ldots, m$, is such that for $\varphi \in PC[[-r, 0], \Omega]$,*

$$V(t^+, \varphi(0) + I_k(\varphi)) \le \psi_k(V(t, \varphi(0))),\ t = t_k,\ k = 1, 2, \ldots,$$

and the inequality

$$D^+_{(2.8)} V(t, \varphi(0)) \le F(t, V(t, \varphi(0))),\ t \ne t_k,\ k = 1, 2, \ldots$$

holds whenever $V(t + s, \varphi(s)) \le V(t, \varphi(0))$ for $-r \le s \le 0$.

Then $\sup_{-r \le s \le 0} V(t_0 + s, \varphi_0(s)) \le u_0$ implies

$$V(t, x(t; t_0, \varphi_0)) \le u^+(t; t_0, u_0),\ t \in [t_0, \infty).$$

For the scalar case $m = 1$ when $F \equiv g$, $g \in PC[\mathbb{R} \times \mathbb{R}_+]$, a theorem similar to Theorem 2.19 follows immediately from Theorem 2.20.

Theorem 2.21. *Let the following conditions hold.*

1. *The conditions of Theorem 2.6 hold.*
2. *Conditions 2–4 of Theorem 2.19 are met.*

3. *The function* $V \in V_0$ *is such that for* $\varphi \in PC[[-r,0],\Omega]$,

$$V(t^+, \varphi(0) + I_k(\varphi)) \leq \tilde{\psi}_k(V(t, \varphi(0))),\ \ t = t_k,\ t_k > t_0,$$

and the inequality

$$D^+_{(2.8)}V(t, \varphi(0)) \leq g(t, V(t, \varphi(0))),\ t \neq t_k,\ t \in [t_0, \infty)$$

holds whenever $V(t+s, \varphi(s)) \leq V(t, \varphi(0))$ *for* $-r \leq s \leq 0$.

Then $\sup\limits_{-r \leq s \leq 0} V(t_0 + s, \varphi_0(s)) \leq u_0$ *implies*

$$V(t, x(t; t_0, \varphi_0)) \leq u^+(t; t_0, u_0),\ t \in [t_0, \infty).$$

We can also formulate the impulsive comparison results in terms of Lyapunov functions $g(t,u) = 0$ for $(t,u) \in \mathbb{R} \times \mathbb{R}_+$ and $\tilde{\psi}_k(u) = u$ for $u \in \mathbb{R}_+$, $k = \pm 1, \pm 2, \ldots$:

Corollary 2.3. *Assume that the function* $V \in V_0$ *is such that for* $\varphi \in PC[[-r,0],\Omega]$,

$$V(t^+, \varphi(0) + I_k(\varphi)) \leq V(t, \varphi(0)),\ \ t = t_k,\ t_k > t_0,$$

and the inequality

$$D^+_{(2.8)}V(t, \varphi(0)) \leq 0,\ t \neq t_k,\ t \in [t_0, \infty)$$

holds whenever $V(t+s, \varphi(s)) \leq V(t, \varphi(0))$ *for* $-r \leq s \leq 0$.

Then

$$V(t, x(t; t_0, \varphi_0)) \leq \sup_{-r \leq s \leq 0} V(t_0^+, \varphi_0(s)),\ t \in [t_0, \infty).$$

Remark 2.3. Analogous comparison results can be proved for impulsive systems [35, 179] in which minimal solutions are used.

Remark 2.4. Similar results can be proved in terms of functions from the classes V_2 and W_0 [284, 289, 290].

Next we shall consider a Bihari and Gronwall type integral inequality in a special case with impulses.

Theorem 2.22 ([35]). *Let the following conditions hold:*

1. *Condition* H2.1 *is met.*
2. *The functions* $m : \mathbb{R} \to \mathbb{R}_+$, $p : \mathbb{R} \to \mathbb{R}_+$ *are continuous on each of the sets* $(t_{k-1}, t_k]$, $t_k > t_0$.
3. $C \geq 0$, $\beta_k \geq 0$ *and*

$$m(t) \leq C + \int_{t_0}^{t} p(s)m(s)ds + \sum_{t_0 < t_k < t} \beta_k m(t_k).$$

Then

$$m(t) \le C \prod_{t_0<t_k<t} (1+\beta_k) e^{\int_{t_0}^{t} p(s)ds}.$$

2.6 Coincidence Degree Lemmas

In Chap. 4, we shall investigate the existence of positive periodic solutions of different classes of Lotka–Volterra models. Our main results are based on coincidence degree theory [118].

Let X, Z be normed vector spaces, $L : \operatorname{dom} L \subset X \to Z$ be a linear mapping, and $N : X \to Z$ be a continuous mapping.

L is said to be a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z.

If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I-Q)$. It follows that $L|\operatorname{dom} L \cap \operatorname{Ker} P : (I-P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P.

The mapping N is said to be L-compact on $\overline{\Omega_1}$ if Ω_1 is an open bounded subset of X, $QN(\overline{\Omega_1})$ is bounded and $K_P(I-\Omega)N : \overline{\Omega_1} \to X$ is compact.

Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $S : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 2.8 (Continuation Theorem [118]). *Let L be a Fredholm mapping of index zero and N be L-compact on* $\overline{\Omega_1}$. *Suppose*

(a) *for each* $\lambda \in (0,1)$*, every solution x of* $Lx = \lambda Nx$ *satisfies* $x \notin \partial\Omega_1$*;*
(b) *for each* $x \in \operatorname{Ker} L \cap \partial\Omega_1$, $\deg_B\{SQN, \ \Omega_1 \cap \operatorname{Ker} L, \ 0\} \ne 0$*, when* $QNx \ne 0$*, where* $\deg_B$ *denotes the Brouwer degree.*

Then the equation $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega_1}$.

We note that $PC[J,\mathbb{R}^n]$ is a Banach space with the norm $||\sigma||_{PC} = \sup_{t\in J} ||\sigma(t)||$ and $PC^1[J,\mathbb{R}^n]$ is also a Banach space with the norm $||\sigma||_{PC^1} = \max\{||\sigma||_{PC}, ||\dot{\sigma}||_{PC}\}$.

Lemma 2.9 ([190]). $H \subset PC[J,\mathbb{R}^n]$ *is relatively compact if and only if the functions in H are uniformly bounded on J and continuous on* $(t_{k-1},t_k]$, $k = 1,2,\ldots,K$*, for any fixed* $K > 1$.

Notes and Comments

The basic fundamental results listed in Sect. 2.1 for impulsive ordinary differential equations are due to Bainov and Simeonov [36] and Stamov [284]. Theorem 2.6 is adapted from Stamova [298]. Analogous results for different classes of impulsive differential systems were obtained in [26, 34, 49, 178, 180–184, 235, 236, 238, 256].

The results on almost periodic sequences and almost periodic functions included in Sect. 2.2 were taken from Samoilenko and Perestuyk [256], and from Stamov [284].

The stability and boundedness definitions in Sect. 2.3 were introduced by Stamov [284] and Stamova [298].

The most efficient tool for the study of the qualitative properties of a given nonlinear system is provided by Lyapunov's second method [214]. Its application to systems with delay has been developed in two directions. The first direction uses Lyapunov functions and the Razumikhin technique [249]. The second method uses Lyapunov–Krasovskii functionals [171]. The method of piecewise continuous Lyapunov functions for impulsive systems was introduced by Bainov and Simeonov [34].

For Theorem 2.18, see [34, 178, 183]. Theorem 2.19 and Corollary 2.2 are results in [178]. Theorems 2.20 and 2.21 are adapted from Stamova [298]. The corresponding comparison theorems for the continuous case were proved in [180, 181]. The applied technique is used by many authors [6, 7, 27, 178, 179, 182–184]. Theorem 2.22 is due to Bainov and Simeonov [35].

Lemma 2.8 is taken from [118] while Lemma 2.9 is a result in [190].

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