CHAPTER 2

Algebra

2001:5. Let n be a positive integer and x a real number not equal to a nonnegative integer. Prove that

$$\frac{n}{x} + \frac{n(n-1)}{x(x-1)} + \frac{n(n-1)(n-2)}{x(x-1)(x-2)} + \cdots + \frac{n(n-1)(n-2)\cdots 1}{x(x-1)(x-2)\cdots(x-n+1)}$$
$$= \frac{n}{x-n+1}.$$

Solution 1. The result holds for $n = 1$ and $x \neq 0$. Suppose, as an induction hypothesis, the result holds for $n = k$ and x equal to any real number that is not a nonnegative integer; note that in this case, the left side has k terms. When $n = k + 1$,

$$\begin{aligned}\frac{k+1}{x} &+ \frac{(k+1)k}{x(x-1)} + \frac{(k+1)k(k-1)}{x(x-1)(x-2)} + \cdots + \frac{(k+1)k(k-1)\cdots 1}{x(x-1)(x-2)\cdots(x-k)} \\ &= \frac{k+1}{x} + \frac{k+1}{x}\left[\frac{k}{x-1} + \frac{k(k-1)}{(x-1)(x-2)} + \cdots + \frac{k(k-1)\cdots 1}{(x-1)\cdots(x-k)}\right] \\ &= \frac{k+1}{x} + \frac{k+1}{x}\left[\frac{k}{x-1-k+1}\right] = \frac{k+1}{x}\left[1 + \frac{k}{x-k}\right] \\ &= \frac{k+1}{x}\left[\frac{x}{x-k}\right] = \frac{k+1}{x-(k+1)+1}\end{aligned}$$

as desired. The result follows by induction.

Solution 2. [P. Gyrya] For real a and positive integer m, let $\binom{a}{m}$ denote $a(a-1)\cdots(a-m+1)/m!$ and let $\binom{a}{0}$ be 1. It is straightforward to establish that, for each positive integer k:

$$\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}$$

E.J. Barbeau, *University of Toronto Mathematics Competition (2001–2015)*, Problem Books in Mathematics,
DOI 10.1007/978-3-319-28106-3_2

and

$$\binom{x}{k} = \sum_{i=0}^{k} \binom{x-i-1}{k-i}.$$

The left side of the required equality is equal to

$$\begin{aligned}\frac{n!}{x\cdots(x-n+1)}&\left[\binom{x-1}{n-1}+\binom{x-2}{n-2}+\cdots+\binom{x-n}{n-n}\right]\\ &= \frac{n!}{x\cdots(x-n+1)}\binom{x}{n-1} = \frac{n}{x-n+1}\end{aligned}$$

as desired.

2001:8 A regular heptagon (polygon with seven equal sides and seven equal angles) has diagonals of two different lengths. Let a be the length of a side, b be the length of a shorter diagonal and c be the length of a longer diagonal of a regular heptagon (so that $a < b < c$). Prove ONE of the following relationships:

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}=6$$

or

$$\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}=5.$$

Solution 1. Let A, B, C, D, E be consecutive vertices of the regular heptagon. Let AB, AC and AD have respective lengths a, b, c, and let $\angle BAC = \theta$. Then $\theta = \pi/7$, the length of BC, of CD and of DE is a, the length of AE is c, $\angle CAD = \angle DAE = \theta$, since the angles are subtended by equal chords of the circumcircle of the heptagon, $\angle ADC = 2\theta$, $\angle ADE = \angle AED = 3\theta$ and $\angle ACD = 4\theta$. Triangles ABC and ACD can be glued together along BC and DC (with C on C) to form a triangle similar to ΔABC, whence

$$\frac{a+c}{b} = \frac{b}{a}. \tag{2.1}$$

Triangles ACD and ADE can be glued together along CD and ED (with D on D) to form a triangle similar to triangle ABC, whence

$$\frac{b+c}{c} = \frac{b}{a}. \tag{2.2}$$

Equation (2.2) can be rewritten as $\frac{1}{b} = \frac{1}{a} - \frac{1}{c}$. whence

$$b = \frac{ac}{c-a}.$$

Substituting this into (2.1) yields

$$\frac{(c+a)(c-a)}{ac} = \frac{c}{c-a}$$

which simplifies to

$$a^3 - a^2c - 2ac^2 + c^3 = 0. \tag{2.3}$$

Note also from (2.1) that $b^2 = a^2 + ac$.

$$\begin{aligned}
\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} - 6 &= \frac{a^4c^2 + b^4a^2 + c^4b^2 - 6a^2b^2c^2}{a^2b^2c^2} \\
&= \frac{a^4c^2 + (a^4 + 2a^3c + a^2c^2)a^2 + c^4(a^2 + ac) - 6a^2c^2(a^2 + ac)}{a^2b^2c^2} \\
&= \frac{a^6 + 2a^5c - 4a^4c^2 - 6a^3c^3 + a^2c^4 + ac^5}{a^2b^2c^2} \\
&= \frac{a(a^2 + 3ac + c^2)(a^3 - a^2c - 2ac^2 + c^3)}{a^2b^2c^2} = 0.
\end{aligned}$$

$$\begin{aligned}
&\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} - 5 \\
&= \frac{(a^4 + 2a^3c + a^2c^2)c^2 + a^2c^4 + a^4(a^2 + ac) - 5a^2c^2(a^2 + ac)}{a^2b^2c^2} \\
&= \frac{a^6 + a^5c - 4a^4c^2 - 3a^3c^3 + 2a^2c^4}{a^2b^2c^2} \\
&= \frac{a^2(a + 2c)(a^3 - a^2c - 2ac^2 + c^3)}{a^2b^2c^2} = 0.
\end{aligned}$$

Solution 2 (of the first result). [J. Chui] Let the heptagon be $ABCDEFG$ and $\theta = \pi/7$. Using the Law of Cosines in the indicated triangles ACD and ABC, we obtain the following:

$$\cos 2\theta = \frac{a^2 + c^2 - b^2}{2ac} = \frac{1}{2}\left(\frac{a}{c} + \frac{c}{a} - \frac{b^2}{ac}\right)$$

$$\cos 5\theta = \frac{2a^2 - b^2}{2a^2} = 1 - \frac{1}{2}\left(\frac{b}{a}\right)^2$$

from which, since $\cos 2\theta = -\cos 5\theta$,

$$-1 + \frac{1}{2}\left(\frac{b}{a}\right)^2 = \frac{1}{2}\left(\frac{a}{c} + \frac{c}{a} - \frac{b^2}{ac}\right)$$

or

$$\frac{b^2}{a^2} = 2 + \frac{a}{c} + \frac{c}{a} - \frac{b^2}{ac}. \tag{2.4}$$

Examining triangles ABC and ADE, we find that $\cos\theta = b/2a$ and $\cos\theta = (2c^2 - a^2)/(2c^2) = 1 - (a^2/2c^2)$, so that

$$\frac{a^2}{c^2} = 2 - \frac{b}{a}. \tag{2.5}$$

Examining triangles ADE and ACF, we find that $\cos 3\theta = a/2c$ and $\cos 3\theta = (2b^2 - c^2)/(2b^2)$, so that

$$\frac{c^2}{b^2} = 2 - \frac{a}{c}. \tag{2.6}$$

Adding Eqs. (2.4), (2.5), (2.6) yields

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 6 + \frac{c^2 - bc - b^2}{ac}.$$

By Ptolemy's Theorem, the sum of the products of pairs of opposite sides of a concylic quadrilaterial is equal to the product of the diagonals. Applying this to the quadrilaterals $ABDE$ and $ABCD$, respectively, yields $c^2 = a^2 + bc$ and $b^2 = ac + a^2$, whence $c^2 - bc - b^2 = a^2 + bc - bc - ac - a^2 = -ac$ and we find that

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 6 - 1 = 5.$$

Solution 3. There is no loss of generality in assuming that the vertices of the heptagon are placed at the seventh roots of unity on the unit circle in the complex plane. Let $\zeta = \cos(2\pi/7) + i\sin(2\pi/7)$ be the fundamental seventh root of unity. Then $\zeta^7 = 1$, $1 + \zeta + \zeta^2 + \cdots + \zeta^6 = 0$ and (ζ, ζ^6), (ζ^2, ζ^5), (ζ^3, ζ^4) are pairs of complex conjugates. We have that

$$a = |\zeta - 1| = |\zeta^6 - 1|$$

$$b = |\zeta^2 - 1| = |\zeta^9 - 1|$$

$$c = |\zeta^3 - 1| = |\zeta^4 - 1|.$$

It follows from this that

$$\frac{b}{a} = |\zeta + 1| \qquad \frac{c}{b} = |\zeta^2 + 1| \qquad \frac{a}{c} = |\zeta^3 + 1|,$$

whence

$$\begin{aligned}
\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} &= (\zeta + 1)(\zeta^6 + 1) + (\zeta^2 + 1)(\zeta^5 + 1) + (\zeta^3 + 1)(\zeta^4 + 1) \\
&= 2 + \zeta + \zeta^6 + 2 + \zeta^2 + \zeta^5 + 2 + \zeta^3 + \zeta^4 \\
&= 6 + (\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6) = 6 - 1 = 5\ .
\end{aligned}$$

Also

$$\frac{a}{b} = |\zeta^4 + \zeta^2 + 1| \qquad \frac{b}{c} = |\zeta^6 + \zeta^3 + 1| \qquad \frac{c}{a} = |\zeta^2 + \zeta + 1|,$$

whence

$$\begin{aligned}\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} &= (\zeta^4+\zeta^2+1)(\zeta^3+\zeta^5+1)+(\zeta^6+\zeta^3+1)(\zeta+\zeta^4+1)\\ &\quad+(\zeta^2+\zeta+1)(\zeta^5+\zeta^6+1)\\ &=(3+2\zeta^2+\zeta^3+\zeta^4+2\zeta^5)+(3+\zeta+2\zeta^3+2\zeta^4+\zeta^6)\\ &\quad+(3+2\zeta+\zeta^2+\zeta^5+2\zeta^6)\\ &=9+3(\zeta+\zeta^2+\zeta^3+\zeta^4+\zeta^5+\zeta^6)=9-3=6.\end{aligned}$$

Solution 4. Suppose that the circumradius of the heptagon is 1. By considering isosceles triangles with base equal to the sides or diagonals of the heptagon and apex at the centre of the circumcircle, we see that

$$\begin{aligned}a &= 2\sin\theta = 2\sin 6\theta = -2\sin 8\theta\\ b &= 2\sin 2\theta = -2\sin 9\theta\\ c &= 2\sin 3\theta = 2\sin 4\theta\end{aligned}$$

where $\theta=\pi/7$ is half the angle subtended at the circumcentre by each side of the heptagon. Observe that

$$\cos 2\theta = \frac{1}{2}(\zeta+\zeta^6) \qquad \cos 4\theta = \frac{1}{2}(\zeta^2+\zeta^5) \qquad \cos 6\theta = \frac{1}{2}(\zeta^3+\zeta^4)$$

where ζ is the fundamental primitive root of unity. We have that

$$\frac{b}{a} = 2\cos\theta = 2\cos 6\theta \qquad \frac{c}{b} = 2\cos 2\theta \qquad \frac{a}{c} = -2\cos 4\theta$$

whence

$$\begin{aligned}\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2} &= 4\cos^2 6\theta+4\cos^2 2\theta+4\cos^2 4\theta\\ &=(\zeta^3+\zeta^4)^2+(\zeta+\zeta^6)^2+(\zeta^2+\zeta^5)^2\\ &=\zeta^6+2+\zeta+\zeta^2+2+\zeta^5+\zeta^4+2+\zeta^3=6-1=5.\end{aligned}$$

Also

$$\begin{aligned}\frac{a}{b} &= \frac{\sin 6\theta}{\sin 2\theta} = 4\cos^2 2\theta-1=(\zeta+\zeta^6)^2-1=1+\zeta^2+\zeta^5\\ -\frac{b}{c} &= \frac{\sin 9\theta}{\sin 3\theta} = 4\cos^2 3\theta-1=4\cos^2 4\theta-1\\ &=(\zeta^2+\zeta^5)^2-1=1+\zeta^4+\zeta^3\\ \frac{c}{a} &= \frac{\sin 3\theta}{\sin\theta} = 4\cos^2 6\theta-1=(\zeta^3+\zeta^4)^2-1=1+\zeta^6+\zeta,\end{aligned}$$

whence

$$\begin{aligned}\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} &= (3+2\zeta^2+\zeta^3+\zeta^4+2\zeta^5)+(3+\zeta+2\zeta^3+2\zeta^4+\zeta^6)\\ &\quad +(3+2\zeta+\zeta^2+\zeta^5+2\zeta^6)\\ &= 9+3(\zeta+\zeta^2+\zeta^3+\zeta^4+\zeta^5+\zeta^6)=9-3=6.\end{aligned}$$

Comment. The regular heptagon is a rich source of interesting relationships. See the paper *Golden fields; a case for the heptagon* by Peter Steinbach in *Mathematics Magazine* 70:1 (February, 1997), 22–31.

2002:2. Angus likes to go to the movies. On Monday, standing in line, he noted that the fraction x of the line was in front of him, while $1/n$ of the line was behind him. On Tuesday, the same fraction x of the line was in front of him, while $1/(n+1)$ of the line was behind him. On Wednesday, the same fraction x of the line was in front of him, while $1/(n+2)$ of the line was behind him. Determine a value of n for which this is possible.

Answer. When $x = 5/6$, he could have $1/7$ of a line of 42 behind him, $1/8$ of a line of 24 behind him and $1/9$ of a line of 18 behind him. When $x = 11/12$, he could have $1/14$ of a line of 84 behind him, $1/15$ of a line of 60 behind him and $1/16$ of a line of 48 behind him. When $x = 13/15$, he could have $1/8$ of a line of 120 behind him, $1/9$ of a line of 45 behind him and $1/10$ of a line of 30 behind him.

Solution. The strategy in this solution is to try to narrow down the search by considering a special case. Suppose that $x = (u-1)/u$ for some positive integer exceeding 1. Let $1/(u+p)$ be the fraction of the line behind Angus. Then Angus himself represents this fraction of the line:

$$1-\left(\frac{u-1}{u}+\frac{1}{u+p}\right)=\frac{p}{u(u+p)},$$

so that there would be $u(u+p)/p$ people in line. To make this an integer, we can arrange that u is a multiple of p. To get an integer for $p = 1, 2, 3$, take u to be any multiple of 6. Thus, we can arrange that x is any of 5/6, 11/12, 17/18, 23/24, and so on.

Comment 1. The solution indicates how we can select x for which the amount of the line behind Angus is represented by any number of consecutive integer reciprocals. For example, in the case of $x = 11/12$, he could also have $1/13$ of a line of 156 behind him. Another strategy might be to look at $x = (u-2)/u$, i.e. successively at $x = 3/5, 5/7, 7/9, \cdots$. In this case, we assume that $1/(u-p)$ of the line is behind him, and need to ensure that $u-2p$ is a positive divisor of $u(u-p)$ for three consecutive values of $u-p$. If u is odd, we can achieve this with u any odd multiple of 15 and with $p=\frac{1}{2}(u-1)$, $\frac{1}{2}(u-3)$, $\frac{1}{2}(u-5)$.

Comment 2. With the same fraction in front on 2 days, suppose that $1/n$ of a line of u people is behind Angus on the first day, and $1/(n+1)$ of a line of v people is behind him on the second day. Then

$$\frac{1}{u} + \frac{1}{n} = \frac{1}{v} + \frac{1}{n+1}$$

so that $uv = n(n+1)(u-v)$. This yields both $(n^2+n-v)u = (n^2+n)v$ and $(n^2+n+u)v = (n^2+n)u$, leading to

$$u - v = \frac{u^2}{n^2+n+u} = \frac{v^2}{n^2+n-v}.$$

Two immediate possibilities are $(n,u,v) = (n, n+1, n)$ and $(n,u,v) = (n, n(n+1), \frac{1}{2}n(n+1))$. To get some more, taking $u - v = k$, we get the quadratic equation

$$u^2 - ku - k(n^2+n) = 0$$

with discriminant

$$\Delta = k^2 + 4(n^2+n)k = [k + 2(n^2+n)]^2 - 4(n^2+n)^2,$$

a pythagorean relationship when Δ is square and the equation has integer solutions. Select α, β, γ so that $\gamma\alpha\beta = n^2+n$ and let $k = \gamma(\alpha^2+\beta^2-2\alpha\beta) = \gamma(\alpha-\beta)^2$; this will make the discriminant Δ equal to a square.

Taking $n = 3$, for example, yields the possibilities $(u,v) = (132, 11)$, (60, 10), (36, 9), (24, 8), (12, 6), (6, 4),(4, 3). In general, we find that $(n,u,v) = (n, \gamma\alpha(\alpha-\beta), \gamma\beta(\alpha-\beta))$ when $n^2+n = \gamma\alpha\beta$ with $\alpha > \beta$. It turns out that $k = u - v = \gamma(\alpha-\beta)^2$.

2003:7. Suppose that the polynomial $f(x)$ of degree $n \geq 1$ has all real roots and that $\lambda > 0$. Prove that the set $\{x \in \mathbb{R} : |f(x)| \leq \lambda|f'(x)|\}$ is a finite union of closed intervals whose total length is equal to $2n\lambda$.

The solution to this problem appears in Chap. 6.

2004:1. Prove that, for any nonzero complex numbers z and w,

$$(|z| + |w|)\left|\frac{z}{|z|} + \frac{w}{|w|}\right| \leq 2|z+w| \ .$$

Solution 1.

$$\begin{aligned}
(|z| + |w|)&\left|\frac{z}{|z|} + \frac{w}{|w|}\right| \\
&= \left|z + w + \frac{|z|w}{|w|} + \frac{|w|z}{|z|}\right| \\
&\leq |z+w| + \frac{1}{|z||w|}|\bar{z}zw + \bar{w}zw| \\
&= |z+w| + \frac{|zw|}{|z||w|}|\bar{z} + \bar{w}| = 2|z+w|.
\end{aligned}$$

Solution 2. Let $z = ae^{i\alpha}$ and $w = be^{i\beta}$, with a and b real and positive. Then the left side is equal to

$$\begin{aligned} |(a+b)(e^{i\alpha}+e^{i\beta})| &= |ae^{i\alpha}+ae^{i\beta}+be^{i\alpha}+be^{i\beta}| \\ &\leq |ae^{i\alpha}+be^{i\beta}|+|ae^{i\beta}+be^{i\alpha}|. \end{aligned}$$

Observe that

$$\begin{aligned} |z+w|^2 &= |(ae^{i\alpha}+be^{i\beta})(ae^{-i\alpha}+be^{-i\beta})| \\ &= a^2+b^2+ab[e^{i(\alpha-\beta)}+e^{i(\beta-\alpha)}] \\ &= |(ae^{i\beta}+be^{i\alpha})(ae^{-i\beta}+be^{-i\alpha})| \end{aligned}$$

from which we find that the left side does not exceed

$$|ae^{i\alpha}+be^{i\beta}|+|ae^{i\beta}+be^{i\alpha}| = 2|ae^{i\alpha}+be^{i\beta}| = 2|z+w|.$$

Solution 3. Let $z = ae^{i\alpha}$ and $w = be^{i\beta}$, where a and b are positive reals. Then the inequality is equivalent to

$$\left|\frac{1}{2}(e^{i\alpha}+e^{i\beta})\right| \leq |\lambda e^{i\alpha}+(1-\lambda)e^{i\beta}|$$

where $\lambda = a/(a+b)$. But this simply says that the midpoint of the segment joining $e^{i\alpha}$ and $e^{i\beta}$ on the unit circle in the Argand diagram representing complex numbers as points in the plane is at least as close to the origin as another point on the segment.

Solution 4. [G. Goldstein] The inequality is equivalent to

$$(|t|+1)\left|\frac{t}{|t|}+1\right| \leq 2|t+1|$$

where $t = z/w$.

Let $t = r(\cos\theta + i\sin\theta)$. Then the inequality becomes

$$\begin{aligned} &(r+1)\sqrt{(\cos\theta+1)^2+\sin^2\theta} \leq 2\sqrt{(r\cos\theta+1)^2+r^2\sin^2\theta} \\ &\quad = 2\sqrt{r^2+2r\cos\theta+1}. \end{aligned}$$

Now,

$$\begin{aligned} &4(r^2+2r\cos\theta+1)-(r+1)^2(2+2\cos\theta) \\ &\qquad = 2r^2(1-\cos\theta)+4r(\cos\theta-1)+2(1-\cos\theta) \\ &\qquad = 2(r-1)^2(1-\cos\theta) \geq 0, \end{aligned}$$

from which the inequality follows.

Solution 5. [R. Mong] Consider complex numbers as vectors in the plane. $q = (|z|/|w|)w$ is a vector of magnitude z in the direction w and $p = (|w|/|z|)z$ is a vector of magnitude w in the direction z. A reflection about the angle bisector of vectors z and w interchanges p and w, q and z. Hence $|p+q| = |w+z|$. Therefore

$$\begin{aligned}(|z|+|w|)\left|\frac{z}{|z|}+\frac{w}{|w|}\right| \\ &= |z+q+p+w| \leq |z+w|+|p+q| \\ &= 2|z+w|.\end{aligned}$$

2004:7. Let a be a parameter. Define the sequence $\{f_n(x) : n = 0, 1, 2, \cdots\}$ of polynomials by

$$f_0(x) \equiv 1$$

and

$$f_{n+1}(x) = xf_n(x) + f_n(ax)$$

for $n \geq 0$.

(a) Prove that, for all n, x,

$$f_n(x) = x^n f_n(1/x).$$

(b) Determine a formula for the coefficient of x^k $(0 \leq k \leq n)$ in $f_n(x)$.

Solution. The polynomial $f_n(x)$ has degree n for each n, and we will write

$$f_n(x) = \sum_{k=0}^{n} b(n,k)x^k.$$

Then

$$x^n f_n(1/x) = \sum_{k=0}^{n} b(n,k)x^{n-k} = \sum_{k=0}^{n} b(n,n-k)x^k.$$

Thus, (a) is equivalent to $b(n,k) = b(n,n-k)$ for $0 \leq k \leq n$.

When $a = 1$, it can be established by induction that $f_n(x) = (x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^n$. Also, when $a = 0$, $f_n(x) = x^n + x^{n-1} + \cdots + x + 1 = (x^{n+1} - 1)(x-1)^{-1}$. Thus, (a) holds in these cases and $b(n,k)$ is respectively equal to $\binom{n}{k}$ and 1.

Suppose, henceforth, that $a \neq 1$. For $n \geq 0$,

$$\begin{aligned}f_{n+1}(x) &= \sum_{k=0}^{n} b(n,k)x^{k+1} + \sum_{k=0}^{n} a^k b(n,k)x^k \\ &= \sum_{k=1}^{n} b(n,k-1)x^k + b(n,n)x^{n+1} + b(n,0) + \sum_{k=1}^{n} a^k b(n,k)x^k \\ &= b(n,0) + \sum_{k=1}^{n}[b(n,k-1) + a^k b(n,k)]x^k + b(n,n)x^{n+1},\end{aligned}$$

whence $b(n+1,0) = b(n,0) = b(1,0)$ and $b(n+1,n+1) = b(n,n) = b(1,1)$ for all $n \geq 1$. Since $f_1(x) = x+1$, $b(n,0) = b(n,n) = 1$ for each n. Also

$$b(n+1,k) = b(n,k-1) + a^k b(n,k) \tag{2.7}$$

for $1 \leq k \leq n$.

We conjecture what the coefficients $b(n,k)$ are from an examination of the first few terms of the sequence:

$$f_0(x) = 1;\ f_1(x) = 1+x;\ f_2(x) = 1+(a+1)x+x^2;$$

$$f_3(x) = 1+(a^2+a+1)x+(a^2+a+1)x^2+x^3;$$

$$f_4(x) = 1+(a^3+a^2+a+1)x+(a^4+a^3+2a^2+a+1)x^2+(a^3+a^2+a+1)x^3+x^4;$$

$$f_5(x) = (1+x^5)+(a^4+a^3+a^2+a+1)(x+x^4)+(a^6+a^5+2a^4+2a^3+2a^2+a+1) \times(x^2+x^3).$$

We make the empirical observation that

$$b(n+1,k) = a^{n+1-k}b(n,k-1) + b(n,k) \tag{2.8}$$

which, with (2.7), yields

$$(a^{n+1-k}-1)b(n,k-1) = (a^k-1)b(n,k)$$

so that

$$b(n+1,k) = \left[\frac{a^k-1}{a^{n+1-k}-1} + a^k\right] b(n,k) = \left[\frac{a^{n+1}-1}{a^{n+1-k}-1}\right] b(n,k)$$

for $n \geq k$. This leads to the conjecture that, when $n > k$,

$$b(n,k) = \left(\frac{(a^n-1)(a^{n-1}-1)\cdots(a^{k+1}-1)}{(a^{n-k}-1)(a^{n-k-1}-1)\cdots(a-1)}\right) b(k,k) \tag{2.9}$$

where $b(k,k) = 1$.

We establish this conjecture. Let $c(n,k)$ be the right side of (2.9) for $1 \leq k \leq n-1$ and $c(n,n) = 1$. Then $c(n,0) = b(n,0) = c(n,n) = b(n,n) = 1$ for each n. In particular, $c(n,k) = b(n,k)$ when $n = 1$.

We show that

$$c(n+1,k) = c(n,k-1) + a^k c(n,k)$$

for $1 \leq k \leq n$, which will, through an induction argument, imply that $b(n,k) = c(n,k)$ for $0 \leq k \leq n$. The right side is equal to

$$\left(\frac{a^n-1}{a^{n-k}-1}\right)\cdots\left(\frac{a^{k+1}-1}{a-1}\right)\left[\frac{a^k-1}{a^{n-k+1}-1} + a^k\right]$$
$$= \frac{(a^{n+1}-1)(a^n-1)\cdots(a^{k+1}-1)}{(a^{n+1-k}-1)(a^{n-k}-1)\cdots(a-1)} = c(n+1,k)$$

as desired. Thus, we now have a formula for $b(n,k)$ as required in (b).

Finally, (a) can be established in a straightforward way, either from the formula (2.9) or using the pair of recursions (2.7) and (2.8).

2005:5. Let $f(x)$ be a polynomial with real coefficients, evenly many of which are nonzero, which is *palindromic*. This means that the coefficients read the same in either direction, i.e. $a_k = a_{n-k}$ if $f(x) = \sum_{k=0}^{n} a_k x^k$, or, alternatively, $f(x) = x^n f(1/x)$, where n is the degree of the polynomial. Prove that $f(x)$ has at least one root of absolute value 1.

The solution of this problem appears in Chap. 6.

2005:7. Let $f(x)$ be a nonconstant polynomial that takes only integer values when x is an integer, and let P be the set of all primes that divide $f(m)$ for at least one integer m. Prove that P is an infinite set.

The solution to this problem appears in Chap. 11.

2006:3. Let $p(x)$ be a polynomial of positive degree n with n distinct real roots $a_1 < a_2 < \cdots < a_n$. Let b be a real number for which $2b < a_1 + a_2$. Prove that

$$2^{n-1}|p(b)| \geq |p'(a_1)(b - a_1)|.$$

Solution. Wolog, let $p(x)$ have leading coefficient 1. Observe that, for $i > 1$,

$$a_i - a_1 = a_i - b + b - a_1 < a_i - b + a_2 - b < 2(a_i - b)$$

and that $p(x) = \prod(x - a_i)$, from which $p'(a_1) = \prod_{i \geq 2}(a_1 - a_i)$. Then

$$\begin{aligned} |p(b)| &= |b - a_1| \prod_{i \geq 2} |b - a_i| \\ &\geq |b - a_1| \prod_{i \geq 2} \frac{1}{2}(a_i - a_1) = |b - a_1||p'(a_i)|2^{-(n-1)}. \end{aligned}$$

2006:9. A high school student asked to solve the surd equation

$$\sqrt{3x - 2} - \sqrt{2x - 3} = 1$$

gave the following answer: *Squaring both sides leads to*

$$3x - 2 - 2x - 3 = 1$$

so $x = 6$. The answer is, in fact, correct.

Show that there are infinitely many real quadruples (a, b, c, d) for which this method leads to a correct solution of the surd equation

$$\sqrt{ax - b} - \sqrt{cx - d} = 1.$$

Solution 1. Solving the general equation properly leads to

$$\sqrt{ax - b} - \sqrt{cx - d} = 1 \Longrightarrow ax - b = 1 + cx - d + 2\sqrt{cx - d}$$
$$\Longrightarrow (a - c)x = (b + 1 - d) + 2\sqrt{cx - d}.$$

To make the manipulation simpler, specialize to $a = c + 1$ and $d = b + 1$. Then the equation becomes

$$x^2 = 4(cx - d) \Longrightarrow 0 = x^2 - 4cx + 4d.$$

Using the student's "method" to solve the same equation gives $ax - b - cx - d = 1$ which yields $x = (1 + b + d)/(a - c) = 2d$. So, for the "method" to work, we need

$$0 = 4d^2 - 8cd + 4d = 4d(d - 2c + 1)$$

which can be achieved by making $2c = d + 1$. So we can take

$$(a, b, c, d) = (t + 1, 2(t - 1), t, 2t - 1)$$

for some real t. The original problem corresponds to $t = 2$.

The equation

$$\sqrt{(t + 1)x - 2(t - 1)} - \sqrt{tx - (2t - 1)} = 1$$

is satisfied by $x = 2$ and $x = 4t - 2$. The first solution works for all values of t, while the second is valid if and only if $t \geq \frac{1}{2}$. The equation $(t + 1)x - 2(t - 1) - tx - (2t - 1) = 1$ is equivalent to $x = 4t - 2$.

Solution 2. [G. Goldstein] *Analysis.* We want to solve simultaneously the equations

$$\sqrt{ax - b} - \sqrt{cx - d} = 1 \tag{2.10}$$

and

$$ax - b - cx - d = 1. \tag{2.11}$$

From (2.10), we find that

$$ax - b = 1 + (cx - d) + 2\sqrt{cx - d}. \tag{2.12}$$

From (2.11) and (2.12), we obtain that $d = \sqrt{cx - d}$, so that $x = (d^2 + d)/c$. From (2.11), we have that $x = (1 + b + d)/(a - c)$.

Select a, c, d so that $d > 0$ and $ac(a - c) \neq 0$, and choose b to satisfy

$$\frac{d^2 + d}{c} = \frac{1 + b + d}{a - c}.$$

Let

$$x = \frac{d^2 + d}{c} = \frac{1 + b + d}{a - c} = \frac{(d^2 + d) + (1 + b + d)}{c + (a - c)} = \frac{(d + 1)^2 + b}{a}.$$

Then

$$\sqrt{ax - b} - \sqrt{cx - d} = \sqrt{(d + 1)^2} - \sqrt{d^2} = (d + 1) - d = 1$$

and

$$ax - b - cx - d = (d + 1)^2 + b - b - d^2 - d - d = 1.$$

Comments. In Solution 2, if we take $c = d = 1$, we get the family of parameters $(a, b, c, d) = (a, 2a - 4, 1, 1)$. R. Barrington Leigh found the set of parameters given in Solution 1. A. Feizmohammadi provided the parameters $(a, b, c, d) = (2c, 0, c, 1)$.

2008:4. Suppose that u, v, w, z are complex numbers for which $u + v + w + z = u^2 + v^2 + w^2 + z^2 = 0$. Prove that

$$(u^4 + v^4 + w^4 + z^4)^2 = 4(u^8 + v^8 + w^8 + z^8).$$

Solution. Let u, v, w, z be the roots of a monic quartic polynomial $f(t)$. Since $u + v + w + z = 0$ and

$$uv+uw+uz+vw+vz+wz = \frac{1}{2}[(u+v+w+z)^2-(u^2+v^2+w^2+z^2)] = 0,$$

we must have that $f(t) = t^4 - at - b$ for some complex numbers a and b.

Suppose that $s_n = u^n + v^n + w^n + z^n$ for positive integers n. Then $s_4 = 4b$ and $s_{k+4} = as_{k+1} + bs_k$ for $k \geq 1$. Therefore, $s_5 = 0$ and $s_8 = as_5 + bs_4 = 4b^2$, so that $4s_8 = (4b)^2 = s_4^2$, as desired.

2008:9. For each positive integer n, let

$$S_n = \sum_{k=1}^{n} \frac{2^k}{k^2}.$$

Prove that S_{n+1}/S_n is not a rational function of n.

Solution 1. Assume that S_{n+1}/S_n is a rational function of n. Then

$$\frac{S_n}{2^n n^{-2}} = \frac{S_n}{S_n - S_{n-1}} = \frac{1}{1 - \frac{S_{n-1}}{S_n}}$$

is a rational function. From this, it follows that $S_n = r(n)2^n n^{-2}$ for some rational function r. Since

$$\frac{2^{n+1}}{(n+1)^2} = S_{n+1} - S_n = r(n+1)\frac{2^{n+1}}{(n+1)^2} - r(n)\frac{2^n}{n^2},$$

we have that

$$2n^2 r(n+1) - (n+1)^2 r(n) = 2n^2.$$

Since this equation holds for infinitely many values of n, we have the corresponding rational function identity. Letting $r(x) = f(x)/g(x)$, where f and g are coprime polynomials, we obtain the equation

$$2x^2 f(x+1)g(x) - (x+1)^2 f(x)g(x+1) = 2x^2 g(x)g(x+1).$$

The polynomial $g(x)$ cannot be a constant (look at the leading coefficients).

There exists an integer n, namely 0, for which the polynomials $g(x)$ and $g(x+n)$ have a common divisor of positive degree. However, if n exceeds the maximum absolute value of a root of $g(x)$, then $g(x)$ and $g(x+n)$ are coprime. Let N be the largest integer for which $g(x)$ and $g(x+N)$ have a common irreducible divisor $u(x)$ of positive degree. Then $u(x-N)$ divides $g(x)$ and so divides $(x+1)^2 f(x)g(x+1)$. Since f and g are coprime, $u(x-N)$ divides $(x+1)^2 g(x+1)$. Suppose that $u(x-N)$ divides $g(x+1)$; then $u(x)$ must divide $g(x+N+1)$, which contradicts the determination of N. Therefore, $u(x-N)$ divides $(x+1)^2$, and so $u(x+1)$ divides $(x+N+2)^2$.

Since $u(x+1)$ divides $g(x+1)$, $u(x+1)$ must divide $2x^2f(x+1)g(x)$. Since $u(x+1)$ does not divide $f(x+1)$, $u(x+1)$ divides $2x^2g(x)$. Since $u(x+1)$ divides $g(x+N+1)$, $u(x+1)$ cannot divide $g(x)$. Therefore, $u(x+1)$ must divide x^2, as well as $(x+N+2)^2$. But this is an impossibility. This contradiction yields the result of the problem.

Solution 2. [C. Ochanine] Since

$$\frac{S_{n+1}}{S_n} - 1 = \frac{S_{n+1} - S_n}{S_n} = \frac{2^{n+1}}{(n+1)^2 S_n},$$

it is enough to show that $S_n/2^n$ is not a rational function. Let $f(n) = S_n/2^n$. Then

$$f(n+1) = \frac{1}{2}f(n) + \frac{1}{(n+1)^2}$$

for every positive integer n.

If f were rational, then we would have

$$f(x+1) = \frac{1}{2}f(x) + \frac{1}{(x+1)^2}$$

for all real x. Since, by the definition, $f(1)$ is finite, so also is $f(0)$. Substituting $x = -1$ in the foregoing equation, we see that -1 is a pole of $f(x)$. Since

$$f(x+2) = \frac{1}{2}f(x+1) + \frac{1}{(x+2)^2} = \frac{1}{4}f(x) + \frac{1}{2(x+1)^2} + \frac{1}{(x+2)^2},$$

we see that -2 is also a pole of $f(x)$. Continuing on, we find that every negative integer is a pole of $f(x)$, contradicting its rationality.

The desired result follows.

[Problem **2008:9** was contributed by Franklin Vera Pacheco.]

2009:2. Let n and k be integers with $n \geq 0$ and $k \geq 1$. Let $\mathbf{x}_0$, $\mathbf{x}_1$, $\ldots$, $\mathbf{x}_n$ be $n+1$ distinct points in $\mathbb{R}^k$ and let y_0, y_1, $\ldots$, y_n be $n+1$ real numbers (not necessarily distinct). Prove that there exists a polynomial p of degree at most n in the coordinates of $\mathbf{x}$ for which $p(\mathbf{x}_i) = y_i$ for $0 \leq i \leq n$.

The solution to this problem appears in Chap. 7.

2010:5. Let m be a natural number, and let $c, a_1, a_2, \cdots, a_m$ be complex numbers for which $|a_i| = 1$ for $i = 1, 2, \cdots, m$. Suppose also that

$$\lim_{n\to\infty} \sum_{i=1}^{m} a_i^n = c.$$

Prove that $c = m$ and that $a_i = 1$ for $i = 1, 2, \cdots, m$.

Solution. If $a_i = e^{i\alpha_i}$, then either the sequence $\{a_i^n\}$ is periodic and assumes the value 1 infinitely often (when α_i is a rational multiple of π) or has a subsequence whose limit is 1 (when α_i is not a rational multiple of π).

[In the latter case, we can find an increasing subsequence $\{n_k\}$ of natural numbers for which $a_i^{n_k}$ converges, so that $a_i^{n_{k+1}-n_k}$ converges to 1.]

We prove that there is a subsequence $\{n_k\}$ of natural numbers for which $\lim_k a_i^{n_k} = 1$ for each $1 \leq i \leq m$. Proceed by induction on m. When $m = 1$, the limit of any subsequence of $\{a_1^n\}$ is equal to the limit of the whole sequence, so that $c = 1$ in this case. In fact, we can go further: $a_1 = \lim a_1^{n+1} = \lim a_1^n = 1$.

Suppose the induction hypothesis holds for $m-1$. Then there is a subsequence S_1 of natural numbers such that $\{a_i^n\}$ has limit 1 along this subsequence for $1 \leq i \leq m-1$. We can find a subsequence S_2 along which the sequence $\{a_m^n\}$ converges to some limit b on the unit circle. Let S_3 be the sequence of differences of consecutive terms in the sequence S_2 (so that the sequence along S_3 consists of quotients of consecutive terms of the sequences along S_2). Then, for $1 \leq i \leq m$, the sequence $\{a_i^n\}$ converges to 1 along S_3.

It follows from this that $c = m$. Also, along S_3, we have that

$$\sum_{i=1}^{m} a_i = \lim \sum_{i=1}^{m} a_i^{n+1} = m.$$

Therefore $\sum_{i=1}^m \text{Re } a_i = m$. But the real part of each a_i does not exceed 1, with equality if and only if $a_i = 1$, it follows that $a_i = 1$ for each i.

[Problem **2010:5** was contributed by Bamdad R. Yahaghi.]

2010:6. Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which

$$f(x)f(x+1) = g(h(x)).$$

Solution 1. [A. Remorov] Let $f(x) = a(x-r)(x-s)$. Then

$$\begin{aligned} f(x)f(x+1) &= a^2(x-r)(x-s+1)(x-r+1)(x-s) \\ &= a^2(x^2+x-rx-sx+rs-r)(x^2+x-rx-sx+rs-s) \\ &= a^2[(x^2-(r+s-1)x+rs)-r][(x^2-(r+s-1)x+rs)-s] \\ &= g(h(x)), \end{aligned}$$

where $g(x) = a^2(x-r)(x-s) = af(x)$ and $h(x) = x^2-(r+s-1)x+rs$.

Solution 2. Let $f(x) = ax^2+bx+c$, $g(x) = px^2+qx+r$ and $h(x) = ux^2+vx+w$. Then

$$\begin{aligned} f(x)f(x+1) = a^2x^4 + 2a(a+b)x^3 + (a^2+b^2+3ab+2ac)x^2 + (b+2c)(a+b)x \\ +c(a+b+c) \end{aligned}$$

$$\begin{aligned} g(h(x)) &= p(ux^2+vx+w)^2 + q(ux^2+vx+w) + r \\ &= pu^2x^4 + 2puvx^3 + (2puw+pv^2+qu)x^2 \\ &\quad + (2pvw+qv)x + (pw^2+qw+r). \end{aligned}$$

Equating coefficients, we find that $pu^2 = a^2$, $puv = a(a+b)$, $2puw+pv^2+qu = a^2+b^2+3ab+2ac$, $(b+2c)(a+b) = (2pw+q)v$ and $c(a+b+c) = pw^2+qw+r$. We need to find just one solution of this system. Let $p = 1$ and $u = a$. Then $v = a+b$ and $b+2c = 2pw+q$ from the second and fourth equations. This yields the third equation automatically. Let $q = b$ and $w = c$. Then from the fifth equation, we find that $r = ac$.

Thus, when $f(x) = ax^2+bx+c$, we can take $g(x) = x^2+bx+ac$ and $h(x) = ax^2+(a+b)x+c$.

Solution 3. [S. Wang] Suppose that

$$f(x) = a(x+h)^2+k = a(t-(1/2))^2+k,$$

where $t = x+h+\frac{1}{2}$. Then $f(x+1) = a(x+1+h)^2+k = a(t+(1/2))^2+k$, so that

$$\begin{aligned} f(x)f(x+1) &= a^2(t^2-(1/4))^2+2ak(t^2+(1/4))+k^2 \\ &= a^2t^4+\left(-\frac{a^2}{2}+2ak\right)t^2+\left(\frac{a^2}{16}+\frac{ak}{2}+k^2\right). \end{aligned}$$

Thus, we can achieve the desired representation with $h(x)=t^2 = x^2+(2h+1)x+\frac{1}{4}$ and $g(x) = a^2x^2+(\frac{-a^2}{2}+2ak)x+(\frac{a^2}{16}+\frac{ak}{2}+k^2)$.

Solution 4. [V. Krakovna] Let $f(x) = ax^2+bx+c = au(x)$ where $u(x) = x^2+dx+e$, where $b = ad$ and $c = ae$. If we can find functions $v(x)$ and $w(x)$ for which $u(x)u(x+1) = v(w(x))$, then $f(x)f(x+1) = a^2v(w(x))$, and we can take $h(x) = w(x)$ and $g(x) = a^2v(x)$.

Define $p(t) = u(x+t)$, so that $p(t)$ is a monic quadratic in t. Then, noting that $p''(t) = u''(x+t) = 2$, we have that

$$p(t) = u(x+t) = u(x)+u'(x)t+\frac{u''(x)}{2}t^2 = t^2+u'(x)t+u(x),$$

from which we find that

$$\begin{aligned} u(x)u(x+1) &= p(0)p(1) = u(x)[u(x)+u'(x)+1] \\ &= u(x)^2+u'(x)u(x)+u(x) = p(u(x)) = u(x+u(x)). \end{aligned}$$

Thus, $u(x)u(x+1) = v(w(x))$ where $w(x) = x+u(x)$ and $v(x) = u(x)$. Therefore, we get the desired representation with

$$h(x) = x+u(x) = x^2+\left(1+\frac{b}{a}\right)x+\frac{c}{a}$$

and

$$g(x) = a^2v(x) = a^2u(x) = af(x) = a^2x^2+abx+ac.$$

Comments. The second solution can also be obtained by looking at special cases, such as when $a = 1$ or $b = 0$, getting the answer and then making a conjecture.

There is an interesting story behind this problem. Originally it was noted that the product of two consecutive oblong numbers (products of consecutive integers) was also oblong, since for each integer n, $(n-1)n \times n(n+1) = (n^2-1)n^2$. The same is true for integers one more than a perfect square, since $[(n-1)^2+1][n^2+1] = (n^2-n+1)^2+1$, and numbers of the form n^2+n+1 since $(n^2-n+1)(n^2+n+1) = n^4+n^2+1$. It was not difficult to conjecture and prove the result that $f(x)f(x+1) = f(x+f(x))$ for monic quadratics $f(x)$.

This problem was also given to high school students participating in a problems correspondence program. One solver, James Rickards, then in Grade 9 at an Ottawa, Canada high school, noted that, if the roots of $f(x)$ were r and s, then those of $f(x+1)$ were $r-1$ and $s-1$. Thus, $f(x)f(x+1)$ was a quartic polynomial two of whose roots had the same sum as the other two. He proved that this root property characterized quartic polynomials representable as the composite of two quadratics.

In fact, he generalized to polynomials of higher degree and showed that a polynomial of degree mn could be written as the composite $g(h(x))$ of polynomials g of degree m and h of degree n if and only if its set of roots could be partitioned into m sets of n elements (not necessarily distinct) for which all the fundamental symmetric functions of degrees up to $m-1$ had the same values. I was unable to find any indication of this result in the literature, and so felt that it should be published in a journal with wide circulation. It was accepted and published under the title: James Rickards, *When is a polynomial a composition of other polynomials? American Mathematical Monthly* 118:4 (April, 2011), 358–363.

2011:5. Solve the system

$$\begin{aligned} x + xy + xyz &= 12 \\ y + yz + yzx &= 21 \\ z + zx + zxy &= 30. \end{aligned}$$

Solution 1. Let $u = xyz$. Then $x+xy = 12-u$ so that $z+z(12-u) = 30$ and $z = 30/(13-u)$. Similarly, $y = 21/(31-u)$ and $x = 12/(22-u)$. Plugging these expressions (with $xyz = u$) into any one of the three equations yields that

$$0 = u^3 - 65u^2 + 1306u - 7560 = (u-10)(u-27)(u-28)\ .$$

We get the three solutions

$$(x, y, z) = (1, 1, 10), \left(-\frac{12}{5}, \frac{21}{4}, -\frac{15}{7}\right), (-2, 7, -2),$$

all of which satisfy the system.

Solution 2. Define u and obtain the expressions for x, y, z as in Solution 1. Substitute these into the equation $xyz = u$. This leads to the quartic equation

$$\begin{aligned} 0 &= u(u-13)(u-22)(u-31) + (12)(21)(30) \\ &= u^4 - 66u^3 + 1371u^2 - 8866u + 7560 \\ &= (u-1)(u^3 - 65u^2 + 1306u - 7560). \end{aligned}$$

Apart from the three values of u already identified, we have $u = 1$. This leads to

$$(x, y, z) = \left(\frac{4}{7}, \frac{7}{10}, \frac{5}{2}\right).$$

While, indeed, $xyz = 1$, we find that $x + xy + xyz = 69/35$, $y + yz + xyz = 69/20$, $z + zx + xyz = 69/14$, so the solution $u = 1$ is extraneous.

[Problem **2011:5** was posed by Stanley Rabinowitz in the Spring, 1982 issue of *AMATYC Review*.]

2011:6. [The problem posed on the competition is not available for publication, as it appeared as Enigma problem 1610 in the August 25, 2010 issue of the *New Scientist*. It concerned the determination of the final score of a three game badminton match where the score of each competitor was in arithmetic progression. The full statement of the problem is available on the internet.]

Solution 1. [K. Ng] Recall the winner of a badminton game must obtain at least 21 points and lead the opponent by at least 2 points. Let the scores for the three games be $a, a+u, a+2u$ for player A and $b, b+v, b+2v$ for player B. Suppose, wolog, that A wins the first game and that B wins the second. Then $a - b \geq 2$ and $(b + v) - (a + u) \geq 2$, whence $v - u \geq 2 + (a - b) \geq 4$. Hence

$$(b + 2v) - (a + 2u) = [(b + v) - (a + u)] + (v - u) \geq 6\ .$$

Thus the third game is won by B by at least 6 points. Therefore, B must have exactly 21 points in the third game. It follows that B has more than 21 points in each of the previous games, as A does in the first game.

Hence $a - b = 2$ and

$$2 = (b + v) - (a + u) = (v - u) - (a - b) = (v - u) - 2,$$

so that $v - u = 4$. Therefore, the difference in the scores of the third game is exactly $2(v - u) - (a - b) = 6$ and the final scores are $(15, 21)$.

Solution 2. There are two different ways the score can present at the conclusion of a game: (a) The winner can have 21 points and the other no more than 19 points; (b) The winner can have at least 22 points and the loser two points fewer than the winner.

It is not possible for the scores in the first and third games to be of type (b). For the signed differences of the scores is also an arithmetic progression of three numbers, the first and third of which are both 2, both -2, or 2 and

-2. In the first two cases, this would mean that the same person won all three games, and in the third case that the scores in the second game were the same and there would be no winner of the second game.

Therefore either the opening or closing game has one person with a score of 21 and the other with a score s less than 20. Let the scores of the first player be $(21, 21+x, 21+2x)$ and of the second be $(s, s+y, s+2y)$, possibly in reverse order. Note that x cannot be negative, since otherwise $s \leq s + 2y = 21 \leq s + y$. Since not all three games are won by the same person, we must have $y > x$, $s + y + 2 = 21 + x$ and $21 + 2x + 2 = s + 2y$. Therefore $s = 19 - (y - x) = 23 - 2(y - x)$, whence $s = 15$. Since the winner of the game with scores $(21, 15)$ is the same as that of the second game and different from that of the third, this game must be the final game of the match. Therefore, the scores of the final game are $(21, 15)$.

Note that $y - x = 4$, and that, whenever $x \geq 1$, sets of scores $(21 + 2x, 23 + 2x), (21 + x, 19 + x), (21, 15)$ all satisfy the conditions.

2011:10. Suppose that p is an odd prime. Determine the number of subsets S contained in $\{1, 2, \ldots, 2p - 1, 2p\}$ for which (a) S has exactly p elements, and (b) the sum of the elements of S is a multiple of p.

Solution. Let ζ be a primitive pth root of unity, so that ζ is a power of $\cos(2\pi/p) + i\sin(2\pi/p)$ with the exponent not divisible by p. Let

$$f(x) = \prod_{i=1}^{2p}(x - \zeta^i) = (x^p - 1)^2 = x^{2p} - 2x^p + 1.$$

We can also write

$$\begin{aligned} f(x) = x^{2p} + \cdots \Big[\sum \{(-1)^p \zeta^{i_1 + \cdots + i_p} : \{i_1, i_2, \cdots, i_p\} \\ \subseteq \{1, 2, \ldots, 2p\} \Big] x^p + \cdots + 1 \\ = x^{2p} + \cdots - (a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a^{p-1}\zeta^{p-1})x^p + \cdots + 1, \end{aligned}$$

for some coefficients a_i. We are required to determine the value of a_0.

Let $g(x) = (a_0 - 2) + a_1 x + a_2 x^2 + \cdots + a_{p-1}x^{p-1}$. Observe that $g(x)$ vanishes when x is any primitive pth root of unity and that the sum defining the coefficient of x^p has $\binom{2p}{p}$ terms, so that $a_0 + a_1 + \cdots + a_{p-1} = \binom{2p}{p}$. Since $g(x)$ is a polynomial of degree $p - 1$ whose zeros are the primitive pth roots of unity, $g(x)$ must be a constant multiple of $1 + x + \cdots + x^{p-1}$, so that

$$a_0 - 2 = a_1 = a_2 = \cdots = a_{p-1}.$$

Thus $a_0 + (p - 1)(a_0 - 2) = \binom{2p}{p}$, so that

$$a_0 = \frac{1}{p}\left[\binom{2p}{p} - 2\right] + 2.$$

[Problem **2011:10** was contributed by Ali Feiz Mohammadi.]

2012:2. Suppose that f is a function defined on the set $\mathbb{Z}$ of integers that takes integer values and satisfies the condition that $f(b) - f(a)$ is a multiple of $b - a$ for every pair a, b, of integers. Suppose also that p is a polynomial with integer coefficients such that $p(n) = f(n)$ for infinitely many integers n. Prove that $p(x) = f(x)$ for every positive integer x.

The solution to this problem appears in Chap. 11.

2012:3. Given the real numbers a, b, c not all zero, determine the real solutions x, y, z, u, v, w for the system of equations:

$$\begin{aligned} x^2 + v^2 + w^2 &= a^2 \\ u^2 + y^2 + w^2 &= b^2 \\ u^2 + v^2 + z^2 &= c^2 \\ u(y + z) + vw &= bc \\ v(x + z) + wu &= ca \\ w(x + y) + uv &= ab. \end{aligned}$$

Solution 1. Evaluating b^2c^2 in two ways, we find that

$$\begin{aligned} 0 &= (u^2 + y^2 + w^2)(u^2 + v^2 + z^2) - (uy + uz + vw)^2 \\ &= (u^2 - yz)^2 + (uv - wz)^2 + (wu - vy)^2. \end{aligned}$$

Hence $u^2 = yz$. Similarly, $v^2 = zx$ and $w^2 = xy$. Inserting these three values into the first three equations yields that

$$\begin{aligned} x(x + y + z) &= a^2 \\ y(x + y + z) &= b^2 \\ z(x + y + z) &= c^2. \end{aligned}$$

It follows that not all of x, y, z can vanish and that $(x+y+z)^2 = a^2+b^2+c^2$. Also

$$x : a^2 = y : b^2 = z : c^2 = 1 : (x + y + z) = (x + y + z) : (a^2 + b^2 + c^2).$$

Therefore

$$(x, y, z, u, v, w) = (a^2d, b^2d, c^2d, bcd, cad, abd)$$

where $d^2(a^2 + b^2 + c^2) = 1$.

Solution 2. [Y. Wu; P.J. Zhao] From the given equations, we see that

$$\begin{aligned} (x, v, w) \cdot (w, u, y) &= w(x + y) + uv = ab = \sqrt{x^2 + v^2 + w^2}\sqrt{w^2 + u^2 + y^2}; \\ (w, u, y) \cdot (v, z, u) &= u(y + z) + vw = bc = \sqrt{w^2 + u^2 + y^2}\sqrt{v^2 + z^2 + u^2}; \\ (v, z, u) \cdot (x, v, w) &= v(x + z) + uw = ca = \sqrt{v^2 + z^2 + u^2}\sqrt{x^2 + v^2 + w^2}. \end{aligned}$$

It follows from the conditions for equality in the Cauchy-Schwarz Inequality that (x, v, w), (w, u, y) and (v, z, u) are all constant multiples of the same unit vector (p, q, r). Indeed

$$(x, v, w) = a(p, q, r); \quad (w, u, y) = b(p, q, r); \; (v, z, u) = c(p, q, r).$$

Thus, $x = ap$, $y = br$, $z = cq$, $u = bq = cr$, $v = aq = cp$ and $w = ar = bp$, whence

$$u^2 = bcqr = yz; \quad v^2 = acpq = xz; \quad w^2 = abpr = xy.$$

The solution now can be completed as in Solution 1.

2013:6. Let $p(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial with rational coefficients. Suppose that $p(x)$ has exactly one real root r. Prove that r is rational.

Solution. Since nonreal roots occur in pairs, $p(x)$ must have an even number of real roots counting multiplicity. Therefore r must be a double or quadruple root. If r is a quadruple root, then $p(x) = (x - r)^4$ and $r = -a/4$ is rational. Suppose, otherwise, that $p(x) = (x - r)^2 q(x)$ where $q(x)$ is an irreducible quadratic. The derivative $p'(x)$ is equal to $(x - r)f(x)$ where $f(x) = 2q(x) + (x - r)q'(x)$. Since $q(r)$ does not vanish, $f(r) \neq q(r)$ so that $f(x)$ and $q(x)$ must be distinct and coprime. The monic greatest common divisor of $p(x)$ and $p'(x)$ must therefore be $x - r$. Since (by the Euclidean algorithm) this is a polynomial with rational coefficients, therefore r is rational.

2014:6. Let $f(x) = x^6 - x^4 + 2x^3 - x^2 + 1$.

(a) Prove that $f(x)$ has no positive real roots.

(b) Determine a nonzero polynomial $g(x)$ of minimum degree for which all the coefficients of $f(x)g(x)$ are nonnegative rational numbers.

(c) Determine a polynomial $h(x)$ of minimum degree for which all the coefficients of $f(x)h(x)$ are positive rational numbers.

Solution. (a) Note that

$$f(x) = (x^2 - 1)(x^4 - 1) + 2x^3 = (x^2 - 1)^2(x^2 + 1) + 2x^3$$

from which we see that $f(x) > 0$ for all $x > 0$. Alternatively,

$$f(x) = x^6 + x^3 + (x^3 - x^4) + (1 - x^2) = x^4(x^2 - 1) + x^2(x - 1) + x^3 + 1$$

from which we see that there are no roots in either of the intervals $[0, 1]$ and $[1, \infty)$.

(b) It is straightforward to see that multiplying $f(x)$ by a linear polynomial will not achieve the goal. We have that

$$f(x)(x^2 + bx + c) = x^8 + bx^7 + (c - 1)x^6 + (2 - b)x^5 + (2b - c - 1)x^4 + (2c - b)x^3 + (1 - c)x^2 + bx + c$$

from which we see that taking $c = 1$ and $1 \leq b \leq 2$ will yield the desired polynomial $g(x)$. Examples of suitable $g(x)$ are $x^2 + x + 1$ and $(x + 1)^2$.

(c) From (b), we note that no quadratic polynomial will serve. However, taking $h(x) = x^3 + 2x^2 + 2x + 1$, we find that

$$f(x)h(x) = x^9 + 2x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + 2x + 1.$$

Comment. It is known that for a given real polynomial $f(x)$, there exists a polynomial $g(x)$ for which all the coefficients of $f(x)g(x)$ are nonnegative (resp. positive) if and only if none of the roots of $f(x)$ are positive (resp. nonnegative).

Another example is $f(x) = x^3 - x + 1$. Since $f(x) = x^3 + (1 - x) = x(x^2 - 1) + 1$, we see that there are no roots in $[0, 1]$ and $[1, \infty)$. No linear polynomial $x + c$ will do for $g(x)$. Since

$$f(x)(x^2 + bx + c) = x^5 + bx^4 + (c - 1)x^3 + (1 - b)x^2 + (b - c)x + c,$$

we require that $1 \le c \le b$ and $0 \le b \le 1$. The only possibility is that $g(x) = x^2 + x + 1$.

For strictly positive coefficients in the product, set $g(x) = ax^3 + bx^2 + cx + d$. Then

$$f(x)g(x) = ax^6 + bx^5 + (c - a)x^4 + (a + d - b)x^3 + (b - c)x^2 + (c - d)x + d.$$

For g to be suitable, we require that $c > a > 0$, $a + d > b$ and $b > c > d > 0$. We can take $g(x) = 3x^3 + 5x^2 + 4x + 3$.

[The examples in Problem **2014:6** are due to Horst Brunotte in Düsseldorf, Germany.]

2015:10. (a) Let

$$g(x, y) = x^2y + xy^2 + xy + x + y + 1.$$

We form a sequence $\{x_0\}$ as follows: $x_0 = 0$. The next term x_1 is the unique solution -1 of the linear equation $g(t, 0) = 0$. For each $n \ge 2$, x_n is the solution other than x_{n-2} of the equation $g(t, x_{n-1}) = 0$.

Let $\{f_n\}$ be the Fibonacci sequence determined by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \ge 2$. Prove that, for any nonnegative integer k,

$$x_{2k} = \frac{f_k}{f_{k+1}} \qquad \text{and} \qquad x_{2k+1} = -\frac{f_{k+2}}{f_{k+1}}.$$

(b) Let

$$h(x, y) = x^2y + xy^2 + \beta xy + \gamma(x + y) + \delta$$

be a polynomial with real coefficients β, γ, δ. We form a bilateral sequence $\{x_n : n \in \mathbf{Z}\}$ as follows. Let $x_0 \neq 0$ be given arbitrarily. We select x_{-1} and x_1 to be the two solutions of the quadratic equation $h(t, x_0) = 0$ in either order. From here, we can define inductively the terms of the sequence for positive and negative values of the index so that x_{n-1} and x_{n+1} are the two solutions of the equation $h(t, x_n) = 0$. We suppose that at each stage, neither of these solutions is zero.

Prove that the sequence $\{x_n\}$ has period 5 (i.e. $x_{n+5} = x_n$ for each index n) if and only if $\gamma^3 + \delta^2 - \beta\gamma\delta = 0$.

(a) *Solution.* Observe that

$$g(x, y) = (xy + 1)(x + y + 1).$$

For each value of $y \neq 0$, the equation $g(x, y) = 0$ has two solutions: $y = -1/x$ and $y = -(x + 1)$. Observe that $g(x, y)$ is symmetrical in x and y, so that, for each consecutive pair x_n, x_{n+1} of terms in the sequence $g(x_n, x_{n+1}) = g(x_{n+1}, x_n) = 0$. Consider the equation $0 = g(t, x_1) = g(t, -1) = (-t + 1)t$. One of its solutions is $x_0 = 0$ and the other is $x_2 = 1$.

For the equation $0 = g(t, x_2)$, we have that $g(x_1, x_2) = g(x_2, x_1) = 0$ and $x_1x_2 + 1 = 0$. Therefore $x_2 + x_3 + 1 = 0$, so that $x_3 = -(x_2 + 1)$. Continuing on in this way, we find that, for each positive integer k, $x_{2k-1}x_{2k} = -1$ and $x_{2k} + x_{2k+1} = -1$, whereupon

$$x_{2k+1} = -1 + \frac{1}{x_{2k-1}} = \frac{1 - x_{2k-1}}{x_{2k-1}}.$$

When $k = 1$, we find that $x_{2k-1} = x_1 = -1 = -f_2/f_1$. Suppose, for $k \geq 1$, we have that $x_{2k-1} = -f_{k+1}/f_k$. Then

$$x_{2k+1} = -1 - \frac{f_k}{f_{k+1}} = -\frac{f_{k+1} + f_k}{f_{k+1}} = -\frac{f_{k+2}}{f_{k+1}}.$$

By induction, we obtain the desired expression for x_{2k+1}. Also $x_{2k} = -1/x_{2k-1} = f_k/f_{k+1}$.

(b) *Solution 1.* Observe that from the sum of the roots of $h(t, x_n) = 0$, we have that

$$x_{n-1} + x_{n+1} = -\left[\frac{x_n^2 + \beta x_n + \gamma}{x_n}\right],$$

or

$$x_{n-1} + x_n + x_{n+1} = -\beta - \frac{\gamma}{x_n}$$

for each n. The sequence will have period 5 if and only if the sum of any five consecutive terms is constant.

Since, for each integer n,

$$x_{n+2} + x_{n+1} + x_n = -\beta - \frac{\gamma}{x_{n+1}}$$

and

$$x_n + x_{n-1} + x_{n-2} = -\beta - \frac{\gamma}{x_{n-1}},$$

we have that

$$x_{n+2} + x_{n+1} + x_n + x_{n-1} + x_{n-2} = -2\beta - \gamma\left(\frac{x_{n+1} + x_{n-1}}{x_{n+1}x_{n-1}}\right) - x_n$$

$$
\begin{aligned}
&= -2\beta + \gamma\left(\frac{x_n^2 + \beta x_n + \gamma}{\gamma x_n + \delta}\right) - x_n \\
&= \frac{-(\beta\gamma + \delta)x_n + (\gamma^2 - 2\beta\delta)}{\gamma x_n + \delta} \\
&= -\left(\beta + \frac{\delta}{\gamma}\right) + \left(\frac{\gamma^3 + \delta^2 - \beta\gamma\delta}{\gamma^2 x_n + \gamma\delta}\right).
\end{aligned}
$$

This sum is independent of n if and only if the term involving x_n vanishes identically, i.e., if and only if the required condition holds.

Solution 2. From the formula for the product of the roots, we obtain that

$$x_{n-1}x_{n+1} = \frac{\gamma x_n + \delta}{x_n}$$

so that

$$x_{n-1}x_n x_{n+1} = \gamma x_n + \delta$$

for each index n. Therefore

$$
\begin{aligned}
x_{n+2}x_{n+1}x_n^2 x_{n-1}x_{n-2} &= (\gamma x_{n+1} + \delta)(\gamma x_{n-1} + \delta) \\
&= \gamma^2\left(\frac{\gamma x_n + \delta}{x_n}\right) - \gamma\delta\left(x_n + \beta + \frac{\gamma}{x_n}\right) + \delta^2 \\
&= (\gamma^3 + \delta^2 - \beta\gamma\delta) - \gamma\delta x_n
\end{aligned}
$$

whence

$$x_{n+2}x_{n+1}x_n x_{n-1}x_{n-2} = -\gamma\delta + \left(\frac{\gamma^3 + \delta^2 - \beta\gamma\delta}{x_n}\right).$$

The result again follows.

Comments. If $x_0 = 0$, then $h(t, x_0)$ is linear and there is a single root $-\delta/\gamma$. We can extend the sequence in only one direction, and it begins with the terms $0, -\delta/\gamma, (\gamma^3 + \delta^2 - \beta\gamma\delta)/(\gamma\delta), \ldots$.

The study of the type of sequence given in this problem began with the recursions given by

$$x_{n+1} = \frac{x_n + c}{x_{n-1}}$$

or, equivalently, if you want to continue the sequence backwards,

$$x_{n-1} = \frac{x_n + c}{x_{n+1}},$$

where c is a parameter and n ranges over the integers. We suppose that the terms of the sequence are selected to avoid division by 0. When $c = 0$ and $c = 1$, any bilateral sequence satisfying the recursion is periodic with respective periods 6 and 5. For other values of c, certain but not all sequences are periodic. The examination of the periodic cases lead to the identification of an invariant for the sequence.

When
$$f_c(x,y) = \frac{x^2y + xy^2 + x^2 + (c+1)x + y^2 + (c+1)y + c}{xy},$$
it turns out that
$$f(x,y) = f\left(y, \frac{y+c}{x}\right)$$
with the result that $f(x_n, x_{n+1})$ is constant with respect to the index n.

The equation $f_c(x,y) = k$ can be rewritten as $h_{c,k}(x,y) = 0$ where
$$\begin{aligned} h_{c,k}(x,y) &= x^2y + xy^2 + x^2 + y^2 - kxy + (c+1)(x+y) + c \\ &= xy(x+y) + (x+y)^2 - (k+2)xy + (c+1)(x+y) + c \\ &= (y+1)x^2 + (y^2 - ky + (c+1))x + (y+1)(y+c)\ . \end{aligned}$$

The function $h_{c,k}(x,y)$ is a symmetric polynomial quadratic in each variable. For each n, the terms x_{n-1} and x_{n+1} are the roots of the quadratic equation $h_{c,k}(x, x_n) = 0$, provided $x_n \neq 1$. Indeed, from the relation between the coefficients and the product of the roots of a quadratic, we corroborate the relation
$$x_{n-1}x_{n+1} = x_n + c. \tag{2.13}$$
Moreover, we have
$$x_{n-1} + x_{n+1} = \frac{x_n^2 - kx_n + c + 1}{x_n + 1}. \tag{2.14}$$
These sequences, for various values of the parameter c, are studied in the paper Ed Barbeau, Boaz Gelbord and Steve Tanny, *Periodicities of solutions of the generalized Lyness recursion*, *J. Difference Equations and Applications* 1 (1995), 291–306.

We look at a slight generalization and point out how two types of recursions are related to each other. Let
$$\begin{aligned} h(x,y) &= x^2y + xy^2 + \alpha(x^2 + y^2) + \beta xy + \gamma(x+y) + \delta \\ &= (y+\alpha)x^2 + (y^2 + \beta y + \gamma)x + (\alpha y^2 + \gamma y + \delta). \end{aligned}$$
As before, we can seed the recursion by specifying x_0 and arranging that, for each integer n, x_{n-1} and x_{n+1} are the solutions of the equation $h(x, x_n) = 0$. To avoid complications, we suppose that x_n never assumes the value $-\alpha$.

Sequences determined in this way satisfy both the recursions:
$$x_{n+1} + x_{n-1} = -\left[\frac{x_n^2 + \beta x_n + \gamma}{x_n + \alpha}\right] \tag{2.15}$$
and
$$x_{n+1}x_{n-1} = \frac{\alpha x_n^2 + \gamma x_n + \delta}{x_n + \alpha}. \tag{2.16}$$

Remarkably, if we are given any sequence that satisfies the first of these recursion relations, we can select a constant δ so that it satisfies the second. Similarly, given any sequences given by the second recursion relationship, we can select a value of β so that it satisfies the first. Thus, either recursion individually allows us to introduce a function $h(x,y)$ that will allow us to generate the sequence. This follows from the following proposition.

We have

$$h\left(\frac{\alpha y^2+\gamma y+\delta}{x(y+\alpha)}, y\right) = \frac{\alpha y^2+\gamma y+\delta}{x^2(y+\alpha)} h(x,y)$$

and

$$h\left(-\left[\frac{\alpha y^2+\beta y+\gamma}{y+\alpha}\right] - x, y\right) = h(x,y)$$

with the result that, when x and y are consecutive terms, $\frac{h(x,y)}{xy}$ is invariant along any recursion satisfying (2.16) and $h(x,y)$ is invariant along any recursion satisfying (2.15). Thus, any recursion satisfying either (2.15) or (2.16) will satisfy the other with a suitable choice of parameters.

Proof.

$$\begin{aligned}
h\left(\frac{\alpha y^2+\gamma y+\delta}{x(y+\alpha)}, y\right) =& (y+\alpha)\frac{(\alpha y^2+\gamma y+\delta)^2}{x^2(y+\alpha)^2} + (y^2+\beta y+\gamma)\frac{\alpha y^2+\gamma y+\delta}{x(y+\alpha)} \\
&+ (\alpha y^2+\gamma y+\delta) \\
=& \frac{\alpha y^2+\gamma y+\delta}{x^2(y+\alpha)}\left[(\alpha y^2+\gamma y+\delta) + x(y^2+\beta y+\gamma) + x^2(y+\alpha)\right] \\
=& \frac{\alpha y^2+\gamma y+\delta}{x^2(y+\alpha)}\left[x^2y + y^2x + \alpha(x^2+y^2) + \beta xy + \gamma(x+y) + \delta\right] \\
=& \frac{\alpha y^2+\gamma y+\delta}{x^2(y+\alpha)} h(x,y).
\end{aligned}$$

Also

$$\begin{aligned}
&h\left(-\left[\frac{y^2+\beta y+\gamma}{y+\alpha}\right] - x, y\right) \\
&\quad = (y+\alpha)\left[x^2 + \frac{2x(y^2+\beta y+\gamma)}{y+\alpha} + \frac{(y^2+\beta y+\gamma)^2}{(y+\alpha)^2}\right] \\
&\qquad\quad - (y^2+\beta y+\gamma)\left[\frac{y^2+\beta y+\gamma}{y+\alpha} + x\right] + (\alpha y^2+\gamma y+\delta) \\
&\quad = x^2y + \alpha x^2 + 2xy^2 + 2\beta xy + 2\gamma x - xy^2 - \beta xy - \gamma x + \alpha y^2 + \gamma y + \delta \\
&\quad = h(x,y).
\end{aligned}$$

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University of Toronto Mathematics Competition
(2001–2015)
Barbeau, E.J.
2016, VIII, 207 p. 5 illus., 1 illus. in color., Hardcover
ISBN: 978-3-319-28104-9