Chapter 2
Scientific Foundations of Stochastic Tribomodeling

Modeling (simulation) is one of the most effective methods of knowledge. The effectiveness of modeling (simulation) as a tool for learning is determined, above all, by the ability to highlight the main (essential) and to abstract the minor (inconsequential).

The solution of the adequate mathematical models in the complex technical problems, such as tribotechnical, is the qualitative analysis primarily based on the one of the main classification systems. Therefore, in the modeling (simulation), we should progress from the object to the model. The current research provides a systematic approach to the study of tribological applications. The importance of that approach rises especially when the probabilistic–statistic methods are applied.

The objects, their properties, and relationships between those two define the system. According to N.P. Fedorchenko, the part of the system objects are the purpose, input, the process, output, the boundaries and the inverse connection. The purpose of such a system is the achievement of optimal functioning of the tribotechnical system and its further maintainability at the level of maximum stability. This goal has to be achieved by complying at least two restrictions: The quantity of materials required for the manufacture of friction units should not exceed the predetermined level, and the operational cost of the entire system must not exceed the certain values.

Through the system analysis, we can determine the system class: deterministic or stochastic (probabilistic). V.P. Trofimov defined that "the system is the rigidly deterministic if under those initial conditions it transfers itself in a single state. Accordingly, the system is stochastic (probabilistic) if under the same initial conditions it can transfer itself into different states, with different probabilities."

The division of system into the rigidly deterministic and stochastic (probabilistic) is very objective. B.V. Gnedenko wrote that "the deterministic systems are quite rare. Moreover, there is an established view in the modern physics that in nature there is no purely deterministic laws, and all laws are of probabilistic nature." This statement is consistent with the fundamental research of V.V. Bolotin for designing the structures using methods of theory of probability and reliability.

The issues of practical application of statistical modeling (simulation) in various fields of scientific and technical research are well documented in numerous literatures.

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2.1 Probability Analysis of Generalized Variables Based on Stochastic Nature of Parameters

Studying the failure of friction bodies using the conventional methods makes the task almost impossible. Therefore, the applications of new mathematical methods such as physical–statistical and probabilistic methods are required for tribology. The significant progress has recently been achieved in the related problems [1–5].

The conventional mathematical methods applied in solving problems for even simple friction processes are virtually ineffective, because besides the complexity of analyzed process, there is also the stochastic nature of its functionality. So the negligence of even a large number of random factors usually leads to the inaccurate test results of the assessed process, even when there is a perfect mathematical model.

At the present time, there is a wide range of literatures that analyze ideas and practical implementation of probabilistic methods [6–9], which can be used to solve many actual problems. The method of statistical test (Monte Carlo) in this sense is of particular importance.

The evaluations of various loading conditions of friction devices and the methods of stochastic description have received a lot of attention currently. The evaluation of real load of friction devices is associated with both solving the problem through theory of reliability and predicting its actual durability and founding the optimal test modes [10].

The statistical characteristics of the loading indicators of machine usually vary significantly for the different operation modes, so it is advisable to consider not the impact of the individual values, but the complex values, called the generalized variables (GV) (the similarity criteria). In reality, the technical systems are mainly stochastically defined, and their parameters are subject to the random changes. The GV, combining those system parameters, will also be subjected to the random variations. The considerable interest is of studying the probability distribution of GV and the definition of probability characteristics. Generally, the GV can be represented as follows [11]:

$$\pi_k = \prod_{i=1}^{n} P_i^{\pm\alpha_i}, \tag{2.1}$$

where P_i—the parameters included in the GV and α_i—the exponent parameters.

The parameters included in the GV are the random variables with their own distributions, and the GV is the function of those random variables. It should be

noted that the probability distribution of the P_i parameters is mainly known as a result of the test or a priori data. To study the probability distribution of π_k, we have to analyze the dependence of the form:

$$\pi_k = \varphi\left(P_1^{\pm\alpha_1}, P_2^{\pm\alpha_2}, \ldots, P_n^{\pm\alpha_n}\right). \tag{2.2}$$

Due to the complexity of analytical solution, it is advisable to investigate the problem through the method of statistical tests on a computer or by the analytical approximation [12, 13].

The essence of statistical test for solving the problem is that parameters included in GV are modelled through the specified distribution, and the array of data is formed for the random values π_k. On the basis of this array, the GV distribution is determined and its statistical characteristics are calculated.

To analyze the expression (2.2), we represent the initial GV as a function of two random variables—X_1 and X_2 and denote the GV by y:$y = \varphi(x_1, x_2)$.

Given the probability density of random variables (X_1, X_2). Then, the distribution function of the random variable Y = GV can be determined based on the probability density function of the individual random variables X_1 and X_2 [14].

$$F(y) = \iint\limits_D f(x_1, x_2)\mathrm{d}x_1\,\mathrm{d}x_2,$$

where D—the region in the plane X_1OX_2 for which $Y < y$.

The probability density is:

$$f(y) = \frac{\mathrm{d}}{\mathrm{d}y}F(y).$$

Consider some typical for GV (consisting of two parameters), the expressions of the density distribution functions of the continuous random variables:

1. $Y = X_1X_2; f(y) = \int \left|\frac{1}{x_1}\right| f_1(x_1) f_2\left(\frac{y}{x_1}\right)\mathrm{d}x_1 = \int \left|\frac{1}{x_2}\right| f_1\left(\frac{y}{x_1}\right) f_2(x_1)\mathrm{d}x_2;$
2. $Y = \frac{X_1}{X_2}; f(y) = \int |x_2| f_1(yx_2) f_2(x_2)\mathrm{d}x_2 = \int \left|\frac{x_1}{y^2}\right| f_1(x_1) f_2\left(\frac{x_1}{y}\right)\mathrm{d}x_1.$

(1) When the allocation of X_1 and X_2 follows the normal law, the density distribution of Y for product [15] is:

$$f(y) = \frac{1}{\pi\sigma_{x_1}\sigma_{x_2}} K_0 \frac{y}{\sigma_{x_1}\sigma_{x_2}},$$

where K_0—the Macdonald function.

(2) When the allocation of X_1 and X_2 follows the uniform law within the interval (a, b), the density distribution for product will be:

$$f(y) = \begin{cases} \frac{1}{(b-a)^2} \ln \frac{y}{a^2}, & a^2 \le y \le ab; \\ \frac{1}{(b-a)^2} \ln \frac{a^2}{y}, & ab \le y \le b^2. \end{cases}$$

(3) When the allocation of X_1 and X_2 follows the normal law, the density distribution of Y for the individual case is:

$$f(y) = \frac{1}{\pi \left[y^2 + (\sigma_{x_1}/\sigma_{x_2})^2 \right]} \cdot \frac{\sigma_{x_1}}{\sigma_{x_2}}$$

which complies with the Cauchy distribution.

If the function depends on many random variables, there is the determinant of Ostrogradski–Jacobi for the transformation of random variables $(X_1, X_2, \ldots, X_n)$ into the random variables $(Y_1, Y_2, \ldots, Y_n)$.

$$f(y) = \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

and if this transformation is bijective, then

$$f_y(y_1, y_2, \ldots, y_n) = |D| f_x(x_1, x_2, \ldots, x_n)$$

where the quantities of $x_1, x_2, \ldots, x_n$ are expressed through $y_1, y_2, \ldots, y_n$.

The task becomes more complex, when we deal with the multidimensional system of random variables and with the calculation of multiple integrals.

Because of the complexity of determining the density distribution $f(y)$, in some cases we can restrict the finding of numerical values. In this case, we assume that the distribution of arguments or their numerical characteristics are given. Then, to find the expectation and the variance of the functions of random variables, the following expression is applied [14].

$$\begin{aligned} m_\varphi = M[\varphi(X_1, X_2, \ldots, X_n)] &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \varphi(x_1, x_2, \ldots, x_n) \times f(x_1, x_2, \ldots, x_n) \mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n \\ D[\varphi(X_1, X_2, \ldots, X_n)] &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left[\varphi(x_1, x_2, \ldots, x_n) - m_\varphi \right]^2 \times f(x_1, x_2, \ldots, x_n) \mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n. \end{aligned} \tag{2.3}$$

Due to the necessity of studying the distribution of GV, it is of considerable interest to determine the distribution of π_k. In this paper, when the number of variables is more than three, the most appropriate method for solving the problem

should be the Monte Carlo method [16]. For the study, the GV of friction was taken from the tribological nodes of oil equipment, obtained in [17].

Figure 2.1 shows the GV and the probability distribution histograms for normal and uniformly distributed parameters. The parameter tolerance values are given within ±(5–15) % of its baseline values. The calculations were performed on PC. The algorithm for the GV probability distributions using the Monte Carlo method is shown in Fig. 2.2. Tables 2.1 and 2.2 show the numerical characteristics of these distributions and the normal and uniform distributions of P_i, respectively, where $\overline{\pi}$—the average value, σ—the dispersion, S_π—the skewness ($S_\pi = \mu_3/\sigma_3$; μ_3—the central point of the third order), E_π—the excess ($E_\pi = \mu_4/\sigma_4$; and μ_4—the fourth-order central moment) for the random variable π.

To calculate the n-dimensional integrals, appearing in the second line of the expressions (2.3), for the expectation and variance of the random variable functions $\varphi(X_1, \ldots, X_n)$, we apply the Monte Carlo method [18, 19], which is used for the approximation of integral

$$I = \int_G f(x_1, \ldots, x_2) p(x_1, \ldots, x_2) \mathrm{d}x_1 \ldots \mathrm{d}x_n, \tag{2.4}$$

where $p(x_1, \ldots, x_n)$—the joint probability density function of the random variables $X_1, \ldots, X_n$, defined in G so that

$$\int_G p(x_1, \ldots, x_2) \mathrm{d}x_1 \ldots \mathrm{d}x_n = 1. \tag{2.5}$$

Here, G—the limited range of variation values $X_1, \ldots, X_n$, since the variables P_i, included in that we consider the GV of the form (2.1), are limited by their physical meaning: $0 < P_i < c_i$.

In (2.3), x_i—the GV P_i, as $f(x_1, \ldots, x_n)$ we adopt the function $\varphi(x_1, \ldots, x_n)$, and the probability density function $p(x_1, \ldots, x_n)$ is denoted by $f(x_1, \ldots, x_n)$.

To evaluate the integral (2.4) through the Monte Carlo method, we consider the random point $Q \in G$ with the density $p(x)$, where $x = (x_1, \ldots, x_n)$—the n-dimensional random variable $X = (X_1, \ldots, X_n)$ and we introduce the random scalar variable $Z = f(Q)$, the mathematical expectation of which is equal to the desired integral (2.4)

$$MZ = \int f(x) p(x) \mathrm{d}x = I, \tag{2.6}$$

M the mathematical expectation. Recall that, by definition, the expectation MZ exists if and only if there is $M|Z|$.

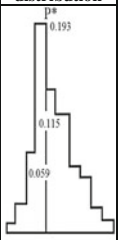
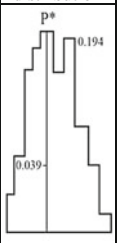
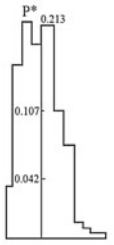
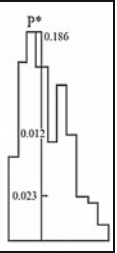
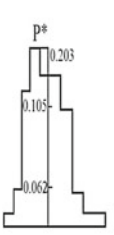
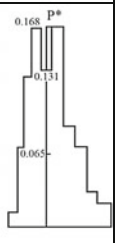
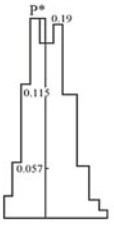
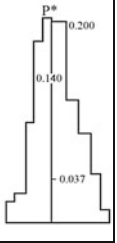
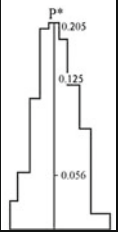
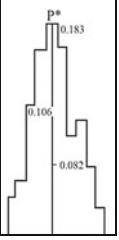
Item №	Generalized variables	Expression	Formal representation	Properties of probability distribution	
				When π has a normal distribution	When π has a uniform distribution
1.	Hardness	$\pi_1 = \frac{HB_1 \cdot HB_2 \cdot K_{r_1}^{2/3} \cdot K_{r_2}^{2/3}}{P^2}$	$\pi_1 = \frac{P_1 \cdot P_2 \cdot P_3^{2/3} \cdot P_4^{2/3}}{P_5^2}$		
2.	Density	$\pi_2 = \frac{P_1 \cdot P_2 \cdot P_3 \cdot \vartheta^6 \cdot K_{r_1} \cdot K_{r_2}}{P^3}$	$\pi_2 = \frac{P_6 \cdot P_7 \cdot P_8 \cdot P_9^6 \cdot P_3 \cdot P_4}{P_5^3}$		
3.	Dynamic viscosity	$\pi_3 = \frac{\eta_1 \cdot \eta_2 \cdot \eta_3 \cdot \vartheta^2 \cdot K_{r_1}^{1/3} \cdot K_{r_2}^{1/3}}{P^2}$	$\pi_3 = \frac{P_{10} \cdot P_{11} \cdot P_{12} \cdot P_9^2 \cdot P_3^{1/3} \cdot P_4^{1/3}}{P_5^2}$		
4.	Macro-geometry	$\pi_4 = \frac{A_{a_1} \cdot A_{a_2}}{K_{r_1}^{2/3} \cdot K_{r_2}^{2/3}}$	$\pi_4 = \frac{P_{13} \cdot P_{14}}{P_3^{2/3} \cdot P_4^{2/3}}$		
5.	Single asperities	$\pi_5 = \frac{r_1 \cdot r_2}{K_{r_1}^{1/3} \cdot K_{r_2}^{1/3}}$	$\pi_5 = \frac{P_{15} \cdot P_{16}}{P_3^{1/3} \cdot P_4^{1/3}}$		

Fig. 2.1 The GV and the probability distribution histograms for normal and uniformly distributed parameters

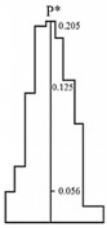
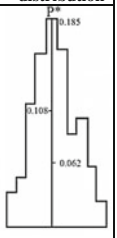
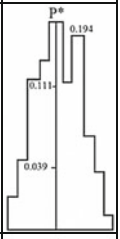
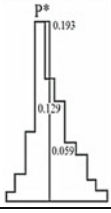
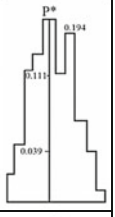
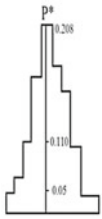
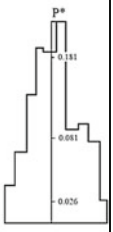
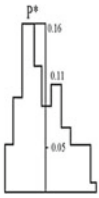
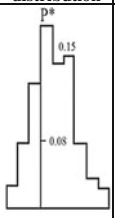
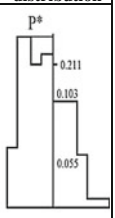
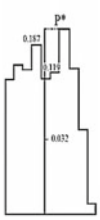
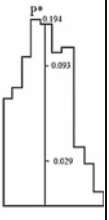
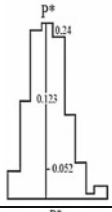
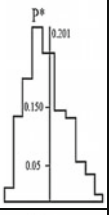
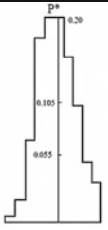
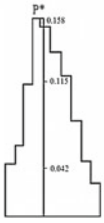
Item №	Generalized variables	Expression	Formal representation	Properties of probability distribution	
				When π has a normal distribution	When π has a uniform distribution
6.	Maximum height of asperity of rough surface	$\pi_6 = \frac{h_1 \cdot h_2}{K_{r_1}^{1/3} \cdot K_{r_2}^{1/3}}$	$\pi_6 = \frac{P_{17} \cdot P_{18}}{P_3^{1/3} \cdot P_4^{1/3}}$		
7.	Modulus of elasticity	$\pi_7 = \frac{E_1 \cdot E_2 \cdot K_{r_1}^{2/3} \cdot K_{r_2}^{2/3}}{P^2}$	$\pi_7 = \frac{P_{19} \cdot P_{20} \cdot P_3^{2/3} \cdot P_4^{2/3}}{P_5^2}$		
8.	Shear strain of films on surface of friction	$\pi_8 = \frac{\tau_{\text{пп}_1} \cdot \tau_{\text{пп}_2} \cdot K_{r_1}^{2/3} \cdot K_{r_2}^{2/3}}{P^2}$	$\pi_8 = \frac{P_{21} \cdot P_{22} \cdot P_3^{2/3} \cdot P_4^{2/3}}{P_5^2}$		
9.	Friction duration	$\pi_9 = \frac{t \cdot v}{K_{r_1}^{1/6} \cdot K_{r_2}^{1/6}}$	$\pi_9 = \frac{P_{23} \cdot P_3}{P_3^{1/6} \cdot P_4^{1/6}}$		
10.	Thermal conductivity	$\pi_{10} = \frac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \vartheta_1^{3/2} \cdot \vartheta_2^{3/2}}{P^3 \cdot v^3 \cdot K_{r_1}^{1/2} \cdot K_{r_2}^{1/2}}$	$\pi_{10} = \frac{P_{24} \cdot P_{25} \cdot P_{26} \cdot P_{27}^{3/2} \cdot P_{28}^{3/2}}{P_5^3 \cdot P_9^3 \cdot P_3^{1/2} \cdot P_4^{1/2}}$		

Fig. 2.1 (continued)

Item №	Generalized variables	Expression	Formal representation	Properties of probability distribution	
				When π has a normal distribution	When π has a uniform distribution
11.	Coefficient of specific heat	$\pi_{11} = \frac{C_1 \cdot C_2 \cdot C_3 \cdot \vartheta_1^{3/2} \cdot \vartheta_2^{3/2}}{v_6^5}$	$\pi_{11} = \frac{P_{29} \cdot P_{30} \cdot P_{31} \cdot P_{27}^{3/2} \cdot p_{28}^{3/2}}{P_9^6}$		
12.	Coefficient of linear expansion	$\pi_{12} = \alpha_1 \cdot \alpha_2 \cdot \vartheta_1 \cdot \vartheta_2$	$\pi_{12} = P_{32} \cdot P_{33} \cdot P_{27} \cdot P_{28}$		
13.	Acceleration	$\pi_{13} = \frac{\omega \cdot K_{r_1} \cdot K_{r_2}}{v^3}$	$\pi_{19} = \frac{P_{34} \cdot P_3 \cdot P_4}{P_9^2}$		
14.	Coefficient of heat transfer	$\pi_{14} = \frac{\sigma_1 \cdot \sigma_2 \cdot K_{r_1}^{2/3} \cdot K_{r_2}^{2/3} \cdot \vartheta_1 \cdot \vartheta_3}{P^2 \cdot v^3}$	$\pi_{14} = \frac{P_{35} \cdot P_{36} \cdot P_3^{2/3} \cdot p_4^{2/3} \cdot P_{27} \cdot P_{28}}{P_5^2 \cdot P_9^2}$		
15.	Work under breaking	$\pi_{15} = \frac{W_{т.п.}}{P \cdot K_{r_1}^{1/6} \cdot K_{r_2}^{1/6}}$	$\pi_{15} = \frac{P_{37}}{P_5 \cdot P_3^{1/6} \cdot P_4^{1/6}}$		

P* - the value of empirical function of density

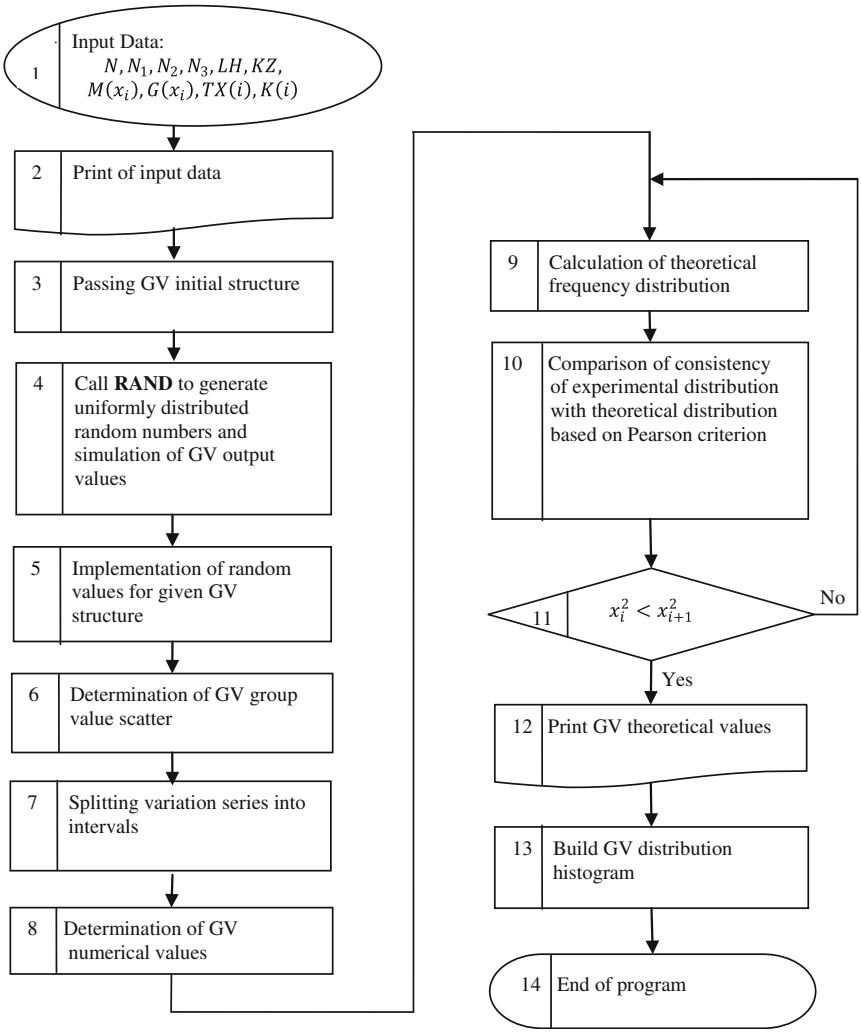


Fig. 2.2 The program algorithm for the GV probability distributions using the Monte Carlo method

Table 2.1 The first 10 values of p_3 (i)

i	1	2	3	4	5	6	7	8	9	10
i triple	1	2	10	11	12	20	21	22	100	101
$p_3(i)$ triple	0.1	0.2	0.01	0.11	0.21	0.02	0.12	0.22	0.001	0.101
$p_3(i)$	1/3	2/3	1/9	4/9	7/9	2/9	5/9	8/9	1/27	10/27

Table 2.2 Numerators $V_j^s (j = 1, \ldots, n)$

I	s																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	3	5	15	17	51	85	255	257	771	1285	3855	4369	13,107	21,845	65,535	65537	196,611	327,685	983,055
3	1	1	7	11	13	61	67	79	465	721	823	4091	4125	4141	28,723	45,311	53,505	250,113	276,231	326,411
4	1	3	1	5	31	29	81	147	433	149	719	3693	3841	11,523	16,641	49,925	16,671	83,229	515,921	482,707
5	1	1	5	3	15	51	125	141	177	759	267	1839	6929	16,241	16,565	17,139	82,207	50,979	252,717	851,901
6	1	3	7	7	9	11	79	63	193	707	1351	1479	6857	1227	32,527	36,863	102,401	241,667	258,055	323,591
7	1	1	3	11	7	31	13	161	309	901	2007	1311	6235	2915	27,745	35,885	34,225	112,177	378,387	246,491
8	1	3	3	13	23	57	11	219	101	687	449	579	3779	717	17,751	20,473	23,627	261,147	140,453	658,927
9	1	1	1	15	11	29	13	85	475	63	1233	337	6577	16,383	24,859	24,845	110,365	175,013	97,387	453,871
10	1	3	5	1	29	61	79	169	293	177	689	2259	2197	6225	22,861	64,589	15,455	107,641	103,861	247,585
11	1	1	7	5	5	3	127	97	47	819	765	3341	3379	15,217	21,513	19,463	15,387	50,237	130,435	790,151
12	1	3	1	11	25	15	9	191	417	631	789	1327	2885	4295	29,021	37,603	62,509	248,899	87,493	716,363
13	1	1	5	7	19	51	51	151	157	141	389	901	5889	16,131	12,039	11,055	90,495	123,227	49,489	574,417

To find the value of I, we choose N independent and identically distributed (i.i.d.) values of $Z_i = f\left(Q^{(i)}\right)(i = 1, \ldots, N)\left(Q^{(i)} = \left(Q_1^{(i)}, \ldots, Q_n^{(i)}\right)\right)$—$i$th value of the n-dimensional random quantity $Q = (Q_1, \ldots, Q_n)$ and we calculate the arithmetic mean

$$\overline{Z}_N = \frac{1}{N}\sum_{i=1}^{N} Z_i, \tag{2.7}$$

Since the sequence of independent identically distributed random variables $\{Z_i\}_{i=\overline{1,N}}$, with the mathematical expectation, follows the law of large numbers (theorem of A.Y. Khinchin [20]), the arithmetic mean of these values converges with the expectation $MZ = I$ when $N \to \infty$, i.e., for any $\varepsilon > 0$

$$P\left\{\left|\overline{Z}_N - I\right| \geq \varepsilon\right\} \to 0. \tag{2.8}$$

Briefly, this is written as

$$\overline{Z}_n \xrightarrow{P} I. \tag{2.9}$$

Thus, for large N, the value of $\overline{Z}_n \approx I$ and the evaluation (2.7) can be used in all cases when there is $MZ = I$, for which, as indicated above, it is necessary and sufficient that the integral

$$M|Z| = \int_G |f|x||p(x)\mathrm{d}x. \tag{2.10}$$

The existence of the integral (2.10) follows from restrictions

$$0 < f(x) \leq c, \tag{2.11}$$

which comes from the definition of GV considered by us $\Pi_k (k = 1, \ldots, 15)$ (see Fig. 2.1).

Let us say that there is the Monte Carlo method for calculating some scalar value a, if there is a random variable η, where its expectation is equal to a:

$$M\eta = a, \tag{2.12}$$

and the estimate for a is the arithmetic mean

$$a \approx \frac{1}{N}\sum_{i=1}^{N} \eta_i, \tag{2.13}$$

Here $\eta_1, \ldots, \eta_N$—the independent values of η. With respect to the integral (2.4), we have: $a = I$, $\eta = Z$, $\eta_i = Z_i$, $Z_i = f(Q_i)(i = 1, \ldots, N)$.

Let us say that there is the *Monte Carlo algorithm* for calculating the value of a, if, besides the formula (2.13), there is formula

$$\eta = \Phi\left(\gamma_1, \ldots, \gamma_{n,\ldots}\right), \tag{2.14}$$

which expresses the desired value of a through the independent uniformly distributed random numbers $\gamma_1, \ldots, \gamma_n, \ldots$.

Definition 1 [19]. If the function Φ depends on n arguments

$$\Phi = \Phi(\gamma_1, \ldots, \gamma_n),$$

then we say that the constructive dimension (c.d.) of the algorithm (2.13)–(2.14) is equal to n.

In this case, for the ith implementation of "test," it is sufficient to choose the n random numbers $\gamma_1^{(i)}, \ldots, \gamma_n^{(i)}$ and based on them to calculate the values of $\eta_i = \Phi\left(\gamma_1^{(i)}, \ldots, \gamma_n^{(i)}\right)$. The constructive dimension of n is the maximum number of random values that may be required to implement a test.

By condition, since each of the independent variables $\gamma_1, \ldots, \gamma_n$ is uniformly distributed in the interval (0; 1), the function $\Phi(\gamma_1, \ldots, \gamma_n)$ is defined in the unit n-dimensional cube

$$K^n = \{0 < y_1 < 1, \ldots, 0 < y_n < 1\}, \tag{2.15}$$

and the random n-dimensional value of $Q = \{\gamma_1, \ldots, \gamma_n\}$ is uniformly distributed in the K^n: Its density is $p_Q(y_1, \ldots, y_n) = 1$ for $(y_1, \ldots, y_n) \in K^n$. Therefore, the desired value of a can be written in the form of the n-dimensional integral over K^n:

$$a = M\eta = M\Phi(Q) = \int_0^1 \int_0^1 \Phi(y_1, \ldots, y_n)\mathrm{d}y_1 \ldots \mathrm{d}y_n. \tag{2.16}$$

Thus, the common interpretation of the Monte Carlo algorithm is as follows: If the structural dimension of the algorithm is n (c.d. = n), then the algorithm is the approximation method for calculating the n-dimensional integral (2.16) with random points $Q^{(i)} = \left(\gamma_1^{(i)}, \ldots, \gamma_n^{(i)}\right)$, evenly distributed in the K^n

$$\int_{K^n} \Phi(Q)\mathrm{d}Q \approx \frac{1}{N}\sum_{i=1}^{N} \Phi\left(Q^{(i)}\right). \tag{2.17}$$

Here and below, for brevity, we use the notation

$$\int_{K^n} \Phi(Q)\mathrm{d}Q \equiv \int_0^1 \ldots \int_0^1 \Phi(y_1, \ldots, y_n)\mathrm{d}y_1 \ldots \mathrm{d}y_n. \tag{2.18}$$

Formula (2.17) is equivalent to (2.13) and (2.14).

We assume that the domain G in (2.4) is the n-dimensional parallelepiped $\Pi^n = \{a_j \le x_j \le b_j, j = 1, \ldots, n\}$, which can be easily reduced to the n-dimensional cube (2.15) through the linear transformations

$$y_j = \frac{x_j - a_j}{b_j - a_j}(j = 1, \ldots, n).$$

In the case of the integral (2.14) (with the proviso that n-dimensional random quantity $Q = (\gamma_1, \ldots, \gamma_n)$ is evenly distributed in the cube K^n), we have $\Phi(Q) = f(Q)$ and the formula (2.16) can be written in the form of

$$I = \int_0^1 \ldots \int_0^1 f(y_1, \ldots, y_n)\mathrm{d}y_1 \ldots \mathrm{d}y_n, \tag{2.19}$$

where $(y_1, \ldots, y_n)$—the value of n-dimensional random variable Q.

The question naturally arises: Is it possible to specify a non-random sequence of points $P^{(1)}, \ldots, P^{(i)}, \ldots$ of K^n such that

$$\int_{K^n} \Phi(Q)\mathrm{d}Q = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \Phi\left(Q^{(i)}\right). \tag{2.20}$$

Definition 2 [19]. The sequence of points $P^{(1)}, \ldots, P^{(i)}, \ldots$ is uniformly distributed in the K^n, if (2.20) holds for any function $\Phi(y_1, \ldots, y_n)$, integrable in K^n according to Riemann.

Weyl introduced this notion in 1916 [21], and also produced examples of the uniformly distributed sequences. Therefore, the sequence $\{P^{(i)}\}$, uniformly distributed by the Definition 2, is also called the *Weyl equidistributed*.

Recall that the Riemann integral is defined only for limited functions.

The comparison of the formulas (2.20) and (2.17) shows that for the implementation of Monte Carlo algorithms with c.d. = n, we can try to use instead of the random points $Q^{(i)}$ the points of sequence $\{P^{(i)}\}$, which are Weyl equidistributed. To do this, at the i-th step of "test" instead of random numbers $\left(\gamma_1^{(i)}, \ldots, \gamma_n^{(i)}\right)$ we use the Cartesian coordinates $y_{i1}, \ldots, y_{in}$ of the point $P^{(i)}$. The relation (2.20) guarantees the convergence of this calculation method for the majority, occurring in practice, of the Monte Carlo algorithms.

The equality (2.20) is not violated if we change any finite number of points in the sequence $\{P^{(i)}\}$. However, the convergence of arithmetic means toward the limit

and, in this case, can become very slow. Therefore, in practice, not every the Weyl equidistributed sequence is reasonable to use as pseudorandom points. Among all uniformly distributed sequences, we should select "good" ones in some sense. Such sequences are the sequence of Holton $P^{*(i)}$ and the ЛПτ sequence of $Q^{*(i)}$, which will be described below.

Let G denote an arbitrary n-dimensional domain that belongs to K^n and through V_G—its volume (n-dimensional). Next, let $S_N(G)$ be the number of points with numbers $1 \le i \le N$, belonging to G.

Theorem 1 (Weyl) *For the sequence of points* $P^{(1)}, \ldots, P^{(i)}, \ldots$ *to be uniformly distributed in the* K^n *(by Definition 2), it is necessary and sufficient that for every* $G \subset K^n$

$$\lim_{N\to\infty}\left[\frac{S_N(G)}{N}\right] = V_G. \tag{2.21}$$

This shows that for the large values of N, the number of points belonging to G, among points $P^{(1)}, \ldots, P^{(N)}$, is approximately proportional to the volume of V_G.

Let us consider a random point $\Gamma = (\gamma_1, \ldots, \gamma_n)$, uniformly distributed in the K^n, and N of its independent realizations $\Gamma^{(1)}, \ldots, \Gamma^{(N)}$. Since the probability is $P(\Gamma \in G) = V_G$, the convergence of the hitting rate for these implementations in G to the probability of hitting G means that

$$\left[\frac{S_N(G)}{N}\right] \xrightarrow{P} V_G.$$

The comparison of this formula with (2.21) shows once again that the points $Q^{(i)} = \left(Q_1^{(i)}, \ldots, Q_n^{(i)}\right)$ of the non-random Weyl equidistributed sequence $\left\{Q^{(i)}\right\}$ are the analogues of independent realizations $\Gamma^{(i)}$ of the uniformly distributed random point $\Gamma = (\gamma_1, \ldots, \gamma_n)$.

The "uniformity" of distribution of the non-random sequences can be evaluated using the value called the deviation. To determine it, we choose in K^n some arbitrary point P and it is denoted by Π_P parallelepiped with the diagonal OP and sides parallel to the coordinate axes (Fig. 2.3).

The deviation of the group of points $P^{(1)}, \ldots, P^{(N)}$ is defined by

$$D_N = \sup|S_N(\Pi_P) - N \cdot V_{\Pi_P}|, \quad P \in K^n. \tag{2.22}$$

Theorem 2 [19]. *For the sequence of points* $P^{(1)}, \ldots, P^{(N)}, \ldots$ *to be uniformly distributed in the* K^n *(in the sense of Weyl), it is necessary and sufficient that*

$$\lim_{N\to\infty}\left[\frac{D_N}{N}\right] = 0.$$

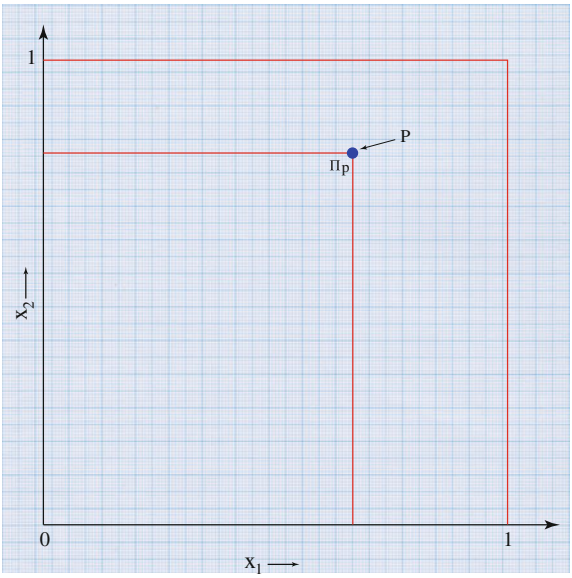


Fig. 2.3 Parallelepiped with the diagonal OP and sides parallel to the coordinate axes

It is obvious that the faster the ratio D_N/N decreases, the more evenly the sequence is distributed. In the literature, the ratio D_N/N is often referred to as a deviation, as this is the upper bound of deviations of the empirical distribution function $S_N(\Pi_P)/N$ for points $P^{(1)}, \ldots, P^{(N)}$ from the theoretical distribution function of random point Γ in K^n, in which the point P is equal to V_{Π_P}.

We can prove that $\frac{1}{2} \leq D_N \leq N$, but it is unclear which is the best order growth of D_N at $N \to \infty$. At the present time, there are only two known classes of sequences of points in K^n, such that for all N

$$D_N = O(\ln^n N). \tag{2.23}$$

This sequence of Holton [22] and the ЛП_τ sequences [23] are denoted by P_i^* and Q_i^*, respectively. Examples of such sequences are listed below. Are there any sequences for which $D_N = O(\ln^n N)$ for all N is unknown. However, for the points $Q^{*(1)}, \ldots, Q^{*(N)}$ in the ЛП_τ sequence with $N = 2^m$, the deviation is $D_N = O\left(\ln^{n-1} N\right)$.

Let us investigate in more detail the convergence of limit in (2.20). The Eq. (2.20) holds for all Riemann integral functions $\Phi(y_1, \ldots, y_n)$. If we consider the more restricted classes of functions, then there can be an estimate error of this formula. For example, the inequality

$$\left| \int_{K^n} \Phi(P) \mathrm{d}P - \frac{1}{N} \sum_{i=1}^{N} \Phi\left(P^{(i)}\right) \right| \leq c(\Phi) \frac{D_N}{N}, \tag{2.24}$$

where $c(\Phi)$ does not depend on neither N, nor points $P^{(i)}$ being valid for all $P^{(1)}, \ldots, P^{(N)}$ and for all functions $\Phi(y_1, \ldots, y_n)$, which are continuous and bounded in K^n along with their partial derivatives containing not more than one differentiation with respect to each variable. All of these derivatives can be written by the formula $\partial^k \Phi / \partial y_{j_1} \ldots \partial y_{j_k}$, where $1 \le j_1 < j_k \le n$ and k can take the values $1, 2, \ldots, n$. The oldest among these derivatives is $\partial^n \Phi / \partial y_1 \ldots \partial y_n$.

By the way, in the integral (2.4) with $\Phi(y_1, \ldots, y_n) = f(y_1, \ldots, y_n)$, the above-mentioned conditions for the smoothness of Φ, where the inequality (2.23) holds, are fulfilled for all considered GV $\pi_k (k = 1, \ldots, 15)$, if their constituent variables P_i satisfy the condition $P_i > 0$ in the domain G.

If the sequence of Holton $P_i = P_i^*$ or ЛПτ sequence $P_i = Q_i^*$ is substituted into (2.24), then according to (2.23) the right-hand side will be the order of $O(N^{-1} \ln^n N)$. Since for all sufficiently large N, the inequality $\ln^n N < N^{\varepsilon}$ (for any fixed $n \ge 1$ and $\varepsilon > 0$) is valid, we can say that the error (2.24) decreases faster than $N^{-(1-\varepsilon)}$ with any $\varepsilon > 0$. The accuracy of formula (2.20) with the "real" random points is equal to $N^{\frac{1}{2}}$, i.e., much worse.

Definition 3 [24]. The numbers $\gamma_1, \ldots, \gamma_n$, which are calculated by any given formula and resemble the random numbers with their statistical properties, are called the *pseudorandom numbers*.

The "good" pseudorandom numbers are the terms of Holton sequence P_i^* or the terms of ЛПτ sequence Q_i^*.

Definition 4 [19]. If in the r-ary $(r \ge 2)$ system the numeration is $i = a_m a_{m-1} \ldots a_2 a_1$, then again in the r-ary system

$$p_r(i) = 0, \quad a_1 a_2 \ldots a_{m-1} a_m.$$

The entire a_s are the r-ary digits, i.e., they are equal to one of the values $0, 1, 2, \ldots, r-1$.

In the decimal system, the last two formulas are as follows:

$$i = \sum_{S=1}^{m} a_S r^{S-1}, \; p_r(i) = \sum_{S=1}^{m} a_S r^{-s}.$$

The first 10 values of $p_3(i)$ are listed in Table 2.1.

Let $r_1, \ldots, r_n$ be the pairwise prime numbers.

Definition 5 [19]. The sequence of Holton is a sequence of points in the unit cube K^n (2.15) with the Cartesian coordinates.

$$(p_{r_1}(i), \ldots, p_{r_n}(i)), \quad i = 1, 2, \ldots.$$

The sequences were designed by Holton [22], and he also obtained values for them (2.23). All such sequences are distributed uniformly in K^n (according to Definition 2).

Usually in practice, as $r_1, \ldots, r_n$, we choose the first n primes $r_1 = 2, r_2 = 3, r_3 = 5, \ldots$ and use the n-dimensional points

$$P_i^* = (p_2(i), p_3(i), p_5(i), \ldots, p_{r_n}(i)), \quad i = 1, 2, \ldots.$$

Definition 6 [19]. The points $Q_i = (q_{i_1}, \ldots, q_{i_1})$ of the unit n-dimensional cube are defined by the formula

$$Q_i = e_1 V_1 \cdot e_2 V_2 \cdot \ldots \cdot e_m V_m. \tag{2.25}$$

where $i = e_m e_{m-1} \ldots e_2 e_1$—the binary representation of i ($e_m \in \{0, 1\}$—the binary digits) and $V_s (s = 1, 2, 3, \ldots, m)$—the guiding points, the denominators of coordinates of which are equal to 2^s, and the numerators are defined in Table 2.2, called the points of $ЛП_\tau$ sequences studied in detail in [23].

Here, (*) denotes the bitwise addition modulo 2 in the binary system. As a rule, all computers have a dedicated function to carry out operations (*): This is the so-called function of "comparison" (in each digit, the numbers are added by the rules $0 + 0 = 1 + 1 = 0$, $0 + 1 = 1 + 0 = 1$).

Here is an example of calculating the coordinates of V_s for $s = 3$ and $n = 13$

$$V_s = \left(\frac{1}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{8}, \frac{5}{8}, \frac{7}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{8}, \frac{5}{8}\right).$$

Here, the denominator is $2^3 = 8$, and the numerators for $s = 3$ are taken from the third column of Table 2.2. If there is a need in a point of lower dimension n, then it should be limited to the first n-numbers of the sth column.

Table 2.2 makes easy to calculate the points $Q_0, Q_1, \ldots, Q_{n-1}$ that for any N form, the "good" integration formula (2.20) can be written as

$$\int_0^1 \ldots \int_0^1 f(y_1, \ldots, y_n) \mathrm{d}y_1 \ldots \mathrm{d}y_n \approx \frac{1}{N} \sum_{i=1}^{N-1} f\left(Q^{(i)}\right). \tag{2.26}$$

For an explanation of the rule (2.25), let us calculate the point Q_{22} in four-dimensional cube. In the binary system, the number 22 is written as 10110. Therefore, in accordance with (2.25), we obtain $Q_{22} = V_2 \cdot V_3 \cdot V_5$.

The coordinates of point $Q_{22} = \left(q_{22,1}, q_{22,2}, q_{22,3}, q_{22,4}\right)$ are as follows:

$$q_{22,1} = \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{1}{32} = 0.01 \cdot 0.001 \cdot 0.00001 = 0.01101 = \frac{13}{32};$$

$$q_{22,2} = \frac{3}{4} \cdot \frac{5}{8} \cdot \frac{17}{32} = 0.11 \cdot 0.101 \cdot 0.10001 = 0.1101 = \frac{29}{32};$$

$$q_{22,3} = \frac{1}{4} \cdot \frac{7}{8} \cdot \frac{13}{32} = 0.01 \cdot 0.111 \cdot 0.01101 = 0.11001 = \frac{25}{32};$$

$$q_{22,4} = \frac{3}{4} \cdot \frac{1}{8} \cdot \frac{31}{32} = 0.11 \cdot 0.001 \cdot 0.11111 = 0.00011 = \frac{3}{32}.$$

And so,

$$Q_{22} = \left(\frac{13}{32}, \frac{29}{32}, \frac{25}{32}, \frac{3}{32}\right).$$

The number of operations made on a computer to calculate Q_i increases along with i, but slowly decreases as $\log_2 i$. It uses only the simplest (logical) operations that are performed on computer faster than arithmetic operations.

The various coordinates of point Q_i are unequal: The coordinates with lower values are better distributed. Therefore, in the integrand function $f(y_1, \ldots, y_n)$, the variables are better to be numbered so that the most significant coordinates have smaller values.

The accuracy of formula (2.26) is better estimated, if instead of points Q_i we use nodes of the Π_τ mesh or the initial parts of ЛП_τ sequence:

$$\|\delta\| \approx \frac{2^{n-1+\tau}}{N^{\frac{1}{p}}}, \tag{2.27}$$

if the function f belongs to the linear space S_p.

The order of convergence (at $N \to \infty$) in (2.27) is best. The space S_p is entered using the function represented by the series of Haar:

$$f(P) = c_1 + \widehat{\sum} \sum_{k_1 \ldots k_s} c^{i_1 \ldots i_s}_{k_1 \ldots k_s} \chi_{k_1}(x_{i_1}) \ldots \chi_{k_s}(x_{i_s}) \tag{2.28}$$

where

$$c_1 = \int_{K^n} f(P)\, \mathrm{d}P,$$

$$c^{i_1 \ldots i_s}_{k_1 \ldots k_s} = \int_{K^n} f(P) \chi_{k_1}(x_{i_1}) \ldots \chi_{k_s}(x_{i_s})\, \mathrm{d}P\,,$$

the indexes $k_1, \ldots, k_s$ vary from 2 to ∞. Each of the quantities in (2.27) located under the sign of $\widehat{\sum}$ depends only on the variables $x_{i_1}, \ldots, x_{i_s}$, and we can assume that it is given on the verge of $K_{i_1 \ldots i_s}$ of cube K^n.

The Haar function $\chi(x)$ of one variable x is defined by the formula

$$f(y) = \begin{cases} 2^{\frac{m-1}{2}} \text{at} x \in l_{mj}^{-} \\ -2^{\frac{m-1}{2}} \text{at} x \in l_{mj}^{+} \\ 0 \text{ at} x \notin l_{mj} \end{cases}, \tag{2.29}$$

where $l_{mj} = \left[\frac{j-1}{2^{m-1}}, \frac{j}{2^{m-1}}\right]$, $(j = 1, \ldots, 2^{m-1})$—the binary segments obtained by dividing the $[0, 1]$ interval into 2^m equal parts $(m = 1, 2, \ldots)$; l_{mj}^{-} and l_{mj}^{+}—the left and right halves of the segment l_{mj}.

Definition 7 [23]. The class $S_P(A_{i_1 \ldots i_s})$ is the set of functions $f(P)$, represented in the form of (2.28), and the coefficients of the Fourier–Haar satisfy the conditions:

$$A_p^{i_1 \ldots i_s}(f) \le A_{i_1 \ldots i_s}, \tag{2.30}$$

In any marked indexes $1 \le i_1 < i_2 < \cdots < i_s$, $1 \le s \le n$; $A_{i_1 \ldots i_s}$—constants; $1 \le p < \infty$; the value $A_p^{i_1 \ldots i_s}(f)$ is associated with each term in the series (2.28)

$$\sum_{k_1 \ldots k_s} c_{k_1 \ldots k_s}^{i_1 \ldots i_s} \chi_{k_1}(x_{i_1}) \ldots \chi_{k_s}(x_{i_s}) \tag{2.31}$$

and it is defined as

$$A_p^{i_1 \ldots i_s}(f) \le \sum_m 2^{\frac{m_1-1}{2} + \cdots + \frac{m_s-1}{2}} \left\{ \sum_j \left| c_k^i \right|^p \right\}^{\frac{1}{p}}, \tag{2.32}$$

where for short $i = (i_1, \ldots, i_s)$, $m = (m_1, \ldots, m_s)$, $j = (j_1, \ldots, j_s)$, $k = (k_1, \ldots, k_s)$.

The constants $A_{i_1 \ldots i_s}$ from (2.30) are called *defining constants* of class $S_p(A_{i_1 \ldots i_s})$.

The union of all classes $S_p(A_{i_1 \ldots i_s})$ for all possible constants $A_{i_1 \ldots i_s}$ and substitutions $i_s \ldots i_s \subset (1, 2, \ldots, n)$ will be denoted by S_p.

If $1 < p < p'$, then it follows:

$$S_1(A_{i_1 \ldots i_s}) \subset S_P(A_{i_1 \ldots i_s}) \subset S_{p^1}(A_{i_1 \ldots i_s}). \tag{2.33}$$

For any function $f(P)$ on $S_P(A_{i_1 \ldots i_s})$, the series (2.28) converges absolutely and uniformly.

As it is known, for the functions $f(x)$ of one variable x the Holder class functions H_α (in particular, the differentiable function $f(x)$ belongs to the class H_α with $\alpha = 1$) are embedded in the class S_P. This property can be generalized to the class of functions $f(x_1, \ldots, x_n)$.

Definition 8 [23]. The class of Holder functions $H_\alpha(L_{i_1 \ldots i_s})$ is the set of functions $f(P)$, which satisfy the following conditions: If $P = (x_1, \ldots, x_n) \in K^n$ and $P + Q \in K^n$, where $Q = (\xi_1, \ldots, \xi_n)$, then for any $1 \le i_1 < i_2 < \cdots < i_n \le n$, $1 \le s \le n$

$$\left|\Delta_{\xi_{i_1}} \ldots \Delta_{\xi_{i_s}} f(P)\right| \le L_{i_1 \ldots i_s} \left|\xi_{i_1} \ldots \xi_{i_2}\right|^{\alpha}, \tag{2.34}$$

where $\Delta_{\xi_{i_s}} f(P) = f\left(x_1, \ldots, x_{i_s} + \xi_{i_s}, \ldots, x_n\right) - f(x_1, \ldots, x_n)$—the increment operator of function $f(P)$ with respect to x_{i_s}. The constants $L_{i_1 \ldots i_s}$ are called *defining constants* of class $H_\alpha(L_{i_1 \ldots i_s})$; the parameter is $0 < \alpha < 1$.

Just as in the one-dimensional case, if $\alpha < \alpha' < 1$, then

$$H_1(L_{i_1 \ldots i_s}) \subset H_{\alpha'}(L_{i_1 \ldots i_s}) \subset H_\alpha(L_{i_1 \ldots i_s}). \tag{2.35}$$

The classes $H_\alpha(L_{i_1 \ldots i_s})$ are the generalization of n-dimensional case of the class of Holder functions with one variable $H_\alpha(L)$. However, it should be emphasized that the condition (2.34) is different from the Holder multidimensional conditions used in the theory of differential equations [25]:

$$|f(P+Q) - f(P)| \le \sum_{i=1}^{n} L_i |\xi_i|^{\alpha}, \tag{2.36}$$

For a function $f(P)$ satisfies (2.36) with $\alpha = 1$, it is sufficient that all its partial derivatives $\partial f / \partial x_i$ were limited: $|\partial f / \partial x_i| \le L_i$. And to ensure that it satisfies (2.34) with $\alpha = 1$, this is not enough: We must require that all partial derivatives $\partial^s f / \partial x_{i_1} \ldots \partial x_{i_s}$ were limited, as if $|\partial^s f / \partial x_{i_1} \ldots \partial x_{i_s}| \le L_{i_1 \ldots i_s}$, then the equality

$$\Delta_{\xi_{i_1}} \ldots \Delta_{\xi_{i_s}} f(P) = \int_0^{\xi_{i_1}} \ldots \int_0^{\xi_{i_s}} \frac{\partial^s f(P+T)}{\partial x_{i_1} \ldots \partial x_{i_s}} \mathrm{d}t_{i_1} \ldots \mathrm{d}t_{i_s},$$

where $T = (0, \ldots, 0, t_{i_1}, 0, \ldots, 0, t_{i_2}, 0, \ldots, 0, t_{i_s}, 0, \ldots, 0)$, and it is easy to obtain (2.34) with $\alpha = 1$.

Theorem 3 [23]. *If* $\alpha p > 1$, *then* $H_\alpha(L_{i_1 \ldots i_s}) \subset S_P(A_{i_1 \ldots i_s})$, *where*

$$A_{i_1 \ldots i_s} = \left(2^{1+\alpha} - 2^{1+\frac{1}{p}}\right)^{-s} L_{i_1 \ldots i_s}. \tag{2.37}$$

We denote the GV $\varphi(x_1, \ldots, x_n)$ in the first of formulas (2.3) through $f(x_1, \ldots, x_n)$, and the probability density function $f(x_1, \ldots, x_n)$ through $p(x_1, \ldots, x_n)$. Assuming that the integrals in (2.3) after the above linear transformations lead to integrals over the unit cube K^n, on the basis of formula (2.19), we have

$$m_\varphi = \int_0^1 \ldots \int_0^1 f(y_1, \ldots, y_n)\, \mathrm{d}y_1 \ldots \mathrm{d}y_n. \tag{2.38}$$

besides the integral in (2.38) can be calculated from the approximate quadrature formula (2.26), where as the nodes $Q_i = \left(y_1^i, \ldots, y_n^i\right)$ of the integration grid points

are taken the points $Q_i^* = \left(y_1^i, \ldots, y_n^i\right)$ of the ЛПТ$_\tau$ sequence (2.25), and not the random points obtained through the random number generator.

Since the function $f(x_1, \ldots, x_n)$ for all 15 GV has the bounded partial derivatives $\partial^s f / \partial y_{i_1} \ldots \partial y_{i_s}$, these functions also satisfy the conditions (2.34) with $\alpha = 1$ and therefore belong to the class S_p, which allows to use the formula (2.27) for the error approximation formula (2.26) in the case of the integral (2.38) for m_φ and the corresponding integral for $D_\varphi = M\left[\left(\varphi - m_\varphi\right)^2\right]$.

Unfortunately, the constants $2^{n-1+\tau(n)}$ increase along with n in the error estimation (2.27). In any case, when $n = 2, 3, 4$, these constants are equal to, respectively, 2, 4, and 16 and can be considered small. Therefore, the use of non-random nodes of the quadrature formula (2.26) in the form of ЛПТ$_\tau$ sequences, defined by (2.25), is recommended in [23] as a good method for the computation of integrals with not too high multiplicity $(n \leq 4)$ from not too smooth functions, which can be also useful if $n > 4$.

As noted in [26], the existing generators of the pseudorandom numbers, developed to model these distributions, perform this function well; however, an attempt to consider the sequences generated by the generator (e.g., RANDU), as the trajectories of a random sequence (or rather a sequence of independent identically distributed random variables with corresponding distribution), is theoretically incorrect and can lead to the false conclusions. The simulated process is obtained by applying a model through filter F to the generator of white noise $X(t)$, and at the same time, the researchers often assume the adequacy of the sequence generator of white noise or the statistical uncorrelatedness of its successive observations $X(t)$.

Generating the pseudorandom numbers of satisfactory quality is a very complicated process, since there are no algorithms that could produce sequences having all the properties of random sequences.

In this regard, the above-described methods for the approximate calculation of multidimensional integrals with the help of grid points in the unit cube forming Q_i^* points of the ЛПТ$_\tau$ sequences are the virtually convenient way to calculate the expectation and variance of the GV from Fig. 2.1.

If we approach the issue more closely, then it will be good to impose following requirements on of pseudorandom numbers:

1. the asymptote of D_N (2.22) is the best (or at least close to the best);
2. the constant in (2.23) is the best (or at least sufficiently small);
3. the value of D_N/N is small even for small N;
4. the algorithm for calculating these points on a computer is simple enough.

Unfortunately, it is not currently possible to check all of these requirements; as for D_N, the best order of growth is not even known. However, the points of P_i^* (terms of the Holton sequence) and the points Q_i^* (terms of the ЛПТ$_\tau$ sequence) fully satisfy the first condition. For small n, the points of Q_i^* satisfy the second and third requirements. Finally, the calculation time for the points Q_i^* is of the same order as the time for standard pseudorandom points Γ_i (unless there is a ready table

for $V_j^{(S)}$). To calculate the points P_i^*, we need only n primes, but in comparison with time to calculate Q_i^*, it is approximately n times greater.

In the calculations performed on the points Q_i^*, the actual error is often on the order less than the probable error

$$r_N = 0.6745\sqrt{\frac{D_\eta}{N}}. \tag{2.39}$$

where η—the random variable in (2.12).

We should not think that if the variance does not find an error, then all methods aimed at reducing the variance [19, 27] are meaningless. Firstly, the algorithms with a less variance correspond to the functions Φ in (2.14) with a little change, which, generally speaking, are better to integrate. Secondly, the algorithms with a less variance often correspond to the smooth functions Φ, satisfying the criteria (2.24).

We can expect higher convergence rate through a deterministic random number by using the Monte Carlo methods [19].

Taking into account that in practice we do not require a high precision of determination of the GV probability distribution, the evaluation approach to these issues [11] becomes obvious. For the purposes to simplify the analysis of the GV probability distribution, we represent the expression (2.1) in the form:

$$\ln \pi_k = \sum_{i=1}^{n} \alpha_i \ln P_i. \tag{2.40}$$

Therefore, the probability distribution π_k will tend to approach the lognormal view; it can be used as the statistical model for the random variables. The value of GV is obtained by multiplying a large number of small errors [28]. This premise can be taken as the basis for studying the patterns of the probability distribution of the parameters that are included in the GV. If deviations ΔP_i do not exceed (10–20) % of their nominal value P_i, then the expression (2.40) can be written as follows:

$$\ln \pi_k = \sum_{i=1}^{n} \alpha_i \ln[1 + (P_i - 1)] \approx \sum_{i=1}^{n} \alpha_i (P_i - 1), \tag{2.41}$$

i.e., to linearize the function (2.40) and take into account only the first term of the expansion (2.41). In this case, the estimate of the variance is easily defined

$$D(\ln \pi_k) = \sum_{i=1}^{n} \alpha_i^2 D(P_i).$$

Since the distribution P_i is accepted as normal, $(\ln \pi_k) \approx \sum \alpha_i (P_i - 1)$ is also distributed approximately normal.

In general, the probability density function π_k will be determined as follows (at $M(\ln \pi_k) = 0$):

$$P(\pi_k < \pi) = F(\pi) = F(\ln \pi_k < \ln \pi) = \Phi\left(\frac{\ln \pi}{\sqrt{\sum_{i=1}^{n} \alpha_i^2 D(P_i)}}\right), \tag{2.42}$$

$$F(\pi) = \frac{dP(\pi_k < \pi)}{d\pi} \approx \frac{1}{\sqrt{2\pi \sum \alpha_i D(P_i)}} \exp + \left[\frac{1}{2} \cdot \frac{\ln \pi}{\sum_{i=1}^{n} \alpha_i D(P_i)}\right]. \tag{2.43}$$

Let us consider the case with $\Delta P_i = \pm 20\ \%$, or

$$\alpha = \frac{1}{15}.$$

$$D(\ln \pi_k) = \frac{1}{225} \sum_{i=1}^{n} \alpha_i^2,$$

and substitute in (2.43) the values of the various structures π_k, shown in Fig. 2.1.

For any GV structure, we can write the corresponding expressions of the probability density function π_k.

For example, for GV 2, 12, and 13

$$\begin{aligned} &F(\pi) \approx \frac{15}{\sqrt{50\pi}} \varphi\left(\frac{15 \ln \pi}{\sqrt{50}}\right); \; F(\pi) \approx \frac{15}{\sqrt{4\pi}} \varphi\left(\frac{15 \ln \pi}{\sqrt{4}}\right); \\ &F(\pi) \approx \frac{15}{\sqrt{7\pi}} \varphi\left(\frac{15 \ln \pi}{\sqrt{7}}\right) \end{aligned} \tag{2.44}$$

These expressions can be used to define the GV range variation and also their probability distributions. For example, asking the value of standard deviation σ, we can find the variation limits of π_k.

For the practical purpose, we set $P = 0.95$ in most cases. The calculations showed that the value of π with probability $P = 0.95$ is in that area.

Since the results of evaluation approach are compared with the results obtained through the statistical tests, we need also to examine the probability distribution of GV depending on the particular set of random numbers. It is obvious that the probability distribution of the same GV, defined by statistical tests twice with the same number of tests, will be somewhat different from each other. The conducted comparative calculations show that the error of the results does not exceed 12 %.

It should be noted that the probability distributions of GV determined by the evaluation procedure are within the range of the distributions obtained by the statistical tests. The analysis of GV probability characteristics (Tables 2.3 and 2.4) showed that the GV distribution curves are close to the lognormal distribution.

Table 2.3 The analysis of GV probability characteristics

GV #	$\overline{\pi}$	σ	S_π	E_π	$\pi_{\min}$	$\pi_{\max}$
1	4.66×10^{2}	3.74×10	0.44	0.027	3.76×10^{2}	5.76×10^{2}
2	2.44×10^{-2}	5.30×10^{-3}	0.47	0.075	1.37×10^{-2}	4.31×10^{-2}
3	8.62×10	8.60	0.282	0.155	6.55×10	1.12×10^{2}
4	2.60×10	7.08×10^{2}	0.074	0.327	2.52×10^{4}	2.90×10^{4}
5	2.78×10^{-6}	1.29×10^{7}	0.044	0.292	2.47×10^{-6}	3.14×10^{-6}
6	3.71×10^{-9}	1.71×10^{-10}	0.044	0.302	3.29×10^{-9}	4.19×10^{-9}
7	2.28×10^{6}	1.82×10^{4}	0.440	0.021	1.83×10^{6}	2.81×10^{6}
8	3.52	0.28	0.440	0.020	2.83	4.35
9	1.44×10^{3}	8.39×10	0.034	0.164	1.23×10^{3}	1.67×10^{6}
10	1.94×10^{-11}	3.11×10^{-12}	0.470	0.544	1.38×10^{-11}	2.76×10^{-11}
11	1.04×10^{16}	1.23×10^{15}	0.136	0.163	6.89×10^{15}	1.38×10^{16}
12	2.04×10^{-4}	7.50×10^{-5}	0.02	0.558	1.31×10^{-4}	3.99×10^{-4}
13	1.17×10^{-9}	8.21×10^{-11}	0.281	0.20	9.55×10^{-10}	1.40×10^{-9}
14	2.11×10^{-10}	2.89×10^{-11}	0.442	0.50	1.32×10^{-10}	3.04×10^{-10}
15	7.44×10^{4}	3.13×10^{2}	0.205	0.455	6.58×10^{4}	8.18×10^{4}

Table 2.4 The GV distribution curves

GV #	$\overline{\pi}$	σ	S_π	E_π	$\pi_{\min}$	$\pi_{\max}$
1	4.69×10^{2}	3.62×10	0.074	−0.679	3.84×10^{2}	5.61×10^{2}
2	2.41×10^{-2}	5.40×10^{-3}	0.472	−0.584	1.48×10^{-2}	3.90×10^{-2}
3	8.57×10	6.02×10^{-1}	0.296	−0.350	6.58×10	1.09×10^{2}
4	2.68×10^{4}	6.66×10^{2}	0.079	−0.346	2.52×10^{4}	2.86×10^{4}
5	2.77×10^{-6}	1.20×10^{-7}	0.209	−0.617	2.50×10^{-6}	3.05×10^{-6}
6	3.69×10^{-9}	1.58×10^{-10}	0.123	−0.518	3.33×10^{-9}	4.07×10^{-9}
7	2.29×10^{6}	1.76×10^{5}	0.074	−0.684	1.87×10^{6}	2.74×10^{6}
8	3.54	2.27×10^{-1}	0.074	−0.684	2.90	4.23
9	1.43×10^{4}	7.77×10	0.148	−0.603	1.26×10^{4}	1.67×10^{4}
10	1.97×10^{-11}	3.00×10^{-13}	0.386	−0.240	1.37×10^{-11}	2.83×10^{-11}
11	1.05×10^{16}	1.27×10^{15}	0.401	−0.287	8.16×10^{15}	1.50×10^{16}
12	1.93×10^{-4}	7.01×10^{-5}	0.047	−1.109	6.73×10^{-5}	3.54×10^{-4}
13	1.17×10^{-3}	8.90×10^{-11}	0.180	−0.642	1.0×10^{-3}	1.40×10^{-9}
14	2.09×10^{-10}	2.75×10^{-11}	0.503	−0.149	1.50×10^{-10}	2.50×10^{-10}
15	7.46×10^{4}	3.27×10^{2}	0.082		6.77×10^{4}	8.24×10^{4}

All distribution curves are characterized by a positive value of the asymmetry coefficient. The asymmetry of distribution curves increases when the parameter tolerance increases and the presence of higher orders in the first parameters. At the relatively lower values of tolerance, the distribution curves lie between normal and

lognormal. The distribution curves are characterized by the structure of GV and do not depend on the numerical values of parameters.

At P_i, with the normal distribution, the GV has six or more parameters, and the kurtosis of distribution curves would be somewhat larger (see Fig. 2.1, for GV 10, 14), when the uniform distribution is even less. When the product of parameters describes the GV, the distribution curves get closer to theoretical normal and uniform shapes, respectively (see Fig. 2.1, for GV 12). The GV distribution curves with similar structures can be used in studying other the physical processes. Applying the method of statistical tests or the evaluation techniques, we can find the influence characteristics of parameters included in the GV of friction processes. Based on the probability distributions of the process GV and applying the known models of failure, it is possible to calculate the reliability and predict the possible behavior of any particular friction unit.

2.2 Varying Generalized Variable When Planning Experiments

The questions on the synthesis of the similarity theory and the mathematical theory of experiment planning are highly relevant and promising [7, 12]. The application of the theory of experimental planning to the analysis of criterion correlations (of the GV) has a number of features associated with the fact that in this case there is no need to operate with separate parameters, like it is done usually, but with generalized parameters (the criterion of similarity) [11]. In those cases when the parameters are only included in one of the GV, the definition of variation pitch and the construction of the planning matrix can be carried out in accordance with [7]. The task becomes much more complicated when the same parameter is entered into the several GV.

Let us consider one of the possible ways of varying the GV and building the planning matrix. The experiment planning matrix (EPM), which contains the GV, can be successfully used for the generalized regression equations in the analysis of tribological problems.

Generally, in order to realize the full or fractional factorial experiment, we must meet the following conditions:

$$\sum_{u=1}^{N} X_{iu} = 0;\ \sum_{u=1}^{N} X_{iu}^2 = N;\ \sum_{u=1}^{N} X_{iu}X_{ju} = 0.$$

where u—the number of columns in matrix and N—the number of experiments. These conditions correspond to the symmetry, normalization, and orthogonality of the planning matrix.

The analysis of these conditions for the EPM, containing GV, shows that in the EPM building, the GV plays the role of factors, consisting of a number of parameters. Of course, while building the planning matrix in the form of GV, all the mentioned conditions are fully hold, i.e.,

$$\sum_{u=1}^{N} \pi_{iu} = 0;\ \sum_{u=1}^{N} \pi_{iu}^2 = N;\ \sum_{u=1}^{N} \pi_{iu}\pi_{ju} = 0.$$

The GV variation is achieved by varying the parameters included in it. If we analyze the matrix columns, relating to one of GV, the following is noted:

$$\sum_{u=1}^{N} X_{iu}\pi_{iu} = 0;\ \sum_{u=1}^{N} (X_{iu}\pi_{iu})^2 = N;\ \sum_{u=1}^{N} X_{iu}\pi_{iu}X_{ju}\pi_{ju} = \pm N.$$

where $X_{iu}\pi_i$—the parameter in the kth of GV, $+N$ corresponds with multiplication, and $-N$ corresponds with division. It will be seen that the EPM has the orthogonal and symmetric properties. In principle, there can be a variety of options of the GV, as by a combination of parameters and as by a combination of structures.

For the convenience of presentation, the proposed method is used as an example of variation of three GV:

$$\pi_1 = \frac{A}{BC};\ \pi_2 = \frac{DM}{A};\ \pi_3 = \frac{A}{NK}.$$

The GV variation limits are defined by the specific deviation parameters. The upper level is:

$$\pi_1 + \Delta\pi_1 = \frac{(A + \Delta A)}{(B + \Delta B)(C - \Delta C)};$$

$$\pi_2 + \Delta\pi_2 = \frac{(D + \Delta D)(M + \Delta M)}{(A - \Delta A)};$$

$$\pi_3 + \Delta\pi_3 = \frac{(A - \Delta A)}{(N - \Delta N)(K - \Delta K)}.$$

The lower level is:

$$\pi_1 - \Delta\pi_1 = \frac{(A - \Delta A)}{(B + \Delta B)(C + \Delta C)};$$

$$\pi_2 - \Delta\pi_2 = \frac{(D - \Delta D)(M - \Delta M)}{(A + \Delta A)};$$

$$\pi_3 + \Delta\pi_3 = \frac{(A - \Delta A)}{(N + \Delta N)(K + \Delta K)}.$$

Let us assume that in the first experiment, all A's must be at the upper level. In this case, the values of π_1 and π_3 have to be at the top level, and the values of π_2 are

at the lower level. Of course, both these conditions are not feasible. In these cases, the proposed parameters, occurring simultaneously in several GV, leave at the basic level. This way, setting of the upper and lower levels is achieved through the proportional changes of other parameters included in the GV. It is the proportionality of changes of the remaining GV allows to objectively maintain the regularity in the relationship between the GV and the target function.

The planning matrix of the full factorial experiment (FFE) for this case is shown in Table 2.5. The symbols used in the planning matrix are:

A, B, C, D, M, N, K—the baseline parameters; $\Delta B'$, $\Delta C'$, $\Delta D'$, $\Delta M'$, $\Delta K'$, $\Delta N'$—the parameter increments to compensate for the necessary change of parameters in order to achieve the levels of the corresponding changes of the GV, i.e., additional increment;
ΔA, ΔB, ΔC, ΔD, ΔM, ΔK, ΔN—the parameter increments corresponding to changes of the GV, i.e., major increments. The implementation of this matrix and the corresponding processing of results will determine the regression equation of the GV.

2.3 Application of Group Method of Data Handling with Respect to Tribotechnical Problems

The method for creating the friction and wear mathematical models, using the theory of similarity, dimensions, and the mathematical planning of experiment, is progressive, since the transition to GV dramatically reduces the number of factors to be considered, reduces the time and labor to carry out the experimental studies, and provides reasonable enough values for output parameters.

Let us illustrate this by comparison. Suppose that we need to give a mathematical description of process with one objective function and 8 variable parameters. For the mathematical definition, in accordance with the theory of experimental planning, it is necessary to have $N = 2^8 = 256$ experiments at the FFE, and an average of two overlapping randomizations, and these are additional 512 experiments. Thus, the total number of required tests becomes 768. We assume that these variables are included in the three criteria of similarity. In this case, the mathematical description will require $N = 2^3 = 8$ experiments. Consider that the duplication has the total number of required experiments equal to 24. Thus, the number of experiments reduces by 32 times. The comparison shows a clear advantage of combining the two methods—the theory of similarity and the experimental planning. The advisability of such a synthesis is obvious, since it leads to the optimal use of the capabilities of each of the methods.

The comprehensive review of all the opportunities arising from the combination of methods of the theory of similarity and the statistical methods seems to us very important.

Table 2.5 The planning matrix of the full factorial experiment

#	π_1			π_2			π_3		
1	+			+			+		
	A	$B-\Delta B-\Delta B'$	$C-\Delta C-\Delta C'$	A	$D+\Delta D+\Delta D'$	$M+\Delta M+\Delta M'$	A	$N-\Delta N-\Delta N$	$K-\Delta K-\Delta K'$
2	–			+			+		
	A	$B+\Delta B+\Delta B'$	$C+\Delta C+\Delta C'$	A	$D+\Delta D+\Delta D'$	$M+\Delta M+\Delta M'$	A	$N-\Delta N-\Delta N'$	$K-\Delta K-\Delta K'$
3	+			–			+		
	$A+\Delta A$	$B-\Delta B$	$C-\Delta C$	$A+\Delta A$	$D-\Delta D$	$M-\Delta M$	$A+\Delta A$	$N-\Delta N$	$K-\Delta K$
4	–			–			+		
	A	$B+\Delta B+\Delta B'$	$C+\Delta C+\Delta C'$	A	$D-\Delta D-\Delta D'$	$M-\Delta M-\Delta M'$	A	$N-\Delta N-\Delta N'$	$K-\Delta K-\Delta K'$
5	+			+			–		
	A	$B+\Delta B+\Delta B'$	$C+\Delta C+\Delta C'$	A	$D+\Delta D+\Delta D'$	$M+\Delta M+\Delta M'$	A	$N+\Delta N+\Delta N'$	$K+\Delta K+\Delta K'$
6	–			+			–		
	$A-\Delta A$	$B+\Delta B$	$C+\Delta C$	$A-\Delta A$	$D+\Delta D$	$M+\Delta M$	$A-\Delta A$	$N+\Delta N'$	$K+\Delta K$
7	+			–			–		
	A	$B+\Delta B-\Delta B'$	$C+\Delta C-\Delta C'$	A	$D-\Delta D-\Delta D'$	$M-\Delta M-\Delta M'$	A	$N+\Delta N+\Delta N'$	$K+\Delta K+\Delta K'$
8	–			–			–		
	A	$B+\Delta B+\Delta B'$	$C+\Delta C+\Delta C'$	A	$D-\Delta D-\Delta D'$	$M-\Delta M-\Delta M'$	A	$N+\Delta N+\Delta N'$	$K+\Delta K+\Delta K'$

2.3.1 Development of Mathematical Model with Initial Variables

The modern statistical methods of planning and analysis of experiments are increasingly used in the scientific research; with their help, we can significantly improve the efficiency of researches, requiring the considerable financial costs and long terms [7, 10, 29, 30]. The effect is especially significant in the study of complex multifactorial processes such as friction and wear processes [1]. The action of a large number of random factors usually leads to a fact that a proceeding of deviation does not always become "low," and they cannot be considered as corrections. The introduction of the various "factors of ignorance" reduces the quality of the calculations.

Most systems operate in accordance with the Pareto principle, which states that in terms of the system performance, there are only few important factors out of many. Indeed, in the most systems, the 20 % of factors define the 80 % of system, and the remaining 80 % of factors determine only 20 % of the system properties. Our task is to identify the significant factors [31]. To solve this problem, it is advisable to use the methods of group data handling (MGDH), a priori factor ranking, the rank correlation, the random balance, and others. The rational choice of an appropriate method is determined by the presence of a priori information about the tested process and the complexity of experiments.

At the present time to establish a connection between the input and output parameters, and to obtain the mathematical model that is adequate to the studied object, there are the widely used regression analysis and the method of group data handling (MGDH) [32]. We set n input variables $x_1, x_2, \ldots, x_n$ and output variable y. The search of the tribotechnical functional dependence $y = f(x_1, x_2, \ldots, x_n)$ is carried out in the class of polynomial functions, producing a sequential scan of the input variables and their various combinations in order to choose the most appropriate which will allow the best way to describe the experimental data. Two criteria determine the quality of description in MGDH, either the regularity criterion or criterion of a minimum offset [32].

We choose the selection criterion matching the requirements of desired model. The regularity is the main criterion, if the high accuracy is required from the model.

The model becomes more stable with respect to the initial experimental data when using the minimum offset criterion; in other words, by gradually increasing the complexity of model while changing the set of experimental data, the coefficients remained unchanged. The final mathematical model was presented as follows:

$$y = A_0 + \sum_{n=1}^{S} A_i Z_i,$$

where y—the output parameter; $A_0, A_1, \ldots, A_S$—the coefficients of mathematical model; Z_i—the generalized factor, a kind of GV, which are included in the test mode settings. The structure of the generalized factor is defined as follows:

$$Z_1 = \prod_{n=1}^{n} x_i^{k_{ij}}; \quad k_{ij} = 0, 1, 2.$$

The experiment is designed in accordance with the selected model and the study objectives. There are several ways of constructing the planning matrix of large dimension [33]. Here, we used a FFE. The planning and implementation of FFE consist of the following main steps: the selection of factors and levels and their variation; the encoding of factors; drafting up the EPM; the randomization of tests; and the implementation of the experiment plan.

There are literatures with a detail description on how to draft the EPM [34, 35]. In this work, we study the effect of sliding speed—V, loading—P, and braking work—W on the coefficient of friction during braking. After selecting variables and domain, there is a need to find a local area for the experiment. This procedure involves the choice of levels of the varying factors.

In order to start encoding, we initially choose the starting area of experiment by setting the upper and lower limits of change for each factor during the experiment. The upper level corresponds to $+1$, the lower level corresponds to -1, and the main is set at 0. It is easily done by using the formula that connects the values of factors in the coded scale (x_i) with ones in the natural scale (X_i):

$$x_i = \frac{X_i - X_{i_0}}{\Delta X_i}.$$

$$X_i - X_{i_0} = \Delta X_i x_i,$$

where $\Delta X_i = X_{i_{\max}} - X_{i_{\min}}/2$—the variation interval; X_{i_0}—the main level.

It should be noted that in the general case, the selection of the variation interval depends on the given problem.

We present the planning matrix and the test results of the experimental planing 2^3 of the friction material ФК-24А (for other materials the planning matrix is established in a similar way, see Table 2.6).

The three major classes of friction pad materials were tested (asbestos–resin, asbestos–rubber, metal–ceramic) pairing with steel 40XH for the descending mode of the drilling tool in accordance with the drafted planning matrix. After processing the experimental data on the computer, the following mathematical model of the studied object is obtained, i.e., the equation relating the coefficient of friction f with predetermining factors:

for the friction material ФК-24А

Table 2.6 Planning matrix and the test results

Factors		v	P	W	f_e	F_c
Main level		3.5	355	7.5		
Upper level		4.2	430	9.5		
Lower level		2.8	280	5.5		
Variation level		0.7	75	2.0		
Code designation		x_1	x_2	x_3	y_e	y_c
	1	+1	+1	+1	0.26	0.301
	2	+1	+1	–1	0.28	0.312
	3	+1	–1	+1	0.30	0.297
	4	+1	–1	–1	0.29	0.309
	5	–1	+1	+1	0.34	0.332
	6	–1	+1	–1	0.33	0.339
	7	–1	–1	+1	0.31	0.320
	8	–1	–1	–1	0.34	0.328
	9	$+R$	0	0	0.35	0.300
	10	$-R$	0	0	0.37	0.314
	11	0	$+R$	0	0.36	0.319
	12	0	$-R$	0	0.32	0.312
	13	0	0	$+R$	0.28	0.300
	14	0	0	$-R$	0.29	0.315
	15	0	0	0	0.30	0.321

$$f_1 = 0.312 - 0.0308\frac{v \cdot W^2}{P} + 0.0045\frac{P}{v^4} + 4.785\frac{v \cdot W}{P^2};$$

for the friction material Б—42

$$f_2 = 0.336 + 0.873\frac{v^2}{P} + 0.000466\frac{v^4 \cdot W}{P};$$

for the friction material МКБ—50

$$f_3 = 0.202 + 0.6017 \times 10^{-7} \cdot P^2 W + 0.92\frac{1}{v \cdot W} + 27.596\frac{1}{P}.$$

The indicator of successful synthesis of modeling of complex processes is the minimum depth of the basic selection criterion. All parameters of the calculated MGDH algorithms, as well as the structure of these algorithms, are chosen by selecting the number of options so as to obtain the deepest minimum. The modeling will not be considered complete if it cannot get the deepest minimum.

As shown in [32], the allowable minimum depends on the intensity of noise and it is 5–10 % for the usually occurring measuring accuracy of variables. The resulting models adequately reproduce the studied braking process, since the error of these equations according to the original sample is within 6–7 %, which is acceptable.

The resulting tribological functional dependences (the mathematical models) for various friction materials allow to reliably predict the behavior of the friction unit in the braking regime. The concept of a generalized factor is introduced, and its structure is formulated.

2.3.2 Development of Mathematical Model with Generalized Variables

Studying the friction and wear of various parts of machines, it is advisable to obtain a mathematical model, which is expressed as a tribotechnical functional relationship of the main regime, design, and operational parameters which determine the nature of the tested process. These dependencies can become main information that will be used by designers to create new designs of friction devices and their further operation.

However, the creation of mathematical models of friction and wear has a number of difficulties: the uncertainty in setting up the initial information; its probabilistic nature forces to conduct a large amount of experiments. In order to systematize the results of experiments, to determine the main characteristics of communication, to combine them into a general quantitative regularities it requires the experimental planning and the determination of minimum number of experiments. The mathematical model of process can be obtained using the modeling techniques and the statistical methods of experiment design.

During the braking process, a number of variables become a crucial factor [1]. With a large number of variables, it is extremely difficult and even practically impossible to bring the results into a specific system or to find the general quantitative regularities. Studying such a complex physical tasks, it is necessary to introduce a lot of dissimilar variables, and each of these quantities is considered as an independent variable.

Based on very general physical considerations, the multiplicity of relationships is not a specific property of the studied problems defined by their physical nature [11]. It is shown that the effect of individual factors, represented by the different values, should not be not considered individually, but as a complex, and that in fact it is necessary to consider not these individual values, but their combination defined for each process. It is known the method of constructing such populations—a method that allows the direct analysis of the problem formulation to find a connection between the individual values and their groups, and to combine them in the well-defined complexes of the specific form [11].

Thus, the new variables are inherently generalized and their application gives the whole analysis a generalized nature. The analysis prerequisite is the completeness of the initial information given by the parameter list, drawn up on the basis of the adopted process model by the researcher.

Since the studied process of friction is random, the task is, using the appropriate GV defining the friction and wear process in the braking units of the oil drilling equipment, to obtain the consolidate tribological functions for the qualitative and quantitative evaluations of the tested processes on the basis of the mathematical models.

The generalized equation obtained using the theory of dimensions for the braking process during the descent of drilling tool is presented in [17]. The explicit form of the generalized equation can be obtained by the statistical processing of the individual results of experiments.

Drawing up the plan and the experimental techniques, the following GV are selected to define the process:

the generalized thermal conductivity

$$\pi_\lambda = \pi_1 = \frac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \vartheta_1^{\frac{3}{2}} \cdot \vartheta_2^{\frac{3}{2}}}{P^3 \cdot v^3 \cdot K_{\Gamma_1}^{-\frac{1}{2}} \cdot K_{\Gamma_2}^{-\frac{1}{2}}};$$

the generalized hardness

$$\pi_{HB} = \pi_2 = \frac{HB_1 \cdot HB_2 \cdot K_{\Gamma_1}^{\frac{2}{3}} \cdot K_{\Gamma_2}^{\frac{2}{3}}}{P^2};$$

the generalized braking work

$$\pi_w = \pi_3 = \frac{W_{\text{T.П.}}}{P \cdot K_{\Gamma_1}^{\frac{1}{6}} \cdot K_{\Gamma_2}^{\frac{1}{6}}};$$

the generalized variable deceleration time

$$\pi_t = \pi_4 = \frac{v \cdot t}{K_{\Gamma_1}^{\frac{1}{6}} \cdot K_{\Gamma_2}^{\frac{1}{6}}}.$$

Here, π_1—the one of the main indicators of the material thermophysical properties; π_2—the hardness parameter, included in the molecular-mechanical model of friction, is the principal value; π_3 and π_4—the parameters defining the operation regime that have a great influence on the brake.

The experiment planning and methodology to identify the quantitative relations of the studied variables are described in detail in [17, 30]. Let us represent the planning matrix and the experimental results for the FFE type 2^4 of the friction

Table 2.7 The generalized mathematical model

Factors	π_1	π_2	π_3	π_4
Main level	2.5×10^3	16×10^6	3.3	11
Upper level	3×10^3	20×10^6	4.0	14
Lower level	2×10^3	12×10^6	2.0	8
Variation level	0.5×10^3	4×10^6	0.7	3
Code designation	x_1	x_2	x_3	x_4

material ФК-24A (the planning matrix for other materials will be similar). The MGDH is explicitly employed to build the generalized tribotechnical functions.

In order to establish the relationship between the friction process and the GV, the test is conducted for the friction pad materials of three major classes (of asbestos resin, of asbestos, rubber, metal–ceramic) paired with the steel 40XH for the descending regime of the drilling tool in accordance with the planning matrix. The generalized mathematical model was adopted as in Sect. 2.3.1 (Table. 2.7).

After processing the experimental data on the computer the following mathematical model is obtained, i.e., the equation relating the coefficient of friction f with its predetermined GV:

for the friction material ФК-24A

$$\begin{aligned} f_1 = {} & 0.294 - 0.1287 \times 10^{-5}\pi_1 \cdot \pi_3 \cdot \pi_4 + 0.29 \times 10^{-9}\pi_2 \cdot \pi_3 \\ & + 21.36\frac{\pi_1 \cdot \pi_4}{\pi_2} + 0.86 \times 10^{-8}\frac{\pi_2^2 \cdot \pi_3^4}{\pi_2} + 0.01546\frac{\pi_1 \cdot \pi_4}{\pi_2} \\ & - 3.21817\frac{\pi_2^2 \cdot \pi_3^3}{\pi_1^5 \cdot \pi_4^5} - 0.239\frac{\pi_1^3 \cdot \pi_3^5 \cdot \pi_4^3}{\pi_3^3} + 0.4 \times 10^{-9}\pi_1^3 \cdot \pi_3; \end{aligned}$$

for the friction material Б-42

$$\begin{aligned} f_2 = {} & 0.413 - 0.135 \times 10^{-6}\frac{\pi_2}{\pi_3^2} - 0.163 \times 10^{-6}\pi_1 \cdot \pi_3 \cdot \pi_4 - 0.368 \times 10^{-8}\frac{\pi_2^3}{\pi_1^2 \cdot \pi_3^2 \cdot \pi_4^2} \\ & - 0.321 \times 10^{-8}\frac{\pi_2^3}{\pi_1^2 \cdot \pi_3^8 \cdot \pi_4^2} + 0.367 \times 10^{-3}\frac{\pi_1^2 \cdot \pi_4}{\pi_2 \cdot \pi_3^3} + 11.138\frac{\pi_2^2}{\pi_1^3 \cdot \pi_3^7 \cdot \pi_4^2} \\ & + 0.825 \times 10^{-7}\frac{\pi_1^6 \cdot \pi_4^2}{\pi_2^3 \cdot \pi_3^6} + 0.6693 \times 10^{-4}\frac{\pi_2}{\pi_1} + 0.189 \times 10^{-5}\frac{\pi_1^7 \cdot \pi_4}{\pi_2^4 \cdot \pi_3^3} \\ & + 0.4275 \times 10^{-8}\frac{\pi_2^2 \cdot \pi_3^3}{\pi_1^2}; \end{aligned}$$

for the friction material МКВ-50

$$\begin{aligned} f_3 = {} & 0.271 + 0.877 \times 10^{-8}\pi_1^2 \cdot \pi_4 + 3.99\frac{1}{\pi_3^2 \cdot \pi_4} - 0.1676 \times 10^{-6}\frac{\pi_1^6}{\pi_3} \\ & + 17.79\frac{\pi_1^1}{\pi_2 \cdot \pi_3^8 \cdot \pi_4^5} - 0.7426 \times 10^{-8}\frac{\pi_1^2 \cdot \pi_4}{\pi_3^2}; \end{aligned}$$

As shown in Sect. 2.3.1, the problem of modeling will not be considered solved if it does not get enough deep minimum, and only upon reaching it, the problem will be solved, with the practical measurement accuracy equal to 5–10 %. The resulting models adequately reproduce the studied friction process during braking, and according to the original sample, the error of these equations is within 3–5 %, which is acceptable.

The resulting generalized tribological functions (the mathematical models) for the different friction materials allow us to reliably predict the behavior of friction unit in the braking mode. They can also be used in the formulation of engineering calculations associated with the estimation of the friction and wear characteristics.

The foregoing also demonstrates the undoubted benefits of combining the methods of theory of similarity and the modeling methods with the experimental design. Of course, such synthesis will be beneficial and, as shown above, lead to the significant improvement of their individual application.

The results of obtained generalized description can be applied to all similar processes.

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