Virtual Power and Pseudobalance Equations for Generalized Continua

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Abstract In this paper the balance equations of linear and angular momentum are deduced from some regularity properties of the system of contact actions and from the law of action and reaction. This approach provides a simple and unifying formulation of the theories of non polar and polar continua. It also allows for a direct deduction of the classical plate and beam theories as special Cosserat continua, obtained by dimensional reduction induced by appropriate geometrical constraints.

Deduction of Balance Equations

1. The traditional, generally accepted approach to Continuum Mechanics is based on Euler's balance laws of linear and angular momentum. During the second half of the past century, this approach was revisited a number of times. In 1963, W. Noll showed that the Euler laws are in fact a consequence of the postulate of the indifference of power [12]. Later, Gurtin and Martins [9] and Šilhavý [14, 15] came to the conclusion that the same laws, until then regarded as balance equations between distance and contact actions, are in fact regularity assumptions on the system of the contact actions alone.
   This conclusion also originated from an idea of Noll. In [13] he showed that, if a system of contact actions is skew-symmetric,[1] it is also additive on the boundaries of disjoint sets.[2] If this is the case, the contact action over the boundary of a part Π of the body, which may also be seen as a volume action,

[1] This assumption corresponds to Newton's law of action and reaction.
[2] That is, if Π and Π' are disjoint sets with a portion S of boundary in common and if $Q(S) = -Q(-S)$ is the contact action interchanged across S, the contact action on $\partial(\Pi \cup \Pi')$ is the sum of the contact actions on $\partial\Pi$ and on $\partial\Pi'$.

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$$F(\Pi) = -Q(\partial\Pi), \tag{1}$$

is additive on disjoint sets.[3] In the presence of sufficient regularity, a *surface density* s can be associated with Q and a *volume density* $b^\dagger$ can be associated with F, and the preceding equation can be given the form

$$\int_\Pi b^\dagger(x)\,\mathrm{d}V = -\int_{\partial\Pi} s(x,\partial\Pi)\,\mathrm{d}A. \tag{2}$$

A system of contact actions which admits both surface and volume densities is called a *Cauchy flux*, and Eq. (2) is called a *pseudobalance equation*. The reason for the name is that, though it looks like a balance equation, *this is not a balance equation, but only an identity between two different representations of the same contact action*.

The pseudobalance equation is all what is needed to prove the dependence of $s(x,\partial\Pi)$ on the normal n to $\partial\Pi$ at x,[4] and the linearity of this dependence.[5] That is, to prove that there is a linear transformation on the vectors, the *Cauchy stress tensor* T, such that

$$s = Tn. \tag{3}$$

Thus, rather than a consequence of the balance of linear momentum, the existence of the Cauchy tensor is a property enjoyed by all sufficiently regular skew-symmetric systems of contact actions. This reduces the importance of the role played by the Euler balance laws, which are usually considered as a fundamental postulate of mechanics. Indeed, to define a *classical continuum* it becomes convenient to take as primitive the concept of *external power*, which is an integral involving the inner products of the *external actions* of contact s and at distance b by a field v of *virtual velocities*[6]

$$P_{ext}(\Pi, v) = \int_\Pi b\cdot v\,\mathrm{d}V + \int_{\partial\Pi} s\cdot v\,\mathrm{d}A. \tag{4}$$

In particular, a *rigid virtual velocity field* is a vector field of the form

$$v(x) = a + \varpi \times x, \tag{5}$$

[3] The minus sign on the right is just matter of convenience.

[4] The dependence of s on the normal was conjectured by Cauchy, and was currently called the *Cauchy postulate*. Only in 1959 Noll proved that this conjecture is true, under the assumption that the internal actions have a local character [11]. Since then, the Cauchy postulate has become the *Noll theorem*.

[5] This is the *tetrahedron theorem* of Cauchy.

[6] Alternatively, one can take as primitives the concept of virtual velocity and the existence of two types of actions, distance and contact.

with a and ϖ arbitrary constant vector fields. Assuming the *indifference of power* under rigid virtual velocity fields,

$$P_{ext}(\Pi, a) = 0, \qquad P_{ext}(\Pi, \varpi \times x) = 0, \tag{6}$$

the balance laws of Euler

$$\int_{\Pi} b \,\mathrm{d}V + \int_{\partial\Pi} s \,\mathrm{d}A = 0, \quad \int_{\Pi} x \times b \,\mathrm{d}V + \int_{\partial\Pi} x \times s \,\mathrm{d}A = 0, \tag{7}$$

easily follow. With the aid of the relation (3) and of the divergence theorem, the surface integral in (4) can be transformed into a volume integral, called the *internal power*

$$P_{int}(\Pi, v) = \int_{\Pi} \big((b + \operatorname{div} T) \cdot v + T \cdot \nabla v\big) \,\mathrm{d}V. \tag{8}$$

The indifference conditions (6) applied to this integral yield the *local forms* of the balance equations

$$\operatorname{div} T + b = 0, \qquad T = T^T. \tag{9}$$

Equating the two expressions (4), (8) of the power, the *equation of virtual power*

$$P_{ext}(\Pi, v) = P_{int}(\Pi, v) \tag{10}$$

is obtained. This is not an equation between two different powers, as it is frequently asserted,[7] but only an identity between two different expressions of the same power.

Substituting the local forms (9) into the internal power (8), a *reduced form* for the power is obtained

$$P_{red}(\Pi, v) = \int_{\Pi} T \cdot \nabla v^S \,\mathrm{d}V. \tag{11}$$

This reduced form characterizes T as the unique *active internal action* present in a classical continuum, and ∇v^S as the corresponding *generalized deformation velocity*.

2. The framework introduced above is easily extended to the generalized continua. A *generalized continuum* is a continuum whose description involves a finite array $\{\xi^\alpha\}$ of primary variables (*state variables*), which can be scalar, vectorial, or tensorial. Coupled with dual variables β^α, σ^α of the same tensorial nature (*bulk and surface external actions*), they determine the external power[8]

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[7] In fact, on this assumption is based of the "method of virtual power" developed by Germain [7, 8] and others.

[8] Here and in the following, repeated indices are summed.

$$P_{ext}(\Pi, v, v^\alpha) = \int_\Pi (b \cdot v + \beta^\alpha \cdot v^\alpha)\, \mathrm{d}V + \int_{\partial\Pi} (s \cdot v + \sigma^\alpha \cdot v^\alpha)\, \mathrm{d}A, \tag{12}$$

in which the v^α are the *virtual velocities* of the state variables. If the contact actions σ^α are Cauchy fluxes, each of them has its own pseudobalance equation

$$\int_{\partial\Pi} \sigma^\alpha(x, \partial\Pi)\, \mathrm{d}A = -\int_\Omega \beta^{\alpha\dagger}(x)\, \mathrm{d}V, \tag{13}$$

and from it, with the aid of Noll's and Cauchy's theorems, follows the existence of a linear transformation Σ^α such that

$$\sigma^\alpha = \Sigma^\alpha n. \tag{14}$$

The divergence theorem then allows the passage from the external to the internal power

$$P_{int}(\Pi, v, v^\alpha) = \int_\Pi \big((b + \operatorname{div} T) \cdot v + T \cdot \nabla v + (\beta^\alpha + \operatorname{div} \Sigma^\alpha) \cdot v^\alpha + \Sigma^\alpha \cdot \nabla v^\alpha\big)\, \mathrm{d}V, \tag{15}$$

from which the balance equations are deduced imposing the indifference to rigid virtual velocity fields. But, unlike in classical continua, the *rigid virtual velocities* are not uniquely defined, since their definition depends on the physical nature of the state variables. In what follows, we consider two classes of generalized continua, *polar* and *non-polar*, with different definitions of rigid virtual velocities. In a non-polar continuum, the state variables describe rearrangements of matter at the microscopic level. A rigid virtual velocity involves no rearrangements, that is, the corresponding virtual velocities v^α are zero. Therefore, the indifference conditions are

$$P_{ext}(\Pi, a, 0) = 0, \qquad P_{ext}(\Pi, Wx, 0) = 0, \tag{16}$$

where W is the skew-symmetric tensor associated with the rotation vector ϖ, defined by the relation

$$Wa = \varpi \times a, \tag{17}$$

for all vectors a. In a polar continuum, the state variables introduce further degrees of freedom for the deformation. Then a rigid rotation is a simultaneous rotation of the macroscopic deformation and of the state variable. In the case of tensorial variables,[9] the virtual velocities v^α are tensor fields, and the appropriate indifference conditions are

$$P_{ext}(\Pi, a, 0) = 0, \qquad P_{ext}(\Pi, Wx, W) = 0. \tag{18}$$

[9]This case includes the *micromorphic continua* [6] and, in particular, the *micropolar continua*, also called *Cosserat continua*.

Thus, the polar and non-polar continua have the same translational indifference condition, but different rotational indifference conditions.

3. An example of a non-polar continuum is met in the theory of *gradient plasticity*. This theory is based on the Kröner-Lee decomposition

$$\nabla f = F^e F^p, \tag{19}$$

according to which the macroscopic deformation gradient ∇f is supposed to be the composition of a microscopic rearrangement F^p and of the local deformation F^e necessary to restore the macroscopic deformation ∇f. This decomposition defines a generalized continuum described by a single state variable, the tensor F^p. For it, the relation (14) has the form

$$S = \mathbb{T}n, \tag{20}$$

where the second-order tensor S is the contact action associated with F^p and the third-order tensor $\mathbb{T}$ is the corresponding internal action. Denoting by L^p the virtual velocity of F^p, the external and internal powers take the form

$$\begin{aligned} P_{ext}(\Pi, v, L^p) &= \int_\Pi (b \cdot v + B \cdot L^p)\, \mathrm{d}V + \int_{\partial\Pi} (s \cdot v + S \cdot L^p)\, \mathrm{d}A, \\ P_{int}(\Pi, v, L^p) &= \int_\Pi \big((b + \mathrm{div}T) \cdot v + T \cdot \nabla v \\ &\qquad + (B + \mathrm{div}\mathbb{T}) \cdot L^p + \mathbb{T} \cdot \nabla L^p\big)\, \mathrm{d}V. \end{aligned} \tag{21}$$

The indifference conditions (16) yield the same restrictions (9) of the classical continuum, and therefore the reduced form of the internal power is

$$P_{red}(\Pi, v, L^p) = \int_\Pi \big(T \cdot \nabla v^S + (B + \mathrm{div}\mathbb{T}) \cdot L^p + \mathbb{T} \cdot \nabla L^p\big)\, \mathrm{d}V. \tag{22}$$

In plasticity it assumed that the Cauchy stress T is a function of the elastic part F^e of the decomposition (19). From this decomposition follows the relation

$$\nabla v = L^e + L^p \tag{23}$$

between the corresponding virtual velocities. Thus, the reduced power takes the form

$$P_{red}(\Pi, v, L^p) = \int_\Pi \big(T \cdot D^e + T^p \cdot L^p + \mathbb{T} \cdot \nabla L^p\big)\, \mathrm{d}V, \tag{24}$$

where D^e is the symmetric part of L^e and T^p is the *plastic stress*

$$T^p = T + B + \mathrm{div}\mathbb{T}. \tag{25}$$

The last equation and the balance equation $(9)_1$ are the differential equations of the equilibrium problem of gradient plasticity.[10] The formulation of the problem is completed by a set of constitutive equations between the internal actions T, T^p, $\mathbb{T}$ and the corresponding generalized deformations, and by appropriate boundary conditions.

4. In a polar continuum, quite frequently the state variables are supposed to be vectorial, and in this case they are called the *directors*. The number of the directors depends on the nature of the continuum. For example, the *liquid crystals* have just one director, in *crystal plasticity* the number of the directors coincides with the number of the slip planes, and a *micromorphic continuuum* is characterized by a triple of linearly independent directors. Just as the macroscopic deformation of the body is described locally by the deformation gradient ∇f, the microscopic deformation of a micromorphic continuuum is described by a second-order tensor F^m, the *microscopic deformation gradient*. Thus, at each point of the continuum the microdeformation has the same geometric structure of the macrodeformation undergone by the whole body.[11]
   In the microdeformation, the directors d^α are mapped into the vectors $F^m d^\alpha$. Denoting by
$$v^\alpha = L^m d^\alpha \tag{26}$$
   the corresponding virtual velocity, substituting into (12), and setting
$$B = \beta^\alpha \otimes d^\alpha, \qquad S = \sigma^\alpha \otimes d^\alpha, \tag{27}$$
   the external and internal powers (21) are re-obtained, with L^m in place of L^p. With the indifference conditions (18), the balance laws are
$$\mathrm{div} T + b = 0, \qquad T + T^m = (T + T^m)^T, \qquad T^m = B + \mathrm{div}\mathbb{T}. \tag{28}$$
   That is, the symmetry of the Cauchy stress T required by the balance laws (9) of the non-polar continua is now replaced by the symmetry of the tensor $T + T^m$. As a consequence, in the integrand function of $(21)_2$ one has
$$\begin{aligned} T \cdot \nabla v + (B + \mathrm{div}\mathbb{T}) \cdot L^m &= T^S \cdot \nabla v^S + T^W \cdot \nabla v^W + T^m \cdot L^m \\ &= T^S \cdot \nabla v^S - T^{mW} \cdot \nabla v^W + T^m \cdot L^m = T^S \cdot \nabla v^S + T^m \cdot \mathcal{L}^m, \end{aligned} \tag{29}$$

[10]We emphasize that (25) is a consequence of the pseudobalance equation (13) and not a new balance equation. In the literature, it is named *balance of micromomentum, microforce balance, equilibrium equation for the macrostress tensor*, and is presented, at least tacitly, as a new axiom of mechanics.

[11]Ericksen and Truesdell [5], Mindlin [10] and Eringen [6].

where

$$\mathcal{L}^m = L^m - \nabla v^W \tag{30}$$

is the virtual velocity of the directors with respect to the body already deformed by the macroscopic deformation. The reduced power then takes the form

$$P_{red}(\Pi, v, L^m) = \int_\Pi \left(T^S \cdot \nabla v^S + T^m \cdot \mathcal{L}^m + \mathbb{T} \cdot \nabla L^m\right) \mathrm{d}V. \tag{31}$$

For a micromorphic continuum, the differential equations of the equilibrium problem are $(9)_1$ and $(28)_3$, and the constitutive equations are relations between the internal actions T^S, T^m, $\mathbb{T}$ and the generalized deformations ∇v^S, $\mathcal{L}^m$, ∇L^m. The Cauchy stress is not symmetric, and its skew-symmetric part T^W does not appear in the expression of the power. It plays the role of a reaction, determined by the relation $(28)_2$, $T^W = -T^{mW}$.

5. [12]In a micromorphic continuum, the deformation of the directors may be subject to geometrical constraints. For example, the *micropolar continua* are micromorphic continua for which the only deformation allowed to the orthonormal triple of directors is a rigid rotation, variable from point to point. Thus, if $R^m U^m$ is the polar decomposition of F^m, the constraint acting on a micropolar continuum is

$$F^m = R^m, \qquad U^m = I. \tag{32}$$

In this case the virtual velocity L^m reduces to its skew-symmetric part W^m, and in the external power we have

$$B \cdot L^m = B^W \cdot W^m = c \cdot \omega, \qquad S \cdot L^m = S^W \cdot W^m = m \cdot \omega, \tag{33}$$

with ω, $c/2$ and $m/2$ the vectors associated with W^m, B^W and S^W by the relation (17)

$$W^m a = \omega \times a, \qquad B^W a = \frac{1}{2} c \times a, \quad S^W a = \frac{1}{2} m \times a. \tag{34}$$

The external power then takes the form

$$\mathcal{P}_{ext}(\Pi, v, \omega) = \int_\Pi \left(b \cdot v + c \cdot \omega\right) \mathrm{d}V + \int_{\partial\Pi} \left(s \cdot v + m \cdot \omega\right) \mathrm{d}A, \tag{35}$$

with ω the vectorial measure of the virtual rotation of the directors, and c and m the *volume couple* and the *surface couple*. If s and m are Cauchy fluxes, they have the representations

[12]For reasons of brevity, from here on most of the statements are given without comments and proofs. More detailed treatments can be found in the paper [2] and in the forthcoming lecture notes [4]. For plate and beam theories, see [3].

$$s = Tn, \qquad m = Mn, \tag{36}$$

with T the Cauchy stress and M the *couple stress tensor*. With the aid of the divergence theorem one obtains the internal power

$$\mathcal{P}_{int}(\Pi, v, \omega) = \int_{\Pi} \big((b + \mathrm{div}T) \cdot v + T \cdot \nabla v + (c + \mathrm{div}M) \cdot \omega + M \cdot \nabla\omega\big)\, \mathrm{d}V. \tag{37}$$

The indifference conditions now give

$$\mathrm{div}T + b = 0, \qquad \mathrm{div}M + c + 2t = 0, \tag{38}$$

with t the vector associated with the skew-symmetric part of T. Substitution into Eq. (37) yields the reduced form

$$\mathcal{P}_{red}(\Pi, v, \omega) = \int_{\Pi} \big(T^S \cdot \nabla v^S - 2\, t \cdot \varphi + M \cdot \nabla\omega\big)\, \mathrm{d}V, \tag{39}$$

where

$$\varphi = \omega - \frac{1}{2}\mathrm{curl}\ v \tag{40}$$

is the vector measure of the relative rotation $W^m - \nabla v^W$. The equilibrium problem now consists of the differential equations (38), of constitutive equations relating the internal actions T^S, t and M with the generalized deformations ∇v^S, φ and $\nabla\omega$, and of a set of boundary conditions.

The *constrained Cosserat continua* are obtained by imposing the supplementary constraint

$$\omega = \frac{1}{2}\mathrm{curl}\ v, \tag{41}$$

which requires that the relative rotation φ be zero.[13] For such continua the indifference conditions still have the form (38), and the reduced power is

$$\mathcal{P}_{red}(\Pi, v) = \int_{\Pi} \big(T^S \cdot \nabla v^S + \frac{1}{2} M \cdot \nabla \mathrm{curl}\ v\big)\, \mathrm{d}V. \tag{42}$$

Here, T^S and M are the only active internal actions. The rotation ω formally disappears from the list of the geometric variables, though its effects are still present in the product $M \cdot \nabla\mathrm{curl}\ v$.[14] As a consequence, t is not anymore an active internal action, and therefore it is not anymore determined by a constitutive equation. In

[13] This constraint corresponds to the *Cauchy-Born hypothesis*, according to which the directors follow the macroscopic deformation.

[14] The presence of a microstructure which does not appear explicitly in the expression of the power characterizes this continuum as a *continuum with latent microstructure* [1].

the equilibrium problem, t is eliminated from the differential equations (38) with the aid of the identity

$$\mathrm{div} T^W = -\mathrm{curl}\, t, \tag{43}$$

thanks to which the two equations merge in the single, higher-order equation

$$\mathrm{div} T^S + \frac{1}{2}\mathrm{curl}\,(\mathrm{div} M + c) + b = 0. \tag{44}$$

Of course, the boundary conditions must be re-formulated accordingly.

6. Other geometrical constraints lead to *dimensional reduction*, providing thereby the classical theories of plates and beams, viewed as two- and one-dimensional Cosserat continua. Assume that the body in its reference configuration has a cylindrical shape, and let $\{e, e^\alpha\}$ be an orthonormal triple of vectors, with e directed as the axis of the cylinder. The constraint

$$v(x) = v_3(x_1, x_2)\, e, \qquad \omega(x) = \omega_\alpha(x_1, x_2)\, e^\alpha, \qquad \alpha \in \{1,\ 2\}, \tag{45}$$

allows for a virtual velocity v parallel to e and for a virtual rotation ω about an axis orthogonal to e. It also requires that both v and ω be constant in the direction e. Under these constraints, the external power (35) reduces to

$$P_{ext}(\Gamma, v_3, \omega_\alpha) = \int_\Gamma (b_3 v_3 + c_\alpha \omega_\alpha)\, \mathrm{d}A + \int_{\partial\Gamma} (s_3 v_3 + m_\alpha \omega_\alpha)\, \mathrm{d}\ell, \tag{46}$$

where the volume element Π is replaced by its perpendicular projection Γ in the direction e, and $\mathrm{d}\ell$ is the line element on the boundary line $\partial\Gamma$.
In the relations (36), by effect of the constraints, the stress tensor T_{ij} degenerates into the vector of the *internal shearing forces* Q_α, and the couple-stress tensor M_{ij} degenerates into the 2×2 tensor of the *internal moments* $M_{\alpha\beta}$

$$s_3 = Q_\alpha n_\alpha, \qquad m_\alpha = M_{\alpha\beta} n_\beta. \tag{47}$$

Then the internal power becomes

$$\begin{aligned}\mathcal{P}_{int}(\Gamma, v_3, \omega_\alpha) = \int_\Gamma \big((q + Q_{\alpha,\alpha}) v_3 + Q_\alpha v_{3,\alpha} \\ +(c_\alpha + M_{\alpha\beta,\beta})\omega_\alpha + M_{\alpha\beta}\omega_{\alpha,\beta}\big)\, \mathrm{d}A,\end{aligned} \tag{48}$$

with the component b_3 of the body force now viewed as a transverse load q. The indifference conditions (18) provide the balance equations

$$Q_{\alpha,\alpha} + q = 0, \qquad M_{\alpha\beta,\beta} + c_\alpha + e_{\alpha\beta} Q_\beta = 0, \tag{49}$$

which, in turn, lead to the reduced power

$$P_{red}(\Gamma, v, \omega) = \int_{\Gamma} (Q_\alpha \varphi_\alpha + M_{\alpha\beta}\omega_{\alpha,\beta})\,\mathrm{d}A, \tag{50}$$

where

$$\varphi_\alpha = v_{3,\alpha} + e_{\alpha\beta}\omega_\beta \tag{51}$$

are the rotations of the directors relative to the deformed surface Γ. Thus, the active internal actions are Q_α and $M_{\alpha\beta}$, and φ_α and $\omega_{\alpha,\beta}$ are the corresponding generalized deformations. The equations (49) are the equilibrium equations of the *Reissner-Mindlin plate theory*. Here, this theory has been deduced from that of the three-dimensional micropolar continuum, simply by imposing the constraint (45).

Moreover, the *Kirchhoff-Love plate theory* is obtained by imposing the additional constraint

$$\varphi_\alpha = 0. \tag{52}$$

Indeed, with this constraint, in the reduced power (50) the first term cancels, and $\omega_{\alpha,\beta}$ is replaced by $e_{\alpha\gamma}v_{3,\gamma\beta}$. Introducing the *modified moment tensor*

$$M^*_{\gamma\beta} = e_{\gamma\alpha}M_{\alpha\beta}, \tag{53}$$

which is the one currently used in the constrained plate theory, the reduced power takes the form

$$P_{red}(\Gamma, v) = -\int_{\Gamma} M^*_{\gamma\beta}v_{3,\gamma\beta}\,\mathrm{d}A. \tag{54}$$

The active internal forces are reduced to the tensor $M^*_{\gamma\beta}$, and the associated generalized deformation is the *curvature tensor* $-v_{3,\gamma\beta}$. As a consequence, the vector Q_α must be eliminated from the balance equations (49). This gives rise to the unique, higher order equation

$$M^*_{\gamma\beta,\gamma\beta} + c^*_{\gamma,\gamma} + q = 0, \tag{55}$$

where $c^*_\gamma = e_{\gamma\alpha}c_\alpha$ is the modified external couple. This is the equilibrium equation of the Kirchhoff-Love plate theory.

7. In a quite similar way, it can be shown that the constraints

$$v(x) = v(x_3), \qquad \omega(x) = \omega(x_3) \tag{56}$$

provide the *Timoshenko beam theory*, and that the additional constraint

$$\omega_\alpha = -e_{\alpha\beta}v'_\beta \tag{57}$$

leads to the *Euler-Bernoulli beam theory*. For a detailed treatment, the interested reader is addressed to the paper [3].

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