Chapter II. Lagrangian Field Theory with non-commuting variations.

5. Introduction.

In this chapter we define and study the **non-commuting variations** for a Lagrangian action $\mathcal{A}_D(s(x))$ defined by a "tensor field" $K^{\alpha}_{i\beta}$ in the 1-jet space $J^1\pi$. We will call tensor $K^{\alpha}_{i\beta}$ the **NC-tensor** and the variations defined by tensor K - the K-twisted variations.

Using the modified first variations we obtain the Euler-Lagrange equations modified by the sources (forces in Mechanics):

$$E_\alpha(L) = \frac{\delta L}{\delta y^\alpha} = f_\alpha,\ j = 1, \dots, m, \tag{5.1}$$

f_α being determined by the NC-tensor K. When $K = K^{\alpha}_{i\beta} = 0$, system 5.1 reduces to the conventional Euler-Lagrange system.

We will develop the formalism of non-commuting variations for the **first order Lagrangians** $L \in C^\infty(J^1\pi)$ - infinitely differential functions of the variables x^i, y^μ, y^μ_i . At the end of this Chapter we extend our approach to the Lagrangian Field Theory of higher order.

Remark 5. Notice that the "NC-tensor" $K^{\alpha}_{i\beta}$ behaves tensorially with respect to the **transformations of the fibred charts** (U, x^i, y^α).

Remark 6. In their series of works, C.Muriel and J.Romero introduced the "λ-twisted symmetries" for a scalar ODE based on the modified prolongation procedure defined by a scalar function λ - λ-prolongation. Later on, G.Gaeta, P.Morando and G.Cicogna extended the method of "λ-twisted symmetries" to the "systems of ODEs and PDEs", see [39, 40]. The method of μ-symmetries "NC-tensor" K used by these authors is introduced as the $gl(n)$-valued horizontal 1-form $\mu = K^{\alpha}_{i\beta} dx^i$. This point of view was very convenient for modifying the geometrical structures on the jet bundles adopting them to the K-modified prolongation procedures. In Chapter 4 we present some basic results of these authors related to our study.

6. **Euler-Lagrange equations with K-twisted variations**.

In this section we introduce and begin to study the modified construction of variations of jet variables (derivatives of dynamical fields y^α), corresponding to the **modified lift of the vertical vector fields** $\xi \in V(\pi)$ to the vertical vector fields $Pr^1_K(\xi)$ in the 1-jet bundle $\pi_1 : J^1(\pi) \to X$:

$$\xi \to Pr^1_K(\xi) = \xi + (d_i\xi^\alpha + K^{\alpha}_{i\beta}\xi^\beta)\partial_{y^\alpha_i}, \tag{6.1}$$

where

$$K^{\alpha}_{i\beta} \in C^\infty(J^1(\pi)) \tag{6.2}$$

are smooth functions of variables $x^i, y^\alpha, y^\alpha_i$ (i.e. functions in the 1-jet space $J^1(\pi)$).

These functions form the NC-tensor $K^{\alpha}_{i\beta}$ (tensor relative to the group of automorphisms of the configurational bundle π, or, equivalently, with respect to the group of local changes of fibred coordinates (x^i, y^μ, y^μ_i)).

Form the variation of the action $A_D(s)$ using the prolongation (6.1) of variational vector field $\xi = \xi^\alpha\partial_\alpha$ to the 1-jet bundle $J^1\pi$, we obtain infinitesimal variations

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of fields and their derivatives. Notice that base (independent) variables x^i are not variated,

$$\begin{cases} y^\alpha \to y^\alpha + \epsilon\xi^\alpha(x,y), \\ y^\alpha_i \to y^\alpha_i + \epsilon(d_i\xi^\alpha + K^\alpha_{i\beta}\xi^\beta). \end{cases}$$

Now we form the variation of the action

$$\begin{aligned}
&\Delta A_D(s)(\epsilon\xi) = A_D(s_\epsilon) - A_D(s) = \\
&= \int_D [L(x, s^\alpha(x) + \epsilon\xi^\alpha(x,s(x)), s^\alpha_{,i}(x) + \epsilon(d_i\xi^\alpha(x,s(x)) + K^\alpha_{i\beta}(x, j^1 s(x))\xi^\beta(x,s(x)))) \\
&\qquad - L(x, s(x), s^\alpha_{,i}(x))]dv = \\
&= \epsilon\int_D [\frac{\partial L}{\partial y^\beta}\xi^\beta(x,s(x)) + \frac{\partial L}{\partial y^\alpha_i}(d_i\xi^\alpha(x,s(x)) + K^\alpha_{i\beta}(x,\beta^1 s(x))\xi^\beta(x,s(x))))]dv + O(\epsilon^2) = \\
&= \epsilon\int_D [\frac{\partial L}{\partial y^\beta} - d_i\left(\frac{\partial L}{\partial y^\beta_i}\right) + K^\alpha_{i\beta}(x, j^1 s(x))\frac{\partial L}{\partial y^\alpha_i}]\xi^\beta(x,s(x))dv + O(\epsilon^2).
\end{aligned}$$

As a result the system of Euler-Lagrange equations takes the form

$$\frac{\partial L}{\partial y^\beta} - d_i\left(\frac{\partial L}{\partial y^\beta_i}\right) = f_\beta = -K^\alpha_{i\beta}(x, j^1 s(x))\frac{\partial L}{\partial y^\alpha_i} = -K^\alpha_{i\beta}\pi^i_\alpha,\ \beta = 1,\dots,m. \quad (6.3)$$

The right side of this equation represents the generalized (non-potential) sources

$$f_\beta = -K^\alpha_{i\beta}(x, j^1 s(x))\frac{\partial L}{\partial y^\alpha_i} = -K^\alpha_{i\beta}\pi^i_\alpha \quad (6.4)$$

acting in the system. Here $\pi^i_\alpha = L_{,y^\alpha_i}$ are the *momenta* corresponding to the Lagrangian L.

Thus, using the non-commuting variations in the conventional variation formalism we get the systems of equations of the form

$$\frac{\delta L}{\delta y^\beta} = f_\beta,\ f_\beta = -K^\alpha_{i\beta}\pi^i_\alpha. \quad (6.5)$$

Example 1. Look at the simplest case of one ordinary differential equation of second order where $n = 1, m = 1$ (i.e. where there is one independent variable x and one dynamical variable y). Let Lagrangian be a function $L = L(x, y, y')$ of x, y and the first derivative $y' = \frac{dy}{dx}$.

Let $K = K(x, y, y')$ be a NC-tensor (scalar in this case). The Euler-lagrange Equation $\frac{\delta L}{\delta y} = -\frac{\partial L}{\partial y'}K$, has the form

$$\frac{\partial L}{\partial y} - d_x\left(\frac{\partial L}{\partial y'}\right) = -\frac{\partial L}{\partial y'}K,$$

where $d_x = \frac{\partial}{\partial x} + y'\frac{\partial L}{\partial y} + y''\frac{\partial L}{\partial y'}$ is the **total derivative** of lagrangian L by x. In coordinates this equation has the form

$$\frac{\partial^2 L}{\partial x\partial y'} + y'\frac{\partial L}{\partial Y} + y''\frac{\partial L}{\partial Y'} - \frac{\partial L}{\partial y} = -K\frac{\partial L}{\partial y'}.$$

that differs from the conventional EL-equation by the force/source in the right side.

Example 2. In a case of Mechanics (n=1), there is one independent variable t and m dynamical variables y^μ. A NC-tensor K^α_β defines K-twisted variations of variables y^μ and their derivatives:

$$\delta y^\mu = \xi^\mu(t,y),\ \delta \dot{y}^\mu = d_t\xi^\mu + K^\mu_\beta \xi^\beta. \tag{6.6}$$

The full prolongation formula contains the second (horizontal) component K^ν (see bellow, Section 16 for the general prolongation procedure). As a result, the K-twisted 1-prolongation of a vector field $v = \xi\partial_t + \xi^\mu \partial_{y^\mu}$ is

$$Pr^1_K v = (\xi\partial_t + \xi^\mu\partial_\mu) + [(d_t(\xi^\mu - \dot{y}^\mu\xi) + (K^\mu_\nu\xi^\nu - K^\mu\xi)]\partial_{\dot{y}^\mu}. \tag{6.7}$$

Corresponding Euler-Lagrange equations (6.3) takes, in the case of "Mechanics", the form:

$$\frac{\partial L}{\partial y^\beta} - d_t\left(\frac{\partial L}{\partial \dot{y}^\beta}\right) = -K^\alpha_\beta(t, y^\alpha, \dot{y}^\alpha)\frac{\partial L}{\partial \dot{y}^\alpha} = -K^\alpha_\beta \pi_\alpha,\ \beta = 1, \ldots, m, \tag{6.8}$$

where $\pi_\alpha = \frac{\partial L}{\partial \dot{y}^\alpha}$ are momenta corresponding to the variables y^α.

Remark 7. Bellow, in Chapter V, we will study the relation between the representation of the forces in the form $f_\beta = K^\alpha_\beta \pi_\alpha$ and the Rayleigh representation, in terms of "dissipative potentials" ([116]).

In Section 10 below we present a variety of examples of Euler Lagrange equations (and systems of Euler-lagrange-equations) with non-commuting variations. NC tensors K defining the twisted properties of variations in these systems reflects specific dynamical and symmetry properties of these equations.

Remark 8. Prolongation procedures of the form (6.1,6.8) of vector fields from the configurational space Y to the jet bundles of arbitrary order $k \geqq 1$ were introduced by C.Muriel and J.L.Romero, [101, 102] in the case where $n = 1$ (ordinary differential equations). Their prolongation procedure was defined by a C^∞-function $\lambda(x, y, \frac{dy}{dx})$ defined on the 1-jet space: $\lambda \in C^\infty(J^1\pi)$. In our notations, for $m = 1$, tensor K^μ_ν in (6.8) reduces to one function K. The procedure of prolongation (6.8) coincides with the λ-prolongation introduced by C.Murial and J.Romero. These authors introduced a new class of symmetries (**λ-symmetries**) for ordinary differential equations. Usage of λ-symmetries allowed the authors to define **new general procedure for the reduction of ODE**. In particulary, they found that some nonlinear ODE having nontrivial Lie derivatives could be integrated using λ-symmetries. Their method of reduction contains, as the special cases, many known procedures of reduction.

Chapter 4 of these notes is a short presentation of the theory of λ-symmetries and its generalization to the case of PDE and systems of PDE by G.Gaeta, G.Gicogna, P.Morando and their collaborators.

Remark 9. Introduction of non-commutative variations modifies the Euler-Lagrange Equations by the sources/forces $f_\beta = -K^\alpha_{i\beta}\pi^i_\alpha$. At the same time, sections $s(x^i) = \{s^\alpha(x)\}$ delivering local extremal values (local or global,l minima or maxima) to the action functional $A_D(s)$ stays (local or global) minimum or maximum although now they satisfy a different system of differential equations.

7. Noether Theorem, Energy-Momentum balance law.

Infinitesimal condition for a Lie group G of (diffeomorphic) transformations of the manifold Y to be a **group of variational symmetries of Lagrangian** L is the relation $Pr^{\{k\}} \cdot L + LDiv(\xi) = 0$ (see (79.5). At the same time,condition for a Lie group G of (diffeomorphic) transformations of the manifold Y to be a **group of divergent symmetries of Lagrangian** L is that for some n-tuple $B = \{B_i\}$ of functions in Y (i.e. functions of variables x^i, y^μ) has the invariant form

$$Pr^k\xi + Ldiv(\xi) = Div(B),$$

(see [106], Sec.4.4). In a case of balance laws this relation is *appropriately modified by the NC-tensor* $K^\nu_{\mu i}$ (see Section 81) or, equivalently, by the source form $f_\mu dy^\mu$. Notice that the defining property of a one-parameter group ϕ^t of symmetries of Lagrangian is that the action of transformations $\phi^t,\ t \in R$ transform solutions of Euler-Lagrange Equations to the solutions.

Therefore, phase fields of such one-parameter groups act as the infinitesimal variations of the action $\mathcal{A}_D(s)$ corresponding to L.

Let $\zeta \in \mathfrak{g}$ be an arbitrary element of Lie algebra $\mathfrak{g}$. Let $Pr^1_K(\zeta)$ be the prolongation of vector field ζ to the 1-jet bundle $J^1(\pi)$ modified by a NC-tensor K (see Appendix II, (79.6)). Then,a vector field ζ is the infinitesimal divergent symmetry of L, and generates the (local) one-parameter group of symmetries of L, if and only if there exists a horizontal 1-form $B = B_k(x^i, y^\mu, y^\mu_i)dx^i$ in $J^1(\pi)$ such that

$$Pr^1_K(\zeta)L + L(div(\bar{\zeta}) = Div(B). \tag{7.1}$$

Divergence is taken using the volume form $d_g v$ defined by the metric g in he base X.

Remark 10. Notice that the symmetry condition contains the term $K^\mu_{\nu i}\zeta^\mu \partial_{y^\mu_i} L$ depending on the NC-tensor K.

Theorem 2. *Let L be a Lagrangian of order k and let*

$$E_\mu(L) = f_\mu = -K^\beta_{i\mu}\pi^i_\beta,\ \mu = 1, \dots, m \tag{7.2}$$

be an Euler-Lagrange system with the Lagrangian L of order k and the sources $f_\mu, \mu = 1, \dots, m$. Let ξ be a vector field in Y - infinitesimal generator of variational symmetry of the Lagrangian L (see Sec.78) and let $Pr^k(\xi)$ be its prolongation of order k.Then there exist an n-tuple of the smooth functions A^i such that for all solutions of the system (80.6) the following equality is fulfilled

$$Div(A + L\xi) = -Q^\mu \cdot f_\mu \tag{7.3}$$

Here $Q = \{Q^\mu = \xi^\mu - y^\mu_i \xi^i\}$ is the characteristic *of the vector field ξ (see Appendix III, Sec.79).*

In the case of **divergent symmetry** corrresponding balance law is modified by a introducing the form B (see Sec.79). As a result, we get, for solution of the Euler-Lagrangian system with NC-variations the **Noether balance law in the presence of non-commuting variations**

$$\begin{cases} \mu Div(P) = -Q^\nu K^i_{i\nu}\pi^i_\mu, \text{ where} \\ P^i = \zeta^\mu L_{,y^\mu_i} + \zeta^i L - \zeta^j y^\nu_j L_{,y^\mu_i} - B. \end{cases} \tag{7.4}$$

Here, Q^ν are components of the characteristic $Q = \{Q^\nu \partial_\nu = (\zeta^\nu - y^\nu_i \zeta^i)\partial_\nu\}$ of the vector field ζ. Balance laws obtained in Theorem 2 will be called **K-twisted conservation or balance laws** for the Lagrangian L.

As an example, illustrating the modified Noether conservation law, we present the canonical Stress-Energy-Momentum balance law.

7.1. **Stress-Energy-Momentum balance law.** Here we write down the canonical stress-energy-momentum (CEM) balance law corresponding to a Lagrangian $L(x^i, y^\alpha, y^\alpha_i)$ - function of independent variables x^i, dynamical fields y^α and their first derivatives $y^\alpha_{,x^i}$. We will be using the approach of [106], modified to produce the balance law, see Appendix III, Sec.81.

Consider a case where vector field $\xi = \xi_k = \partial_{x^k}$ is lifted to the space Y by a connection Γ in the bundle $\pi : Y \to X$: $\partial_{x^k} \to \hat{\xi}_k = \partial_{x^k} + \Gamma^\mu_k \partial_\mu$. Introduce the *characteristic* Q of the vector field $\hat{\xi}_k$ - the quantity playing the principal role in the prolongation of vector fields and in studying symmetries and conservation laws associated with the differential equations (see [106] and sections 72,75 of Appendix. Characteristic of the vector field $\hat{\xi}_k$ has the components $Q^\mu = \Gamma^\mu_k - y^\mu_k$ and we define the Energy-Momentum tensor as

$$T^i_k = L\delta^i_k - (\Gamma^\mu_k - y^\mu_k)L_{,y^\mu_i}. \tag{7.5}$$

As a result the balance equation (see Appendix III, Sec.81, equation 81.4), corresponding to this vector field (*stress-energy-momentum balance law*) has the form

$$\begin{aligned} d_i T^i_k = d_i\left(L\delta^i_k - y^\mu_k L_{,y^\mu_i}\right) &= -\frac{\partial L}{\partial x^k}_{expl} - (\Gamma^\mu_k - y^\mu_k)K^\nu_{i\mu}(x, \mathrm{j}^1 s(x))\frac{\partial L}{\partial y^\nu_i} = \\ &= -\frac{\partial L}{\partial x^k}_{expl} + (\Gamma^\mu_k - y^\mu_k)f_\mu,\ k = 0,1,2,3. \end{aligned} \tag{7.6}$$

where $f_\nu = -K^\mu_{i\nu}\pi^i_\mu$.

The CEM-tensor $T^i_k = \left(L\delta^i_k - y^\mu_k L_{,z^\mu_i}\right)$ has the standard form ([80]).

In particular, in a case where connection Γ is trivial (Γ^μ_k=0), for $k = 0$ we get the energy balance law in the form

$$d_i T^i_0 = d_i\left(L\delta^i_0 - y^\sigma_0 L_{,y^\sigma_i}\right) = -\frac{\partial L}{\partial x^0}_{expl} - y^\nu_0 K^\mu_{i\nu}(x, j^1 s(x))\frac{\partial L}{\partial y^\mu_i} = -\frac{\partial L}{\partial x^0}_{expl} - y^\nu_0 \cdot K^\mu_{i\nu}\pi^i_\mu. \tag{7.7}$$

Notice the linear dependence of the second force term on the velocities y^μ_0 and on the momenta π^i_μ. This terms allows the system to have positive or negative dissipation.

Introduce the following notion:

Definition 2. *A source $f_\mu dy^\mu$ is called conservative if the are no energy dissipation in the process described by the Euler-Lagrange equations with the source $f_\mu dy^\mu$.*

From the energy-momentum balance law 7.6 it follows that, if through any point $(x^i, y^\mu, y^\mu_i) \in J^1(\pi)$ there passes a solution of the system (6.9), a source $f_\mu dy^\mu$ is conservative if and only if

$$K^\mu_{\nu i} : y^\nu_0 \pi^i_\mu = y^\nu_0 f_\nu = 0. \tag{7.8}$$

Remark 11. Notice that the energy dissipation $K^{\mu}_{\nu i} : y^{\nu}_0 \pi^i_\mu = y^{\nu}_0 f_\nu$ has the bilinear form where terms y^{ν}_0 have the meaning of velocities (rate of change of the field y^ν) while the components f_ν are related to the "forces" acting on the system. As a result, expression for the energy dissipation is similar to the expression for the entropy production in the Continuum Thermodynamics, see "Onzager Principle" in [55, 94].

Example 3. Let the bundle π be the tensor-like bundle and tensor K have the canonical form (9.8), see Sec.9). Then, the energy balance laws has the form

$$d_\mu T^\mu_0 = -\frac{\partial L}{\partial x^0}_{expl} - \dot{y}^i f_i$$

.

Exercise. Consider an inner variation $x^i \rightarrow x^i + \xi^i(x)$, its prolongation to the space Y defined by a **connection** Γ **(see Sec.17, Ch. 3)** and lift ζ of this vector field to the 1-jet bundle $J^1\pi$ induced by an Ehresmann connection K (more specifically, by its component K^μ_{ki}), see Definition 8, Sec.17. Work out the variation equation $\frac{\delta A_D}{\delta y^\mu}(\zeta) = 0$.

Show that if the connection Γ is trivial and if vertical connection K is such that $K^\mu_{ik} = y^\nu_k K^\mu_{\nu i}$, then the obtained equation coincide with the Energy-Momentum balance law with the K-induced dissipation (7.6).

Remark 12. Compare this case of the vertical connection K with the first λ, μ -prolongation in Chapter 4.

8. On the non-unicity of NC-representation (6.5).

By introducing the presentation (6.4) of the source (force) terms in the systems 6.3, it is natural to inquire, for which sources $\{f_\beta\}$ does a such representation exists, under which conditions "such" representation is unique. Finally, if such a representation is not unique, it is interesting to determine the degree of such non-unicity. This information may be useful for the choice of the most convenient representation (6.6), for instance, the one leading to the minimal dissipation.

More then this, it is interesting to study these questions by qualifying the sources f_β and the NC-tensors $K^\alpha_{i\beta}$ in terms of the type and order of derivatives of the fields y^α they may depend on. Answers to these questions depend on the order of derivatives entering f_β and NC-tensor $K^\alpha_{i\beta}$ as well as on the form of this dependence.

Here we discuss the case where sources f_β and NC-tensor K depend on the points of the space Y and on the first derivatives of dynamical fields y^α, i.e when $f_\beta, K^\alpha_{i\beta} \in C^\infty(J^1(\pi))$.

Let there be two different representations of the form (6.4) for the sources f_β:

$$f_\beta = K^\alpha_{i\beta}\pi^i_\alpha = P^\alpha_{i\beta}\pi^i_\alpha.$$

Subtracting these equations we have, for the difference $Q^\alpha_{i\beta} = K^\alpha_{i\beta} - P^\alpha_{i\beta}$, the equality

$$Q^\alpha_{i\beta}\pi^i_\alpha = 0. \tag{8.1}$$

Denote by $\mathcal{R}$ the vector space of all solutions $Q^\alpha_{i\beta}$ of this equation with the components in the space $C^\infty(J^1(\pi))$. If, for a given sources $f_\beta \in C^\infty(\pi)$ there exist a representation (6.4) with the NC-tensor $K^\alpha_{i\beta}\pi^i_\alpha$ $f_\beta = K^\alpha_{i\beta}\pi^i_\alpha$, then the set of **all possible** NC-tensors representing sources f_β as $f_\beta = K^\alpha_{i\beta}\pi^i_\alpha$ has the form

$\{K^\alpha{}_i\beta + Q^\alpha_{i\beta}\}$ where $Q^\alpha_{i\beta} \in \mathcal{R}$. Notice that the space $\mathcal{R}$ does not depend on the choice of the sources $f_{j\beta}$ but only on the Lagrangian L. In addition to this, variables x^i, y^α appear in the equation 8.1 as parameters only.

We describe the space $\mathcal{R}$ in the case where $L(x, y, y^\alpha_i)$ is a regular Lagrangian of the first order, i.e. let the matrix $\frac{\partial^2 L}{\partial y^\alpha_i \partial y^\beta_j}$ be non-degenerate. In such a case, Legendre transformation $(x, y, y^\alpha_i) \to (x, y, \pi^i_\alpha)$ is (local) diffeomorphism and the momenta π^i_α define the coordinates in the fibres of the bundle $\pi_{10} : J^1(\pi) \to Y$.

Consider now a formal analog of the system 8.1. Let R^k be the k-dim vector space endowed with the conventional Euclidian metric. Let u^i be orthogonal Cartesian coordinates in R^k. Consider the radial vector field $\partial_r = \sum_i u^i \partial_{,u^i}$. A vector field $F = f^i(u^j)\partial_{u^i}$ is orthogonal to the radial vector field ∂_r if and only if

$$\sum_i f_i(u)u^i = 0. \tag{8.2}$$

This condition is equivalent to the condition that for any point $x \in R^k$, vector field $F(x)$ is tangent to the sphere with center at the origin $\mathbf{0}$ and radius $|r|$.

Since the tangent bundle to the unit sphere (and, therefore, to all spheres centered at the origin) are generated by the vector fields $F^{ij} = u^i\partial_{u^j} - u^j\partial_{u^i}$, all vector fields F satisfying to the equation (8.2) have the form

$$F(u) = f_{ij}(u^k)(u^i\partial_{u^j} - u^j\partial_{u^i}).$$

Notice that the components of vector fields $F^{ij} = F^{ijs}\partial_{u^s}$ are defined by the condition

$$F^{ijs} = \begin{cases} -u^j \text{ if } s = i, \\ = u^i \text{ if } s = j, \\ 0 \text{ if } s \neq i, j. \end{cases} \tag{8.3}$$

Using Legendre transformation and applying these arguments to the fibers of the 1-jet bundle $J^1(\pi)$ with the coordinates $u^j = \pi^i_\alpha$ and the metric defined by the condition that the vector fields $\partial_{\pi^i_\alpha}$ form the orthonormal (Cartesian) basis we prove the following

Lemma 1. *Let L be a **regular Lagrangian**, i.e. let the matrix $\frac{\partial^2 L}{\partial y^\alpha_i \partial y^\beta_j}$ be non-degenerate. Then all the components Q^β of the solutions of the system of equations 8.1 have the form*

$$K = \sum q_{ij}()F^{ijs}, \tag{8.4}$$

where functions F^{ijs} are defined in 8.3 and q_{ij} are arbitrary functions from $C^\infty(J^1(\pi))$.

This result gives the description of all possible representations of the source form $f = f_\alpha dy^\alpha$ in the form (6.6) provided one such representation exists. Below we will show that such a representation exists on all the natural bundles $\pi : Y \to X$.

On the other hand, in Section (22) below, we will show that if there exists a representation (6.4) with a NC-tensor $K \in C^\infty(Y)$, such representation **is unique**.

Example 4. Case: Components $K^\alpha_{i\beta}$ are linear by y^α_i.

Tensors $K^\alpha_{i\beta}$ linear by first order derivatives

$$K^\alpha_{i\beta} = L^{\alpha j}_{i\beta\kappa}(x, y)y^\kappa_j$$

(*linear functions of 1-jet variables*) were used by different authors, see papers by H.Kleinert, P.Fiziev and A.Pelster,[35, 65], works of Dj.Djukich and B.Vujanovic and T.Atanaskovich, see (VJ,Vu, At,At2). In the notations of the papers of H.Kleinert and his coworkers on the variational form of Mechanics in the Cartan space-time, $K^{\alpha}_{t\beta} = 2S^{\alpha}_{\kappa\beta}y^{\kappa}_t$ where $S^{i}_{\kappa\beta}$ is the torsion tensor of an affine connection in the Cartan space-time (see Ch.5, below). When the torsion of this connection vanish, variations (4.10) reduce to the conventional variation (3.1).

Consider a case where $n = 1$ and let a Lagrangian have the form $L(x, y^{\alpha}, \dot{y}^{\alpha}) = a_{\alpha}(x, y^{\beta})\dot{y}^{\alpha}$. In this case the equality (8.1) takes the form

$$Q^{\alpha}_{\kappa}(x, y)a_{\alpha}(x, y)\dot{y}^{\kappa} = 0.$$

This equation is equivalent to the system of linear equations $Q^{\alpha}_{\kappa}(x, y)a_{\alpha}(x, y) = 0,\ \kappa = 1, \ldots, m$. The association $(x, y) \to a^{\alpha}(x, y)$ defines the vertical vector field $a = a^{\alpha}\partial_{\alpha}$ in Y and any 1-form $\nu = p_{\alpha}dy^{\alpha}$ annulating the vector field a delivers the (1.1) tensor $Q = \partial_{y^{\alpha}} \otimes \nu$ satisfying to the previous equality.

Turning to the question of existence of a representation (6.4) we notice that the tensor field $K^{\alpha}_{i\beta}(x^k, y^{\alpha}, y^{\alpha}_k)$ can be considered as the mapping of vector bundles with the finite-dimensional fibers $K : \pi^{*}_{10}V(\pi) \to V(\pi_{10})$ over $J^1(\pi)$ sending the vertical over X vector field in Y $\xi^{\alpha}\partial_{\alpha}$ to the vertical over Y vector field in $J^1(\pi)$ $K^{\alpha}_{i\beta}\xi^{\beta}\partial_{y^{\alpha}_i}$. Conjugate mapping of bundles

$$K^{*} : V(\pi_{10})^{*} \to \pi^{*}_{10}V(\pi)^{*} \tag{8.5}$$

is such that

$$K^{*}d_{v_{10}}L = f_{\beta}dy^{\beta}. \tag{8.6}$$

$d_{v_{10}}L$ here is the vertical differential of the function L.

Vice versa, if K^{*} is **any** bundle mapping 8.5 satisfying the condition 8.6, then its conjugate K^{**} of this bundle mapping $\pi^{*}_{10}V(\pi) \to V(\pi_{10})$ is such that taking $K = K^{**}$ in defining the variation of jet variables (4.1) we get the Euler-Lagrange system (6.4).

Locally, over a domain in $J^1(\pi)$ where both bundles in 8.5 are trivial, this can be done simply, by defining K^{*} to satisfy (8.6) and to be an arbitrary mapping in a subspace of $V(\pi_{10})^{*}$ complemental to the linear span of the vertical covector $d_{v_{10}}L$ (taking, for instance, an orthogonal complement in the fibers with respect to some natural metric).

Thus, locally, near the points where $d_{v_{10}}L \neq 0$, tensor K can be defined in such a way that the corresponding mapping (8.5) has the "rank one". In the next section we formalize these arguments and construct the canonical NC-tensor $K^{\alpha}_{i\beta}$ on the bundles $\pi : Y \to X$ having the property of "metric prolongation" or on the bundles.

9. Case of natural (tensor-like) bundles: Canonical NC-tensor $K^{i}_{\mu j}$

. In this section we consider the case where the configurational bundle $\pi : Y \to X$ has some properties similar to the property of tensor and tensor density bundles of "metric prolongation" (see Appendix I). More specifically, we assume that the configurational space Y is endowed with the metric q such that the projection $\pi : Y \to X$ is the isometry. Tangent bundle $T(Y)$ splits as the direct q-orthogonal sum

$$T(Y) = T(V)(\pi) \oplus H,\ \ H_y = V_y(\pi)^{\perp}. \tag{9.1}$$

In a case of "normal bundles", including bundles of tensors and tensor densities, the metric g in X defines the metric q_x on the fibers Y_x of the bundle π. In local fibred chart (x^i, y^α), where the basis of the tangent bundle to the fiber Y_x at a point $(x,y) \in Y_x$ can be taken to be $\partial_\alpha, \alpha = 1, \dots, m$,

$$< \partial_\alpha, \partial_\beta >_{(x,y)} = q_{(x,y)\ \alpha\beta}.$$

In particular, the vertical subbundle $\nu : V(\pi) \to Y$ of the tangent bundle $T(Y) \to Y$ is endowed with the induced metric.

At the same time, metric g in X defines the covariant metric $g^{\sigma\lambda}$ on the fibres of the cotangent bundle $\tau^* : T^*(X) \to X$.

Let Γ be a connection on the bundle π defined by the decomposition (9.1). If π is a tensor or tensor density bundle, we can take Γ to be prolongation of Levi-Civita connection Γ^g to $\pi : Y \to X$ using products and operation of conjugation, see ([33]).

Connection Γ defines (and is defined by) the section $j_\Gamma : Y \to J^1(Y)$. This section defines an original point in affine fibers $J^1(\pi)_y$ over points $y \in Y$. As a result, it defines the isomorphism **of the bundles over** Y

$$J^1(\pi) \simeq \pi^* T^*(X) \otimes V(\pi). \tag{9.2}$$

Metrics given above in the fibers of bundles $T^*(X) \to X$ and in $V(\pi) \to Y$, define the metric G in the fibers of the bundle (9.2) over Y. For instance, one can choose local frames $\{e_j, j = 1, \dots, n\}; \{f_\mu, \mu = 1, \dots, m\}$ orthonormal in the fibers of respective bundles and define the metric G **requesting the frame** $\{e_\mu \otimes f_k\}$ **formed by sections of 9.2 to be** G**-orthonormal**.

In terms of local frames dx^i, ∂_μ and identifying vector spaces - fibers of the bundle (4.8) over Y with the tangent space to these fibers this metric has the form

$$G(dx^i \otimes \partial_\mu, dx^j \otimes \partial_\nu) = g^{ij} q_{Y\ \mu\nu}. \tag{9.3}$$

Coming back to the $J^1(\pi)$ and using the fact that the π_{10}-vertical tangent bundle of $J^1(\pi) \to Y$ is the vector bundle with the basis $\partial_{y^i_\mu}$, we get the metric in the fibres of this bundle defined by

$$G(\partial_{y^{mu}_i}, \partial_{y^\nu_j}) = g^{ij} q_{Y\ \mu\nu}. \tag{9.4}$$

The dual metric in the fibers of the bundle $V(\pi_{10})^*$ have the form

$$G^*(dy^\mu_i, dy^\nu_j) = G^{\mu\nu}_{ij} = g_{ij} q^{\mu\nu}_Y. \tag{9.5}$$

Lagrangian L defines the section $d_{v_{10}} L = L_{,y^\mu_i} dy^\mu_i$ of the bundle $V(\pi_{10})^* \to J^1(\pi)$ (where the "*vertical differential*" with respect to the variables in the fiber of the bundle $\pi_{10} : J^1(\pi) \to Y$ was used) To the section $d_{v_{10}} L$ there corresponds the orthogonal complement $dL^\perp$ in the metric G^* .

Define the morphism $K^* : V(\pi_{10})^* \to V(\pi)^*$ between the dual bundles of vertical bundles ν_{10} and ν_π by the conditions

$$\begin{cases} K^*(d_{v_{10}} L) = f_\mu dy^\mu, \\ K^*|_{dL^\perp} = 0. \end{cases} \tag{9.6}$$

This defines the tensor K^* and, therefore, the dual tensor $K = \{K^\alpha_{i\beta}\}$ that can be considered as the unique mapping $K : V(\pi) \to V(\pi_{10})$. More concretely, the

G^*-orthogonal projector P to the element $\pi = d_{v_{10}}L = \pi^i_\mu dy^\mu_i$ has the form

$$Pk = \frac{G^*(k, d_{v_{10}}L)}{\|d_{v_{10}}L\|^2_{G^*}} d_{v_{10}}L = \frac{G^{\mu\nu}_{ik} k^k_\nu \pi^i_\mu}{\|\pi\|^2_{G^*}} d_{v_{10}}L.$$

The corresponding bundle mapping $K^* : V(\pi_{10})^* \to V(\pi)^*$ has the form

$$\mathcal{K}^* k = \frac{G^{\mu\nu}_{ik} k^\nu_k \pi^i_\mu}{\|\pi\|^2_{G^*}} f_\alpha dy^\alpha.$$

the dual mapping $\mathcal{K} : V(\pi) \to V(\pi_{10})$, defined by the condition

$$< K^* k, \xi^\mu \partial_\mu >=< k, K(\xi^\mu \partial_\mu) >,$$

is

$$K(\xi^\mu \partial_\mu) = (\xi^\mu f_\nu) \frac{\pi^i_\mu}{\|\pi\|} G^{\nu\kappa}_{ik} \partial_{y^\kappa_k}. \tag{9.7}$$

As a result, **the "canonical" tensor** $\mathcal{K}$ has the form

$$K^\mu_{\nu i} = \left(\frac{\pi^j_\alpha}{\|\pi\|^2} G^{\mu\alpha}_{ij} \right) f_\nu. \qquad (9.7').$$

This tensor is defined at all points $(x^i, y^\mu, y^\mu_j) \in J^1(\pi)$ except those where $\pi = d_{v_{10}}L = 0$.

Theorem 3. *Let π be a tensor type bundle and let Γ be a connection in the bundle π. Let G be the metric in the fibers of the vertical bundle ν_{10} in the bundle $J^1(\pi) \to Y$. Then, the tensor*

$$K^\mu_{\nu i} = \left(\frac{\pi^j_\alpha}{\|\pi\|^2} G^{\mu\alpha}_{ij} \right) f_\nu. = \frac{\partial ln(\|\pi\|)}{\partial \pi^i_\mu} f_\nu = \chi^\mu_i f_\nu, \tag{9.8}$$

where

$$\chi^\mu_i = \frac{\partial ln(\|\pi\|)}{\partial \pi^i_\mu} \tag{9.9}$$

was introduced, is such that the Euler-Lagrange system with the variations of derivatives, modified by the tensor K has the form (6.4) of the Euler-Lagrange system with Lagrangian L and the source covector (force if $n = 1$) $f = f_\beta dy^\beta$.

Bundle mapping $K^* : V(\pi_{10})^* \to V(\pi)^*$ of the form

$$\mathcal{K}^* k = \frac{G^{\mu\nu}_{ik} k^\nu_k \pi^i_\mu}{\|\pi\|^2_{G^*}} f_\alpha dy^\alpha.$$

has minimal norm between all the mappings of these bundles such that $K^*(d_{v_{10}}L) = f_\beta dy^\beta$.

Remark 13. If the source functions f_β do not depend on the derivatives, i.e. if $f_\beta \in C^\infty(Y)$, then the tensor K has the "quasi-potential form" form

$$K^\mu_{j\nu i} = \frac{\partial}{\partial \pi^i_\mu} (ln(\|\pi\|) \cdot f_\nu). \tag{9.10}$$

Example 5. (n=1). If we take

$$K^\mu_{i\nu} = \frac{\partial ln(\|\pi\|)}{\partial \pi^i_\mu} Q_{\beta\nu} \dot{y}^\beta, \tag{9.11}$$

with a (0,2)-tensor $Q_{\beta\nu}$ in $J^1(\pi)$, the Euler-Lagrange Equations for a Lagrangian $L \in C^\infty(J^1(\pi))$ takes the form

$$\dot{y}^\beta = Q^{\beta\mu}\frac{\delta L}{\delta y^\mu}. \tag{9.12}$$

Following this scheme one can realize a large group of reaction-diffusion equations () as Euler-Lagrange equations of an appropriate Variational Principle.

Remark 14. Being universal, canonical form (9.7-9.7') of the NC-tensor K, defining non-commuting variations, is rarely the most convenient. As the examples presented in this text shows, most natural expressions for the non-commutativity tensor K are constructed using the specific quantities in the systems to study. Yet, in some examples, this form appears naturally in the construction of forces (source terms). For instance, linear connections participating in the "geometrization" of a mechanical system with non-potential forces (see Ch.V., Sec. 5.1) are directly related with the canonical form of the NC-tensor K.

10. **Examples.**

In this Section we present several examples of Lagrangian systems with non-commuting variations and classes of such systems. These examples illustrate the range and flexibility of method of NC-variations.

Example 1. Harmonic oscillator.

Here $n = 1$, t-time is the only independent variable, $x = x(t)$ is the only dynamical variable. There is only one 1-jet variable $\dot{x}$. Lagrangian L has the form

$$L = \frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2.$$

Vertical variations have the form $\xi = \xi^1(t,x)\partial_x$. Being modified by a "NC-tensor" (here - scalar) $K(t,x,\dot{x})$ lift to 1-jet space has the form

$$\xi^{(1)}_K = \xi^1\partial_x + [d_t\xi^1 + K^t, x, \dot{x})\xi^1]\partial_{\dot{x}}.$$

Euler-Lagrange equation of harmonic oscillator with a non-commutative variation(s) has the form

$$L_{,x} - d_t(L_{,\dot{x}}) = -KL_{,\dot{x}} \Leftrightarrow \ddot{x} + (-K)\dot{x} + \omega^2 x = 0. \tag{10.1}$$

This is the standard harmonic oscillator with the (positive) dissipation for $K < 0$ and negative dissipation for $K > 0$. the energy balance law here has the form

$$d_t E = d_t(\frac{1}{2}\dot{x}^2 + \frac{\omega^2}{2}x^2) = K\dot{y}^2.$$

Example 2. Linear system with dissipation of rate type. In a linear system for a vector function $u : T \to R^m$,

$$\ddot{u} = Au + B\dot{u}, \tag{10.2}$$

with a symmetric $m \times m$-matrix A and negative definite symmetrical matrix B energy dissipates provided matrix B is negative definite : $B < 0$ (see [139]).

This system has the form (6.4) with the Lagrangian $L = \frac{1}{2}\|\dot{u}\|^2 - \frac{1}{2}(Au, u)$ (where standard Euclidian metric in the fibers R^m of the configurational bundle is used) and the tensor-potential $K^{\mu}_{i\nu} = -\delta^0_i B^{\mu}_{\nu}$.

Example 3. Heat equation.

Let

$$\theta_{,t} - D\Delta\theta = 0 \tag{10.3}$$

be a standard heat equation in the 4-dim Galilean space-time with the Euclidian metric and Cartesian coordinates $(x^0 = t, x^A,\ A = 1, 2, 3.$ Let $\theta(t, x)$ be the absolute temperature in the 4-dim domain $\Omega = [t_0, t_1] \times V \subset R^4$. Temperature is the only dynamical field.

Consider an action

$$A_\Omega(\theta) = \int_{t_0}^{t_1} \int_V \left[\frac{\partial\theta}{\partial t} + \frac{D}{2}\|\nabla\theta\|^2\right] dvdt. \tag{10.4}$$

Introduce the tensor K of the type $K^{\mu}_{i\nu}$, see Section 6 above, with the components $K^{\theta}_{i\theta} = \theta_{,t}\delta^0_i, \mu = 0, 1, 2, 3$. Variations of the derivatives defined as in (6.1) have the form

$$\begin{cases} \delta \frac{\partial\theta}{\partial x^A} = \frac{\partial}{\partial x^A}\delta\theta, \\ \delta \frac{\partial\theta}{\partial t} = \frac{\partial}{\partial t}\delta\theta + \frac{\partial\theta}{\partial t}\delta\theta. \end{cases} .$$

It is easy to see that the Euler-Lagrange equation (6.9) takes the form (10.3).

Example 4. Laplace Equation.

Consider the case where $m = 1, n > 1$, base X is the Euclidian vector space E^n and the function $u(x^1, \dots, x^m)$ satisfies to a EL-equation with the Lagrangian $L(u) = \frac{1}{2}\sum_{i=1}^{i=n}(u_{,x^i})^2$. NC-variation is defined by a "NC-tensor"" K^i . EL-equation (6.9) with NC variations has the form

$$\Delta u + K^i \partial_{x^i} u = 0. \tag{10.5}$$

Example 5. Burgers Equation.

Burgers equation

$$u_{,t} + uu_{,x} = \alpha u_{,xx},$$

can be realized, modifying the construction of Example 2, as the Euler-Lagrange Equations with the Lagrangian $L = u_{,t} + \frac{\alpha}{2}(u_{,x})^2$ and the action

$$\mathcal{A}(u) = \int_{t_0}^{t_1} \int_{V^3} [u_{,t} + \frac{\alpha}{2}(u_{,x})^2] dvdt$$

and the commuting relations for variations with

$$K^u_{iu} = \begin{cases} u_{,t},\ i = 0, \\ u,\ i = 1 \end{cases}$$

Moments π^i have the form: $\pi^0 = 1,\ \pi^1 = \alpha u_{,x}$. Energy density has the form $E = \frac{\alpha}{2}(u_{,x})^2$ and the energy balance law takes the form

$$d_t(\frac{\alpha}{2}(u_{,x})^2) + d_x(-\alpha(u_x)^2) = -(u_{,t}^2 + uu_{,t}u_{,x}). \tag{10.6}$$

Similarly, modifying the construction of Example 2, one can realize the Burgers-KdV equation, Kuramoto-Sivashinsky equation and the Ginzburg-Landau equation (see [81]).

Example 6. Maxwell equations.

In this example we are using notations usual in classical electrodynamics, [80]. In particular, indices of fields are the same as indices of space-time coordinates since the fields here are tensor fields in the space-time.

Let $\xi \to M^4$ be the complex line bundle over the pseudo-riemannian manifold (M, g) with the space-time coordinates $x^\mu,\ \mu = 0.1.2.3$. $A = A_\mu dx^\mu$ is a connection form in the bundle ξ, $F = dA = F_{\mu\nu} dx^\mu \wedge dx^\nu, F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ - corresponding curvature form. Operator $*$ is the "star operator" corresponding to the metric g, $d_g v$ - corresponding volume form. Action for the connection A is

$$\mathcal{A}_{D^4}(A) = \int_D \|F\|^2_g d_g v = \int_D F \wedge *F.$$

Maxwell equations in empty space are

$$\begin{cases} dF = 0, \\ d * F = 0. \end{cases}$$

Let $A_\sigma \to A_\sigma + \epsilon \xi^\sigma$ be a a variation of A corresponding to the vertical vector field $\xi = \xi^\sigma \partial_{A_\sigma}$. Let $K^\nu_{\mu\sigma}$ be tensor (corresponding to a vertical connection). Variation of the derivatives $z^\sigma_\mu = A_{\sigma,\mu}$ corresponding to the above variation of A_σ is $z^\sigma_\mu \to z^\sigma_\mu + (d_\mu \xi^\sigma + K^\sigma_{\lambda\mu} \xi^\lambda)$. Since only antisymmetric combinations of derivatives $A_{\sigma,\mu}$ enter Lagrangian, we may assume that tensor K is antisymmetric by corresponding variables.

Then, Euler-Lagrange equations for action $\mathcal{A}_{D^4}(A)$ with the variation rule (4.1) have the form

$$d * F = K * F \Leftrightarrow, \text{ in components } - \quad d_\alpha (*F)^{\alpha\beta} = -K^\beta_{\gamma\delta} (*F)^{\gamma\delta}. \tag{10.7}$$

The energy-momentum balance law takes the form

$$d_\mu (\|F\|^2_g \delta^\mu_\sigma - 2F^\mu_\sigma) = -K^{\nu\lambda}_\mu F_{\nu\sigma} (*F)^\mu_\lambda. \tag{10.8}$$

Maxwell equations in a media with the dissipation of energy (generated heat) quadratic by the field force are well known in the Electrodynamics of Continuum, see, for example, [48], Ch.13.

If we introduce complex coefficients of dielectric (ϵ) and magnetic (μ) permutability into the tensor K in such a way that it is possible to separate electrical and magnetic terms we can describe a situation where the energy dissipation of a way of frequency ω (in the right side of 10.8) will have the form

$$\frac{\omega}{4\pi} (Im(\epsilon) \|E\|^2 + Im(\mu) \|H\|^2)$$

coinciding with the expression of energy dissipated **in a wave of frequency** ω in dispersive media, [79], Sec.61. NC-tensor K constructed in such way carries information about the dispersive properties of media.

Example 7. Rate type dissipation.

Let $n = 4, i = 0, 1, 2, 3$. Take $K^\mu_{i\nu} = k\delta^0_i \delta^\mu_{\mu_0} \delta^{\nu_0}_\nu$. Then, the lift of a vertical variation $\xi = \xi^\mu \partial_\mu$ to $J^1(\pi)$ has the form

$$\tilde{\xi} = \xi + (d_i \xi^\mu + k\delta^0_i \delta^\mu_{\mu_0} \xi^{\nu_0}) \partial_{z^\mu_i} = \xi^1 + k\xi^{\nu_0} \partial_{z^{\mu_0}_0}.$$

As a result the only *noncommutativity* here is the one of the time derivative ∂_t and the variation of μ_0-th dynamical field y^{μ_0}. The noncommutativity term is proportional to the variation of y^{ν_0}:

$$\delta\partial_t y^{\mu_0} - \partial_t \delta y^{\mu_0} = k\delta y^{\nu_0}.$$

The Euler-Lagrange system has the form

$$L_{y^\nu} - d_i(L_{,z^\nu_i}) = -k\delta^{\nu_0}_\nu L_{z^{\mu_0}_0} = -k\delta^{\nu_0}_\nu \pi^0_{\mu_0}. \tag{10.9}$$

Thus, the force appears only in the ν_0-th equation and is proportional to the μ_0**-th component of linear momentum** $\pi^0_{\mu_0}$. The simplest case is, of course, when $\mu_0 = \nu_0$ and the force that appears in the equation for y^{μ_0} is proportional to π_{μ_0}.

Energy-momentum law takes the following form

$$d_i T^i_0 = -\frac{\partial L}{\partial x^0}_{expl} - k\dot{y}^{\mu_0}\pi^0_{\nu_0} =^{\mu_0=\nu_0} -\frac{\partial L}{\partial x^0}_{expl} - k\dot{y}^{\mu_0}\pi^0_{\mu_0}. \tag{10.10}$$

In a classical case when Lagrangian L is the sum of kinetic energy $K = \frac{1}{2}m_{\mu\nu}\dot{y}^\mu\dot{y}^\nu$ and potential energy (often - a function of fields y^μ only) and where we take $\mu_0 = \nu_0$, the dissipation term $k\dot{y}^{\mu_0}L_{,\dot{y}^{\mu_0}}$ is proportional to the square of the rate of change of the field y^{μ_0}.

Example 8. Gradient type dissipation. If we take $K^i_{0j} = 0$, the dissipative force in the right side of Euler-Lagrange equations (6.6-6.9) takes the form (6.9), $-\sum_{A=1,2,3} K^\mu_{A\nu}\pi^A_\mu$ and the second term in the energy balance (7.6) becomes equal to $-\dot{y}_\nu K^\mu_{A\nu}\pi^A_\mu$. As a result, this dissipation term is proportional to the spacial gradient of the fields y^i, more specifically, their momenta π^A_μ.

Example 9. Controlled oscillator. In the next example, evolution of one dynamical variable is influenced by the rate of change of the other variable. Consider a system of two oscillators with the variables $y^\mu, \mu = 1,2$ and the Lagrangian $L = \frac{m_1}{2}(\dot{y}^1)\ ^2 + \frac{m_2}{2}\dot{y}^2)\ ^2 - U(y^1, y^2)$. Let K^μ_ν be of the type $K^\mu_\nu = \partial_\nu P^\mu$ with some functions $P^\mu(y^\mu, y^2)$.

Then, the EL-equations with the force induced by "tensor" K takes the form (see Sec.)

$$\begin{cases} m_1\ddot{y}^1 + U_{,y^1} = \partial_{y^1}(m_1P^1\dot{y}^1 + m_2P^2\dot{y}^2), \\ m_2\ddot{y}^2 + U_{,y^2} = \partial_{y^2}(m_1P^1\dot{y}^1 + m_2P^2\dot{y}^2), \end{cases} \tag{10.11}$$

Consider the case where $P^1 = 0,\ m_2P^2 = \phi(y^\mu)$. Then the last system takes the form

$$\begin{cases} m_1\ddot{y}^1 + U_{,y^1} = \phi_{,y^1}\dot{y}^2, \\ m_2\ddot{y}^2 + U_{,y^2} = 0. \end{cases} \tag{10.12}$$

Thus, the second oscillator influences the dynamical behavior of the first one through the rate of change $\dot{y}^2$ of the variable y^2.

Remark 15. In Sec. (17) it will be shown that such a tensor K corresponds to a *vertical connection* in the bundle $J^1(\pi) \to Y$ with zero curvature.

11. Weak and strong minimizers for the systems with NC-variations.

In this section we compare week and strong minimizers for Lagrangian systems with and without NC-variations. We show that in both these cases strong minimizers are always the same and weak minimizers are the same provided the NC-tensor K is bounded from above and below. We refer reader to the fundamental monograph [47], Vol.2 Chapter 5 for more details and examples.

Local minimizers are sections $s(x)$ of the configurational bundle that delivers minimum to the action functional in some neighborhood of section $s(x)$. In order to specify a type of local minimizers one has to specify the *type of neighborhoods* U of "potential local minimizer" s so that from such neighborhoods of s we take the sections $p(x)$ for comparison of value of action $\mathcal{A}_D(p)$ with the value $\mathcal{A}_D(s)$.

This is done by the choice of metric (or norm) in the space of sections $s(x)$ used in the problem.

In this section we assume that the configurational bundle $\pi : Y \to X$ is a vector bundle with the standard fiber $F \equiv R^m$ (see Appendix I,Sec.64). In particular, in the domain U of a fibred chart (U, x^i, y^μ) (over the domain $\bar{U} = \pi(U) \subset X$) where bundle π is trivial: $\pi_U \equiv (\bar{U} \times F \to \bar{U}$, sections $s : \bar{U} \to Y$ defines and are defined by the mappings $u : \bar{U} \to F$.

Sections $s(x)$ whose components $y^\alpha(x)$ have continuous derivatives $D_I y^\alpha$. up to the order k, $|I| \leqq k$ form the Banach space $C^k(\bar{U})$ with the norm

$$\|s\|_k = \underset{x\in U}{Sup}\{\|D_I u(x)\|, |0 \leqq \alpha \leqq k|.$$

Here $I = (i_1, i_2, \ldots, i_n)$ is a multiindex (i_k are natural numbers) and $D^I y^\alpha = \partial_1^{i_1} \cdot \partial_n^{i_n}$ - corresponding partial dervative of order $|I| = i_1 + i_2 + \ldots + i_n$ of the function $y^\alpha(x)$.

Let $L(x^i, y^\mu, y^\mu_i)$ be a first order Lagrangian and let $D \subset X$ be an open subset with the closure $\bar{D}$. Action $A_D(s)$ is defined for the sections $s \in C^1(\bar{D}, Y)$ ($C^1(\bar{D}, F)$ in the domain of a fibred chart) of the class C^1 *up to the boundary.* This means that for any chart $(U \subset D, x^i, y^\mu)$ components $y^\mu(x))$ of a section $s(x)$ are functions of the class C^1 up to the border ∂D.

Now we define

Definition 3. *For a section $s \in C^1(\bar{D}, Y)$ and for some $\epsilon > 0$, the set*

$$N_\epsilon(s) = \{p \in C^1(\bar{D}, Y) : \|s - p\|_{C^0} < \epsilon\} \tag{11.1}$$

is called a (standard) ***strong neighborhood of section*** *s.*

On the other hand,

Definition 4. *For a section $s \in C^1(\bar{D}, Y)$ and for some $\epsilon > 0$, the set*

$$N^1_\epsilon(s) = \{p \in C^1(\bar{D}, Y) : \|p - s\|_{C^1} < \epsilon\} \tag{11.2}$$

is called a (standart) ***weak neighborhood of section*** *s.*

It is clear that $N^1_\epsilon(s)$ is the subset of $N_\epsilon(s)$ and that $N^1_\epsilon(s) \subset N_\epsilon(s)$ is the (strict) inclusion.

Now we define the notions of **weak and strong minimizers**.

Definition 5. (1) *A section $s : D \to Y$ of the class $C^1(\bar{D},\pi)$ is called a* ***weak local minimizer*** *of action $\mathcal{A}_D(s)$ if there exists a weak ϵ-neighborhood $N^1_\epsilon(s)$ of section s such that*

$$\mathcal{A}_D(s) \leqq \mathcal{A}_D(p), \text{ for all sections } p \in N^1_\epsilon(s). \tag{11.3}$$

(2) *A section $s : D \to Y$ of the class $C^1(\bar{D},\pi)$ is called a* ***strong local minimizer*** *of action $\mathcal{A}_D(s)$ if there exists a strong ϵ-neighborhood $N_\epsilon(s)$ of s such that*

$$\mathcal{A}_D(s) \leqq \mathcal{A}_D(p), \text{ for all sections } p \in N_\epsilon(s). \tag{11.4}$$

It is obvious that any strong (local) minimizer is, at the same time, the weak (local) minimizer. Next example shows that the opposite statement would be wrong.

Example 6. -(Exercise) ([118],Sec.2.1.)). Prove that zero function $u(x) = 0$ is a weak minimizer but is not **the strong** minimizer of the functional

$$\mathcal{A}(u) = \int_0^\pi (y^2(x))(1 - y'^2(x))dx,$$

with the initial condition $y(0) = y'(0) = 0$.

Hint: consider $y(x) = \frac{1}{\sqrt{n}} sin(nx)$.

If the boundary ∂D is piecewise smooth, conditions for a section s to be a strong or weak minimizer (see Definition 5) are equivalent to the following **strong and weak minimum properties**.

Definition 6. (1) *A section $s : D \to Y$ of the class $C^1(\bar{D},\pi)$ satisfies to the* ***strong minimum property*** *if for some $\epsilon > 0$, such that inequality $\mathcal{A}_D(s) \leqq \mathcal{A}_D(s + \phi)$ holds for all $\phi \in C^1(\bar{D})$ such that $\|\phi\|_{C^0(\bar{D})} < \epsilon$.*

(2) *A section $s : D \to Y$ of the class $C^1(\bar{D},\pi)$ satisfies to the* ***weak minimum property*** *if for some $\epsilon > 0$, such that inequality $\mathcal{A}_D(s) \leqq \mathcal{A}_D(s^* + \phi)$ holds for all $\phi \in C^1(\bar{D},\pi)$ such that $\|\phi\|_{C^1(\bar{D})} < \epsilon$.*

11.1. **Minimizers for systems with NC-variations.** Let the configurational bundle $\pi : Y \to X$ be a vector bundle (See Appendix I).

Properties of a section $s : D \to Y$ to satisfy the strong or weak minimum property is formulated in terms of variations $s \to s + \phi$ of sections s. For the strong minimum property variation ϕ is estimated in the norm $\|\phi\|_{C^0}$ and, as a result, is calculated using the metric (norm in the case of a vector bundle) in fibers Y_x of the configurational bundle π and the Sup norm for the continuous sections of π.

A strong ϵ-neighborhood of a section s is defined by the condition: $p(D) \subset O_\epsilon(s)$ if

$$\|p - s\|_{C^0(D)} = Sup_{x\in D}|p(x) - s(x)| \leqq \epsilon$$

,

This neighborhood does not depend on a prolongation of section s to the 1-jet bundle. As a result, **strong ϵ-neighborhoods are the same for conventional and NC-modified variations**. Therefore, **strong minimizers are the same whether one is using conventional or NC-modified prolongations**.

For the weak minimum property, the norm of variation ϕ is calculated using the prolongation of sections s and $s + \phi$ to the 1-jet bundle $\pi_1 : J^1(\pi) \to Y \to X$ and estimating the C^1-norm of variation $\|\phi\|$.

If we are using a non-commutative variation approach, prolongation of variation ϕ differs from the conventional one. As a result, weak ϵ-neighborhoods for the system with NC-variations could be different from those in the case of conventional variations. To study this difference we need to look more attentively at the form of C^1-neighborhoods of 1-jet prolongations of sections of the configurational bundle.

More specifically, in the definition of weak ϵ-neighborhood of a section s, one has to use (in the case of a vector bundle π) variations $s(x) \to s(x) + \phi(x)$ such that

$$\|\phi\|_{C^1}(D) = \|Pr^1_K(\phi)\|_{C^0(D)} < \epsilon. \tag{11.5}$$

This condition contains the Sup norms of the components $d_\mu \phi^i + K^i_{\mu j}\phi^j$ of vector field $Pr^1_K\phi$ in $J^1(\pi)$ (weak variation components).

In the case of a vector bundle, ϵ-neighborhood $O_\epsilon(s)$ (in sense of any of our norms) of a section s has the form $s + O_\epsilon(0)$ where $O_\epsilon(0)$ is the ϵ-neighborhood of zero section of the bundle π. Thus, to compare minimality properties of sections s, it is sufficient to consider the case where s is zero section.

Considering a more general setting, assume that the bundle π is endowed with a Riemannian metric (in the fibers Y_x of the bundle π) and introduce a **tubular neighborhood** $O_\epsilon(s_*)$ of a section $s_* : D \to Y$ (see [6], Sec.8.1 or [86]) and the description of tubular neighborhood below) of the image $s_*(D) \subset Y$ of section s (notice that $s(D)$ is diffeomorphic to D and is the regular submanifold of Y) **generated by geodesics of metric in the fibers** Y_x.

A tubular neighborhood is obtained by using the exponential mappings $exp : T_{s_*(x)}(Y_x) \to Y$: $\gamma_v(t) : exp(tv)$ for unit vectors v $\|v\| = 1$. Then, for small ϵ the subsets

$$\{(x, \gamma_v(t))|v \in T_{s_*(x)}(Y_{s_*(x)}),\ 0 \leqq t \leqq \epsilon\}$$

scan the ϵ-neighborhood of $s_*(D)$. To prove this, it is sufficient to notice that the geodesic are shortest curves for small ϵ, and, if a section $p : D \to Y$ is such that $\|s_* - p\| \leqq \epsilon$, then $p(D) \subset \mathcal{O}_\epsilon(s)$.

To describe weak neighborhoods of a section s we will extend the description of strong tubular ϵ-neighborhoods of the image $s(X)$ of a section $s : D \to Y$ above to the variations of 1-jets $j^1 s$ of sections s.

To do this we have to employ a metric in the fibers of double bundle $J^1(\pi) \to Y \to X$. To introduce a metric, we recall that the bundle $\pi_{10} : J^1(\pi) \to Y$ is the vector bundle provided the configurational bundle $\pi : Y \to X$ is. Namely, a connection ν in the vector bundle π defines (and is defined by) the section $q_\nu : Y \to J^1(\pi)$ (See Appendix I, Sec.75). This section defines the origin $0_y = q_\nu(y)$ at any fiber $J^1_y(\pi)$ specifying the structure of vector space in this fiber and the splitting

$$T(J^1(\pi)) = V(\pi_{10}) \oplus q_{\nu *}(T(Y))$$

at the points of $q(Y)$. The action of a linear group acting in the fibers of $\pi_{10} : J^1(\pi) \to Y$ extends this decomposition to the whole 1-jet bundle.

Choose a norm in the fibers of the vector bundle $J^1(\pi) \to Y$ smoothly depending on (x, y). Together with the norm in the fibers Y_x of the bundle π this delivers the norm in the fibers of the vector bundle $J^1(\pi) \to Y \to X$ over X. This norm (smoothly depending on $x \in X$) defines the Riemannian metric in the fibers $J^1(\pi)_x$ of $\pi_1 : J^1(\pi) \to X$.

Now we repeat arguments used for the previous construction to the case of 1-jets $j^1(p)$ of sections p in the neighborhoods of the section $s_*(x)$.

Take an infinitesimal variation presented by a π-vertical vector field $\xi = \xi^i\partial_i$ and let ϕ^t be the flow of this vector field. This vector field induced the variation $\phi^t(x)s_*(x)$ of the section $s_*(x)$.

Let $Pr^1(\xi) = \xi^i\partial_{y^i} + d^i_\mu\xi^i\partial_{y^i_\mu}$ be the flow prolongation of $\xi^i\partial_{y^i}$- vector field in the 1-jet space $J^1(\pi)$ whose phase flow is acting on the 1-jet j^1s_* of section s_*

$$j^1\phi^t s = \widehat{\phi}^t j^1 s_* = e^{t(\xi^i\partial_i + d^i_\mu\xi^i\partial_{y^i_\mu})} j^1 s_*(x) \tag{11.6}$$

Such prolongations for all vector fields ξ (normal to the surface $s_*(D)\subset Y$) and for small t, say for $t\leqq\epsilon$, scan the neighborhood of 1-jet section j^1s_*.

Using the Sup-norm for sections of 1-jet bundle $\pi_1 : J^1(\pi)\to X$ we obtain the (equivalent of) C^1-norm for the sections $s_*(x)$ of the bundle π.

For a chosen $\epsilon > 0$, there exists $\delta > 0$ such that if $t\leqq\delta$, the distance between $\phi_t s_*$ and s_* is less then ϵ in the C^1-norm. Taking ϵ small enough we prove that to check the condition of weak minimum for section s_*, it is sufficient to use weak neighborhoods obtained this way - by using flows of variational vector fields.

In the case of a non-commuting variations formalism defined by a tensor $K^\nu_{i\mu}$, expression (11.6) for the flows scanning the weak neighborhood of a potential weak minimizer s_* has the form

$$Pr^1\phi^t_K j^1 s_* = e^{t(\xi^i\partial_i + (d^i_\mu\xi^i + K^i_{\mu j}\xi^j)\partial_{y^i_\mu})} j^1 s_*(x). \tag{11.7}$$

Calculate now the decline of 1-jets of variated sections from the 1-jet of the section s_* itself:

$$Pr^1\phi^t_K j^1 s_* - j^1 s_* \approx (t(\xi^i(s(x)) + (d^i_\mu\xi^i + K^i_{\mu j}\xi^j)(j^1 s_*(x))) + O(t^2). \tag{11.8}$$

Estimating the terms in the right side we get

$$\|Pr^1\phi^t_K j^1 s - j^1 s\|_{C^1(D)} \approx t\sum_i Sup_{x\in D}|\xi^i(s(x))| + tSup_{x\in D}\sum_i |(d_\mu\xi^i + K^i_{j\mu}\xi^j)| + o(t). \tag{11.9}$$

Now we notice that in a case where $|K^i_{j\mu}|\leqq C$, the right hand side is less then or equal to

$$\begin{aligned} t\sum_i Sup_{x\in D}|\xi^i(s(x))| + tSup_{x\in D}\sum_i |(d_\mu\xi^i))| + Ct\|\xi\|_{C^0} \leqq \\ \leqq (1+C)t\|\xi\|_{C^0} + t\sum_i \|(d_\mu\xi^i))\|_{C^0} \leqq (1+C)t\|Pr^1(\xi)(j^1 s(x))\|. \end{aligned} \tag{11.10}$$

and, therefore, it is less or equal to $(1+C)\|Pr\phi^t j^1 s - j^1 s\|_{C^0}$. Thus, any K-weak C^1-ϵ-neighborhood of a section s is contained in the conventional weak C^1-$\frac{\epsilon}{1+C}$=neighborhood of a section s. As a result, if the weak minimum property is fulfilled for the section s in the conventional sense, it is, at the same time, fulfilled in the $K^\nu_{i\mu}$-modified sense.

Along the phase curves of variational vector fields, the tensor K can behave as $t^{-\alpha}$ with $\alpha > 1$. In such a case, no diminishing of delta can prevent the difference of the norms of jets of section s_* and the variated one to leave the weak ϵ-neighborhood of section s_* - weak minimizer for the case of commuting variations.

Thus, we get to the conclusion:

Proposition 1. *For a Lagrangian variational system with a Lagrangian L of the first order and the non-commuting variations defined by a NC-tensor $K^{\nu}_{i\mu}$,*

1. *Strong minimizers of the action $\mathcal{A}_D(s)$ with Lagrangian L and* with commuting variations *and only them are strong minimizers of action with the same Lagrangian and a non-commutativity tensor $K^{\nu}_{i\mu}$.*
2. *If tensor $K^i_{\mu j}$ of the non-commutativity of variations is bounded: $|K^i_{j\mu}(s)| \leqq C$, then the weak minimizers of action with a first order Lagrangian L and the conventional variations is, at the same time, local weak minimizer in the K-modified formalism.*

Conjecture:

1. If tensor $K^{\nu}_{i\mu}$ has singularities, weak minimizers in the conventional sense may cease to be weak minimizers provided the NC-variations are used,

2. If $c \leqq \|K^{\nu}_{i\mu}\| \leqq C$ for positive constants $c < C$, then the weak variations are the same for the conventional variations and for NC-variations defined by the tensor $K^{\nu}_{i\mu}$.

12. Second variation.

The Euler-Lagrange Equations for a system with a Lagrangian L and a non-commutativity tensor K state that the *first variation of action functional* $\delta\mathcal{A}(s)$ is zero along a solution $s(x)=\{y^\mu(x)\}$. *The second variation* along a solution $s(x)$ - $\delta^2\mathcal{A}(s)$ may then be used to recognize a solution $s(x)$ as a local minimum or local maximum of the action functional (Legendre and Jacobi necessary conditions, see [44], Ch.5). In addition, study of the second variation leads to the sufficient conditions of weak extremum ([44], Ch.5). See also [47], V.I,Ch.3 for a more modern exposition of these results.

Thus, it is natural to investigate what form the results mentioned above take in the case of non-commuting variations.

In the case for a Lagrangian L of order one, we have for an action $\mathcal{A}_D(s+\delta s)$ calculated for a a solution $s(x)=\{y^\mu(x)\}$ of the Euler-Lagrange system subject to a variation $\delta y^\mu+\delta y^\mu_i$ of the form $\epsilon[\xi^\mu\partial_{y^\mu}+(d_i\xi^\mu+K^\mu_\sigma\xi^\sigma)\partial_{y^\mu_i}]$,

$$\begin{aligned}
&\mathcal{A}_D(s+\delta s)=\int_D L(x^i,y^\mu+\delta y^\mu,y^\mu_i+\delta y^\mu_i)d^nx=\\
&=\int_D[L(j^1s(x))+\epsilon L_{,y^\mu}(j^1s(x))\xi^\mu(s(x))+\epsilon L_{,y^\mu_i}(j^1s(x))(d_i\xi^\mu+K^\mu_{i\kappa}\xi^\kappa)(j^1s(x))+\\
&+\frac{1}{2}\epsilon^2[L_{y^\mu y^\nu}\xi^\mu\xi^\nu+L_{y^\mu y^\nu_i}\xi^\mu(d_i\xi^\nu+K^\nu_{i\kappa}\xi^\kappa)+L_{,y^\mu_i y^\nu_j}(d_i\xi^\mu+K^\mu_{i\kappa}\xi^\kappa)(d_j\xi^\nu+K^\nu_{j\kappa}\xi^\kappa)]\\
&+HOT(\epsilon)]d^nx=\mathcal{A}_D(s)+\epsilon\int_D\left[L_{,y^\mu}-d_iL_{,y^\mu_i}+K^\kappa_{i\mu}L_{,y^\kappa_i})\right](j^1s(x))\xi^\mu(s(x))d^nx+\\
&\qquad+\frac{\epsilon^2}{2}\int_D[(L_{,y^\mu y^\nu}+L_{,y^\mu y^\kappa_i}K^\kappa_{\nu i}+L_{y^\kappa_i y^\lambda_j}K^\kappa_{i\mu}K^\lambda_{j\nu})\xi^\mu\xi^\nu+\\
&\qquad+(L_{y^\mu y^\nu_i}\xi^\mu d_i\xi^\nu+L_{,y^\kappa_i y^\nu_j}K^\kappa_{i\mu}\xi^\mu d_j\xi^\nu+L_{,y^\mu_i y^\kappa_j}K^\kappa_{j\nu}\xi^\nu d_i\xi^\mu)+\\
&\qquad\qquad+L_{,y^\mu_i y^\nu_j}d_i\xi^\mu d_j\xi^\nu]d^nx+HOT(\epsilon).
\end{aligned}\tag{12.1}$$

As a result, the **second variation** of the action functional $A_D(s)$ is the quadratic form of the arguments $(\xi^\mu,d_i\xi^\mu)$ the form

$$\begin{aligned}
\delta^2A(\epsilon\xi^\mu,\epsilon d_i\xi^\mu)=&\frac{\epsilon^2}{2}\int_D[(L_{,y^\mu y^\nu}+L_{,y^\mu y^\sigma_i}K^\sigma_{\nu i}+L_{y^\sigma_i y^\zeta_\nu}K^\sigma_{i\mu}K^\zeta_{j\nu})\xi^\mu\xi^\nu+\\
&+(L_{y^\mu y^\nu_i}\xi^\mu d_i\xi^\nu+L_{,y^\sigma_i y^\nu_j}K^\sigma_{i\mu}\xi^\mu d_j\xi^\nu+L_{,y^\mu_i y^\sigma_j}K^\sigma_{j\nu}\xi^\nu d_i\xi^\mu)+\\
&+L_{,y^\mu_i y^\nu_j}d_i\xi^\mu d_j\xi^\nu]d^nx.
\end{aligned}\tag{12.2}$$

In a case where $n=1$ (Mechanics), the second variation has the form

$$\begin{aligned}
\delta^2A(\xi^\mu,d_t\xi^\mu)(\epsilon\xi,\epsilon\xi)=&\frac{\epsilon^2}{2}\int_a^b[(L_{,y^\mu y^\nu}+L_{,y^\mu\dot y^\sigma}K^\sigma_\nu+L_{,\dot y^\sigma\dot y^\zeta}K^\sigma_\mu K^\zeta_\nu)\xi^\mu\xi^\eta+\\
&+(L_{,y^\mu\dot y^\nu}\xi^\mu d_t\xi^\nu+L_{,\dot y^\sigma\dot y^\nu}K^\sigma_\mu\xi^\mu d_t\xi^\nu+L_{,\dot y^\mu\dot y^\sigma}K^\sigma_\nu\xi^\nu d_t\xi^\mu)+\\
&+L_{,\dot y^\mu\dot y^\nu}d_t\xi^\mu d_t\xi^\nu]dt.
\end{aligned}\tag{12.3}$$

If, additionally, $m = 1$, the second variation has the following form (comp [44], Ch.V)

$$\delta^2 A(\xi, d_t\xi) =$$
$$= \frac{\epsilon^2}{2}\int_a^b [(L_{,yy} + L_{,y\dot{y}}K + L_{,\dot{y}\dot{y}}K^2)\xi\xi + 2(L_{,y\dot{y}} + L_{,\dot{y}\dot{y}}K)\xi d_t\xi + L_{,\dot{y}\dot{y}} d_t\xi d_t\xi]dt. \tag{12.4}$$

Now we integrate by parts the expression for the second variation (12.3) in the case where $n = 1$ (Mechanics). We assume that variations ξ^i vanish on the boundary ∂D of the domain of integration D. We also assume, as is done in [44], Sec. 29, that $L_{y^\mu \dot{y}^\nu} = L_{y^\nu \dot{y}^\mu}$. Finally, to simplify the obtained expression we add the condition that $L_{\dot{\sigma}\dot{\mu}}K^\sigma_\nu = L_{\dot{\sigma}\dot{\nu}}K^\sigma_\mu$ has to be valid for all μ, ν. As a result, we get the following expression for the second variation

$$\delta^2 A(\xi^\mu, d_t\xi^\mu)(\epsilon\xi, \epsilon\xi) =$$
$$= \frac{\epsilon^2}{2}\int_D \{[L_{,\dot{y}^\mu\dot{y}^\nu}]d_t\xi^\mu d_t\xi^\nu +$$
$$+[(L_{,y^\mu y^\nu} + L_{,y^\mu\dot{y}^\sigma}K^\sigma_\nu + L_{,\dot{y}^\sigma\dot{y}^\zeta}K^\sigma_\mu K^\zeta_\nu) - \frac{1}{2}d_t(L_{,y^\mu\dot{y}^\nu} + L_{,\dot{y}^\sigma\dot{y}^\nu}K^\sigma_\mu + L_{,\dot{y}^\mu\dot{y}^\sigma}K^\sigma_\nu)]\xi^\mu\xi^\nu)]\}dt =$$
$$= \epsilon^2\int_a^b [P_{\mu\nu}\dot{\xi}^\mu\dot{\xi}^\nu + Q_{\mu\nu}\xi^\mu\xi^\nu]dt. \tag{12.5}$$

Here,

$$\begin{cases} P_{\mu\nu} = \frac{1}{2}L_{,\dot{y}^\mu\dot{y}^\nu}, \\ Q_{\mu\nu} = \frac{1}{2}[(L_{,y^\mu y^\nu} - d_t L_{,y^\mu\dot{y}^\nu}) + (L_{,y^\mu\dot{y}^\sigma}K^\sigma_\nu + L_{,\dot{y}^\sigma\dot{y}^\zeta}K^\sigma_\mu K^\zeta_\nu - d_t(L_{,y^\mu\dot{y}^\nu} + L_{,\dot{y}^\sigma\dot{y}^\nu}K^\sigma_\mu + L_{,\dot{y}^\mu\dot{y}^\sigma}K^\sigma_\nu)]. \end{cases} \tag{12.6}$$

Notice that the (0,2)-form P does not depend on the non-commutativity tensor K. If we assume, as it is suggested in [44], Sec.29, that the matrix $L_{y^\mu \dot{y}^\nu}$ **is symmetric**, then both matrices $P_{\mu\nu}, Q_{\mu\nu}$ are symmetric and the classical theory of sufficient conditions of Legendre-Jacobi is applied immediately with the corresponding modification of the Jacobi's equation.

In particular, necessary Legendre condition has the usual form:

Theorem 4. *A necessary condition for the quadratic functional*

$$\xi \to \int_a^b [(P\dot{\xi}, \dot{\xi}) + (Q\xi, \xi)]dt$$

to be non-negative for all $\xi(t)$ such that $\xi(t)_{t=a,b} = 0$, is that the matrix P_{ij} ***is non-negative definite.***

12.1. **Jacobi equation.** Expression (12.5) defines the quadratic functional

$$J(h, h) = \epsilon^2\int_a^b [P_{\mu\nu}\dot{h}^\mu\dot{h}^\nu + Q_{\mu\nu}h^\mu h^\nu]dt$$

for a vector function $h : R \to R^m$. Euler-Lagrange system of this functional is called the **Jacobi equation**:

$$-\frac{d}{dt}P_{\mu\nu}\dot{h}^\nu + Q_{\mu\nu}h^\nu = 0, \ \mu = 1, \ldots, m. \tag{12.7}$$

Remark 16. The principal part of Jacobi equations do not depend on the tensor $K^i_{\mu j}$ and is the same as in classical theory. At the same time, zero order terms in Jacobi equations depend on the tensor K linearly and quadratically.

The classical theory of quadratic functionals, the Jacobi equation and the conjugate points ([44]) are applied here without any restrictions. In particular, the following two classical result are valid here:

Theorem 5. *(Legendre necessary condition for $\mathcal{A}(s)$, case $n = 1$). If the vector function (solution of Euler-Lagrange equations) $s(t) = \{y^\mu(t)\}$ yields a* ***local minimum*** *of the functional $\mathcal{A}_D(s)$, then the matrix function*

$$P_{\mu\nu}(s(t)) = \frac{1}{2} L_{,\dot{y}^\mu \dot{y}^{j\nu}}(s(t))$$

is nonnegative definite ($P_{ij}(s(t)) \geqq 0$) along the solution $s(x)$.

Theorem 6. *Jacobi necessary condition for $\mathcal{A}(s)$).*

If the regular *(i.e.such that the condition $det(\frac{\partial^2 L}{\partial \dot{y}^\mu \partial \dot{y}^\nu})(s(t)) \neq 0$ is fulfilled along all the curve $s(t)$) extremal $s(t) \in C^2([a,b] \to R^m)$ of variational problem yields a local minimum for the action functional $\mathcal{A}(s)$with $L(t, y^\mu, \dot{y}^\nu) \in C^3(J^1(\pi))$, then the interval (a,b) does not contain points conjugate to the point a.*

Example 7. Consider the harmonic oscillator with dissipation (Example 1, Sec. 10) with the NC-tensor (being a scalar function $K(t,x,\dot{x})$) K. Jacobi equation for a function h is

$$\ddot{h} + \omega^2 - 2d_t K - K^2 = 0. \tag{12.8}$$

If K is constant and $K^2 - \omega^2 > 0$, then, there are no no points $t > 0$ conjugate to the point $t = 0$ and the lower position of equilibrium is stable.

Remark 17. Necessary and sufficient second variation criteria for weak minimizers were studied in terms of spectral properties (minimal eigenvalue) of Jacobi operator - linearization of Euler-Lagrange operator, starting with the works of H.A.Schwartz (1885), see [47], Part II, Ch.5. In the paper [5], T.Atanascovich obtained a sufficient second variation condition for local extremum in a mechanical system with non-conservative forces for the variational principle with the non-commuting variations. This condition has the form $\lambda_{min} + \gamma > 0$ where λ_{min} is the minimal eigenvalue of the appropriate Jacobi operator while γ is the constant participation in the positive definiteness condition of the quadratical form of second variation with coefficients depending on the non-conservative forces.

13. **Hamiltonian systems and the NC-variations.**

In this section we present the Hamiltonian systems corresponding to the Euler-Lagrange equations with non-commuting variations. We will follow the classical approach of H.Rund ([117]).

Let L be a regular Lagrangian of the first order (i.e. such that determinant $Det(\frac{\partial^2 L}{\partial y^\mu_i y^\nu_j})$ of Hessian of the function L is nonzero) and let $K^\mu_{i\nu}$ be a NC-tensor as in Section 6. Regularity condition allows to apply Legendre transformation

$$\begin{array}{ccc} J^1(\pi) & \xrightarrow{\mathcal{L}} & J^{1*}(\pi) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{=} & Y \end{array}$$

from the 1-jet space to the dual space $J^{1*}(\pi)$ of variables (xi, y^μ, π^i_μ):

$$(x^i, y^\mu, y^\mu_i) \to (x^i, y^\mu, \pi^i_\mu = L_{,y^\mu_i}).$$

One can consider this transformation as defining the new local chart in $J^1(\pi)$ (Legendre coordinates, see [44]). Due to the regularity condition, this transformation is invertible, i.e., in the domain of original fibred coordinates one has

$$y^\mu_i = \phi^\mu_i(x, y, \pi) \tag{13.1}$$

with the smooth functions $\phi \in C^\infty(J^{1*}(\pi))$.

Introduce the Hamiltonian function

$$H(x, y, \pi) = \pi^i_\mu \phi^\mu_i(x^i, y^\mu; \pi^j_k) - L(x^i, y^\mu; \phi^\mu_i(x, y, \pi)). \tag{13.2}$$

Using 13.1 we calculate (using, at the last equality, the definition of π)

$$\frac{\partial H}{\partial \pi^j_\nu} = -\frac{\partial L}{\partial \phi^\mu_i}\frac{\partial \phi^\mu_i}{\partial \pi^j_\nu} + \pi^i_\mu \frac{\partial \phi^\mu_i}{\partial \pi^j_\nu} + \phi^\nu_j = \phi^\nu_j = y^\nu_j. \tag{13.3}$$

Similarly, one proves that

$$\frac{\partial H}{\partial y^\mu} = -\frac{\partial L}{\partial y^\mu},\ \mu = 1, \ldots, m, \tag{13.4}$$

and

$$\frac{\partial H}{\partial x^i} = -\frac{\partial L}{\partial x^i},\ i = 1, \ldots, n.$$

Consider now the Euler-Lagrange Equations with the sources f_ν:

$$d_\mu(\frac{\partial L}{\partial y^\nu_i}) - \frac{\partial L}{\partial y^\nu} = f_\nu,\ \nu = 1, \ldots, m,$$

substitute $\pi^i_\nu = L_{,y^\mu_i}$ here and use (13.4). This equation takes the form

$$d_i \pi^i_\nu = -\frac{\partial H}{\partial y^\nu} + f_\nu. \tag{13.5}$$

Combining this equation with (13.3) we will get Hamiltonian (canonical) system of equations equivalent to the Euler-Lagrange system with the sources f_ν

$$\begin{cases} y^\mu_{,i} = \frac{\partial H}{\partial \pi^i_\mu}, \\ d_i \pi^i_\nu = -\frac{\partial H}{\partial y^\nu} + f_\nu. \end{cases} \tag{13.6}$$

If the sources f_ν in (13.5) are generated by the tensor K, Hamiltonian system (13.6) takes the form

$$\begin{cases} y^\mu_{,i} = \frac{\partial H}{\partial \pi^i_\mu}, \\ d_i \pi^i_\nu = -\frac{\partial H}{\partial y^\nu} + K^\mu_{i\nu}\pi^i_\mu. \end{cases} \tag{13.7}$$

Consider the case where $n = 1$, i.e. the case of finite-dimensional dynamical systems, and use the notation t for the independent variable. In this case we have (using the notation π_ν for momenta instead of the traditional p_ν)

$$E_\nu(L) = \frac{d\pi_\nu}{dt} + \frac{\partial H}{\partial y^j \nu}. \tag{13.8}$$

Thus, the Euler-Lagrange equations $E_\nu(L) = f_\nu$ are equivalent to the Hamiltonian dynamical system

$$\begin{cases} y^\nu_{,t} = \phi^\nu_{,t} = \frac{\partial H}{\partial \pi_\nu} \\ \frac{d\pi_\nu}{dt} = -\frac{\partial H}{\partial y^\nu} + f_\nu =^{NC-case} = -\frac{\partial H}{\partial y^\nu} + K^\mu_\nu \pi_\mu. \end{cases} \tag{13.9}$$

To get the energy balance for Hamiltonian system (13.9) we calculate $\frac{d}{dt}H$ and get the **energy balance law**

$$\frac{d}{dt}H = \frac{\partial H}{\partial t} - \dot{y}^\sigma f_\sigma =^{NC-case} \frac{\partial H}{\partial t} - \dot{y}^\sigma K^\mu_\sigma \pi_\mu = \frac{\partial H}{\partial t} - K^\mu_\sigma \dot{y}^\sigma L_{,\dot{y}^\mu}, \tag{13.10}$$

with the energy dissipation term

$$-K^\mu_\sigma \dot{y}^\sigma L_{,\dot{y}^\mu} = -K^\mu_\sigma \dot{y}^\sigma \pi_\mu. \tag{13.11}$$

Let $J \in C^\infty(J^*(\pi))$ be a function. This function is the **first integral of the dynamical system 13.9** if (and only if) $d_t J = 0$ along the solution of system (13.9).

Calculate this total derivative of a function J we get

$$d_t J = \frac{\partial J}{\partial t} + \frac{\partial J}{\partial y^\mu}\frac{\partial H}{\partial \pi_\mu} + \frac{\partial J}{\partial \pi_\nu}\left(-\frac{\partial H}{\partial y^\nu} + f_\nu\right) = (\frac{\partial J}{\partial t} + f_\nu \frac{\partial J}{\partial \pi_\nu}) + \left(\frac{\partial J}{\partial y^\mu}\frac{\partial H}{\partial \pi_\mu} - \frac{\partial J}{\partial \pi_\nu}\frac{\partial H}{\partial y^\nu}\right). \tag{13.12}$$

Using the Poisson bracket of the functions on the (symplectic) fibers of the Hamiltonian bundles $J^*(\pi) \to R_t$: $\{F, G\} = \frac{\partial F}{\partial y^\mu}\frac{\partial G}{\partial \pi_\mu} - \frac{\partial G}{\partial y^\mu}\frac{\partial F}{\partial \pi_\mu}$ we write (13.12) for any function $F \in C^\infty(J^*(\pi))$ in the form

$$d_t F = \left(\frac{\partial F}{\partial t} + f_\nu \frac{\partial F}{\partial \pi_\nu}\right) + \{F, H\} \tag{13.13}$$

In particular,

Lemma 2. *A function $J \in C^\infty(J^*(\pi))$ is the first integral of the Hamiltonian system (13.9) with a Hamiltonian H and the "force" f_ν (13.9) if and only if*

$$\left(\frac{\partial J}{\partial t} + f_\nu \frac{\partial J}{\partial \pi_\nu}\right) + \{J, H\} = 0. \tag{13.14}$$

13.1. **Comparison with the metriplectic model.** Here we would like to compare, in the case where $n = 1$, the NC-variations Lagrangian model of introduction of forces into a Hamiltonian dynamical system with the other geometrical model having similar purpose - "Metriplectic systems" (known also under the names: double bracket systems (in Control Theory), GENERIC systems (in Europe) and, in more general setting as the dynamics on "Leibniz manifolds",[108].

Rewrite the system (13.7) in the form

$$\frac{d}{dx^i}\begin{pmatrix} y^\mu \\ \pi^i_\nu \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla_{\begin{pmatrix} y \\ \pi \end{pmatrix}} H + \begin{pmatrix} 0 \\ K^\sigma_{i\nu}\pi^i_\sigma \end{pmatrix}. \tag{13.15}$$

Introduce the kinetic energy written in terms of the momenta π^i_σ: $T = \frac{1}{2}G^{ls}_{ij}(x,y)\pi^i_l\pi^j_s$ and present the previous system in the form

$$\frac{d}{dt}\begin{pmatrix} y^\mu \\ \pi^i_\nu \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla_{\begin{pmatrix} y \\ \pi \end{pmatrix}} H + \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} T_{,y^l} \\ T_{,\pi^\nu_s} \end{pmatrix} \tag{13.16}$$

with some symmetrical matrix function M_{ls}. Calculating derivatives in the second term of the right side we see that non-Hamiltonian term in the last system has the form

$$\begin{pmatrix} 0 \\ M_{js}(x^i, y^\mu, y^\mu_i)G^{sl}(x,y)\pi_l. \end{pmatrix}$$

Introduce now the tensor

$$K^l_j = M_{js}(x^i, y^\mu, y^\mu_i)G^{sl}(x,y)\pi_l. \tag{13.17}$$

Then, the metriplectic dynamical system (13.12) with the degenerate metric $\tilde{M} = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}$ in the space $T^*(Y)$ is the representation of the Legendre transformation of the Lagrangian system with the non-commutative variations defined by the tenor K. Formula (13.17) establishes a relationship between the tensor K and the degenerate (covariant) metric M characterizing the dissipative conditions in the phase space of metriplectic system (10.12).

14. Hamilton-Jacobi Equation.

The first order nonlinear Hamilton-Jacobi PDE ([44, 117] and [47], vol II,Ch.9) plays a prominent role in the explicit determination of solutions of variational problems. Combined with the method of separating of variables, the Hamilton-Jacobi equation represents, by the opinion of many classical mathematicians, the best tool for construction of explicit solutions in Classical and Quantum Mechanics, Control theory and other domains.

That is why it is interesting to see, if this method works in the presence of non-commutative variations and if not, if it is possible to modify it to be able to use Hamilton-Jacobi method for K-twisted Euler-Lagrange or/and equivalent Hamiltonian systems (see previous Section).

Recall that the interest in the development of Hamilton-Jacobi Theory in a non-classical settings led to a series of works extending, in different forms, the Hamilton-Jacobi Theory to the Classical Field Theory (classical works of Caratheodory, Weyl, Rund, etc., see [117]), to the singular Lagrangian systems and to the non-holonimic systems (see, for example, [85, 84]).

Refering the reader to the sources cited here and below, we restrict our presentation to two simple aspects. First we show that the classical Hamilton-Jacobi Theory is not applicable to a Lagrangian mechanical system with NC-variations or, more generally, to the Euler-Lagrange equations with the generic sources. More specifically, while the first Jacobi Theorem holds with some modifications, the proof of the second Jacobi Theorem (see [44], Sec.23, Theorem 2 fails in the case of Hamiltonian system with a non-conservative forces.

We suggest to our reader an Exercise to construct an example of such Hamiltonian system with failing Jacoby Theorem 2.

At the next step we show, that, following the idea of "embedding surfaces of solutions" similar to the use of "geodesic fields" in H.Weyl's Theory of multiple integrals ([117, 47]) some classes of solutions of Hamiltonian systems with sources can be constructed starting with the general solutions of appropriately constructed partial differential equations. We follow the method suggested by B.Vujanovich([133], Chapter 6) for dynamical systems (with one independent variable), a method that, in a variety of cases, replaces the Hamilton-Jacobi method in a case of non-conservative, dissipative, dynamical systems. We present his method below, refering reader to the monograph ([133]) and the cited papers for more details.

14.1. **Case $n=1$. Formula for variations.** Consider an action functional for the curves $[t_0,t_1]\rightarrow Q$ in the space Q of variables $y^i, i=1,\dots,m$

$$A(s=\{y^i(t)\})=\int_{t_0}^{t_1} L(t,y^i(t),\dot{y}^i(t))dt. \tag{14.1}$$

We start with the case $n=1$ using notation t for an independent variable and consider the interval $[t_0,t_1]$ as the domain D of integration. We introduce the Hamilton-Jacobi equation using the variational formula for action. Here we follow the [44], Sec.13, 23.

We begin with the formula for variation of action including the variations of endpoints ([44], Sec.13, (7))

$$\delta\mathcal{A} = \int_{t_0}^{t_1} \sum_{i=1}^{m} \left(L_{,y^i} - \frac{d}{dt} L_{,\dot{y}^i} + K_i^k(j^1 s(t)) L_{,\dot{y}^k} \right) \xi^i dt + \sum_{i=1}^{m} F_{,\dot{y}^i} \delta y^i |_{t=t_0}^{t=t_1}$$
$$+ \left(L - \sum_{i=1}^{m} \dot{y}^i L_{,\dot{y}^i} \right) \delta t|_{t=t_0}^{t=t_1}, \qquad (14.2)$$

The term with the tensor K_i^k appeared when we calculate the slope of variations on the ends of the interval - domain of the variated curve, see [44],Sec.13, p.55.

The notation $\delta t|_{t=t^j} = \delta t^j;\ \delta y^i|_{t=t_j} = \delta y_i^j, j = 0, 1$ are used here.

Introduce the dual (momentum) variables $p_i = L_{,y^i}$ and assume that the Jacobian $Det(\frac{\partial p_i}{\partial \dot{y}^j}) \neq 0$.

Then we can solve for $\dot{y}^i$ and get

$$\dot{y}^i = \psi^i(y^k, p_l)$$

.

Introduce the Hamiltonian function

$$H = \sum_{i=1}^{i=m} \dot{y}^i p_i = L. \qquad (14.3)$$

Here $\dot{y}^i$ are considered as the functions of y^k, p_l.

Performing the Legendre transformation (i.e. using the canonical variables $(t, y^i, p_j))$ in the expression for Hamiltonian variations (Sec.13) we get

$$\delta\mathcal{A} = \int_{t_0}^{t_1} \sum_{i=1}^{m} \left(L_{,y^i} - \frac{d}{dt} L_{,\dot{y}^i} + K_i^k L_{,\dot{y}^k} \right)_{(t,y^s(y),p_k(t))} \xi^i(t, y^s(t))dt + \left(\sum_{i=1}^{m} p_i \delta y^i - H\delta t \right) |_{t=t_0}^{t=t_1}. \qquad (14.4)$$

Now let the curve $s(t) = \{y^i(t)\}$ be a solution of Euler-Lagrange Equations with the NC-variations defined by the tensor K and connecting points $A = (t_0, y_0^i)$ and $B = (t_1, y_1^i)$. Then the first term in (14.3) vanishes and the variation $\delta\mathcal{A}$ takes (in canonical variables) the standard form (see [44], Sec.13)

$$\delta\mathcal{A} = \left(\sum_{i=1}^{m} p_i \delta y^i - H\delta t \right) |_{t=t_0}^{t=t_1} \qquad (14.5)$$

14.2. **Case n=1, Hamilton-Jacoby equation and failure of Second Jacoby Theorem.** Here we follow (up to a point) [44], Sec. 23.

Consider a domain W in the space Q of variables y^i such that for any two points of W there exists a unique extremal in the domain W connecting these two points. Consider the functional $\mathcal{A}(s(t) = \{y^i(t)\}$ and fix the initial point $A = s(t_0)$. As a result, we get the function of the endpoint B

$$S = \int_{t_0}^{t_1} L(t, y^i, \dot{y}^j)dt \qquad (14.6)$$

evaluated along the extremal γ joining the points $A = (t_0, y_i^0)$ and $B = (t_1, y_i^1)$ - a *geodetic distance* ([44],Sec.23) between A and B. Fix the initial point and consider S as the function of final point B.

Calculating partial derivatives $\frac{\partial S}{\partial t}, \frac{\partial S}{\partial y^\alpha}$ of function $S(t, y^k(t))$ we get the relation $dS = \delta\mathcal{A}$ where variation of action is calculated at the fixed extremal γ. Using formula (14.4) for the general variation (including variations at endpoints) of an extremal we get

$$\frac{\partial S}{\partial t} = H,\ \frac{\partial S}{\partial y^i} = p_i = L_{,\dot{y}^i}, \tag{14.7}$$

where $H = H[t, y^i; p_k(t, y^k)]$ is the Hamilton function introduced in Sec.13 (see (13.2)). From these two equations it follows that the function $S(t, y^i)$ as the function of endpoint B satisfies the (conventional) Hamilton-Jacoby Equation (HJ-equation)

$$\frac{\partial S}{\partial t} + H(t, y^1, \dots, y^m, \frac{\partial S}{\partial y^1}, \dots, \frac{\partial S}{\partial y^m}) = 0. \tag{14.8}$$

Basic relations between solutions of the Hamilton-Jacoby equation and the integrals of the Euler-Lagrange system to be modified in the case of non-commuting variations. Namely,

Theorem 7. *(Jacoby) Let $S = S(t, y^i, \alpha_1, \dots, \alpha_k)$ be a solution of HJ-equation depending on $k \leqq m$ parameters α_j. Then, each derivative $\frac{\partial S}{\partial \alpha_s}$ is the first integral for the solutions $(t, y^i(t), p_l(t))$ of the (K-modified) Hamiltonian system:*

$$\frac{d}{dt}\frac{\partial S}{\partial \alpha_s} = \frac{\partial^2 S}{\partial y^j \partial \alpha_s} f_j = 0. \tag{14.9}$$

Proof. Proof of this theorem repeats literally the proof in the non-twisted case ([44], Sec.23, Thm.1) due to the fact that the first part of Hamilton equations in K-twisted case coincide with the classical one. □

Consider now the second (basic) Theorem of Jacoby.

Theorem 8. *(Jacoby) Let $S = S(x, y^\mu, \alpha_i,\ i = 1, \dots, m)$ be a* ***complete integral of the Hamilton-Jacoby Equation*** *depending on m parameters α_i.Let the determinant of $m \times m$ matrix*

$$\left(\frac{\partial^2 S}{\partial \alpha_i \partial y^\mu}\right)$$

be nonzero and let $\beta_j,\ j = 1, \dots, m$ be arbitrary constants.

Then the functions $y^\mu = y^\mu(x, \alpha_i, \beta_j)$ defined from the relations

$$\frac{\partial}{\partial \alpha_i} S(x, y, \alpha) = \beta_i, \imath = 1, \dots, m, \tag{14.10}$$

together with the functions

$$p_k = \frac{\partial}{\partial y^k} S(x, y, \alpha) \tag{14.11}$$

where y are defined at (), constitute the general solution of Hamitonian Equations

$$y^\mu_{,x} = \frac{\partial H}{\partial p_\mu},\ p_{k,x} = -\frac{\partial H}{\partial y^k}.$$

It is easy to see that the conventional proof ([44]) does not work in the present situation where $K \neq 0$. More specifically, starting with a general solution of the Hamilton-Jacoby Equation (14.8) and following the proof we get the conventional Hamiltonian system rather then the K-modified one.

As a result, in its classical form, Hamiton-Jacoby Theory can not be applied to the K-twisted Euler-Lagrange systems.

Remark 18. It is interesting to see if some modification of the HJ-equation and/or procedure of construction of functions y^μ and p_ν could lead to the K-modified hamiltonian system.

14.3. **Basic Field Equation for non-conservative dynamical systems.** In the works (), B. Vujanovich suggested a method that in a variety of cases, replaces for the dynamical systems (one independent variable) the Hamilton-Jacoby method. He called the PDE replacing the Hamitlon-Jacoby Equation the “Basic Field Equation” and applied it to a variety of dynamical systems: modified Hamilton equations with non-conservative forces, equations of vibrations, non-conservative coupled oscillators, diffusion in a Tubular reactor and others, see [133].

We will present here a short description of this method and two examples of its application.

We start with the general idea of choosing an "embedding" -surface in the configurational space where solutions of a dynamical system take values.

Let

$$\dot{x}^i = X_i(t, x^k),\ i = 1, \dots, m \tag{14.12}$$

be a dynamical system for a vector function $x(t) = (x^1(t), \dots x^m(t)$.

Look for solutions of this system satisfying the condition

$$\Phi(t, x^1, \dots, x^m) = 0, \tag{14.13}$$

i.e. belonging to the hypersurface $\Phi \subset Y$.

Assume now that $\frac{\partial \Phi}{\partial x^1} \neq 0$. Then we can (at least locally) rewrite condition (14.12) in the form

$$x^1 = \Psi(t, x^2, \dots, x^m). \tag{14.14}$$

Taking the total derivative by time and using the equations (14.12) for $i > 1$ we get the first order PDE for solutions of system (14.12) "embedded" into the surface Φ

$$\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x^i} X_i(t, \Psi, x^k, k > 1) = X_1(t, \Psi, x^k, k > 1) = 0. \tag{14.15}$$

We will call this equation the **basic field equation** (BFE) defined by the surface Φ.

This equation is a quasi-linear PDE and as such, is simpler then the Hamiton-Jacoby Equation.

14.4. **Basic Field Equation for a Hamiltonian system with nonconservative forces.** Let

$$\begin{cases} \dot{x}^i = \frac{\partial H}{\partial p_l},\ i = 1, \dots, m \\ \dot{p}_j = -\frac{\partial H}{\partial x^j} + f_j(t, x^i . p_l), j = 1, \dots, m. \end{cases} \tag{14.16}$$

be a Hamiltonian system with arbitrary forces f_j.

This system can be obtained from (and is actually equivalent to) the modification of D'Alambert variational principle

$$\frac{d}{dt}(p_l \delta x^i) - \delta(p_i \dot{x}^i - H) - f_i \delta x^i = 0,\ i = 1, \dots, m. \tag{14.17}$$

Let $U(t, x^i, p_l,\ l = 2, \dots, m)$ be a function of the listed variables. Adding and subtracting the variation $U\delta x^1$ in this equation, we write it in the form

$$\frac{d}{dt}(p_l \delta x^i - U\delta x^1) - \delta(p_i \dot{x}^i - H) - f_i \delta x^i + \frac{d}{dt}(U\delta x^1) = 0,\ i = 1, \dots, m. \tag{14.18}$$

Now we **define** the first component of momentum

$$p_1 = U(t, x^i, p_l,\ l = 2, \ldots, m). \tag{14.19}$$

This is the "embedding condition" mentioned in the beginning to this Section. From now on it is convenient to use the notation z_a for p_a with $a = 2, \ldots, m$.

Using this in the previous variational equation we write it in the form

$$\frac{d}{dt}(z_a \delta x^a) - \delta(U\dot{x}^1 - z_a \dot{x}^a - H) - f_i \delta x^1 - f_a \delta x^a + \frac{d}{dt}(U \delta x^1) = 0,\ i = 1, \ldots, m. \tag{14.20}$$

Separating the coefficients of independent variations $\delta x^1, \delta x^a, \delta z_a$ we get the system of equations

$$\begin{cases} \delta x^1 : \ \frac{dU}{dt} + \frac{\partial H}{\partial x^1} + \frac{\partial H}{\partial U}\frac{\partial U}{\partial x^a} + \frac{\partial U}{\partial x^a}\dot{x}^a + \frac{\partial U}{\partial z_a}\dot{z}_a - f_i = 0. \\ \delta x^a : \ \dot{z}_a - \dot{x}^1 \frac{\partial U}{\partial x^a} + \frac{\partial H}{\partial x^a} + \frac{\partial H}{\partial U}\frac{\partial U}{\partial x^a} - f_a = 0, \\ \delta z_a : \ \dot{x}^a + \dot{x}^1 \frac{\partial U}{\partial z_a} - \frac{\partial H}{\partial U}\frac{\partial U}{\partial z_a} - \frac{\partial H}{\partial z_a} = 0. \end{cases} \tag{14.21}$$

Calculating derivatives $\dot{x}^a$ and $\dot{z}_a$ from the last two equations and substituting them into the first equation we get the field equation (BFE) for the scalar function U:

$$\frac{dU}{dt} + \frac{\partial H}{\partial x^1} + \frac{\partial H}{\partial U}\frac{\partial U}{\partial x^1} + \frac{\partial H}{\partial z_a}\frac{\partial U}{\partial x^a} + \left(f_a - \frac{\partial H}{\partial x^a}\right) - f_1 = 0. \tag{14.22}$$

Remark 19. This way to work out "basic field equation", being based on the D"Alambert equation, proves the invariant character of obtained equation.

Remark 20. Notice that any solution of a Hamiltonian dynamical system with non-conservative sources (14.16) satisfies to the condition (14.19) with some function U. Therefore, if starting with a general solution of (14.22) we construct corresponding solutions of (14.16). As a result, we find ALL solutions of system (14.16).

14.5. **Complete solution of BFE and related conservation laws.**

Definition 7. *A general solution of the BFE is the relation*

$$p_1 = U(t, x^1, \ldots, x^n; z_2, \ldots, z_n; C_1, \ldots, C_{2m}), \tag{14.23}$$

where C_i are 2m arbitrary constants such that if we substitute it to the BFE-equation 14.22 it is satisfied identically.

Every complete solution satisfies the property: if we eliminate all 2mn constants C_i from 2m+1 relations: 14.23, $\frac{\partial U}{\partial t}, \frac{\partial U}{\partial x^i}, \frac{\partial U}{\partial z_a}, i = 1, \ldots, m; a = 2, \ldots, m$, we obtain the basic field equation 14.22.

Next we notice that the general solution 14.23 contains in itself 2m conservation laws for the dynamical system (14.16). We can recover these equations by fixing values of 2m-1 constants C_i and leaving values of the left constant arbitrary. For instance if we fix $C_i = C_2 = \ldots = C_{k-1} = C_{k+1} = \ldots, C_{2m} = 0$ for $k = 1, \ldots, 2m$ and obtain $2n$ relations

$$p_1 = U_k(t, x^1, \ldots, x^m; z_2, \ldots, z_m; C_k),$$

and solve these relations for C_k then, assuming that these relations are mutually independent (i.e. under the condition that $\frac{\partial U_k}{\partial C_k} \neq 0$), $2m$ conservation laws

$$\Phi_k(t, x^1, \ldots, x^m; z_1, \ldots, z_m) = C_k \tag{14.24}$$

represent a complete set of conservation laws for the dynamical system (14.16).

14.6. **General solution of modified Hamiltonian system (14.16) from the general solution of BFE..** Let (14.23) be a complete solution of the BFE equation. Choose now one of the constants in this solution, say, consider C_1 as an arbitrary function of other constants $C_2, \ldots, C_{2m}$: $C_1 = C_1(C_2, \ldots, C_{2m})$. Taking the derivative of the BFE by C_A where $A = 2, \ldots, 2m$, we get $2m - 1$ functional equations

$$\frac{\partial U}{\partial C_A} + \frac{\partial U}{\partial C_1} D_A = 0, \tag{14.25}$$

where

$$D_A = \frac{\partial C_1}{\partial C_A},\ A = 2, \ldots, 2n \tag{14.26}$$

are new constants.

Now, having a general solution of BFE in the form 14.23, solve it for C_1 to get

$$C_1 = \Psi_1(t, x^1, \ldots, x^m, U, z_2, \ldots, z_m; C_2, \ldots C_{2m}). \tag{14.27}$$

Find constants D_A from () and then substitute the function C_1 just obtained in these expressions. We get the resulting equations

$$\begin{cases} \Psi_2(t, x^1, \ldots, x^m, U, z_2, \ldots, z_m; C_2, \ldots C_{2m}) = D_2, \\ \ldots \\ \Psi_{2m}(t, x^1, \ldots, x^m, U, z_2, \ldots, z_m; C_2, \ldots C_{2m}) = D_{2m}. \end{cases} \tag{14.28}$$

Relations (14.27) and(14.28) together **form the (implicitly written) general solutions of the dynamical system** (14.16)

Parameters $C_A, A = 2, \ldots, 2m$ can take arbitrary values in their relevant domain. by substituting some values and returning to the variables $U = p_1,\ z_2 = p_2, \ldots, z_m = p_m$ we write system of relations (14.27,14.28) delivering (although implicitly) the general solution of the system (14.16) in the final form

$$\begin{cases} \Psi_1(t, x^1, \ldots, x^m, p_1, p_2, \ldots, p_m; C_2, \ldots C_{2m}) = C_1, \\ \Psi_2(t, x^1, \ldots, x^m, U, p_2, \ldots, p_m; C_2, \ldots C_{2m}) = D_2, \\ \ldots \\ \Psi_{2m}(t, x^1, \ldots, x^m, p_2, \ldots, p_m; C_2, \ldots C_{2m}) = D_{2m}. \end{cases} \tag{14.29}$$

To illustrate described method we apply it to the following example

Example 8. Consider the 2-dim dynamical system on the time interval $[1, \infty)$

$$\begin{cases} \dot{x}^1 = x^2, \\ \dot{x}^2 = \frac{{x^2}^2}{x^1} - \frac{x^2}{t}, \end{cases} \tag{14.30}$$

with the initial data

$$\begin{cases} x^1(1) = a, \\ x^2(1) = b. \end{cases}$$

Take $x^2 = U(t, x^1)$ as the **basic field** we find the basic field equation (14.22) to be

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x^1} - \frac{U^2}{x^1} + \frac{U}{t} = 0. \tag{14.31}$$

Separate variables x^1 and t by the assumption $U(t,x^1)=\frac{F(x^1)}{f(t)}$ from which we find that

$$\frac{\dot{f}}{f}=(F'-\frac{F}{x^1})\frac{1}{f}-\frac{a}{t}. \tag{14.32}$$

The overdot here is the time derivative while prime is the derivative with respect to x^1. This relation can be written in the form

$$\dot{f}-\frac{f}{t}=(F'-\frac{F}{x^1}).$$

Equating both expressions to constant 1 and integrating both obtained expressions by corresponding variables we find the general solution of the BFE equation

$$x_2=\frac{x^1}{t}\frac{ln(x^1)+A}{ln(t)+B}, \tag{14.33}$$

where A,B are constants.

Substituting (14.33) to the dynamical system (14.30)with given initial conditions we get $A=B\frac{b}{a}-ln(a)$. Putting this back into (14.33) we obtain the conditional form solution of BFE

$$x^2=\bar{U}(t,x^1,a,b,B)=\frac{x^1}{t}\frac{ln(x^1/a)+B(b/a)}{ln(t)+B}, \tag{14.34}$$

where B is arbitrary.

Using the relation $\frac{\partial\bar{U}}{\partial B}=0$ we see that under the condition $(ln(t)+M)\neq 0$, the last equation gives

$$x^1=at^{b/a}.$$

Substituting this into the expression for x^2 we get the second component of the solution

$$x^2=bt^{(b/a)-1}.$$

To find the general solution of the starting dynamical system using the above described method, we choose the relation between arbitrary constants $A=A(B)$ to be true. Equation $\frac{\partial x^1}{\partial B}=0$ is now equivalent to the relation

$$ln(x^1)=\frac{dA}{dB}(ln(t)+B)-A. \tag{14.35}$$

Using this in () we find

$$x^2=\frac{x^1}{t}\frac{dA}{dB}=\frac{x^1}{t}D_1,\ D_1=\frac{dA}{dB} \tag{14.36}$$

and equation (0 now gives

$$\ln(x^1)=D_1ln(t),\ D_2=B\frac{dA}{dB}-A. \tag{14.37}$$

Last two equations represent the general solution of dynamical system (14.30).

Remark 21. (**Conjecture:Case of Field Theory.**) Now, let

$$\partial_t y^\mu+\sum_{i=1}^{i=3}\partial_{x^i}F^i_\mu=\Pi_\mu,\ \mu=1,\dots,m \tag{14.38}$$

be a system of balance equations, where flux components F^i_μ and sources Π_μ are functions of $t=x^0,x^i,i=1,2,3$ and of the fields y^μ.

Place the embedding condition

$$w_1 = \Phi(t, x^i, w_\nu \nu = 2, \dots, m) \tag{14.39}$$

Acting as in the case of dynamical systems we get for a function Φ the BFE

$$\frac{\partial \Phi}{\partial t} + \sum_{\mu>1} \frac{\partial \Phi}{\partial w^\mu}(\Pi_\mu - \partial_{x^i} F^i_\mu) - (\Pi_1 - \partial_{x^i} F^i_1) = 0. \tag{14.40}$$

This quasilinear BFE for Field Theory can probably be used in the same way as the equation 14.22 in the case of Mechanics.

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