1. Preface

The aim of this text is to present and study the method of so-called "**non-commuting variations (shortly, NC-variations)**" in Variational Calculus. To present this method we recall one of the basic rules of Variational Calculus - the rule defining the variations of derivatives $\frac{\partial y^\mu}{\partial x^i}$ of dynamical variables $y^\mu(x)$ (fields in the Field Theory) corresponding to a variation ξ of dynamical variables (fields): "variation of a derivative equals to the derivative of variation". In Mechanics, this rule takes the form $\delta \dot{y}^\mu = \frac{d}{dt}\delta y^\mu$. In Classical Field theory this rule takes the form

$$\delta \frac{\partial y^\mu}{\partial x^i} = \frac{\partial}{\partial x^i}\delta y^\mu.$$

This rule can be formulated as follows: "taking of variations of dynamical variables $y^i(x)$ commute with the taking of derivatives."

This rule was universally adopted in the XVIII and XIX centuries but, as early as in 1887, this rule was questioned by Vito Volterra, see [130, 131]. Studying non-holonomic mechanical systems, V.Volterra noticed that the use of the conventional rule of defining variations of derivatives does not allows us to obtain equations of motion for non-holonomic systems by variational methods. Further developments including works of L.Boltzman,[8] G.Hamel,[57], T.Levi Civita and U.Amaldi,[83] led to the conflicting points of view at the range of applicability of the conventional rule of defining variations (see a Historical Review between Chapters 1 and 2 below). Finally, the status of this, conventional, rule and its relation to the alternative rules - the use of "non-commuting variations" in Non-Holonomic Mechanics were clarified in works of J.Neimark and his coauthors in the 1950s of XX century ([104, 105]) and by A.Lurie in 1961, [88].

Later on, the non-commuting variations were used in the works of B.Vujanovich and T.Atanackovic on dynamical systems with non-conservative forces ([133, 134, 4, 5]), in Elasticity Theory, and in works of H.Kleinert, P.Fiziev and A.Pelster on the dynamics in Cartan-Riemann spaces ([35, 65]).

While studying the application of non-commuting variations in classical field theory we noticed that the usage of non-commuting rules to define variations of derivatives **is equivalent to the use of a non-trivial vertical connections** to modify the procedure of flow prolongation of variational vector fields in the space Y of the configurational bundle $\pi : Y \to X$ of a physical system to the 1-jet bundle $J^1(\pi) \to Y$ over π, [112]. This led us to the study of the geometrical structures underlying the method of non-commuting variations of derivatives in Lagrangian formalism. In particular,a natural variety of questions that arises here is: which of the basic methods of Variational Calculus - Theory of second variations, Hamiltonian systems and Legendre transformation, conservation laws (including Noether theorems), Hamilton-Jacoby Equations, etc. - are preserved in this modified scheme and which parts require modifications to stay true. These and some other related questions are studied in the present work.

We will also show that any system of PDE of the form: "Euler- Lagrange equations with sources"

$$E_j(L) = f_j \tag{1.1}$$

can be realized by the Lagrangian formalism with a conventional action functional $\mathcal{A}(L)$ and the non-commuting variations defined by an appropriately chosen (defined by the sources f_j) tensor K of NC-variations. We show that the basic methods of conventional Lagrangian formalism - Noether Theorem, second variation technique, Hamiltonian equations, Weyl fields preserve their form in Lagrangian formalism with NC-variations. We study the relations between the properties of sources f_j and the curvature $\tilde{R}$ of the vertical connection tensor K.

We demonstrate that a variety of geometrical structures that appeared in the study of dynamics in some physical systems - dissipative potentials, non-holonomic transformations, torsion of zero curvature connections (absolute parallelism), material time and thermasy (= heat displacement), introduced by H.Helmholtz and studied by D.van Dantzig ([135, 136, 110]) are special cases or are closely related to the use of non-commutative variations defined by a vertical connection in the conventional Lagrangian formalism.

Our perspective in this work, supported by the results of the geometrical (bundle) form of Variational Calculus, is that the conventional rules of taking variations of the derivatives of dynamical variables (fields) (underlying the flow prolongation method) have important mathematical advantages (preservation of Cartan distribution, preservation of Lie bracket, etc.) making them more fundamental. Yet, a more general approach allows inclusion into the framework of Variational Calculus, the physical systems that can not be described by the conventional Lagrangian formalism.

In that we adopt the point of view of B. Vujanovich and that of A.Lurie's that in difference to the variations of fields y^i, variations of their derivatives $y^i_{,\mu}$ *are not only kinematical, but dynamical notions* and should be dealt with as such. In particular, this allows us to introduce geometrical factors that have dynamical meaning into the definition of variations of derivatives that have dynamical meaning. This allows us to describe dissipative processes in the system.

These notes are based on the Lectures delivered by author at the 15th Summer School in Global Analysis at Masaruk University,Brno, CZ on August 8-12, 2011. I am using this case to thank participants of this school and, especially, its organizer Professor Demeter Krupka and Dr.Marcella Palese for useful discussions during the school.

In particular, during this school Marcella Palese informed me about a non-conventional procedures of the non-commuting variations introduced by C.Murial and J.Romero in Spain and used by the group of specialists in Spain and Italy with the goal to extend the range of symmetry groups of Lagrangian systems. Their goals were different from ours but their constructions (of λ and μ-prolongations) are similar, but not identical, to our approach of using vertical connections. We have included a condensed exposition of their work and the relations with our scheme in the present text (see Chapter 4).

Preliminary results on the NC-variations Lagrangian formalism where published in the Proceedings of the GCM2008, [112].

2. Introduction.

In Chapter I we give a short sketch of classical Lagrangian formalism. Here we tried to make a presentation as simple as possible. Yet, we introduce in the beginning of Chapter 1, some basic invariant notions, whose more detailed description reader can find in the Appendix (and, in more details, in the literature refered

http://www.springer.com/978-3-319-28321-0

Non-commuting Variations in Mathematics and Physics
A Survey
Preston, S.
2016, XIV, 235 p. 11 illus., Softcover
ISBN: 978-3-319-28321-0