Preface

For undergraduate students, the transition from calculus to analysis is often disorienting and mysterious. What happened to the beautiful calculus formulas? Where did ϵ-δ and open sets come from? It is not until later that one integrates these seemingly distinct points of view. When teaching "advanced calculus," I always had a difficult time answering these questions.

Now, every mathematician knows that analysis arose naturally in the nineteenth century out of the calculus of the previous two centuries. Believing that it was possible to write a book reflecting, explicitly, this organic growth, I set out to do so.

I chose several of the jewels of classical eighteenth- and nineteenth-century analysis and inserted them near the end of the book, inserted the axioms for reals at the beginning, and filled in the middle with (and only with) the material necessary for clarity and logical completeness. In the process, every little piece of one-variable calculus assumed its proper place, and theory and application were interwoven throughout.

Let me describe some of the unusual features in this text, as there are other books that adopt the above point of view. First is the systematic avoidance of ϵ-δ arguments. Continuous limits are defined in terms of limits of sequences, limits of sequences are defined in terms of upper and lower limits, and upper and lower limits are defined in terms of sup and inf. Everybody thinks in terms of sequences, so why do we teach our undergraduates ϵ-δ's? (In calculus texts, especially, doing this is unconscionable.)

The second feature is the treatment of integration. We follow the standard treatment motivated by geometric measure theory, with a few twists thrown in: The area is two-dimensional Lebesgue measure, defined on all subsets of $\mathbf{R}^2$, the integral of an arbitrary[1] nonnegative function is the area under its graph, and the integral of an arbitrary integrable function is the difference of the integrals of its positive and negative parts.

[1] Not necessarily measurable.

In dealing with arbitrary subsets of $\mathbf{R}^2$ and arbitrary functions, only a few basic properties can be derived; nevertheless, surprising results are available; for example, the integral of an arbitrary integrable function over an interval is a continuous function of the endpoints.

Arbitrary functions are considered to the extent possible not because of generality for generality's sake, but because they fit naturally within the context laid out here. For example, the density theorem and maximal inequality are valid for arbitrary sets and functions, and the first fundamental theorem is valid for an integrable f if and only if f is measurable.

In Chapter 4 we restrict attention to the class of continuous functions, which is broad enough to handle the applications in Chapter 5, and derive the second fundamental theorem in the form

$$\int_a^b f(x)\,dx = F(b-) - F(a+).$$

Here a, b, $F(a+)$ or $F(b-)$ may be infinite, extending the immediate applicability, and the continuous function f need only be nonnegative or integrable.

The third feature is the treatment of the theorems involving interchange of limits and integrals. Ultimately, all these theorems depend on the monotone convergence theorem which, from our point of view, follows from the Greek mathematicians' Method of Exhaustion. Moreover, these limit theorems are stated only after a clear and nontrivial need has been elaborated. For example, differentiation under the integral sign is used to compute the Gaussian integral.

The treatment of integration presented here emphasizes geometric aspects rather than technicality. The most technical aspects, the derivation of the Method of Exhaustion in §4.5, may be skipped upon first reading, or skipped altogether, without affecting the flow.

The fourth feature is the use of real-variable techniques in Chapter 5. We do this to bring out the elementary nature of that material, which is usually presented in a complex setting using transcendental techniques. For example, included is:

- A real-variable derivation of Gauss' AGM formula motivated by the unit circle map
$$x' + iy' = \frac{\sqrt{a}x + i\sqrt{b}y}{\sqrt{a}x - i\sqrt{b}y}.$$
- A real-variable computation of the radius of convergence of the Bernoulli series, derived via the infinite product expansion of $\sinh x/x$, which is in turn derived by combinatorial real-variable methods.

- The zeta functional equation is derived via the theta functional equation, which is in turn derived via the connection to the parametrization of the AGM curve.

The fifth feature is our emphasis on computational problems. Computation, here, is often at a deeper level than expected in calculus courses and varies from the high school quadratic formula in §1.4 to $\exp(-\zeta'(0)) = \sqrt{2\pi}$ in §5.8.

Because we take the real numbers as our starting point, basic facts about the natural numbers, trigonometry, or integration are rederived in this context, either in the body of the text or as exercises. For example, while the trigonometric functions are initially defined via their Taylor series, later it is shown how they may be defined via the unit circle.

Although it is helpful for the reader to have seen calculus prior to reading this text, the development does not presume this. We feel it is important for undergraduates to see, at least once in their four years, a nonpedantic, purely logical development that really does start from scratch (rather than pretends to), is self-contained, and leads to nontrivial and striking results.

Applications include a specific transcendental number; convexity, elementary symmetric polynomial inequalities, subdifferentials, and the Legendre transform; Machin's formula; the Cantor set; the Bailey–Borwein–Plouffe series

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n}\left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right);$$

continued fractions; Laplace and Fourier transforms; Bessel functions; Euler's constant; the AGM iteration; the gamma and beta functions; Stirling identity

$$\exp\left(\int_s^{s+1} \log \Gamma(x)\,dx\right) = s^s e^{-s}\sqrt{2\pi}, \qquad s > 0;$$

the entropy of the binomial coefficients; infinite products and Bernoulli numbers; theta functions and the AGM curve; the zeta function; the zeta series

$$\log(x!) = -\gamma x + \zeta(2)\frac{x^2}{2} - \zeta(3)\frac{x^3}{3} + \zeta(4)\frac{x^4}{4} - \dots, \qquad 1 \geq x > -1;$$

primes in arithmetic progressions; the Euler–Maclaurin formula; and the Stirling series.

After the applications, Chapter 6 develops the fundamental theorems in their general setting, based on the sunrise lemma. This material is at a more advanced level, but is included to point the reader toward twentieth-century developments.

As an aid to self-study and assimilation, there are 450 problems with all solutions at the back of the book. Every exercise can be solved using only previous material from this book. Chapters 1–4 provide the basis for a calculus or beginner undergraduate analysis course. Chapters 4 and 5 provide the basis for an undergraduate computational analysis course, while Chapters 4 and 6 provide the basis for an undergraduate real analysis course.

Philadelphia, PA, USA Omar Hijab
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Hijab, O.
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