Chapter 2
Bivariate Right Fractional Polynomial Monotone Approximation

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Abstract Let $f \in C^{r,p}\left([0,1]^2\right)$, $r,p \in \mathbb{N}$, and let $\overline{L}$ be a linear right fractional mixed partial differential operator such that $\overline{L}(f) \geq 0$, for all (x,y) in a critical region of $[0,1]^2$ that depends on $\overline{L}$. Then there exists a sequence of two-dimensional polynomials $Q_{\overline{m_1},\overline{m_2}}(x,y)$ with $\overline{L}\left(Q_{\overline{m_1},\overline{m_2}}(x,y)\right) \geq 0$ there, where $\overline{m_1}, \overline{m_2} \in \mathbb{N}$ such that $\overline{m_1} > r$, $\overline{m_2} > p$, so that f is approximated right fractionally simultaneously and uniformly by $Q_{\overline{m_1},\overline{m_2}}$ on $[0,1]^2$. This restricted right fractional approximation is accomplished quantitatively by the use of a suitable integer partial derivatives two-dimensional first modulus of continuity.

2.1 Introduction

The topic of monotone approximation started in [5] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [2] the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 2.1. *Let h, k, p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1,1]$ with modulus of continuity $\omega\left(f^{(p)}, x\right)$ there. Let $a_j(x)$, $j = h, h+1, \ldots, k$ be real functions, defined and bounded on $[-1,1]$ and assume $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1,1]$. Consider the operator*

$$L = \sum_{j=h}^{k} a_j(x)\left[\frac{d^j}{dx^j}\right]$$

and suppose, throughout $[-1,1]$,

$$L(f) \geq 0. \tag{2.1}$$

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Then, for every integer $n \geq 1$, there is a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1, 1]$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq Cn^{k-p} \omega \left(f^{(p)}, \frac{1}{n} \right),$$

where C is independent of n or f.

We need

Definition 2.2 (D.D. Stancu [6]). Let $f \in C\left([0,1]^2\right)$, $[0,1]^2 = [0,1] \times [0,1]$, where (x_1, y_1), $(x_2, y_2) \in [0,1]^2$ and $\delta_1, \delta_2 \geq 0$. The first modulus of continuity of f is defined as follows:

$$\omega_1(f, \delta_1, \delta_2) = \sup_{\substack{|x_1 - x_2| \leq \delta_1 \\ |y_1 - y_2| \leq \delta_2}} |f(x_1, y_1) - f(x_2, y_2)|.$$

Definition 2.3. Let f be a real-valued function defined on $[0,1]^2$ and let m, n be two positive integers. Let $B_{m,n}$ be the Bernstein (polynomial) operator of order (m, n) given by

$$B_{m,n}(f; x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} f\left(\frac{i}{m}, \frac{j}{n}\right) \cdot \binom{m}{i} \cdot \binom{n}{j} \cdot x^i \cdot (1-x)^{m-i} \cdot y^j \cdot (1-y)^{n-j}. \tag{2.2}$$

For integers $r, s \geq 0$, we denote by $f^{(r,s)}$ the differential operator of order (r, s), given by

$$f^{(r,s)}(x, y) = \frac{\partial^{r+s} f(x, y)}{\partial x^r \partial y^s}.$$

We use

Theorem 2.4 (Badea and Badea [3]). *It holds that*

$$\begin{aligned} &\left\| f^{(k,l)} - (B_{m,n} f)^{(k,l)} \right\|_\infty \\ &\quad \leq t(k, l) \cdot \omega_1\left(f^{(k,l)}; \frac{1}{\sqrt{m-k}}, \frac{1}{\sqrt{n-l}} \right) \\ &\qquad + \max\left\{ \frac{k(k-1)}{m}, \frac{l(l-1)}{n} \right\} \cdot \left\| f^{(k,l)} \right\|_\infty, \end{aligned} \tag{2.3}$$

where $m > k \geq 0$, $n > l \geq 0$ *are integers,* f *is a real-valued function on* $[0,1]^2$ *such that* $f^{(k,l)}$ *is continuous, and* t *is a positive real-valued function on* $\mathbb{Z}_+ = \{0,1,2,\ldots\}$. *Here* $\|\cdot\|_\infty$ *is the supremum norm on* $[0,1]^2$.

Denote $C^{r,p}\left([0,1]^2\right) := \{f : [0,1]^2 \to \mathbb{R}; f^{(k,l)}$ is continuous for $0 \leq k \leq r$, $0 \leq l \leq p\}$.

In [1] the author proved the following main motivational result.

Theorem 2.5. *Let* h_1, h_2, v_1, v_2, r, p *be integers,* $0 \leq h_1 \leq v_1 \leq r$, $0 \leq h_2 \leq v_2 \leq p$ *and let* $f \in C^{r,p}\left([0,1]^2\right)$. *Let* $\alpha_{i,j}(x,y)$, $i = h_1, h_1+1, \ldots, v_1$; $j = h_2, h_2+1, \ldots, v_2$ *be real-valued functions, defined and bounded in* $[0,1]^2$ *and assume* $\alpha_{h_1 h_2}$ *is either* $\geq \alpha > 0$ *or* $\leq \beta < 0$ *throughout* $[0,1]^2$. *Consider the operator*

$$L = \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y) \frac{\partial^{i+j}}{\partial x^i \partial y^j} \tag{2.4}$$

and suppose that throughout $[0,1]^2$,

$$L(f) \geq 0.$$

Then for integers m, n *with* $m > r$, $n > p$, *there exists a polynomial* $Q_{m,n}(x,y)$ *of degree* (m,n) *such that* $L(Q_{m,n}(x,y)) \geq 0$ *throughout* $[0,1]^2$ *and*

$$\left\| f^{(k,l)} - Q_{m,n}^{(k,l)} \right\|_\infty \leq \frac{P_{m,n}(L,f)}{(h_1-k)!\,(h_2-l)!} + M_{m,n}^{k,l}(f), \tag{2.5}$$

all $(0,0) \leq (k,l) \leq (h_1,h_2)$. *Furthermore we get*

$$\left\| f^{(k,l)} - Q_{m,n}^{(k,l)} \right\|_\infty \leq M_{m,n}^{k,l}(f), \tag{2.6}$$

for all $(h_1+1, h_2+1) \leq (k,l) \leq (r,p)$. *Also (2.6) is true whenever* $0 \leq k \leq h_1$, $h_2+1 \leq l \leq p$ *or* $h_1+1 \leq k \leq r$, $0 \leq l \leq h_2$. *Here*

$$M_{m,n}^{k,l} \equiv M_{m,n}^{k,l}(f) \equiv t(k,l) \cdot \omega_1\left(f^{(k,l)}; \frac{1}{\sqrt{m-k}}, \frac{1}{\sqrt{n-l}}\right) + \max\left\{\frac{k(k-1)}{m}, \frac{l(l-1)}{n}\right\} \cdot \left\| f^{(k,l)} \right\|_\infty \tag{2.7}$$

and

$$P_{m,n} \equiv P_{m,n}(L,f) \equiv \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} \cdot M_{m,n}^{i,j}, \tag{2.8}$$

where t *is a positive real-valued function on* $\mathbb{Z}_+^2$ *and*

$$l_{ij} \equiv \sup_{(x,y)\in[0,1]^2} \left| \alpha_{h_1 h_2}^{-1}(x,y) \cdot \alpha_{ij}(x,y) \right| < \infty. \tag{2.9}$$

In this article we extend Theorem 2.5 to the right fractional level. Indeed here L is replaced by $\overline{L}$, a linear right Caputo fractional mixed partial differential operator. Now the monotonicity property is only true on a critical region of $[0,1]^2$ that depends on $\overline{L}$ parameters. Simultaneous right fractional convergence remains true on all of $[0,1]^2$.

We need

Definition 2.6 (See [4]). Let $\alpha_1, \alpha_2 > 0$; $\alpha = (\alpha_1, \alpha_2)$, $f \in C\left([0,1]^2\right)$ and let $x = (x_1, x_2)$, $t = (t_1, t_2) \in [0,1]^2$. We define the right mixed Riemann–Liouville fractional two-dimensional integral of order α:

$$\left(I_{1-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1}^{1} \int_{x_2}^{1} (t_1 - x_1)^{\alpha_1 - 1} (t_2 - x_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2,$$

with $x_1, x_2 < 1$. Here Γ stands for the Gamma function.

Notice here

$$I_{1-}^{\alpha}(|f|) < \infty. \tag{2.10}$$

Definition 2.7. Let $\alpha_1, \alpha_2 > 0$ with $\lceil \alpha_1 \rceil = m_1$, $\lceil \alpha_2 \rceil = m_2$, ($\lceil \cdot \rceil$ ceiling of the number). Let here $f \in C^{m_1, m_2}\left([0,1]^2\right)$. We consider the right (Caputo type) fractional partial derivative:

$$D_{1-}^{(\alpha_1, \alpha_2)} f(x) := \frac{(-1)^{m_1 + m_2}}{\Gamma(m_1 - \alpha_1)\Gamma(m_2 - \alpha_2)} \cdot \int_{x_1}^{1} \int_{x_2}^{1} (J_1 - x_1)^{m_1 - \alpha_1 - 1} (J_2 - x_2)^{m_2 - \alpha_2 - 1} \frac{\partial^{m_1 + m_2} f(J_1, J_2)}{\partial J_1^{m_1} \partial J_2^{m_2}} dJ_1 dJ_2, \tag{2.11}$$

$\forall\, x = (x_1, x_2) \in [0,1]^2$.

We set

$$D_{1-}^{(0,0)} f(x) := f(x),$$

$$D_{1-}^{(m_1, m_2)} f(x) := (-1)^{m_1 + m_2} \frac{\partial^{m_1 + m_2} f(x)}{\partial x_1^{m_1} \partial x_2^{m_2}}, \ \forall\, x \in [0,1]^2. \tag{2.12}$$

Definition 2.8. We also set

$$D_{1-}^{(0, \alpha_2)} f(x) := \frac{(-1)^{m_2}}{\Gamma(m_2 - \alpha_2)} \int_{x_2}^{1} (J_2 - x_2)^{m_2 - \alpha_2 - 1} \frac{\partial^{m_2} f(x_1, J_2)}{\partial J_2^{m_2}} dJ_2, \tag{2.13}$$

$$D_{1-}^{(\alpha_1, 0)} f(x) := \frac{(-1)^{m_1}}{\Gamma(m_1 - \alpha_1)} \int_{x_1}^{1} (J_1 - x_1)^{m_1 - \alpha_1 - 1} \frac{\partial^{m_1} f(J_1, x_2)}{\partial J_1^{m_1}} dJ_1, \tag{2.14}$$

and

$$D_{1-}^{(m_1,\alpha_2)} f(x) := \frac{(-1)^{m_2}}{\Gamma(m_2-\alpha_2)} \int_{x_2}^{1} (J_2 - x_2)^{m_2-\alpha_2-1} \frac{\partial^{m_1+m_2} f(x_1, J_2)}{\partial x_1^{m_1} \partial J_2^{m_2}} dJ_2, \tag{2.15}$$

$$D_{1-}^{(\alpha_1,m_2)} f(x) := \frac{(-1)^{m_1}}{\Gamma(m_1-\alpha_1)} \int_{x_1}^{1} (J_1 - x_1)^{m_1-\alpha_1-1} \frac{\partial^{m_1+m_2} f(J_1, x_2)}{\partial J_1^{m_1} \partial x_2^{m_2}} dJ_1. \tag{2.16}$$

2.2 Main Result

We present our main result

Theorem 2.9. *Let h_1, h_2, v_1, v_2, r, p be integers, $0 \le h_1 \le v_1 \le r$, $0 \le h_2 \le v_2 \le p$ and let $f \in C^{r,p}\left([0,1]^2\right)$. Let $\alpha_{ij}(x,y)$, $i = h_1, h_1+1, \dots, v_1$; $j = h_2, h_2+1, \dots, v_2$ be real-valued functions, defined and bounded in $[0,1]^2$ and assume $\alpha_{h_1 h_2}$ is either $\ge \alpha > 0$ or $\le \beta < 0$ throughout $[0,1]^2$.*

Let integers $\overline{m_1}, \overline{m_2}$ with $\overline{m_1} > r$, $\overline{m_2} > p$. Set

$$l_{ij} := \sup_{(x,y)\in[0,1]^2} \left|\alpha_{h_1 h_2}^{-1}(x,y) \cdot \alpha_{ij}(x,y)\right| < \infty. \tag{2.17}$$

Also set ($\lceil \alpha_{1i} \rceil = i$, $\lceil \alpha_{2j} \rceil = j$, $\lceil \cdot \rceil$ ceiling of number)

$$\begin{aligned} M_{\overline{m_1},\overline{m_2}}^{i,j} := M_{\overline{m_1},\overline{m_2}}^{i,j}(f) := & \frac{1}{\Gamma(i-\alpha_{1i}+1)\,\Gamma\left(j-\alpha_{2j}+1\right)} \\ & \times \left\{ t(i,j)\, \omega_1\left(f^{(i,j)}; \frac{1}{\sqrt{\overline{m_1}-i}}, \frac{1}{\sqrt{\overline{m_2}-j}}\right) \right. \\ & \left. + \max\left\{\frac{i(i-1)}{\overline{m_1}}, \frac{j(j-1)}{\overline{m_2}}\right\} \cdot \left\|f^{(i,j)}\right\|_\infty \right\}, \end{aligned} \tag{2.18}$$

$i = h_1, \dots, v_1$; $j = h_2, \dots, v_2$.

Here t is a positive real-valued function on $\mathbb{Z}_+^2$, $\|\cdot\|_\infty$ is the supremum norm on $[0,1]^2$. Call

$$P_{\overline{m_1},\overline{m_2}} := P_{\overline{m_1},\overline{m_2}}(f) = \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} \cdot M_{\overline{m_1},\overline{m_2}}^{i,j}. \tag{2.19}$$

Let

$$\begin{aligned} & 0 \le \alpha_{1h_1} \le h_1 < \alpha_{11} < h_1 + 1 < \alpha_{12} < h_1 + 2 < \alpha_{13} \\ & < h_1 + 3 < \cdots < \alpha_{1v_1} < v_1 < \cdots < \alpha_{1r} < r, \end{aligned}$$

with $\lceil \alpha_{1h_1} \rceil = h_1$*;*

$$0 \leq \alpha_{2h_2} \leq h_2 < \alpha_{21} < h_2 + 1 < \alpha_{22} < h_2 + 2 < \alpha_{23}$$
$$< h_2 + 3 < \cdots < \alpha_{2v_2} < v_2 < \cdots < \alpha_{2p} < p,$$

with $\lceil \alpha_{2h_2} \rceil = h_2$*. Here* $h_1 + h_2 = 2m$*,* $m = 0, 1, 2, \ldots$*.*

Consider the right fractional bivariate operator

$$\overline{L} := \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x, y) D_{1-}^{(\alpha_{1i}, \alpha_{2j})}. \tag{2.20}$$

Then there exists a polynomial $Q_{\overline{m_1}, \overline{m_2}}(x, y)$ *of degree* $(\overline{m_1}, \overline{m_2})$ *on* $[0, 1]^2$ *such that*

$$\left\| D_{1-}^{(\alpha_{1k}, \alpha_{2l})}(f) - D_{1-}^{(\alpha_{1k}, \alpha_{2l})}(Q_{\overline{m_1}, \overline{m_2}}) \right\|_\infty$$
$$\leq \frac{P_{\overline{m_1}, \overline{m_2}}}{(h_1 - k)!(h_2 - l)!} \left\{ \left[\sum_{\theta=0}^{h_1 - k} \binom{h_1 - k}{\theta} \frac{\Gamma(h_1 - k - \theta + 1)}{\Gamma(h_1 - \alpha_{1k} - \theta + 1)} \right] \right.$$
$$\left. \left[\sum_{\rho=0}^{h_2 - l} \binom{h_2 - l}{\rho} \frac{\Gamma(h_2 - l - \rho + 1)}{\Gamma(h_2 - \alpha_{2l} - \rho + 1)} \right] \right\} + M_{\overline{m_1}, \overline{m_2}}^{k,l}, \tag{2.21}$$

for $(0, 0) \leq (k, l) \leq (h_1, h_2)$*.*

If $(h_1 + 1, h_2 + 1) \leq (k, l) \leq (r, p)$*, or* $0 \leq k \leq h_1$*,* $h_2 + 1 \leq l \leq p$*, or* $h_1 + 1 \leq k \leq r$*,* $0 \leq l \leq h_2$*, then*

$$\left\| D_{1-}^{(\alpha_{1k}, \alpha_{2l})}(f) - D_{1-}^{(\alpha_{1k}, \alpha_{2l})}(Q_{\overline{m_1}, \overline{m_2}}) \right\|_\infty \leq M_{\overline{m_1}, \overline{m_2}}^{k,l}. \tag{2.22}$$

If $\overline{L}(f(0, 0)) \geq 0$*, then* $\overline{L}(Q_{\overline{m_1}, \overline{m_2}}(0, 0)) \geq 0$*.*

Let $0 < x < 1$*,* $0 < y < 1$*, with* $\alpha_{1h_1} \neq h_1$ *and* $\alpha_{2h_2} \neq h_2$*, such that*

$$\begin{aligned} 1 - x &\geq \Gamma(h_1 - \alpha_{1h_1} + 1)^{\frac{1}{(h_1 - \alpha_{1h_1})}}, \\ 1 - y &\geq \Gamma(h_2 - \alpha_{2h_2} + 1)^{\frac{1}{(h_2 - \alpha_{2h_2})}}, \end{aligned} \tag{2.23}$$

equivalently,

$$\begin{aligned} x &\leq 1 - \Gamma(h_1 - \alpha_{1h_1} + 1)^{\frac{1}{(h_1 - \alpha_{1h_1})}}, \\ y &\leq 1 - \Gamma(h_2 - \alpha_{2h_2} + 1)^{\frac{1}{(h_2 - \alpha_{2h_2})}}), \end{aligned} \tag{2.24}$$

and

$$\overline{L}(f(x, y)) \geq 0.$$

Then

$$\overline{L}\left(Q_{\overline{m_1},\overline{m_2}}(x,y)\right) \geq 0.$$

To prove Theorem 2.9 it takes some preparation. We need

Definition 2.10. Let f be a real-valued function defined on $[0,1]^2$ and let $\overline{m_1}, \overline{m_2} \in \mathbb{N}$. Let $B_{\overline{m_1},\overline{m_2}}$ be the Bernstein (polynomial) operator of order $(\overline{m_1}, \overline{m_2})$ given by

$$B_{\overline{m_1},\overline{m_2}}(f; x_1, x_2) := \sum_{i_1=0}^{\overline{m_1}} \sum_{i_2=0}^{\overline{m_2}} f\left(\frac{i_1}{\overline{m_1}}, \frac{i_2}{\overline{m_2}}\right) \binom{\overline{m_1}}{i_1} \binom{\overline{m_2}}{i_2} \\ x_1^{i_1} (1-x_1)^{\overline{m_1}-i_1} x_2^{i_2} (1-x_2)^{\overline{m_2}-i_2}. \tag{2.25}$$

We need the following simultaneous approximation result.

Theorem 2.11 (Badea and Badea [3]). *It holds that*

$$\left\| f^{(k_1,k_2)} - \left(B_{\overline{m_1},\overline{m_2}} f\right)^{(k_1,k_2)} \right\|_\infty \\ \leq t(k_1,k_2)\, \omega_1 \left(f^{(k_1,k_2)}; \frac{1}{\sqrt{\overline{m_1}-k_1}}, \frac{1}{\sqrt{\overline{m_2}-k_2}} \right) \\ + \max\left\{ \frac{k_1(k_1-1)}{\overline{m_1}}, \frac{k_2(k_2-1)}{\overline{m_2}} \right\} \cdot \left\| f^{(k_1,k_2)} \right\|_\infty, \tag{2.26}$$

where $\overline{m_1} > k_1 \geq 0$, $\overline{m_2} > k_2 \geq 0$ are integers, f is a real-valued function on $[0,1]^2$*, such that $f^{(k_1,k_2)}$ is continuous, and t is a positive real-valued function on $\mathbb{Z}_+^2$. Here $\|\cdot\|_\infty$ is the supremum norm on* $[0,1]^2$.

Remark 2.12. We assume that $\overline{m_1} > m_1 = \lceil \alpha_1 \rceil$, $\overline{m_2} > m_2 = \lceil \alpha_2 \rceil$, where $\alpha_1, \alpha_2 > 0$.

We consider also

$$D_{1-}^{(\alpha_1,\alpha_2)}\left(B_{\overline{m_1},\overline{m_2}} f\right)(x_1,x_2) = \frac{(-1)^{m_1+m_2}}{\Gamma(m_1-\alpha_1)\,\Gamma(m_2-\alpha_2)} \\ \cdot \int_{x_1}^{1} \int_{x_2}^{1} (t_1-x_1)^{m_1-\alpha_1-1} (t_2-x_2)^{m_2-\alpha_2-1} \\ \frac{\partial^{m_1+m_2}\left(B_{\overline{m_1},\overline{m_2}} f\right)(t_1,t_2)}{\partial t_1^{m_1} \partial t_2^{m_2}} dt_1 dt_2, \tag{2.27}$$

$\forall\, (x_1,x_2) \in [0,1]^2$.

Proposition 2.13. *Let* $\alpha_1, \alpha_2 > 0$ *with* $\lceil \alpha_1 \rceil = m_1$, $\lceil \alpha_2 \rceil = m_2$, $f \in C^{m_1, m_2}\left([0,1]^2 \right)$, *where* $\overline{m_1}, \overline{m_2} \in \mathbb{N} : \overline{m_1} > m_1$, $\overline{m_2} > m_2$. *Then*

$$\left\| D_{1-}^{(\alpha_1,\alpha_2)} f - D_{1-}^{(\alpha_1,\alpha_2)} \left(B_{\overline{m_1},\overline{m_2}} f \right) \right\|_\infty \leq \frac{1}{\Gamma(m_1-\alpha_1+1)\,\Gamma(m_2-\alpha_2+1)} \cdot \left\{ t(m_1,m_2)\,\omega_1\left(f^{(m_1,m_2)}; \frac{1}{\sqrt{\overline{m_1}-m_1}}, \frac{1}{\sqrt{\overline{m_2}-m_2}} \right) + \max\left\{ \frac{m_1(m_1-1)}{\overline{m_1}}, \frac{m_2(m_2-1)}{\overline{m_2}} \right\} \cdot \left\| f^{(m_1,m_2)} \right\|_\infty \right\}, \tag{2.28}$$

Proof. We observe the following:

$$\left| D_{1-}^{(\alpha_1,\alpha_2)} f(x_1,x_2) - D_{1-}^{(\alpha_1,\alpha_2)} \left(B_{\overline{m_1},\overline{m_2}} f \right)(x_1,x_2) \right|$$

$$= \frac{1}{\Gamma(m_1-\alpha_1)\,\Gamma(m_2-\alpha_2)} \left| \int_{x_1}^{1}\int_{x_2}^{1} (t_1-x_1)^{m_1-\alpha_1-1}(t_2-x_2)^{m_2-\alpha_2-1} \cdot \left(\frac{\partial^{m_1+m_2} f(t_1,t_2)}{\partial t_1^{m_1}\partial t_2^{m_2}} - \frac{\partial^{m_1+m_2}\left(B_{\overline{m_1},\overline{m_2}} f \right)(t_1,t_2)}{\partial t_1^{m_1}\partial t_2^{m_2}} \right) dt_1 dt_2 \right| \tag{2.29}$$

$$\leq \frac{1}{\Gamma(m_1-\alpha_1)\,\Gamma(m_2-\alpha_2)} \int_{x_1}^{1}\int_{x_2}^{1} (t_1-x_1)^{m_1-\alpha_1-1}(t_2-x_2)^{m_2-\alpha_2-1} \tag{2.30}$$

$$\cdot \left| \frac{\partial^{m_1+m_2} f(t_1,t_2)}{\partial t_1^{m_1}\partial t_2^{m_2}} - \frac{\partial^{m_1+m_2}\left(B_{\overline{m_1},\overline{m_2}} f \right)(t_1,t_2)}{\partial t_1^{m_1}\partial t_2^{m_2}} \right| dt_1 dt_2$$

$$\overset{(2.26)}{\leq} \left\{ t(m_1,m_2)\,\omega_1\left(f^{(m_1,m_2)}; \frac{1}{\sqrt{\overline{m_1}-m_1}}, \frac{1}{\sqrt{\overline{m_2}-m_2}} \right) + \max\left\{ \frac{m_1(m_1-1)}{\overline{m_1}}, \frac{m_2(m_2-1)}{\overline{m_2}} \right\} \cdot \left\| f^{(m_1,m_2)} \right\|_\infty \right\} \tag{2.31}$$

$$\cdot \frac{1}{\Gamma(m_1-\alpha_1)\,\Gamma(m_2-\alpha_2)} \int_{x_1}^{1}\int_{x_2}^{1} (t_1-x_1)^{m_1-\alpha_1-1}(t_2-x_2)^{m_2-\alpha_2-1} dt_1 dt_2$$

$$= \frac{(1-x_1)^{m_1-\alpha_1}(1-x_2)^{m_2-\alpha_2}}{\Gamma(m_1-\alpha_1+1)\,\Gamma(m_2-\alpha_2+1)} \cdot \left\{ t(m_1,m_2)\,\omega_1\left(f^{(m_1,m_2)}; \frac{1}{\sqrt{\overline{m_1}-m_1}}, \frac{1}{\sqrt{\overline{m_2}-m_2}} \right) + \max\left\{ \frac{m_1(m_1-1)}{\overline{m_1}}, \frac{m_2(m_2-1)}{\overline{m_2}} \right\} \left\| f^{(m_1,m_2)} \right\|_\infty \right\}, \tag{2.32}$$

$\forall\, (x_1,x_2) \in [0,1]^2$.

Proof (Proof of Theorem 2.9).

We need for $(0,0) \leq (k,l) \leq (h_1,h_2)$ to calculate

$$D_{1-}^{(\alpha_{1k},\alpha_{2l})}\left(\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right) = \frac{(-1)^{k+l}}{\Gamma(k-\alpha_{1k})\,\Gamma(l-\alpha_{2l})} \cdot \int_x^1\int_y^1 (J_1-x)^{k-\alpha_{1k}-1}(J_2-y)^{l-\alpha_{2l}-1}\frac{J_1^{h_1-k}}{(h_1-k)!}\frac{J_2^{h_2-l}}{(h_2-l)!}dJ_1dJ_2 \tag{2.33}$$

$$= \frac{(-1)^{k+l}}{(h_1-k)!\,(h_2-l)!}\left\{\left(\frac{1}{\Gamma(k-\alpha_{1k})}\int_x^1 (J_1-x)^{k-\alpha_{1k}-1}J_1^{h_1-k}dJ_1\right) \cdot \left(\frac{1}{\Gamma(l-\alpha_{2l})}\int_y^1 (J_2-y)^{l-\alpha_{2l}-1}J_2^{h_2-l}dJ_2\right)\right\}. \tag{2.34}$$

We find that

$$\int_x^1 J_1^{h_1-k}(J_1-x)^{k-\alpha_{1k}-1}dJ_1$$

$$= (-1)^{h_1-k}\int_x^1 (-J_1)^{h_1-k}(J_1-x)^{k-\alpha_{1k}-1}dJ_1$$

$$= (-1)^{h_1-k}\int_x^1 ((1-J_1)-1)^{h_1-k}(J_1-x)^{k-\alpha_{1k}-1}dJ_1$$

$$= (-1)^{h_1-k}\int_x^1\left(\sum_{\theta=0}^{h_1-k}\binom{h_1-k}{\theta}(1-J_1)^{h_1-k-\theta}(-1)^{\theta}\right) \times (J_1-x)^{k-\alpha_{1k}-1}dJ_1 \tag{2.35}$$

$$= \sum_{\theta=0}^{h_1-k}\binom{h_1-k}{\theta}(-1)^{h_1-k+\theta} \times \int_x^1 (1-J_1)^{(h_1-k-\theta+1)-1}(J_1-x)^{k-\alpha_{1k}-1}dJ_1$$

$$= \sum_{\theta=0}^{h_1-k}\binom{h_1-k}{\theta}(-1)^{h_1-k+\theta} \times\left\{\frac{\Gamma(h_1-k-\theta+1)\,\Gamma(k-\alpha_{1k})}{\Gamma(h_1-\alpha_{1k}-\theta+1)}(1-x)^{h_1-\alpha_{1k}-\theta}\right\}. \tag{2.36}$$

Similarly we find

$$\int_y^1 (J_2 - y)^{l-\alpha_{2l}-1} J_2^{h_2-l} dJ_2$$
$$= \sum_{\rho=0}^{h_2-l} \binom{h_2-l}{\rho} (-1)^{h_2-l+\rho}$$
$$\times \left\{ \frac{\Gamma(h_2 - l - \rho + 1)\,\Gamma(l - \alpha_{2l})}{\Gamma(h_2 - \alpha_{2l} - \rho + 1)} (1-y)^{h_2-\alpha_{2l}-\rho} \right\}. \tag{2.37}$$

Therefore

$$D_{1-}^{(\alpha_{1k},\alpha_{2l})}\left(\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right) = \frac{1}{(h_1-k)!\,(h_2-l)!}$$
$$\cdot \left\{ \left[\sum_{\theta=0}^{h_1-k} \binom{h_1-k}{\theta} (-1)^{h_1+\theta} \left\{ \frac{\Gamma(h_1-k-\theta+1)}{\Gamma(h_1-\alpha_{1k}-\theta+1)} (1-x)^{h_1-\alpha_{1k}-\theta} \right\} \right] \right.$$
$$\left. \cdot \left[\sum_{\rho=0}^{h_2-l} \binom{h_2-l}{\rho} (-1)^{h_2+\rho} \left\{ \frac{\Gamma(h_2-l-\rho+1)}{\Gamma(h_2-\alpha_{2l}-\rho+1)} (1-y)^{h_2-\alpha_{2l}-\rho} \right\} \right] \right\}. \tag{2.38}$$

Here we use a lot Proposition 2.13.

Case (i). Assume that on $[0,1]^2$, $\alpha_{h_1h_2} \geq \alpha > 0$.
Call

$$Q_{\overline{m_1},\overline{m_2}}(x,y) := B_{\overline{m_1},\overline{m_2}}(f;x,y) + P_{\overline{m_1},\overline{m_2}} \frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}. \tag{2.39}$$

Then by (2.28) we get

$$\left\| D_{1-}^{(\alpha_{1k},\alpha_{2l})}\left(f + P_{\overline{m_1},\overline{m_2}} \frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right) - D_{1-}^{(\alpha_{1k},\alpha_{2l})}\left(Q_{\overline{m_1},\overline{m_2}}(x,y)\right) \right\|_\infty \leq M^{k,l}_{\overline{m_1},\overline{m_2}}, \tag{2.40}$$

for all $0 \leq k \leq r$, $0 \leq l \leq p$.

When $(0,0) \leq (k,l) \leq (h_1,h_2)$, using (2.38) inequality (2.40) becomes

$$\left\| D_{1-}^{(\alpha_{1k},\alpha_{2l})}(f) + P_{\overline{m_1},\overline{m_2}} \frac{1}{(h_1-k)!\,(h_2-l)!} \right.$$
$$\cdot \left\{ \left[\sum_{\theta=0}^{h_1-k} \binom{h_1-k}{\theta} (-1)^{h_1+\theta} \left\{ \frac{\Gamma(h_1-k-\theta+1)}{\Gamma(h_1-\alpha_{1k}-\theta+1)} (1-x)^{h_1-\alpha_{1k}-\theta} \right\} \right] \right.$$

$$\cdot \left[\sum_{\rho=0}^{h_2-l} \binom{h_2-l}{\rho} (-1)^{h_2+\rho} \left\{ \frac{\Gamma(h_2-l-\rho+1)}{\Gamma(h_2-\alpha_{2l}-\rho+1)} (1-y)^{h_2-\alpha_{2l}-\rho} \right\} \right] \Bigg\}$$

$$-D_{1-}^{(\alpha_{1k},\alpha_{2l})} \left(Q_{\overline{m_1},\overline{m_2}}(x,y)\right) \Bigg\|_\infty \leq M_{\overline{m_1},\overline{m_2}}^{k,l}, \tag{2.41}$$

for all $(0,0) \leq (k,l) \leq (h_1,h_2)$, $x, y \in [0,1]$.

Using (2.41) and triangle inequality we obtain for $(0,0) \leq (k,l) \leq (h_1,h_2)$ that

$$\left\| D_{1-}^{(\alpha_{1k},\alpha_{2l})}(f) - D_{1-}^{(\alpha_{1k},\alpha_{2l})}\left(Q_{\overline{m_1},\overline{m_2}}\right) \right\|_\infty \leq \frac{P_{\overline{m_1},\overline{m_2}}}{(h_1-k)!\,(h_2-l)!}$$
$$\cdot \left\{ \left[\sum_{\theta=0}^{h_1-k} \binom{h_1-k}{\theta} \left\{ \frac{\Gamma(h_1-k-\theta+1)}{\Gamma(h_1-\alpha_{1k}-\theta+1)} \right\} \right] \right.$$
$$\left. \cdot \left[\sum_{\rho=0}^{h_2-l} \binom{h_2-l}{\rho} \left\{ \frac{\Gamma(h_2-l-\rho+1)}{\Gamma(h_2-\alpha_{2l}-\rho+1)} \right\} \right] \right\} + M_{\overline{m_1},\overline{m_2}}^{k,l} \tag{2.42}$$

proving (2.21).

Next if $(h_1+1, h_2+1) \leq (k,l) \leq (r,p)$, or $0 \leq k \leq h_1$, $h_2+1 \leq l \leq p$, or $h_1+1 \leq k \leq r$, $0 \leq l \leq h_2$, we get by (2.40) that

$$\left\| D_{1-}^{(\alpha_{1k},\alpha_{2l})}(f) - D_{1-}^{(\alpha_{1k},\alpha_{2l})}\left(Q_{\overline{m_1},\overline{m_2}}\right) \right\|_\infty \leq M_{\overline{m_1},\overline{m_2}}^{k,l}, \tag{2.43}$$

proving (2.22).

Furthermore, if (x,y) in critical region, see (2.23), we get

$$\begin{aligned} \alpha_{h_1h_2}^{-1}(x,y)\,\overline{L}\left(Q_{\overline{m_1},\overline{m_2}}(x,y)\right) &= \alpha_{h_1h_2}^{-1}(x,y)\,\overline{L}(f(x,y)) \\ &+ P_{\overline{m_1},\overline{m_2}} \frac{(1-x)^{h_1-\alpha_{1h_1}}(1-y)^{h_2-\alpha_{2h_2}}}{\Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right)} \\ &+ \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} \alpha_{h_1h_2}^{-1}(x,y)\,\alpha_{ij}(x,y) \\ &\cdot D_{1-}^{(\alpha_{1i},\alpha_{2j})}\left[Q_{\overline{m_1},\overline{m_2}}(x,y) - f(x,y) - P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right] \\ &\overset{(2.40)}{\geq} P_{\overline{m_1},\overline{m_2}} \frac{(1-x)^{h_1-\alpha_{1h_1}}(1-y)^{h_2-\alpha_{2h_2}}}{\Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right)} \\ &- \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} l_{ij} M_{\overline{m_1},\overline{m_2}}^{i,j} \end{aligned} \tag{2.44}$$

$$= P_{\overline{m_1},\overline{m_2}} \frac{(1-x)^{h_1-\alpha_{1h_1}} (1-y)^{h_2-\alpha_{2h_2}}}{\Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right)} - P_{\overline{m_1},\overline{m_2}}$$

$$= P_{\overline{m_1},\overline{m_2}} \left[\frac{(1-x)^{h_1-\alpha_{1h_1}} (1-y)^{h_2-\alpha_{2h_2}}}{\Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right)} - 1\right] \tag{2.45}$$

$$= P_{\overline{m_1},\overline{m_2}}$$

$$\cdot \left[\frac{(1-x)^{h_1-\alpha_{1h_1}} (1-y)^{h_2-\alpha_{2h_2}} - \Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right)}{\Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right)}\right]$$

$$=: (*) . \tag{2.46}$$

We know $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex and positive on $(0, \infty)$. Here $0 \leq h_1 - \alpha_{1h_1} < 1$ and $0 \leq h_2 - \alpha_{2h_2} < 1$, hence $1 \leq h_1 - \alpha_{1h_1} + 1 < 2$, $1 \leq h_2 - \alpha_{2h_2} + 1 < 2$. Thus $0 < \Gamma(h_1 - \alpha_{1h_1} + 1)$, $\Gamma(h_2 - \alpha_{2h_2} + 1) \leq 1$, and $1 - \Gamma(h_1 - \alpha_{1h_1} + 1)\Gamma(h_2 - \alpha_{2h_2} + 1) \geq 0$. Clearly acting as in (2.44)–(2.46), when $\overline{L}(f(0,0)) \geq 0$, we get $\overline{L}(Q_{\overline{m_1},\overline{m_2}}(0,0)) \geq 0$.

Also clearly here on the critical region (2.23) we have $(*) \geq 0$. That is, there $\overline{L}(Q_{\overline{m_1},\overline{m_2}}(x,y)) \geq 0$.

Case (ii). Assume that throughout $[0,1]^2$, $\alpha_{h_1h_2} \leq \beta < 0$. Consider

$$Q^{-}_{\overline{m_1},\overline{m_2}}(x,y) :\equiv B_{\overline{m_1},\overline{m_2}}(f; x, y) - P_{\overline{m_1},\overline{m_2}} \frac{x^{h_1}}{h_1!} \frac{y^{h_2}}{h_2!}.$$

Then by (2.28) we get

$$\left\| D_{1-}^{(\alpha_{1k},\alpha_{2l})} \left(f - P_{\overline{m_1},\overline{m_2}} \frac{x^{h_1}}{h_1!} \frac{y^{h_2}}{h_2!}\right) - D_{1-}^{(\alpha_{1k},\alpha_{2l})} \left(Q^{-}_{\overline{m_1},\overline{m_2}}(x,y)\right)\right\|_{\infty} \leq M^{k,l}_{\overline{m_1},\overline{m_2}}, \tag{2.47}$$

all $0 \leq k \leq r$, $0 \leq l \leq p$.

Working similarly as earlier in this proof we derive again (2.21), (2.22). Furthermore, if (x,y) in critical region, see (2.23), we get

$$\alpha^{-1}_{h_1h_2}(x,y)\,\overline{L}\left(Q^{-}_{\overline{m_1},\overline{m_2}}(x,y)\right)$$

$$= \alpha^{-1}_{h_1h_2}(x,y)\,\overline{L}(f(x,y)) - P_{\overline{m_1},\overline{m_2}} \frac{(1-x)^{h_1-\alpha_{1h_1}} (1-y)^{h_2-\alpha_{2h_2}}}{\Gamma(h_1-\alpha_{1h_1}+1)\,\Gamma(h_2-\alpha_{2h_2}+1)}$$

$$+ \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha^{-1}_{h_1h_2}(x,y)\,\alpha_{ij}(x,y)$$

$$\cdot D_{1-}^{(\alpha_{1i},\alpha_{2j})} \left[Q^{-}_{\overline{m_1},\overline{m_2}}(x,y) - f(x,y) + P_{\overline{m_1},\overline{m_2}} \frac{x^{h_1}}{h_1!} \frac{y^{h_2}}{h_2!}\right] \tag{2.48}$$

$$\overset{(2.47)}{\leq} -P_{\overline{m_1},\overline{m_2}} \frac{(1-x)^{h_1-\alpha_{1h_1}} (1-y)^{h_2-\alpha_{2h_2}}}{\Gamma(h_1-\alpha_{1h_1}+1)\,\Gamma(h_2-\alpha_{2h_2}+1)}$$

$$+ \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} M^{i,j}_{\overline{m_1},\overline{m_2}} \tag{2.49}$$

$$= P_{\overline{m_1},\overline{m_2}} \left(1 - \frac{(1-x)^{h_1-\alpha_{1h_1}} (1-y)^{h_2-\alpha_{2h_2}}}{\Gamma(h_1-\alpha_{1h_1}+1)\,\Gamma(h_2-\alpha_{2h_2}+1)}\right) \tag{2.50}$$

$$= P_{\overline{m_1},\overline{m_2}}$$

$$\cdot \left(\frac{\Gamma(h_1-\alpha_{1h_1}+1)\,\Gamma(h_2-\alpha_{2h_2}+1) - (1-x)^{h_1-\alpha_{1h_1}} (1-y)^{h_2-\alpha_{2h_2}}}{\Gamma(h_1-\alpha_{1h_1}+1)\,\Gamma(h_2-\alpha_{2h_2}+1)}\right)$$

$$=: (**)\,. \tag{2.51}$$

We know $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex and positive on $(0,\infty)$. Here $0 \leq h_1 - \alpha_{1h_1} < 1$ and $0 \leq h_2 - \alpha_{2h_2} < 1$, hence $1 \leq h_1 - \alpha_{1h_1} + 1 < 2$, $1 \leq h_2 - \alpha_{2h_2} + 1 < 2$. Thus $0 < \Gamma(h_1-\alpha_{1h_1}+1)$, $\Gamma(h_2-\alpha_{2h_2}+1) \leq 1$, and

$$\Gamma(h_1-\alpha_{1h_1}+1)\,\Gamma(h_2-\alpha_{2h_2}+1) - 1 \leq 0.$$

Clearly acting as in (2.48)–(2.51), when $\overline{L}(f(0,0)) \geq 0$, we get

$$\overline{L}\left(Q^{-}_{\overline{m_1},\overline{m_2}}(0,0)\right) \geq 0.$$

Also clearly here on the critical region (2.23) we get $(**) \leq 0$. That is, there $\overline{L}\left(Q^{-}_{\overline{m_1},\overline{m_2}}(x,y)\right) \geq 0$.

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