Chapter 2
Operating Regimes of an Active Two-Pole. Display of Projective Geometry

2.1 Volt–Ampere Characteristics of an Active Two-Pole. Affine and Projective Transformations of Regime Parameters

2.1.1 Affine Transformations

It is known that any linear circuit (an active two-pole A) relative to load terminals is replaced by a voltage source V_0 in series with an internal resistance R_i [1, 6]. Let us consider two cases of the load.

Case 1

A variable voltage source V_L is the load of an active two-pole A in Fig. 2.1. The voltage V_L (independent quantity) and load current I_L are the parameters of operating or running regime.

At change of the load voltage from the short circuit SC $(V_L = 0)$ to open circuit OC $(V_L^{OC} = V_0)$, a load straight line is given by (1.1)

$$I_L = \frac{V_0}{R_i} - \frac{V_L}{R_i} = I_L^{SC} - \frac{V_L}{R_i}, \tag{2.1}$$

where I_L^{SC} is the SC current. This straight line is shown in Fig. 2.2.

The characteristic regime parameters I_L^{SC}, V_0 determine the different scales for the axes of coordinates.

We want to represent a running regime parameter of the load by a certain quantity, which would have an identical value for various actual regime parameters such as voltage and current. To do this, we use a geometrical interpretation of change or "kinematics" of a circuit regime.

As the load characteristic is defined by linear expression (2.1), a similar expression in geometry defines an affine transformation, conformity, or mapping of

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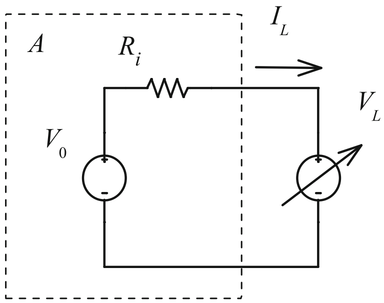


Fig. 2.1 Active two-pole with a load voltage source

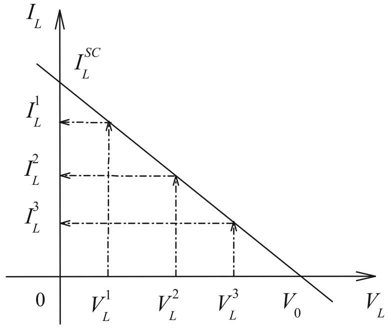


Fig. 2.2 Load straight line of an active two-pole

the voltage axis to current axis $V_L \to I_L$ [7]. The values I_L^{SC}, R_i are parameters of this affine transformation. The mechanism of this mapping is shown by parallel lines with arrows for three voltage values or points $V_L^1,\ V_L^2,\ V_L^3$.

An affine transformation is characterized by an invariant of three points

$$\varphi(V_L^1, V_L^2, V_L^3) = \varphi(I_L^1, I_L^2, I_L^3).$$

The invariant represents a certain expression, which only contains the chosen values of voltage or corresponding values of current; the parameters of the affine transformation do not enter in this expression.

To obtain the invariant expression, it is necessary to exclude the two parameters from Eq. (2.1) by means of three equations. For arbitrary three voltage values $V_L^1,\ V_L^2,\ V_L^3$ we have the system of equations

$$\left\{ I_L^1 = I_L^{SC} - \frac{V_L^1}{R_i}, \quad I_L^2 = I_L^{SC} - \frac{V_L^2}{R_i}, \quad I_L^3 = I_L^{SC} - \frac{V_L^3}{R_i}. \right. \tag{2.2}$$

Using the first and second equations, we exclude I_L^{SC}

$$I_L^1 - I_L^2 = + \frac{V_L^2}{R_i} - \frac{V_L^1}{R_i}. \tag{2.3}$$

Similarly, we have

$$I_L^1 - I_L^3 = +\frac{V_L^3}{R_i} - \frac{V_L^1}{R_i}. \tag{2.4}$$

$$I_L^2 - I_L^3 = +\frac{V_L^3}{R_i} - \frac{V_L^2}{R_i}. \tag{2.5}$$

Let us exclude the parameter R_i. To do this, we use expressions (2.3) and (2.5). Then, the required invariant or affine ratio is obtained as follows:

$$\frac{I_L^1 - I_L^2}{I_L^2 - I_L^3} = \frac{V_L^2 - V_L^1}{V_L^3 - V_L^2}. \tag{2.6}$$

If we use expressions (2.3) and (2.4), we get the other invariant

$$\frac{I_L^1 - I_L^3}{I_L^1 - I_L^2} = \frac{V_L^3 - V_L^1}{V_L^2 - V_L^1}. \tag{2.7}$$

Application of the concrete invariant is determined by a physical sense of a parameter of running regime, as it will be shown later.

Let us consider expression (2.6). We accept values $I_L^1,\ V_L^1$ corresponding to the running regime. In turn, values $I_L^2,\ V_L^2$ correspond to the open-circuit regime, $I_L^2 = 0,\ V_L^2 = V_0$; values $I_L^3,\ V_L^3$ correspond to the short circuit regime, $I_L^3 = I_L^{\mathrm{SC}}, V_L^3 = 0$. The pairs of the respective points $I_L^2,\ V_L^2$ and $I_L^3,\ V_L^3$ are the base or extreme points; the points $I_L^1,\ V_L^1$ are the dividing points.

Then, expression (2.6) takes the form

$$\begin{gathered} \frac{I_L^1 - 0}{0 - I_L^{SC}} = \frac{V_0 - V_L^1}{0 - V_0}, \\ \frac{I_L^1 - 0}{I_L^{SC} - 0} = \frac{V_0 - V_L^1}{V_0 - 0}. \end{gathered} \tag{2.8}$$

The affine ratio n^1 can be represented by the formal view

$$n^1 = (I_L^1\ 0\ I_L^{SC}) = (V_L^1\ V_0\ 0),$$

where

$$(I_L^1\ 0\ I_L^{SC}) = \frac{0 - I_L^1}{0 - I_L^{SC}} = \frac{I_L^1}{I_L^{SC}}, \tag{2.9}$$

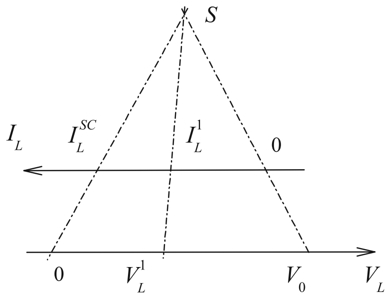


Fig. 2.3 Affine transformation $V_L \to I_L$

$$(V_L^1 \;\; V_0 \;\; 0) = \frac{V_0 - V_L^1}{V_0 - 0}. \tag{2.10}$$

Further, we consider the sense of the value n^1. We obtain the quantity that has an identical value for current and voltage. For the current, this value n^1 is simply the normalized value (at the expense of a particular choice of the base points).

Therefore, expression (2.8) coincides with Eq. (2.1) for the normalized values

$$\frac{I_L^1}{I_L^{SC}} = 1 - \frac{V_L^1}{V_0}. \tag{2.11}$$

The more convenient representation of affine transformation (2.1) is given in Fig. 2.3 for the actual values of the current and voltage. There is a projection center S and straight lines V_L, I_L are parallel. The affine conformity is also given by two pairs of respective base points.

As it was already shown, as the respective base points, it is convenient to use the points of characteristic regimes, which can be easily determined at a qualitative level; that is, the short circuit and open-circuit regimes. The sense of this affine transformation is visible in Fig. 2.3. The projection center S has a final coordinate because of different scales or base points of the current and voltage lines.

If the projection center is $S \to \infty$, the projection is carried out by parallel lines. This mapping corresponds to the Euclidean transformation; that is, the parallel translation of a segment in Fig. 2.4. Such a case conforms to Eq. (2.11).

Now, we consider changes of regime; that is, how to set this change in the invariant form too. Let an initial regime be given by values I_L^1, V_L^1 and subsequent regime be set by I_L^2, V_L^2. Let us express the subsequent value of current through the initial value. Using Eq. (2.3), we get

$$I_L^2 = I_L^1 + \frac{V_L^1}{R_i} - \frac{V_L^2}{R_i} = I_L^1 + I_L^{21}. \tag{2.12}$$

The obtained transformation with a parameter I_L^{21} translates the initial regime I_L^1 point into the subsequent regime I_L^2 point; that is, $I_L^1 \to I_L^2$ shown in Fig. 2.5.

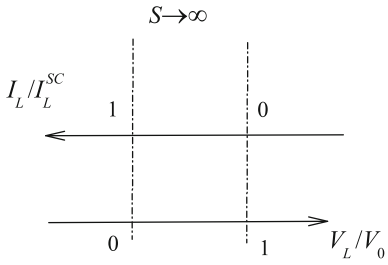


Fig. 2.4 Euclidean transformation $V_L \to I_L$

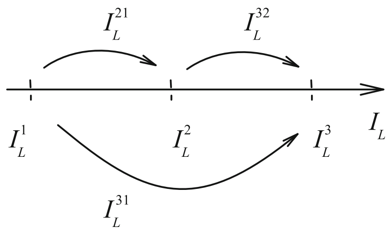


Fig. 2.5 Translation of point $I_L^1 \to I_L^2 \to I_L^3$ along a line

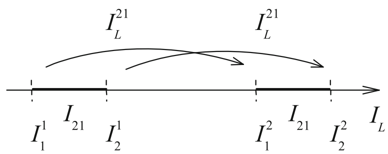


Fig. 2.6 Movement of a segment I_{21}

Further, transformation (2.12) with a parameter I_L^{32} translates the point $I_L^2 \to I_L^3$; that is,

$$I_L^3 = I_L^2 + I_L^{32}.$$

For example, let I_L^{32} be equal I_L^{21}, as it is shown in Fig. 2.5. It now follows that

$$I_L^3 = I_L^1 + I_L^{21} + I_L^{32} = I_L^1 + I_L^{31}.$$

Therefore, we have the resultant transformation $I_L^1 \to I_L^3$ with a parameter I_L^{31}.

Thus, a set of transformations (2.12) is a group.

Let an initial regime be given by two values I_1^1, I_2^1. These values correspond to a segment I_{21} in Fig. 2.6.

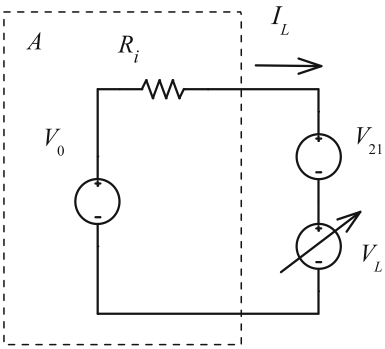


Fig. 2.7 Signal V_{21} and bias V_L voltage is the load of an active two-pole

For example, this segment or increment of current I_{21} corresponds to a signal voltage V_{21} in series with a variable bias voltage V_L in Fig. 2.7.

If we apply transformation (2.12) to the initial points I_1^1, I_2^1, we obtain the subsequent points

$$I_1^2 = I_1^1 + I^{21}, \quad I_2^2 = I_2^1 + I^{21}. \tag{2.13}$$

Obviously, translation (2.12) or (2.13) is characterized by the invariant of two points $\varphi(I_1^1, I_2^1) = I_2^1 - I_1^1$, because

$$I_2^2 - I_1^2 = I_2^1 - I_1^1. \tag{2.14}$$

Thus, this invariant is the Euclidean distance between two points. Therefore, geometry of this transformation group is geometry of a Euclidean straight line.

Also, we can consider the dual variant of segment and its movement in Fig. 2.8.

Let an initial regime be given by two values I_1^1, I_1^2 or segment $I_1^2 I_1^1$. We use the translation with a parameter I_{21}. Then, the subsequent points

$$I_2^1 = I_1^1 + I_{21}, \quad I_2^2 = I_2^1 + I_{21}.$$

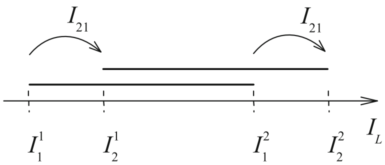


Fig. 2.8 Movement of segment $I_1^2 I_1^1 \rightarrow I_2^2 I_2^1$

Obviously, there is an invariant of two points

$$I_1^2 - I_1^1 = I_2^2 - I_2^1.$$

Using (2.11), we get the similar invariant for the normalized voltage and current values

$$\frac{I_L^2}{I_L^{SC}} - \frac{I_L^1}{I_L^{SC}} = \frac{V_L^1}{V_0} - \frac{V_L^2}{V_0} = n^{21}. \tag{2.15}$$

This regime change corresponds to a relative change in "percent". Note that the invariants of transformation (2.12) coincide with the known principle of superposition for linear circuits.

Remark Let us introduce a regime change by ratio of currents. Using expression (2.11), we get

$$\frac{I_L^2}{I_L^{SC}} \div \frac{I_L^1}{I_L^{SC}} = \frac{I_L^2}{I_L^1} = \frac{V_0 - V_L^2}{V_0 - V_L^1}.$$

We can note that the ratio of currents does not contain a transformation parameter. But, this ratio of voltages contains the transformation parameter V_0. Hence, the condition of invariant is not executed.

Case 2

A variable conductivity Y_L is a load of the simplest circuit in Fig. 2.9; an internal resistance of this active two-pole $R_i = 0$.The equation of this circuit

$$I_L = V_0 Y_L. \tag{2.16}$$

The conductivity Y_L (independent quantity) and load current are the parameters of a running regime.

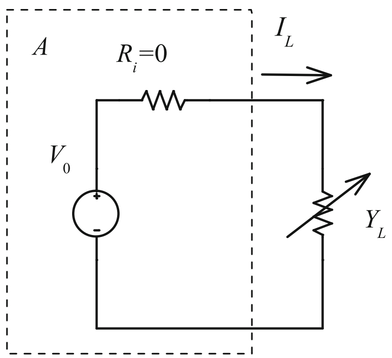


Fig. 2.9 Active two-pole with $R_i = 0$

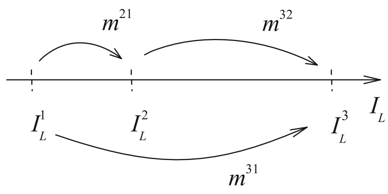


Fig. 2.10 Transformation of a point $I_L^1 \to I_L^2 \to I_L^3$

For this circuit, the regime has only an absolute value; it is impossible to state a relative expression in view of the absence of a scale.

Now, we consider changes of regime. The initial conductivity value equals Y_L^1 and subsequent one equals Y_L^2. Then, we have the two values of currents

$$I_L^1 = V_0 Y_L^1, \quad I_L^2 = V_0 Y_L^2. \tag{2.17}$$

Using the ratio of these equations, we get

$$\frac{I_L^2}{I_L^1} = \frac{Y_L^2}{Y_L^1} = m^{21}. \tag{2.18}$$

Then, the subsequent current is obtained as

$$I_L^2 = m^{21} I_L^1. \tag{2.19}$$

This transformation with a parameter m^{21} translates a point of an initial regime I_L^1 to a point of a subsequent regime I_L^2; that is, $I_L^1 \to I_L^2$. The transformation is a group transformation because

$$I_L^3 = m^{32} I_L^2 = m^{32} m^{21} I_L^1 = m^{31} I_L^1.$$

This group is the dilation group. For example, let m^{32} be equal m^{21}, as shown in Fig. 2.10; a usual distance between currents is increased.

Let an initial regime be given by two values I_1^1, I_2^1. These values correspond to segment $I_1 I_2$ in Fig. 2.11. If we apply transformation (2.19) to the initial points I_1^1, I_2^1 or to the segment, we obtain the subsequent points

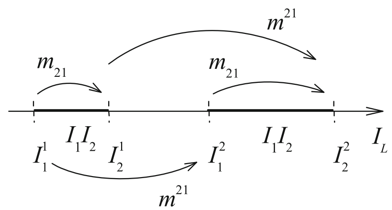


Fig. 2.11 Movement of segment $I_1 I_2$

$$I_1^2 = m^{21} I_1^1, \quad I_2^2 = m^{21} I_2^1.$$

Obviously, transformation (2.19) has the invariant of two points

$$\varphi(I_2^2, I_1^2) = \frac{I_2^2}{I_1^2} = \frac{m^{21} I_2^1}{m^{21} I_1^1} = \frac{I_2^1}{I_1^1} = m_{21} = \varphi(I_2^1, I_1^1). \tag{2.20}$$

This invariant m_{21} is the invariable "length" of segment $I_1 I_2$. Therefore, we obtain the segment end

$$I_2^1 = m_{21} I_1^1.$$

These two examined cases correspond to the known similarity property.

And so, the consideration of regime changes as geometrical transformations gives a methodical foundation for the valid introduction of regime changes for more complex cases.

2.1.2 *Projective Transformations*

Let us consider a circuit with a variable load resistance in Fig. 2.12.

The load straight line is also given by expression (2.1)

$$I_L = \frac{V_0}{R_i} - \frac{V_L}{R_i} = I_L^{SC} - \frac{V_L}{R_i}. \tag{2.21}$$

This straight line is shown in Fig. 2.13.

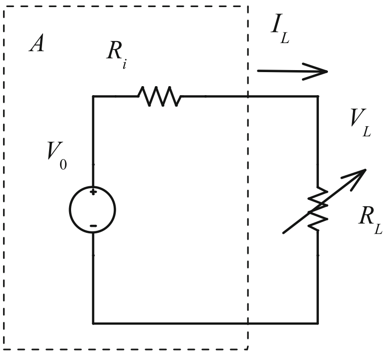


Fig. 2.12 Circuit with a load resistance

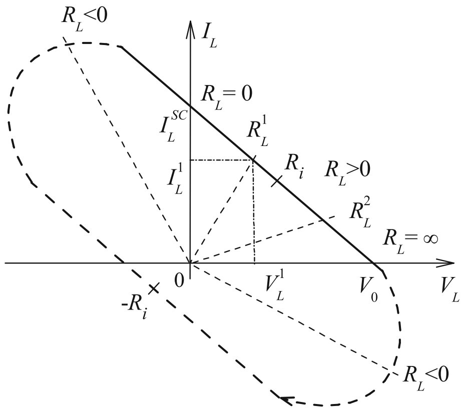


Fig. 2.13 Load straight line of a circuit with a load resistance

The bunch of straight lines with a parameter R_L corresponds to this straight line. The bunch center is the point 0. The equation of this bunch is given as

$$I_L = \frac{1}{R_L} V_L. \tag{2.22}$$

Further, it is possible to calibrate the load straight line in the load resistance values [10, 13, 15]. The internal area $R_L > 0$ at the load change corresponds to regime of energy consumption by the load. If to continue the calibration for the negative values $R_L < 0$, the regime passes into the external area, which physically means return of energy to the voltage source V_0.

Therefore, at the infinitely remote point $R_L = -R_i$, the calibrations of the load straight line will coincide for the areas $V_L > V_0, V_L < 0$. So, this straight line is closed; that is typical property of projective geometry.

The load resistance value determines the nonhomogeneous coordinate for a point of load straight line. In turn, the two values V_L, I_L are the homogeneous coordinates because $R_L = \rho V_L \div \rho I_L$, where ρ is any nonzero real number. Homogeneous coordinates have finite values. Further, the equation $V_L(R_L)$ has the characteristic fractionally linear view

$$V_L = V_0 \frac{R_L}{R_i + R_L}. \tag{2.23}$$

Of course, similar expressions take place for currents and resistances for different branches of active circuits and are used for measurement of circuit parameters [2].

That gives the solid grounds for considering the map $R_L \rightarrow V_L$ as a projective transformation of projective geometry [4, 5, 7]. In general, a projective transformation of points of one straight line into points of another line is set by a center S of projection or three pairs of respective points in Fig. 2.14. Therefore, the infinitely remote point $R_L = \infty$ corresponds to the finite point $V_L = V_0$. Also, fractionally linear expressions of type (2.23) as projective transformations are used for

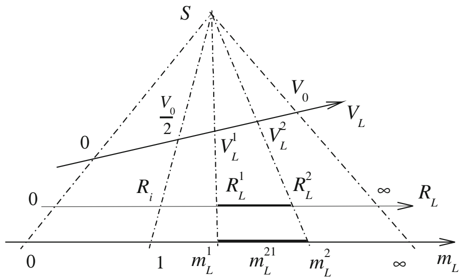


Fig. 2.14 Projective transformation of points $R_L \rightarrow V_L$

measuring instruments [8, 9]. In addition, a class of infinite "ideal" points of projective geometry is introduced to represent infinite resistances [3].

As the pairs of respective points, it is convenient to use the points of the characteristic regimes, which can be easily determined at a qualitative level; that is, the short circuit, open circuit, and maximum load power. In turn, the point of a running regime is the fourth point.

We want to represent the running regime of load in the relative form regarding to these three characteristic or base points. To do this, we may use a cross-ratio of four points.

The projective transformations preserve a cross-ratio of four points. For values R_L^1, V_L^1, I_L^1 of initial or running regime, the cross-ratio m_L has the view

$$m_L^1 = (0\ R_L^1\ R_i\ \infty) = \frac{R_L^1 - 0}{R_L^1 - \infty} : \frac{R_i - 0}{R_i - \infty} = \frac{R_L^1}{R_i}, \tag{2.24}$$

$$m_L^1 = \left(0\ V_L^1\ \frac{V_0}{2}\ V_0\right) = \frac{V_L^1 - 0}{V_L^1 - V_0} \div \frac{\frac{V_0}{2} - 0}{\frac{V_0}{2} - V_0} = \frac{V_L^1}{V_0 - V_L^1}, \tag{2.25}$$

$$m_L^1 = \left(I_L^{SC}\ I_L^1\ \frac{I_L^{SC}}{2}\ 0\right) = \frac{I_L^1 - I_L^{SC}}{I_L^1 - 0} \div \frac{\frac{I_L^{SC}}{2} - I_L^{SC}}{\frac{I_L^{SC}}{2} - 0} = \frac{I_L^{SC} - I_L^1}{I_L^1}. \tag{2.26}$$

We note that expression (2.26) corresponds to (2.7).

The cross-ratio in geometry underlies the definition of the "distance" between points $R_L^1, R_L = R_i$ concerning the extreme or base values $0,\ \infty$. The point R_i is a scale or a unit point. Similarly, we have the base points $0, V_0$, and a unit point $V_0/2$ of voltage. Thus, a projective coordinate of running regime point is set by the value m_L, which is defined in an identical (invariant) manner through various regime parameters as R_L, V_L, I_L.

In turn, a regime change $R_L^1 \rightarrow R_L^2$ (respectively $V_L^1 \rightarrow V_L^2$, $I_L^1 \rightarrow I_L^2$) can be expressed similarly

$$m_L^{21} = (0\ R_L^2\ R_L^1\ \infty) = \frac{R_L^2}{R_L^1}, \tag{2.27}$$

$$m_L^{21} = (0\ V_L^2\ V_L^1\ V_0) = \frac{V_L^2}{V_0 - V_L^2} \div \frac{V_L^1}{V_0 - V_L^1}, \tag{2.28}$$

$$m_L^{21} = (I_L^{SC}\ I_L^2\ I_L^1\ 0) = \frac{I_L^{SC} - I_L^2}{I_L^2} \div \frac{I_L^{SC} - I_L^1}{I_L^1}. \tag{2.29}$$

The cross-ratio has the next quality

$$m_L^{12} = (0\ R_L^1\ R_L^2\ \infty) = \frac{R_L^1}{R_L^2} = \frac{1}{m_L^{21}}.$$

A group property of the cross-ratio realizes

$$m_L^2 = m_L^{21} m_L^1. \tag{2.30}$$

For the next regime change $R_L^2 \to R_L^3$, the group property is given by

$$m_L^3 = m_L^{32} m_L^2 = m_L^{32} m_L^{21} m_L^1 = m_L^{31} m_L^1.$$

Let us express the subsequent voltage value V_L^2 by the initial value V_L^1 and regime change value m_L^{21}. Using (2.28), we get the recalculation formula

$$\frac{V_L^2}{V_0} = \frac{\frac{V_L^1}{V_0} m_L^{21}}{\frac{V_L^1}{V_0}(m_L^{21} - 1) + 1}. \tag{2.31}$$

This formula allows finding a subsequent voltage value by an initial voltage and transformation parameter m_L^{21}. Also, this expression can be obtained from (2.23). For an initial R_L^1 and subsequent R_L^2 value, we get the following system of the equations:

$$\begin{cases} V_L^1 = V_0 \frac{R_L^1}{R_i + R_L^1} \\ V_L^2 = V_0 \frac{R_L^2}{R_i + R_L^2} \end{cases}.$$

Excepting R_i, we get expression (2.31).

In general, transformation (2.31) translates an initial point V_L^1 to subsequent point V_L^2. Therefore, we can set the identical regime changes for the different initial regimes shown in Fig. 2.15 for the straight line V_L as a closed projective straight line.

The identical transformation parameter m_L^{21} forms a segment of invariable "length" (in sense of projective geometry) and we observe the movement of this segment. Here, the change of the Euclidean (usual) length is visible. For the base points, the Euclidean length is decreasing to zero.

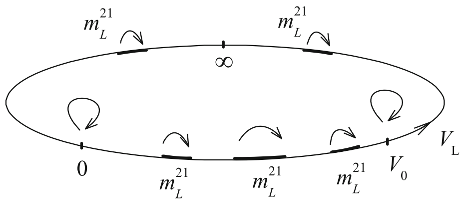


Fig. 2.15 Identical regime changes for the different initial regimes

In the theory of the projective transformations, the fixed points play an important role. For their finding, Eq. (2.31) is solved as $V_L^1 = V_L^2$. It turns out the two real roots, $V_L = 0, V_L = V_0$, which define the hyperbolic transformation of hyperbolic (Lobachevski) geometry. Physically, the fixed point means such a regime when a variable V_L does not depend on the initial or subsequent value R_L. It is evident for SC, OC regimes.

If the roots of equation coincide, the fixed point defines the parabolic transformation and, respectively, parabolic (Euclidean) geometry. If the roots are imaginary, geometry is elliptic (Riemannian).

In geometry, it is established that these three kinds of transformations (projective, affine, and Euclidean) exhaust possible variants of group transformations, which underlie the definition of the metrics of a straight line. Thus, the geometrical approach allows validating regimes determination, and both definitions of a regime and its change are coordinated by structure of expressions and ensure the performance of group properties.

Reasoning from such a geometrical interpretation, it is possible to give the following definition [12, 13]:

- a circuit regime is a coordinate of point on load straight lines and axes of coordinates;
- a regime change is a movement of point on all the straight lines, which defines a segment of corresponding length.

In this connection, it is possible to accept the following requirement (likewise to metric space axioms):

- independence or invariance of regimes and their changes from variables (regime parameters) as type R, V, I;
- the additive postulate of regimes changes;
- assignment of equal regime change for various initial regimes.

2.2 Volt–Ampere Characteristics of an Active Two-Pole with a Variable Element

2.2.1 *Thévenin Equivalent Circuit with the Variable Internal Resistance*

Let us consider the Thévenin equivalent circuit with the variable internal resistance R_i in Fig. 2.16.

In this case, a bunch of load straight lines with a parameter R_i is obtained at a center G in Fig. 2.17. The unified equation of this bunch is given by

$$I_L = \frac{1}{R_i}(-V_L + V_0). \tag{2.32}$$

The coordinate of the center G corresponding to V_0 does not depend on R_i.

Physically, it means that the current across this element is equal to zero. The element R_i can accept the two base or characteristic values, as $0, \infty$. The third characteristic value is not present for R_i.

Also, the bunch of straight lines with a parameter R_L corresponds to these load straight lines. The point 0 is the bunch center. Therefore, R_L can accept the two base or characteristic values, as $0, \infty$. Hereafter, we say about the two bunches with the parameter R_i and R_L correspondingly.

Let relative regimes be considered for this circuit.

Case 1

Let the internal resistance R_i be equal to R_i^1 and the load resistance varies from R_L^1 to R_L^2. In this case, a point of initial regime is $C_1 \to C_2$. If R_i is equal to R_i^2, a point of initial regime $B_1 \to B_2$. Therefore, the regime change, which is determined by the load change (an own change), is expressed similar to (2.27)

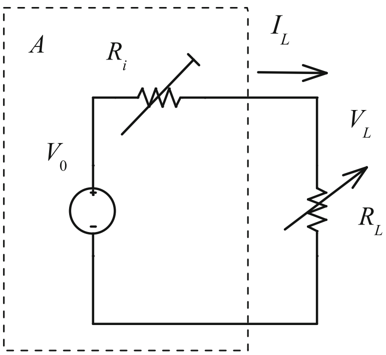


Fig. 2.16 Thévenin equivalent with the variable internal resistance

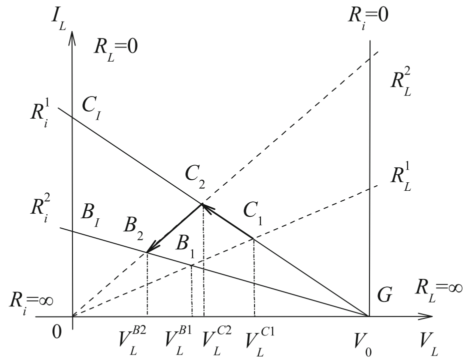


Fig. 2.17 Bunch of load straight lines for a voltage source

$$m_L^{21} = (C_I\ C_2\ C_1\ G) = (B_I\ B_2\ B_1\ G) = (0\ R_L^2\ R_L^1\ \infty) = \frac{R_L^2}{R_L^1}. \tag{2.33}$$

This determination of a regime change does not depend on R_i. Therefore, we must use only the load voltage for this calculation, as (2.28)

$$m_L^{21} = (0\ V_L^{C2}\ V_L^{C1}\ V_0) = \frac{V_L^{C2} - 0}{V_L^{C2} - V_0} \div \frac{V_L^{C1} - 0}{V_L^{C1} - V_0}. \tag{2.34}$$

In this case, the base points C_I, B_I correspond to the common value $V_L = 0$.

Case 2

Similarly, the regime change, which is determined by R_i change (a mutual change), is given by

$$m_i^{21} = (0\ B_2\ C_2\ A_2) = (0\ B_1\ C_1\ A_1) = (\infty\ R_i^2\ R_i^1\ 0) = \frac{R_i^1}{R_i^2}. \tag{2.35}$$

This determination of a regime change does not depend on R_L. Therefore, we must use only load voltage for calculation, as (2.34)

$$m_i^{21} = (0\ V_L^{B2}\ V_L^{C2}\ V_0) = \frac{V_L^{B2} - 0}{V_L^{B2} - V_0} \div \frac{V_L^{C2} - 0}{V_L^{C2} - V_0}. \tag{2.36}$$

In expressions (2.34) and (2.36) the identical base points, which are the centers of the two bunches of straight lines R_L,R_i, are used.

Case 3

If the regime is changed as point $C_1 \to C_2 \to B_2$, it is possible to speak about the general or compound change

$$m^{21} = m_i^{21} m_L^{21} = \frac{R_i^1}{R_i^2} \frac{R_L^2}{R_L^1}. \tag{2.37}$$

Then

$$m^{21} = \left(\frac{V_L^{B2} - 0}{V_L^{B2} - V_0} \div \frac{V_L^{C2} - 0}{V_L^{C2} - V_0} \right) \cdot \left(\frac{V_L^{C2} - 0}{V_L^{C2} - V_0} \div \frac{V_L^{C1} - 0}{V_L^{C1} - V_0} \right).$$

Finally, we obtain

$$m^{21} = \frac{V_L^{B2} - 0}{V_L^{B2} - V_0} \div \frac{V_L^{C1} - 0}{V_L^{C1} - V_0} = (0 \; V_L^{B2} \; V_L^{C1} \; V_0). \tag{2.38}$$

Analogous to (2.31), we get the recalculation formula

$$\frac{V_L^{B2}}{V_0} = \frac{\frac{V_L^{C1}}{V_0} m^{21}}{\frac{V_L^{C1}}{V_0} (m^{21} - 1) + 1}. \tag{2.39}$$

The compound regime and voltage changes, as point moving, are shown in Fig. 2.18.

We may note that values R_L, R_i have not any scales.

2.2.2 Norton Equivalent Circuit with the Variable Internal Conductivity

Let us consider the Norton equivalent circuit with the variable internal resistance or conductance Y_i and load conductance Y_L in Fig. 2.19.

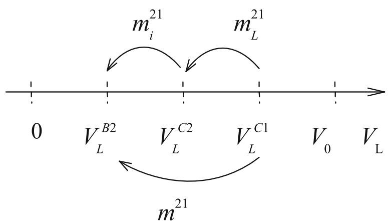


Fig. 2.18 Compound voltage changes

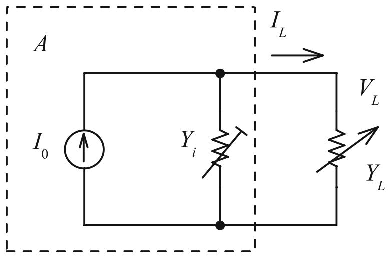
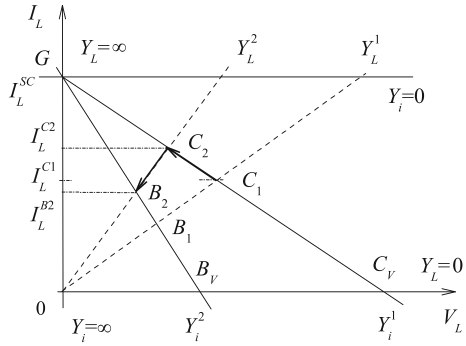


Fig. 2.19 Norton equivalent with the variable internal conductance

Fig. 2.20 Bunch of load straight lines for a current source



In this case, a bunch of load straight lines Y_i is obtained at the center G in Fig. 2.20.

The unified equation of this bunch is given as

$$I_L - I_0 = -Y_i V_L. \tag{2.40}$$

The coordinate of the center G corresponding to $I_0 = I_L^{SC}$ does not depend on the value Y_i. Physically, it means that the current across this element is equal to zero. The element Y_i can accept the two base or characteristic values, as $0, \infty$. The third characteristic value is not present for Y_i.

Let the relative regimes be considered for this case.

Case 1

Let the internal conductance Y_i be equal to Y_i^1 and the load conductance varies from Y_L^1 to Y_L^2. In this case, a point of initial regime $C_1 \to C_2$. If Y_i is equal to Y_i^2, a point of initial regime $B_1 \to B_2$. Therefore, the regime change, which is determined by the load change (an own change), is expressed similarly to (2.33)

$$\begin{aligned} m_L^{21} &= (C_V\ C_2\ C_1\ G) = (B_V\ B_2\ B_1\ G) \\ &= (0\ Y_L^2\ Y_L^1\ \infty) = \frac{Y_L^2}{Y_L^1}. \end{aligned} \tag{2.41}$$

This determination of a regime change does not depend on Y_i. Therefore, we must use only the load current for calculation; that is,

$$m_L^{21} = (0\ I_L^{C2}\ I_L^{C1}\ I_0) = \frac{I_L^{C2} - 0}{I_L^{C2} - V_0} \div \frac{I_L^{C1} - 0}{I_L^{C1} - I_0}. \tag{2.42}$$

In this case, the base points $C_V,\ B_V$ correspond to the common value $I_L = 0$

Case 2

Similarly, the regime change, which is determined by Y_i change (a mutual change), is given as

$$\begin{aligned} m_i^{21} &= (0\ B_2\ C_2\ F_2) = (0\ B_1\ C_1\ F_1) \\ &= (\infty\ Y_i^2\ Y_i^1\ 0) = \frac{Y_i^1}{Y_i^2}. \end{aligned} \tag{2.43}$$

This determination of a regime change does not depend on Y_L. Therefore, we must use only the load current for calculation

$$m_i^{21} = (0\ I_L^{B2}\ I_L^{C2}\ I_0) = \frac{I_L^{B2} - 0}{I_L^{B2} - I_0} \div \frac{I_L^{C2} - 0}{I_L^{C2} - I_0}. \tag{2.44}$$

Case 3

If regime is changed as $C_1 \to C_2 \to B_2$, the compound change is

$$m^{21} = m_i^{21} m_L^{21} = \frac{Y_i^1}{Y_i^2}\frac{Y_L^2}{Y_L^1}. \tag{2.45}$$

Then

$$m^{21} = \frac{I_L^{B2} - 0}{I_L^{B2} - I_0} \div \frac{I_L^{C1} - 0}{I_L^{C1} - I_0} = (0\ I_L^{B2}\ I_L^{C1}\ I_0). \tag{2.46}$$

The recalculation formula

$$\frac{I_L^{B2}}{I_0} = \frac{\frac{I_L^{C1}}{I_0} m^{21}}{\frac{I_L^{C1}}{I_0}(m^{21} - 1) + 1}. \tag{2.47}$$

The compound regime and current changes are shown in Fig. 2.21.

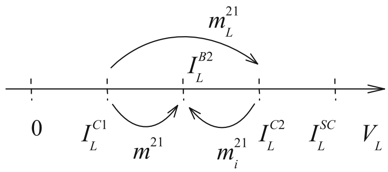


Fig. 2.21 Compound current changes

We may note that values Y_L, Y_i have not any scales also.

The above-mentioned arguments make it possible to confront regimes of compared circuits and give the basis for analysis of the general case of circuit.

2.3 Regime Symmetry for a Load-Power

In the above Sect. 2.1, the examined regime parameters of the type V, I, R are expressed among themselves by linear and fractionally linear expressions (2.1), (2.23) and have projective properties. In turn, such an important energy characteristic, as a load-power, represents quadratic expression (1.6) and determines a parabola in Fig. 1.6. This quadratic curve has similar projective properties that permit to compare the regime of different circuits, to determine the deviation from the power matching [11, 12, 14]. Let us consider these properties in detail.

To do this, we use the circuit in Fig. 2.12 and rewrite Eq. (2.21) of load straight line in the following relative form

$$\frac{I_L}{I_L^{SC}} = I = 1 - \frac{V_L}{V_0} = 1 - K_V, \tag{2.48}$$

where K_V is the voltage transfer ratio. Also, we rewrite Eq. (1.6) of load power in the similar relative form

$$P = \frac{P_L}{P_0^{SC}} = K_V - (K_V)^2, \tag{2.49}$$

where P_0^{SC} is the maximum power of the voltage source V_0 for SC regime.

Therefore, the task of equal regimes does not cause a problem; that is simply corresponding equality of values K_V, I, P. But a deeper analysis will be useful, which allows generalizing the justification of the equality of regimes and will be used for considering a more complex case, the efficiency of two-ports.

2.3.1 Symmetry of Consumption and Return of Power

Let us consider load straight line (2.48) in Fig. 2.22.

In the first quadrant, a positive load consumes energy; there is the maximum load power point P_{LM}^{+} for $R_L = R_i$. At *SC* and *OC* regime points, $R_L = 0, \infty$, load power (2.49) is equal to zero, as shown in Fig. 2.23.

We remind if a negative load returns energy into the voltage source V_0, then the load voltage $V_L < 0$ and $V_L > V_0$. In this case, the load resistance $R_L < 0$; the load power increases. In the final analysis, the load power $P_{LM}^{-} = \infty$ for the resistance $R_L = -R_i$. Therefore, the load straight line is closed and the parabola is a closed oval curve too, which concerns the infinitely remote straight line TP_{LM}^{-} at the point P_{LM}^{-}.

Let us use expression (2.24) of the cross-ratio

$$m_L^1 = (0\ R_L^1\ R_i\ \infty) = \frac{R_L^1}{R_i}. \tag{2.50}$$

For the point $R_L = -R_i$, cross-ratio (2.50)

$$m_L(-R_i) = \frac{-R_i}{R_i} = -1.$$

This special case of the fourth point $-R_i$ determines the property of the harmonic conjugacy of four points, which determines the symmetry of points $-R_i$, R_i relatively to base points 0, ∞.

Fig. 2.22 Conformity of points for positive and negative load

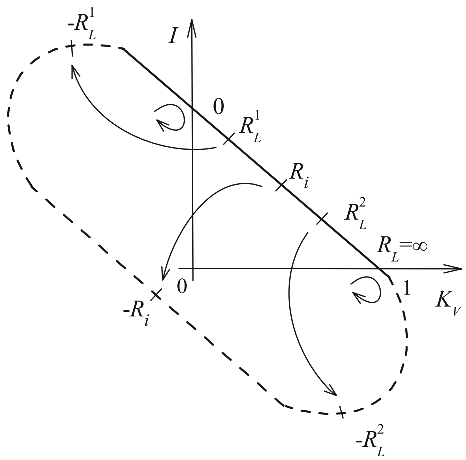
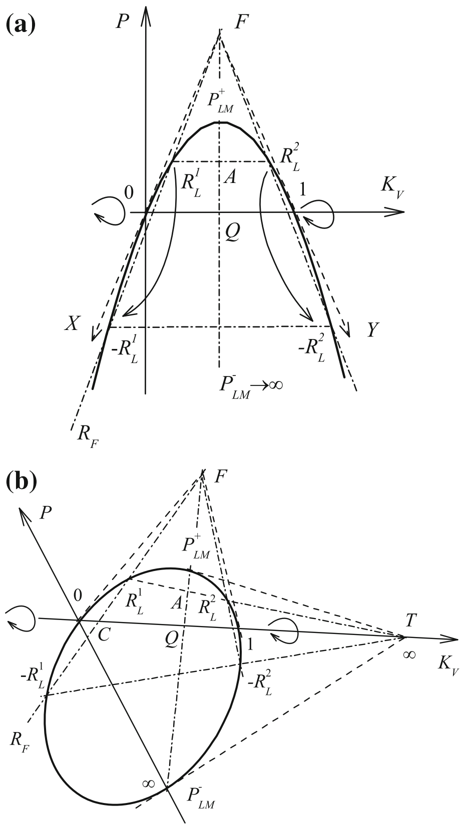


Fig. 2.23 **a** Parabola of power for the Cartesian coordinates, **b** this parabola into the projective coordinates is as a closed curve



For an arbitrary running load R_L, we obtain, at once, the corresponding conjugate load resistance equals $-R_L$ because cross-ratio (2.50)

$$(0 \ -R_L \ R_L \ \infty) = -1. \tag{2.51}$$

So, we have the symmetry of points $-R_L$, R_L relatively to the base points 0, ∞ too. The points $R_i, -R_i$; $R_L^1, -R_L^1$ and so on pass into each other, as it is shown by arrows. Physically, this symmetry corresponds to mapping of the region of power consumption by a load on the region of return. Then, the mapping of points of parabola of the region $P > 0$ onto the region $P < 0$ is realized from a point F. In this case, the points 0, 1 of the axis K_V are fixed. Therefore, we obtain the following condition:

$$(0 \ K_V(-R_L) \ K_V(R_L) \ 1) = \frac{K_V(-R_L) - 0}{K_V(-R_L) - 1} \div \frac{K_V(R_L) - 0}{K_V(R_L) - 1} = -1. \tag{2.52}$$

From here, we get

$$\frac{K_V(R_L) \cdot K_V(-R_L)}{K_V(R_L) + K_V(-R_L)} = 0.5. \tag{2.53}$$

The point F is formed due to the intersection of the tangential FX, FY at the fixed points. This point F is called a pole, and a straight line, passing through the fixed points 0, 1 is a polar $0T$. The indicated symmetry is obtained relatively to the polar.

We may pass to the projective system of coordinates YFX. In this coordinate system, a polar is considered as the infinitely remote straight line. Therefore, our initial parabola will be already a hyperbola, and the coordinate axes FX, FY are asymptotes in Fig. 2.24. The point P_{LM}^{-} has a finite value.

In this case, a point on hyperbola is assigned as the rotation of radius-vector $R_F F$ from the initial position at the point P_{LM}^{+}.

The non-Euclidean distance $R_1 P_M^{-}$ is determined by a hyperbolic arc length of a hyperbola; it will be later on shown in Sect. 4.4.

2.3.2 Symmetry Relatively to the Maximum Power Point

Also, the symmetry of points R_L^1, R_L^2, as the points of equal power, is manifested by arrows relatively to the point R_i or relatively to the straight line $P_{LM}^{+} P_{LM}^{-}$ in Figs. 2.25 and 2.26. The points $+R_i, -R_i$ are the fixed points. This symmetry corresponds to points K_V^1, K_V^2 also.

Using (2.49), we get the following condition:

$$K_V^1 + K_V^2 = 1. \tag{2.54}$$

Also, the mapping of points $R_L^1 \to R_L^2$, relatively to the straight P_M^{-}, P_M^{+} of the parabola, leads to an additional system of pole and polar; the point T is a pole, and the straight line P_M^{-} P_M^{+} is a polar. Similarly, the regime or the point of the parabola,

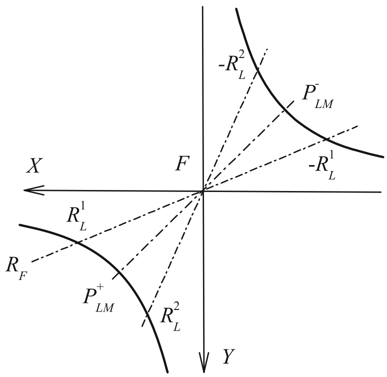


Fig. 2.24 Rotation of a radius-vector $R_F F$

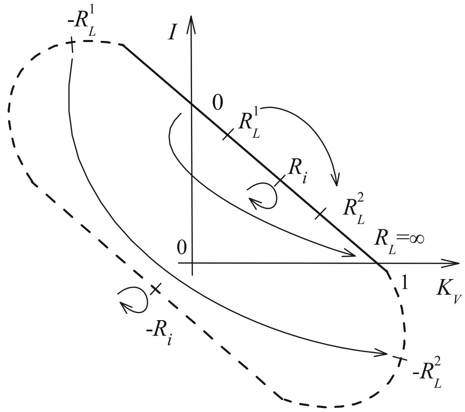


Fig. 2.25 Mapping of points relatively to the maximum load power point

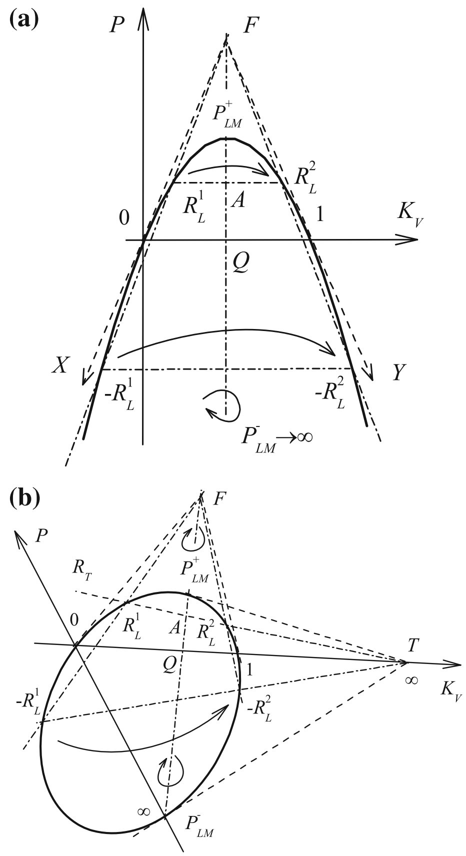


Fig. 2.26 Mapping of a parabola relatively to the maximum load power line. **a** Cartesian coordinates, **b** projective coordinates

is assigned as the rotation of radius-vector $R_T T$ from the initial position at the point 0 or point 1. In the Cartesian coordinate system, this radius-vector $R_T T$ is the parallel line to the axis K_V.

Similar to (2.50), we introduce the other value of running regime

$$m_L^1 = (R_i\ R_L^1\ 0\ \ -R_i) = \frac{R_L^1 - R_i}{R_L^1 + R_i} \div \frac{0 - R_i}{0 + R_i} = \frac{-R_L^1 + R_i}{R_L^1 + R_i}. \tag{2.55}$$

The point $R_L = 0$ is a unit point.

Similar to (2.51), we obtain the harmonic conjugate point using the condition

$$(R_i\ R_L^1\ R_L^2\ \ -R_i) = \frac{R_L^1 - R_i}{R_L^1 + R_i} \div \frac{R_L^2 - R_i}{R_L^2 + R_i} = -1.$$

From that, we get

$$R_L^1 R_L^2 = (R_i)^2. \tag{2.56}$$

2.3.3 Two Systems of Characteristic Points

Thus, we have obtained a "kinematics" diagram of the regime deviations relatively to the selected base points and initial point. Also, we get two conjugate systems of pole and polar. For example, there are four harmonic conjugate points $0, Q, 1, T$ onto the polar $0T$ of the pole F. Reciprocally, there are four harmonic conjugate points $P_M^-, Q,\ P_M^+, F$ onto the polar $P_M^-\ P_M^+$ of the pole T. Let us consider this harmonic conjugacy in detail.

For the pole T, the following correspondences take place. We believe the harmonic conjugate points $0,\ 1$ relatively to the base points Q, T of the polar $0T$. Then, these points correspond to four points $0, 1, K_V(Q) = 0.5, K_V(T) = \infty$ of the axis K_V. The mutual mapping of the points $0, 1$ relatively to fixed points $K_V(T), K_V(Q)$ is shown by arrows in Fig. 2.27.

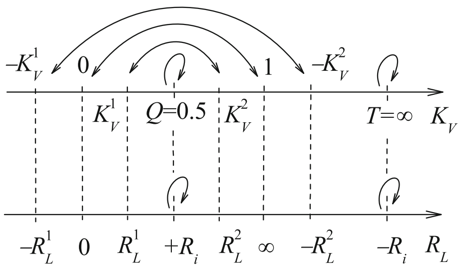


Fig. 2.27 Mutual mapping of points relatively to the fixed points onto the straight line TR_1

In turn, there are such harmonic conjugate points R_L^1, R_L^2 relatively to the base points A, T of running straight line TR_1. The harmonic conjugate points $K_V^1, K_V^2, 0.5, \infty$ of the axis K_V correspond to these points of the line TR_1.

The mutual mapping of points K_V^1, K_V^2 relatively to the fixed or base points $\infty, 0.5$ is shown by arrows in Fig. 2.27 too. Therefore, we can constitute the cross-ratio

$$(0.5\ K_V^1\ K_V^2\ \infty) = \frac{K_V^1 - 0.5}{K_V^1 - \infty} \div \frac{K_V^2 - 0.5}{K_V^2 - \infty} = \frac{K_V^1 - 0.5}{K_V^2 - 0.5} = -1.$$

From this, we get expression (2.54) that confirms the accepted geometrical model.

Similarly, for the pole F, the following correspondences take place. We believe the harmonic conjugate points P_{LM}^+, P_{LM}^- relatively to the base points F, Q of the polar $P_{LM}^+ P_{LM}^-$. In turn, there are such harmonic conjugate points $R_L^1, -R_L^1$ relatively to the base points F, C of running straight line FR_1. The harmonic conjugate points $K_V(C), K_V(R_1), 0.5, K_V(-R_1)$ of the axis K_V correspond to these points of the line FR_1.

The mutual mapping of points $K_V(R_L^1), K_V(-R_L^1)$ relatively to the points $K_V(C), 0.5$ is shown by arrows in Fig. 2.28.

Now, it is necessary to find the common fixed or base points for these two systems of harmonic conjugate points.

We note at once that points $K_V(R_L^1), K_V(-R_L^1)$ are the harmonic conjugate points relatively to the points 0, 1 in accordance with (2.52). Obviously, the points T, Q are the harmonic conjugate points relatively to the points 0, 1 too. These correspondences are shown by dash arrows in Fig. 2.29. Therefore, we may choose the points 0, 1 as the base points.

Thus, the suggested geometrical approach allows to reunite all the points of characteristic regimes into one system and to prove the choice of base points and a unit point. Therefore, it is possible to express any running regime point by a cross-ratio of type (2.50). All the values of cross-ratios for the characteristic and running points are shown on the axis m in Fig. 2.29.

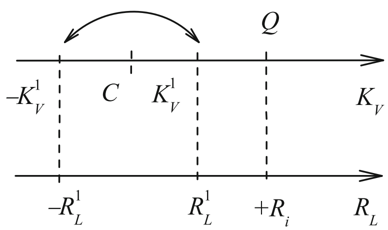


Fig. 2.28 Mutual mapping of points of the straight line FR_1

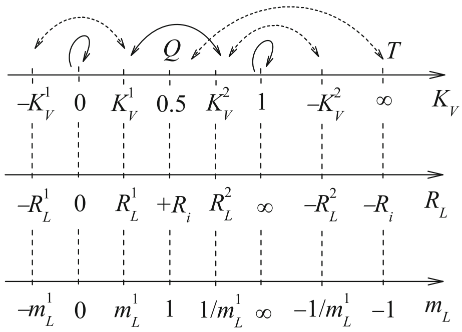


Fig. 2.29 Mutual mapping of points relative to the fixed points and correspondence of the different values

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