Chapter 5
Nonlocal Minimal Surfaces

In this chapter, we introduce nonlocal minimal surfaces and focus on two main results, a Bernstein type result in any dimension and the non-existence of nontrivial s-minimal cones in dimension 2. Moreover, some boundary properties will be discussed at the end of this chapter. For a preliminary introduction to some properties of the nonlocal minimal surfaces, see [135].

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, and $E \subset \mathbb{R}^n$ be a measurable set, fixed outside Ω. We will consider for $s \in (0, 1/2)$ minimizers of the H^s norm

$$\begin{aligned} ||\chi_E||^2_{H^s} =& \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\chi_E(x) - \chi_E(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \\ =& 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\chi_E(x)\chi_{E^{\mathcal{C}}}(y)}{|x-y|^{n+2s}} \, dx \, dy. \end{aligned}$$

Notice that only the interactions between E and $E^{\mathcal{C}}$ contribute to the norm.

In order to define the fractional perimeter of E in Ω, we need to clarify the contribution of Ω to the H^s norm here introduced. Namely, as E is fixed outside Ω, we aim at minimizing the "Ω-contribution" to the norm among all measurable sets that "vary" inside Ω. We consider thus interactions between $E \cap \Omega$ and $E^{\mathcal{C}}$ and between $E \setminus \Omega$ and $\Omega \setminus E$, neglecting the data that is fixed outside Ω and that does not contribute to the minimization of the norm (see Fig. 5.1). We define the interaction $I(A, B)$ of two disjoint subsets of $\mathbb{R}^n$ as

$$\begin{aligned} I(A,B) :=& \int_A \int_B \frac{dx \, dy}{|x-y|^{n+2s}} \\ =& \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\chi_A(x)\chi_B(x)}{|x-y|^{n+2s}} \, dx \, dy. \end{aligned} \tag{5.1}$$

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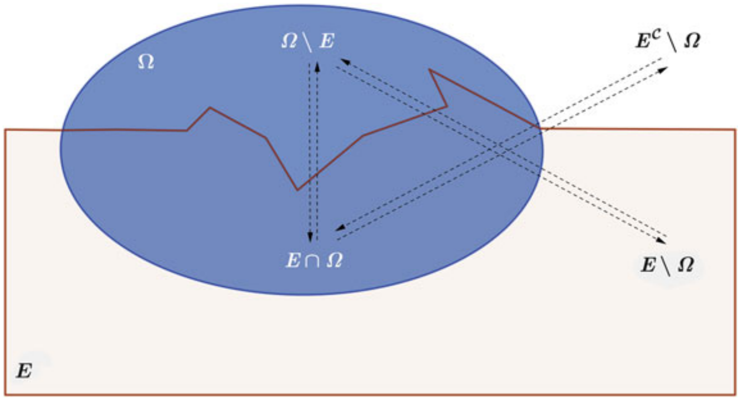


Fig. 5.1 Fractional perimeter

Then (see [28]), one defines the nonlocal s-perimeter functional of E in Ω as

$$\mathrm{Per}_s(E, \Omega) := I(E \cap \Omega, E^{\mathscr{C}}) + I(E \setminus \Omega, \Omega \setminus E). \tag{5.2}$$

Equivalently, one may write

$$\mathrm{Per}_s(E, \Omega) = I(E \cap \Omega, \Omega \setminus E) + I(E \cap \Omega, E^{\mathscr{C}} \setminus \Omega) + I(E \setminus \Omega, \Omega \setminus E).$$

Definition 5.1 Let Ω be an open domain of $\mathbb{R}^n$. A measurable set $E \subset \mathbb{R}^n$ is s-minimal in Ω if $\mathrm{Per}_s(E, \Omega)$ is finite and if, for any measurable set F such that $E \setminus \Omega = F \setminus \Omega$, we have that

$$\mathrm{Per}_s(E, \Omega) \le \mathrm{Per}_s(F, \Omega).$$

A measurable set is s-minimal in $\mathbb{R}^n$ if it is s-minimal in any ball B_r, where $r > 0$.

When $s \to \frac{1}{2}$, the fractional perimeter Per_s approaches the classical perimeter, see [21]. See also [45] for the precise limit in the class of functions with bounded variations, [34, 35] for a geometric approach towards regularity and [6, 117] for an approach based on Γ-convergence. See also [136] for a different proof and Theorem 2.22 in [103] and the references therein for related discussions. A simple, formal statement, up to renormalizing constants, is the following:

Theorem 5.2 *Let $R > 0$ and E be a set with finite perimeter in B_R. Then*

$$\lim_{s \to \frac{1}{2}} \left(\frac{1}{2} - s\right) Per_s(E, B_r) = Per\,(E, B_r)$$

for almost any $r \in (0, R)$.

The behavior of Per_s as $s \to 0$ is slightly more involved. In principle, the limit as $s \to 0$ of Per_s is, at least locally, related to the Lebesgue measure (see e.g. [106]). Nevertheless, the situation is complicated by the terms coming from infinity, which, as $s \to 0$, become of greater and greater importance. More precisely, it is proved in [58] that, if $\mathrm{Per}_{s_o}(E,\Omega)$ is finite for some $s_o \in (0, 1/2)$, and the limit

$$\beta_E := \lim_{s\to 0} 2s \int_{E\setminus B_1} \frac{dy}{|y|^{n+2s}} \tag{5.3}$$

exists, then

$$\lim_{s\to 0} 2s\,\mathrm{Per}_s(E,\Omega) = (|\partial B_1| - \beta_E)\;|E\cap\Omega| + \beta_E\,|\Omega\setminus E|. \tag{5.4}$$

We remark that, using polar coordinates,

$$0 \le \beta_E \le \lim_{s\to 0} 2s \int_{\mathbb{R}^n\setminus B_1} \frac{dy}{|y|^{n+2s}} = \lim_{s\to 0} 2s\,|\partial B_1| \int_1^{+\infty} \rho^{-1-2s}\,d\rho = |\partial B_1|,$$

therefore $\beta_E \in [0, |\partial B_1|]$ plays the role of a convex interpolation parameter in the right hand-side of (5.4) (up to normalization constants).

In this sense, formula (5.4) may be interpreted by saying that, as $s \to 0$, the s-perimeter concentrates itself on two terms that are "localized" in the domain Ω, namely $|E \cap \Omega|$ and $|\Omega \setminus E|$. Nevertheless, the proportion in which these two terms count is given by a "strongly nonlocal" interpolation parameter, namely the quantity β_E in (5.3) which "keeps track" of the behavior of E at infinity.

As a matter of fact, to see how β_E is influenced by the behavior of E at infinity, one can compute β_E for the particular case of a cone. For instance, if $\Sigma \subseteq \partial B_1$, with $\frac{|\Sigma|}{|\partial B_1|} =: b \in [0,1]$, and E is the cone over Σ (that is $E := \{tp,\ p\in\Sigma,\ t\ge 0\}$), we have that

$$\beta_E = \lim_{s\to 0} 2s\,|\Sigma| \int_1^{+\infty} \rho^{-1-2s}\,d\rho = |\Sigma| = b\,|\partial B_1|,$$

that is β_E gives in this case exactly the opening of the cone.

We also remark that, in general, the limit in (5.3) may not exist, even for smooth sets: indeed, it is possible that the set E "oscillates" wildly at infinity, say from one cone to another one, leading to the non-existence of the limit in (5.3).

Moreover, we point out that the existence of the limit in (5.3) is equivalent to the existence of the limit in (5.4), except in the very special case $|E\cap\Omega| = |\Omega\setminus E|$, in which the limit in (5.4) always exists. That is, the following alternative holds true:

- if $|E\cap\Omega| \ne |\Omega\setminus E|$, then the limit in (5.3) exists if and only if the limit in (5.4) exists,

- if $|E \cap \Omega| = |\Omega \setminus E|$, then the limit in (5.4) always exists (even when the one in (5.3) does not exist), and

$$\lim_{s \to 0} 2s \operatorname{Per}_s(E, \Omega) = |\partial B_1|\, |E \cap \Omega| = |\partial B_1|\, |\Omega \setminus E|.$$

The boundaries of s-minimal sets are referred to as *nonlocal minimal surfaces*.

In [28] it is proved that s-minimizers satisfy a suitable integral equation (see in particular Theorem 5.1 in [28]), that is the Euler-Lagrange equation corresponding to the s-perimeter functional Per_s. If E is s-minimal in Ω and ∂E is smooth enough, this Euler-Lagrange equation can be written as

$$\int_{\mathbb{R}^n} \frac{\chi_E(x_0 + y) - \chi_{\mathbb{R}^n \setminus E}(x_0 + y)}{|y|^{n+2s}}\, dy = 0, \tag{5.5}$$

for any $x_0 \in \Omega \cap \partial E$.

Therefore, in analogy with the case of the classical minimal surfaces, which have zero mean curvature, one defines the *nonlocal mean curvature* of E at $x_0 \in \partial E$ as

$$H^s_E(x_0) := \int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^{\mathscr{C}}}(y)}{|y - x_0|^{n+2s}}\, dy. \tag{5.6}$$

In this way, Eq. (5.5) can be written as $H^s_E = 0$ along ∂E.

It is also suggestive to think that the function $\tilde{\chi}_E := \chi_E - \chi_{E^{\mathscr{C}}}$ averages out to zero at the points on ∂E, if ∂E is smooth enough, since at these points the local contribution of E compensates the one of $E^{\mathscr{C}}$. Using this notation, one may take the liberty of writing

$$\begin{aligned} H^s_E(x_0) &= \frac{1}{2} \int_{\mathbb{R}^n} \frac{\tilde{\chi}_E(x_0 + y) + \tilde{\chi}_E(x_0 - y)}{|y|^{n+2s}}\, dy \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \frac{\tilde{\chi}_E(x_0 + y) + \tilde{\chi}_E(x_0 - y) - 2\tilde{\chi}_E(x_0)}{|y|^{n+2s}}\, dy \\ &= \frac{-(-\Delta)^s \tilde{\chi}_E(x_0)}{C(n, s)}, \end{aligned}$$

using the notation of (1.1). Using this suggestive representation, the Euler-Lagrange equation in (5.5) becomes

$$(-\Delta)^s \tilde{\chi}_E = 0 \text{ along } \partial E.$$

We refer to [3] for further details on this argument.

It is also worth recalling that the nonlocal perimeter functionals find applications in motions of fronts by nonlocal mean curvature (see e.g. [32, 40, 91]), problems in which aggregating and disaggregating terms compete towards an equilibrium (see e.g. [78] and [56]) and nonlocal free boundary problems (see e.g. [29] and [61]). See also [106] and [139] for results related to this type of problems.

In the classical case of the local perimeter functional, it is known that minimal surfaces are smooth in dimension $n \leq 7$. Moreover, if $n \geq 8$ minimal surfaces are smooth except on a small singular set of Hausdorff dimension $n-8$. Furthermore, minimal surfaces that are graphs are called minimal graphs, and they reduce to hyperplanes if $n \leq 8$ (this is called the Bernstein property, which was also discussed at the beginning of the Chap. 4). If $n \geq 9$, there exist global minimal graphs that are not affine (see e.g. [86]).

Differently from the classical case, the regularity theory for s-minimizers is still quite open. We present here some of the partial results obtained in this direction:

Theorem 5.3 *In the plane, s-minimal sets are smooth. More precisely:*

(a) If E is an s-minimal set in $\Omega \subset \mathbb{R}^2$, then $\partial E \cap \Omega$ is a C^∞-curve.
(b) Let E be s-minimal in $\Omega \subset \mathbb{R}^n$ and let $\Sigma_E \subset \partial E \cap \Omega$ denote its singular set. Then $\mathcal{H}^d(\Sigma_E) = 0$ for any $d > n-3$.

See [124] for the proof of this results (as a matter of fact, in [124] only $C^{1,\alpha}$ regularity is proved, but then [9] proved that s-minimal sets with $C^{1,\alpha}$-boundary are automatically C^∞). Further regularity results of the s-minimal surfaces can be found in [35]. There, a regularity theory when s is near $\frac{1}{2}$ is stated, as we see in the following Theorem:

Theorem 5.4 *There exists $\epsilon_0 \in \left(0, \frac{1}{2}\right)$ such that if $s \geq \frac{1}{2} - \epsilon_0$, then*

(a) if $n \leq 7$, any s-minimal set is of class C^∞,
(b) if $n = 8$ any s-minimal surface is of class C^∞ except, at most, at countably many isolated points,
(c) any s-minimal surface is of class C^∞ outside a closed set Σ of Hausdorff dimension $n-8$.

5.1 Graphs and s-Minimal Surfaces

We will focus the upcoming material on two interesting results related to graphs: a Bernstein type result, namely the property that an s-minimal graph in $\mathbb{R}^{n+1}$ is flat (if no singular cones exist in dimension n); we will then prove that an s-minimal surface whose prescribed data is a subgraph, is itself a subgraph.

The first result is the following theorem:

Theorem 5.1.1 *Let $E = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}$ s.t. $t < u(x)\}$ be an s-minimal graph, and assume there are no singular cones in dimension n (that is, if $\mathcal{K} \subset \mathbb{R}^n$ is an s-minimal cone, then $\mathcal{K}$ is a half-space). Then u is an affine function (thus E is a half-space).*

To be able to prove Theorem 5.1.1, we recall some useful auxiliary results. In the following lemma we state a dimensional reduction result (see Theorem 10.1 in [28]).

Lemma 5.1 *Let $E = F \times \mathbb{R}$. Then if E is s-minimal if and only if F is s-minimal.*

We define then the blow-up and blow-down of the set E are, respectively

$$E_0 := \lim_{r\to 0} E_r \quad \text{and} \quad E_\infty := \lim_{r\to+\infty} E_r, \quad \text{where} \quad E_r = \frac{E}{r}.$$

A first property of the blow-up of E is the following (see Lemma 3.1 in [79]).

Lemma 5.2 *If E_∞ is affine, then so is E.*

We recall also a regularity result for the s-minimal surfaces (see [79] and [9] for details and proof).

Lemma 5.3 *Let E be s-minimal. Then:*

(a) If E is Lipschitz, then E is $C^{1,\alpha}$.
(b) If E is $C^{1,\alpha}$, then E is C^∞.

We give here a sketch of the proof of Theorem 5.1.1 (see [79] for all the details).

Proof (Sketch of the proof of Theorem 5.1.1) If $E \subset \mathbb{R}^{n+1}$ is an s-minimal graph, then the blow-down E_∞ is an s-minimal cone (see Theorem 9.2 in [28] for the proof of this statement). By applying the dimensional reduction argument in Lemma 5.1 we obtain an s-minimal cone in dimension n. According to the assumption that no singular s-minimal cones exist in dimension n, it follows that necessarily E_∞ can be singular only at the origin.

We consider a bump function $w_0 \in C^\infty(\mathbb{R}, [0, 1])$ such that

$$\begin{aligned}
&w_0(t) = 0 \text{ in } \left(-\infty, \frac{1}{4}\right) \cup \left(\frac{3}{4}, +\infty\right)\\
&w_0(t) = 1 \text{ in } \left(\frac{2}{5}, \frac{3}{5}\right)\\
&w(t) = w_0(|t|).
\end{aligned}$$

The blow-down of E is

$$E_\infty = \big\{(x', x_{n+1}) \text{ s.t. } x_{n+1} \le u_\infty(x')\big\}.$$

For a fixed $\sigma \in \partial B_1$, let

$$F_t := \big\{(x', x_{n+1}) \text{ s.t. } x_{n+1} \le u_\infty\big(x' + t\theta w(x')\sigma\big) - t\big\}$$

be a family of sets, where $t \in (0, 1)$ and $\theta > 0$. Then for θ small, we have that

$$F_1 \text{ is below } E_\infty. \tag{5.7}$$

Indeed, suppose by contradiction that this is not true. Then, there exists $\theta_k \to 0$ such that

$$u_\infty\big(x'_k + \theta_k w(x'_k)\sigma\big) - 1 \geq u_\infty(x'_k). \tag{5.8}$$

But $x'_k \in \operatorname{supp} w$, which is compact, therefore $x'_\infty := \lim_{k \to +\infty} x'_k$ belongs to the support of w, and $w(x'_\infty)$ is defined. Then, by sending $k \to +\infty$ in (5.8) we have that

$$u_\infty(x'_\infty) - 1 \geq u_\infty(x'_\infty),$$

which is a contradiction. This establishes (5.7).

Now consider the smallest $t_0 \in (0, 1)$ for which F_t is below E_∞. Since E_∞ is a graph, then F_{t_0} touches E_∞ from below in one point $X_0 = (x'_0, x^0_{n+1})$, where $x'_0 \in \operatorname{supp} w$. Since E_∞ is s-minimal, we have that the nonlocal mean curvature (defined in (5.6)) of the boundary is null. Also, since F_{t_0} is a C^2 diffeomorphism of E_∞ we have that

$$H^s_{F_{t_0}}(p) \simeq \theta t_0, \tag{5.9}$$

and there is a region where E_∞ and F_{t_0} are well separated by t_0, thus

$$\Big|\big(E_\infty \setminus F_{t_0}\big) \cap \big(B_3 \setminus B_2\big)\Big| \geq c t_0,$$

for some $c > 0$. Therefore, we see that

$$H^s_{F_{t_0}}(p) = H^s_{F_{t_0}}(p) - H^s_E(p) \geq c t_0.$$

This and (5.9) give that $\theta t_0 \geq c t_0$, for some $c > 0$ (up to renaming it). If θ is small enough, this implies that $t_0 = 0$.

In particular, we have proved that there exists $\theta > 0$ small enough such that, for any $t \in (0, 1)$ and any $\sigma \in \partial B_1$, we have that

$$u_\infty\big(x' + t\theta w(x')\sigma\big) - t \leq u_\infty(x').$$

This implies that

$$\frac{u_\infty\big(x' + t\theta w(x')\sigma\big) - u_\infty(x')}{t\theta} \leq \frac{1}{\theta},$$

hence, letting $t \to 0$, we have that

$$\nabla u_\infty(x')w(x')\sigma \leq \frac{1}{\theta}, \text{ for any } x \in \mathbb{R}^n \setminus \{0\}, \text{ and } \sigma \in B_1.$$

We recall now that $w = 1$ in $B_{3/5} \setminus B_{2/5}$ and σ is arbitrary in ∂B_1. Hence, it follows that

$$|\nabla u_\infty(x)| \leq \frac{1}{\theta}, \text{ for any } x \in B_{3/5} \setminus B_{2/5}.$$

Therefore u_∞ is globally Lipschitz. By the regularity statement in Lemma 5.3, we have that u_∞ is C^∞. This says that u is smooth also at the origin, hence (being a cone) it follows that E_∞ is necessarily a half-space. Then by Lemma 5.2, we conclude that E is a half-space as well.

We introduce in the following theorem another interesting property related to s-minimal surfaces, in the case in which the fixed given data outside a domain is a subgraph. In that case, the s-minimal surface itself is a subgraph. Indeed:

Theorem 5.1.2 *Let Ω_0 be an open and bounded subset of $\mathbb{R}^{n-1}$ with boundary of class $C^{1,1}$ and let $\Omega := \Omega_0 \times \mathbb{R}$. Let E be an s-minimal set in Ω. Assume that*

$$E \setminus \Omega = \{x_n < u(x'),\, x' \in \mathbb{R}^{n-1} \setminus \Omega_0\} \tag{5.10}$$

for some continuous function $u: \mathbb{R}^{n-1} \to \mathbb{R}$. Then

$$E \cap \Omega = \{x_n < v(x'),\, x' \in \Omega_0\}$$

for some function $v: \mathbb{R}^{n-1} \to \mathbb{R}$.

The reader can see [64], where this theorem and the related results are proved; here, we only state the preliminary results needed for our purposes and focus on the proof of Theorem 5.1.2. The proof relies on a sliding method, more precisely, we take a translation of E in the nth direction, and move it until it touches E from above. If the set $E \cap \Omega$ is a subgraph, then, up to a set of measure 0, the contact between the translated E and E, will be E itself.

However, since we have no information on the regularity of the minimal surface, we need at first to “regularize” the set by introducing the notions of supconvolution and subconvolution. With the aid of a useful result related to the sub/supconvolution of an s-minimal surface, we proceed then with the proof of the Theorem 5.1.2.

The supconvolution of a set $E \subseteq \mathbb{R}^n$ (Fig. 5.2) is given by

$$E^\sharp_\delta := \bigcup_{x \in E} \overline{B_\delta(x)}.$$

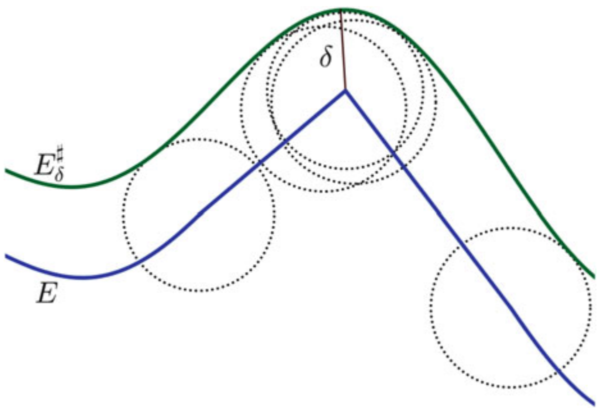


Fig. 5.2 The supconvolution of a set

In an equivalent way, the supconvolution can be written as

$$E_\delta^\sharp = \bigcup_{\substack{v\in\mathbb{R}^n\\ |v|\le\delta}} (E+v).$$

Indeed, we consider $\delta > 0$ and an arbitrary $x \in E$. Let $y \in \overline{B_\delta(x)}$ and we define $v := y - x$. Then

$$|v| \le |y-x| \le \delta \quad \text{and} \quad y = x + v \in E + v.$$

Therefore $\overline{B_\delta(x)} \subseteq E + v$ for $|v| \le \delta$. In order to prove the inclusion in the opposite direction, one notices that taking $y \in E + v$ with $|v| \le \delta$ and defining $x := y - v$, it follows that

$$|x-y| = |v| \le \delta.$$

Moreover, $x \in (E+v) - v = E$ and the inclusion $E + v \in \overline{B_\delta(x)}$ is proved.

On the other hand, the subconvolution is defined as

$$E_\delta^\flat := \mathbb{R}^n \setminus \left((\mathbb{R}^n \setminus E)_\delta^\sharp \right).$$

Now, the idea is that the supconvolution of E is a regularized version of E whose nonlocal minimal curvature is smaller than the one of E, i.e.:

$$\int_{\mathbb{R}^n} \frac{\chi_{\mathbb{R}^n\setminus E_\delta^\sharp}(y) - \chi_{E_\delta^\sharp}(y)}{|x-y|^{n+2s}}\, dy \le \int_{\mathbb{R}^n} \frac{\chi_{\mathbb{R}^n\setminus E}(y) - \chi_E(y)}{|\tilde{x}-y|^{n+2s}}\, dy \le 0, \tag{5.11}$$

for any $x \in \partial E^\sharp_\delta$, where $\tilde{x} := x - v \in \partial E$ for some $v \in \mathbb{R}^n$ with $|v| = \delta$. Then, by construction, the set $E + v$ lies in $E^\sharp_\delta$, and this implies (5.11).

Similarly, one has that the opposite inequality holds for the subconvolution of E, for any $x \in \partial E^\flat_\delta$

$$\int_{\mathbb{R}^n} \frac{\chi_{\mathbb{R}^n \setminus E^\flat_\delta}(y) - \chi_{E^\flat_\delta}(y)}{|x-y|^{n+2s}}\, dy \geq 0, \tag{5.12}$$

By (5.11) and (5.12), we obtain:

Proposition 5.1.3 *Let E be an s-minimal set in Ω. Let $p \in \partial E^\sharp_\delta$ and assume that $\overline{B_\delta(p)} \subseteq \Omega$. Assume also that $E^\sharp_\delta$ is touched from above by a translation of $E^\flat_\delta$, namely there exists $\omega \in \mathbb{R}^n$ such that*

$$E^\sharp_\delta \subseteq E^\flat_\delta + \omega$$

and

$$p \in (\partial E^\sharp_\delta) \cap (\partial E^\flat_\delta + \omega).$$

Then

$$E^\sharp_\delta = E^\flat_\delta + \omega.$$

Proof (Proof of Theorem 5.1.2) One first remark is that the s-minimal set does not have spikes which go to infinity: more precisely, one shows that

$$\Omega_0 \times (-\infty, -M) \subseteq E \cap \Omega \subseteq \Omega_0 \times (-\infty, M) \tag{5.13}$$

for some $M \geq 0$. The proof of (5.13) can be performed by sliding horizontally a large ball, see [64] for details.

After proving (5.13), one can deal with the core of the proof of Theorem 5.1.2. The idea is to slide E from above until it touches itself and analyze what happens at the contact points. For simplicity, we will assume here that the function u is uniformly continuous (if u is only continuous, the proof needs to be slightly modified since the subconvolution and supconvolution that we will perform may create new touching points at infinity). At this purpose, we consider $E_t = E + te_n$ for $t \geq 0$. Notice that, by (5.13), if $t \geq 2M$, then $E \subseteq E_t$. Let then t be the smallest for which the inclusion $E \subseteq E_t$ holds. We claim that $t = 0$. If this happens, one may consider

$$v = \inf\{\tau \text{ s.t. } (x,\tau) \in E^{\mathscr{C}}\}$$

and, up to sets of measure 0, $E \cap \Omega_0$ is the subgraph of v.

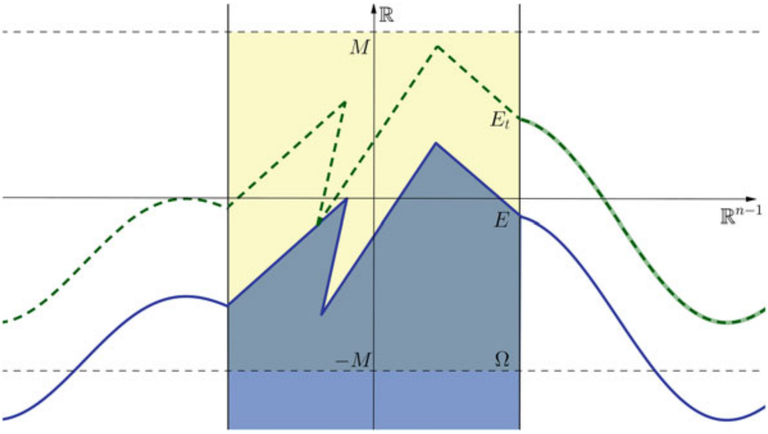


Fig. 5.3 Sliding E until it touches itself at an interior point

The proof is by contradiction, so let us assume that $t > 0$. According to (5.10), the set $E \setminus \Omega$ is a subgraph, hence the contact points between ∂E and ∂E_t must lie in $\overline{\Omega}_0 \times \mathbb{R}$. Namely, only two possibilities may occur: the contact point is interior (it belongs to $\Omega_0 \times \mathbb{R}$), or it is at the boundary (on $\partial\Omega_0 \times \mathbb{R}$). So, calling p the contact point, one may have[1] that

$$\text{either } p \in \Omega_0 \times \mathbb{R} \quad \text{or} \tag{5.14}$$

$$p \in \partial\Omega_0 \times \mathbb{R}. \tag{5.15}$$

We deal with the first case in (5.14) (an example of this behavior is depicted in Fig. 5.3). We consider $E_\delta^\sharp$ and $E_\delta^\flat$ to be the supconvolution, respectively the subconvolution of E. We then slide the subconvolution until it touches the supconvolution. More precisely, let $\tau > 0$ and we take a translation of the subconvolution, $E_\delta^\flat + \tau e_n$. For τ large, we have that $E_\delta^\sharp \subseteq E_\delta^\flat + \tau e_n$ and we consider τ_δ to be the smallest for

[1] As a matter of fact, the number of contact points may be higher than one, and even infinitely many contact points may arise. So, to be rigorous, one should distinguish the case in which all the contact points are interior and the case in which at least one contact point lies on the boundary.

Moreover, since the surface may have vertical portions along the boundary of the domain, one needs to carefully define the notion of contact points (roughly speaking, one needs to take a definition for which the vertical portions which do not prevent the sliding are not in the contact set).

Finally, in case the contact points are all interior, it is also useful to perform the sliding method in a slightly reduced domain, in order to avoid that the supconvolution method produces new contact points at the boundary (which may arise from vertical portions of the surfaces).

Since we do not aim to give a complete proof of Theorem 5.1.2 here, but just to give the main ideas and underline the additional difficulty, we refer to [64] for the full details of these arguments.

which such inclusion holds. We have (since t is positive by assumption) that

$$\tau_\delta \geq \frac{t}{2} > 0.$$

Moreover, for δ small, the sets $\partial E_\delta^\sharp$ and $\partial(E_\delta^\flat + \tau_\delta e_n)$ have a contact point which, according to (5.14), lies in $\Omega_0 \times \mathbb{R}$. Let p_δ be such a point, so we may write

$$p_\delta \in (\partial E_\delta^\sharp) \cap \partial(E_\delta^\flat + \tau_\delta e_n) \quad \text{and} \quad p_\delta \in \Omega_0 \times \mathbb{R}.$$

Then, for δ small (notice that $\overline{B_\delta(p)} \subseteq \Omega$), Proposition 5.1.3 yields that

$$E_\delta^\sharp = E_\delta^\flat + \tau_\delta e_n.$$

Considering δ arbitrarily small, one obtains that

$$E = E + \tau_0 e_n, \quad \text{with} \quad \tau_0 > 0.$$

But E is a subgraph outside of Ω, and this provides a contradiction. Hence, the claim that $t = 0$ is proved.

Let us see that we also obtain a contradiction when supposing that $t > 0$ and that the second case (5.15) holds. Let

$$p = (p', p_n) \quad \text{and} \quad p \in (\partial E) \cap (\partial E_t).$$

Now, if one takes sequences $a_k \in \partial E$ and $b_k \in \partial E_t$, both that tend to p as k goes to infinity, since $E \setminus \Omega$ is a subgraph and $t > 0$, necessarily a_k, b_k belong to Ω. Hence

$$p \in \overline{(\partial E) \cap \Omega} \cap \overline{(\partial E_t) \cap \Omega}. \tag{5.16}$$

Thanks to Definition 2.3 in [28], one obtains that E is a variational subsolution in a neighborhood of p. In other words, if $A \subseteq E \cap \Omega$ and $p \in \overline{A}$, then

$$0 \geq \mathrm{Per}_s(E, \Omega) - \mathrm{Per}_s(E \setminus A, \Omega) = I(A, E^{\mathscr{C}}) - I(A, E \setminus A)$$

(we recall the definition of I in (5.1) and of the fractional perimeter Per_s in (5.2)). According to Theorem 5.1 in [28], this implies in a viscosity sense (i.e. if E is touched at p from outside by a ball), that

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{\mathbb{R}^n \setminus E}(y)}{|p-y|^{n+2s}}\, dy \geq 0. \tag{5.17}$$

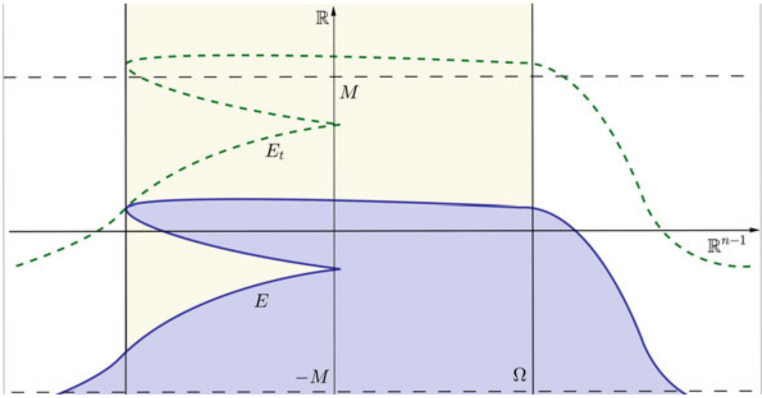


Fig. 5.4 Sliding E until it touches itself at a boundary point

In order to obtain an estimate on the fractional mean curvature in the strong sense, we consider the translation of the point p as follows:

$$p_t = p - te_n = (p', p_n - t) = (p', p_{n,t}).$$

Since $t > 0$, one may have that either $p_n \neq u(p')$, or $p_{n,t} \neq u(p')$.

These two possibilities can be dealt with in a similar way, so we just continue with the proof in the case $p_n \neq u(p')$ (as is also exemplified in Fig. 5.4). Taking $r > 0$ small, the set $B_r(p) \setminus \Omega$ is contained entirely in E or in its complement. Moreover, one has from [27] that $\partial E \cap B_r(p)$ is a $C^{1,\frac{1}{2}+s}$-graph in the direction of the normal to Ω at p. That is: in Fig. 5.4 the set E is $C^{1,\frac{1}{2}+s}$, hence in the vicinity of $p = (p', p_n)$, it appears to be sufficiently smooth.

So, let $\nu(p) = (\nu'(p), \nu_n(p))$ be the normal in the interior direction, then up to a rotation and since Ω is a cylinder (hence $\nu_n(p) = 0$), we can write $\nu(p) = e_1$. Therefore, there exists a function Ψ of class $C^{1,\frac{1}{2}+s}$ such that $p_1 = \Psi(p_2, \ldots, p_n)$ and, in the vicinity of p, we can write ∂E as the graph $G = \{x_1 = \Psi(x_2, \ldots, x_n)\}$.

Given (5.16), we deduce that there exists a sequence $p_k \in G$ such that $p_k \in \Omega$ and $p_k \to p$ as $k \to \infty$. From this it follows that there exists a sequence of points $p_k \to p$ such that

$$\partial E \text{ in the vicinity of } p_k \text{ is a graph of class } C^{1,\frac{1}{2}+s} \tag{5.18}$$

and

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^{\mathscr{C}}}(y)}{|p_k - y|^{n+2s}}\, dy = 0. \tag{5.19}$$

From (5.18) and (5.19), and using a pointwise version of the Euler-Lagrange equation (see [64] for details), we have that

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^{\mathscr{C}}}(y)}{|p-y|^{n+2s}}\, dy = 0.$$

Now, $E \subset E_t$ for t strictly positive, hence

$$\int_{\mathbb{R}^n} \frac{\chi_{E_t}(y) - \chi_{E_t^{\mathscr{C}}}(y)}{|p-y|^{n+2s}}\, dy > 0. \tag{5.20}$$

Moreover, we have that the set $\partial E_t \cap B_{\frac{r}{4}}(p)$ must remain on one side of the graph G, namely one could have that

$$E_t \cap B_{\frac{r}{4}}(p) \subseteq \{x_1 \leq \Psi(x_2, \dots, x_n)\} \text{ or}$$
$$E_t \cap B_{\frac{r}{4}}(p) \supseteq \{x_1 \geq \Psi(x_2, \dots, x_n)\}.$$

Given again (5.16), we deduce that there exists a sequence $\tilde{p}_k \in \partial E_t \cap \Omega$ such that $\tilde{p}_k \to p$ as $k \to \infty$ and $\partial E_t \cap \Omega$ in the vicinity of $\tilde{p}_k$ is touched by a surface lying in E_t, of class $C^{1,\frac{1}{2}+s}$. Then

$$\int_{\mathbb{R}^n} \frac{\chi_{E_t}(y) - \chi_{E_t^{\mathscr{C}}}(y)}{|\tilde{p}_k-y|^{n+2s}}\, dy \leq 0.$$

Hence, making use of a pointwise version of the Euler-Lagrange equation (see [64] for details), we obtain that

$$\int_{\mathbb{R}^n} \frac{\chi_{E_t}(y) - \chi_{E_t^{\mathscr{C}}}(y)}{|p-y|^{n+2s}}\, dy \leq 0.$$

But this is a contradiction with (5.20), and this concludes the proof of Theorem 5.1.2.

On the one hand, one may think that Theorem 5.1.2 has to be well-expected. On the other hand, it is far from being obvious, not only because the proof is not trivial, but also because the statement itself almost risks to be false, especially at the boundary. Indeed we will see in Theorem 5.3.2 that the graph property is close to fail at the boundary of the domain, where the s-minimal surfaces may present vertical tangencies and stickiness phenomena (see Fig. 5.11).

5.2 Non-existence of Singular Cones in Dimension 2

We now prove the non-existence of singular s-minimal cones in dimension 2, as stated in the next result (from this, the more general statement in Theorem 5.3 follows after a blow-up procedure):

Theorem 5.2.1 *If E is an s-minimal cone in $\mathbb{R}^2$, then E is a half-plane.*

We remark that, as a combination of Theorems 5.1.1 and 5.2.1, we obtain the following result of Bernstein type:

Corollary 5.2.2 *Let $E = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \text{ s.t. } t < u(x)\}$ be an s-minimal graph, and assume that $n \in \{1,2\}$. Then u is an affine function.*

Let us first consider a simple example, given by the cone in the plane

$$\mathcal{K} := \left\{(x,y) \in \mathbb{R}^2 \text{ s.t. } y^2 > x^2\right\},$$

see Fig. 5.5.

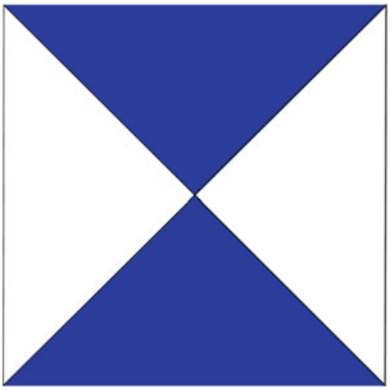
Proposition 5.2.3 *The cone $\mathcal{K}$ depicted in Fig. 5.5 is not s-minimal in $\mathbb{R}^2$.*

Notice that, by symmetry, one can prove that $\mathcal{K}$ satisfies (5.5) (possibly in the viscosity sense). On the other hand, Proposition 5.2.3 gives that $\mathcal{K}$ is not s-minimal. This, in particular, provides an example of a set that satisfies the Euler-Lagrange equation in (5.5), but is not s-minimal (i.e., the Euler-Lagrange equation in (5.5) is implied by, but not necessarily equivalent to, the s-minimality property).

Proof (Proof of Proposition 5.2.3) The proof of the non-minimality of $\mathcal{K}$ is due to an original idea by Luis Caffarelli.

Suppose by contradiction that the cone $\mathcal{K}$ is minimal in $\mathbb{R}^2$. We add to $\mathcal{K}$ a small square adjacent to the origin (see Fig. 5.6), and call $\mathcal{K}'$ the set obtained. Then $\mathcal{K}$ and $\mathcal{K}'$ have the same s-perimeter. This is due to the interactions considered in

Fig. 5.5 The cone $\mathcal{K}$



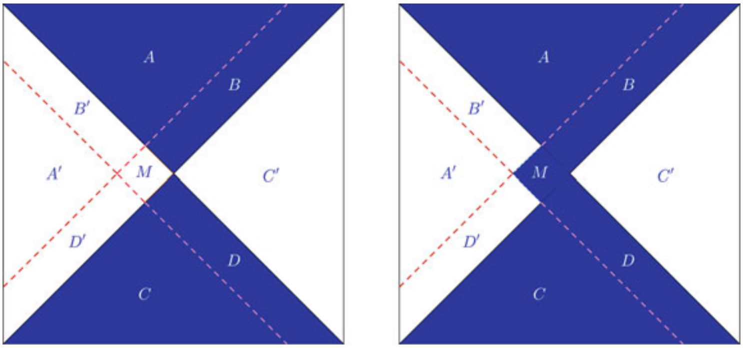


Fig. 5.6 Interaction of M with $A, B, C, D, A', B', C', D'$

the s-perimeter functional and the unboundedness of the regions. We remark that in Fig. 5.6 we represent bounded regions, of course, sets A, B, C, D, A', B', C' and D' are actually unbounded.

Indeed, we notice that in the first image, the white square M interacts with the dark regions A, B, C, D, while in the second the now dark square M interacts with the regions A', B', C', D', and all the other interactions are unmodified. Therefore, the difference between the s-perimeter of $\mathscr{K}$ and that of $\mathscr{K}'$ consists only of the interactions $I(A, M)+I(B, M)+I(C, M)+I(D, M)-I(A', M)-I(B', M)-I(C', M)-I(D', M)$. But $A \cup B = A' \cup B'$ and $C \cup D = C' \cup D'$ (since these sets are all unbounded), therefore the difference is null, and the s-perimeter of $\mathscr{K}$ is equal to that of $\mathscr{K}'$. Consequently, $\mathscr{K}'$ is also s-minimal, and therefore it satisfies the Euler-Lagrange equation in (5.5) at the origin. But this leads to a contradiction, since the dark region now contributes more than the white one, namely

$$\int_{\mathbb{R}^2} \frac{\chi_{\mathscr{K}'}(y) - \chi_{\mathbb{R}^2 \setminus \mathscr{K}'}(y)}{|y|^{2+s}}\, dy > 0.$$

Thus $\mathscr{K}$ cannot be s-minimal, and this concludes our proof.

This geometric argument cannot be extended to a more general case (even, for instance, to a cone in $\mathbb{R}^2$ made of many sectors, see Fig. 5.7). As a matter of fact, the proof of Theorem 5.2.1 will be completely different than the one of Proposition 5.2.3 and it will rely on an appropriate domain perturbation argument.

The proof of Theorem 5.2.1 that we present here is actually different than the original one in [124]. Indeed, in [124], the result was proved by using the harmonic extension for the fractional Laplacian. Here, the extension will not be used; furthermore, the proof follows the steps of Theorem 4.2.1 and we will recall here just the main ingredients.

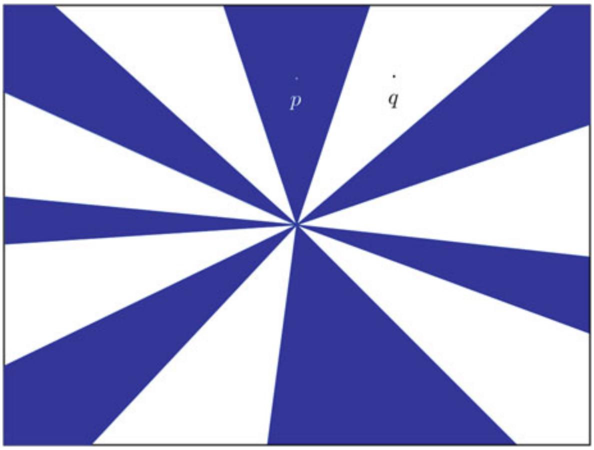


Fig. 5.7 Cone in $\mathbb{R}^2$

Proof (Proof of Theorem 5.2.1)

The idea of the proof is the following: if $E \subset \mathbb{R}^2$ is an s-minimal cone, then let $\tilde{E}$ be a perturbation of the set E which coincides with a translation of E in $B_{R/2}$ and with E itself outside B_R. Then the difference between the energies of $\tilde{E}$ and E tends to 0 as $R \to +\infty$. This implies that also the energy of $E \cap \tilde{E}$ is arbitrarily close to the energy of E. On the other hand if E is not a half-plane, the set $\tilde{E} \cap E$ can be modified locally to decrease its energy by a fixed small amount and we reach a contradiction.

The details of the proof go as follows. Let

$$u := \chi_E - \chi_{\mathbb{R}^2 \setminus E}.$$

From definition (4.35) we have that

$$u(B_R, B_R) = 2I(E \cap B_R, B_R \setminus E)$$

and

$$u(B_R, B_R^{\mathscr{C}}) = I(B_R \cap E, E^{\mathscr{C}} \setminus B_R) + I(B_R \setminus E, E \setminus B_R),$$

thus

$$\mathrm{Per}_s(E, B_R) = \mathscr{K}_R(u), \tag{5.21}$$

where $\mathscr{K}_R(u)$ is given in (4.33) and $\mathrm{Per}_s(E, B_R)$ is the s-perimeter functional defined in (5.2). Then E is s-minimal if u is a minimizer of the energy $\mathscr{K}_R$ in any ball B_R, with $R > 0$.

Now, we argue by contradiction, and suppose that E is an s-minimal cone different from the half-space. Up to rotations, we may suppose that a sector of E

has an angle smaller than π and is bisected by e_2. Thus there exists $M \geq 1$ and $p \in E \cap B_M$ on the e_2-axis such that $p \pm e_1 \in \mathbb{R}^2 \setminus E$ (see Fig. 5.7).

We take $\varphi \in C_0^\infty(B_1)$, such that $\varphi(x) = 1$ in $B_{1/2}$. For R large (say $R > 8M$), we define

$$\Psi_{R,+}(y) := y + \varphi\left(\frac{y}{R}\right) e_1.$$

We point out that, for R large, $\Psi_{R,+}$ is a diffeomorphism on $\mathbb{R}^2$.

Furthermore, we define $u_R^+(x) := u(\Psi_{R,+}^{-1}(x))$. Then

$$\begin{aligned} & u_R^+(y) = u(y - e_1) && \text{for } p \in B_{2M} \\ \text{and} \quad & u_R^+(y) = u(y) && \text{for } p \in \mathbb{R}^2 \setminus B_R. \end{aligned}$$

We recall the estimate obtained in (4.37), that, combined with the minimality of u, gives

$$\mathscr{K}_R(u_R^+) - \mathscr{K}_R(u) \leq \frac{C}{R^2} \mathscr{K}_R(u).$$

But u is a minimizer in any ball, and by the energy estimate in Theorem 4.1.2 we have that

$$\mathscr{K}_R(u_R^+) - \mathscr{K}_R(u) \leq C R^{-2s}.$$

This implies that

$$\lim_{R \to +\infty} \mathscr{K}_R(u_R^+) - \mathscr{K}_R(u) = 0. \tag{5.22}$$

Let now

$$v_R(x) := \min\{u(x), u_R^+(x)\} \qquad \text{and} \qquad w_R(x) := \max\{u(x), u_R^+(x)\}.$$

We claim that v_R is not identically u nor u_R^+. Indeed

$$\begin{aligned} & u_R^+(p) = u(p - e_1) = (\chi_E - \chi_{\mathbb{R}^2 \setminus E})(p - e_1) = -1 \quad \text{and} \\ & u(p) = (\chi_E - \chi_{\mathbb{R}^2 \setminus E})(p) = 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} & u_R^+(p + e_1) = u(p) = 1 \quad \text{and} \\ & u(p + e_1) = (\chi_E - \chi_{\mathbb{R}^2 \setminus E})(p + e_1) = -1. \end{aligned}$$

By the continuity of u and u_R^+, we obtain that

$$v_R = u_R^+ < u \text{ in a neighborhood of } p \tag{5.23}$$

and

$$v_R = u < u_R^+ \text{ in a neighborhood of } p + e_1. \tag{5.24}$$

Now, by the minimality property of u,

$$\mathscr{K}_R(u) \leq \mathscr{K}_R(v_R).$$

Moreover (see e.g. formula (38) in [114]),

$$\mathscr{K}_R(v_R) + \mathscr{K}_R(w_R) \leq \mathscr{K}_R(u) + \mathscr{K}_R(u_R^+).$$

The latter two formulas give that

$$\mathscr{K}_R(v_R) \leq \mathscr{K}_R(u_R^+). \tag{5.25}$$

We claim that

$$v_R \text{ is not minimal for } \mathscr{K}_{2M} \tag{5.26}$$

with respect to compact perturbations in B_{2M}. Indeed, assume by contradiction that v_R is minimal, then in B_{2M} both v_R and u would satisfy the same equation. Recalling (5.24) and applying the Strong Maximum Principle, it follows that $u = v_R$ in B_{2M}, which contradicts (5.23). This establishes (5.26).

Now, we consider a minimizer u_R^* of $\mathscr{K}_{2M}$ among the competitors that agree with v_R outside B_{2M}. Therefore, we can define

$$\delta_R := \mathscr{K}_{2M}(v_R) - \mathscr{K}_{2M}(u_R^*).$$

In light of (5.26), we have that $\delta_R > 0$.

The reader can now compare Step 3 in the proof of Theorem 4.2.1. There we proved that

$$\delta_R \text{ remains bounded away from zero as } R \to +\infty. \tag{5.27}$$

Furthermore, since u_R^* and v_R agree outside B_{2M} we obtain that

$$\mathscr{K}_R(u_R^*) + \delta_R = \mathscr{K}_R(v_R).$$

Using this, (5.25) and the minimality of u, we obtain that

$$\delta_R = \mathscr{K}_R(v_R) - \mathscr{K}_R(u_R^*) \le \mathscr{K}_R(u_R^+) - \mathscr{K}_R(u).$$

Now we send R to infinity, recall (5.22) and (5.27), and we reach a contradiction. Thus, E is a half-space, and this concludes the proof of Theorem 5.2.1.

As already mentioned, the regularity theory for s-minimal sets is still widely open. Little is known beyond Theorems 5.3 and 5.4, so it would be very interesting to further investigate the regularity of s-minimal surfaces in higher dimension and for small s.

It is also interesting to recall that if the s-minimal surface E is a subgraph of some function $u : \mathbb{R}^{n-1} \to \mathbb{R}$ (at least in the vicinity of some point $x_0 = (x_0', u(x_0')) \in \partial E$) then the Euler-Lagrange (5.5) can be written directly in terms of u. For instance (see formulas (49) and (50) in [9]), under appropriate smoothness assumptions on u, formula (5.5) reduces to

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \frac{\chi_{\mathbb{R}^n \setminus E}(x_0 + y) - \chi_E(x_0 + y)}{|y|^{n+2s}} \, dy \\ &= \int_{\mathbb{R}^{n-1}} F\left(\frac{u(x_0' + y') - u(x_0')}{|y'|} \right) \frac{\zeta(y')}{|y'|^{n-1+2s}} \, dy' + \Psi(x_0'), \end{aligned}$$

for suitable F and Ψ, and a cut-off function ζ supported in a neighborhood of x_0'.

Regarding the regularity problems of the s-minimal surfaces, let us mention the recent papers [47] and [48]. Among other very interesting results, it is proved there that suitable singular cones of symmetric type are unstable up to dimension 6 but become stable in dimension 7 for small s (these cones can be seen as the nonlocal analogue of the Lawson cones in the classical minimal surface theory, and the stability property is in principle weaker than minimality, since it deals with the positivity of the second order derivative of the functional).

This phenomenon may suggest the conjecture that the s-minimal surfaces may develop singularities in dimension 7 and higher when s is sufficiently small.

In [48], interesting examples of surfaces with vanishing nonlocal mean curvature are provided for s sufficiently close to $1/2$. Remarkably, the surfaces in [48] are the nonlocal analogues of the catenoids, but, differently from the classical case (in which catenoids grow logarithmically), they approach a singular cone at infinity, see Fig. 5.8.

Also, these nonlocal catenoids are highly unstable from the variational point of view, since they possess infinite Morse index (differently from the standard catenoid, which has Morse index equal to one, i.e. it is, roughly speaking, a minimizer in any functional direction with the exception of one).

Moreover, in [48], there are also examples of surfaces with vanishing nonlocal mean curvature that can be seen as the nonlocal analogues of two parallel hyperplanes. Namely, for s sufficiently close to $1/2$, there exists a surface of revolution made of two sheets which are the graph of a radial function $f = \pm f(r)$. When r

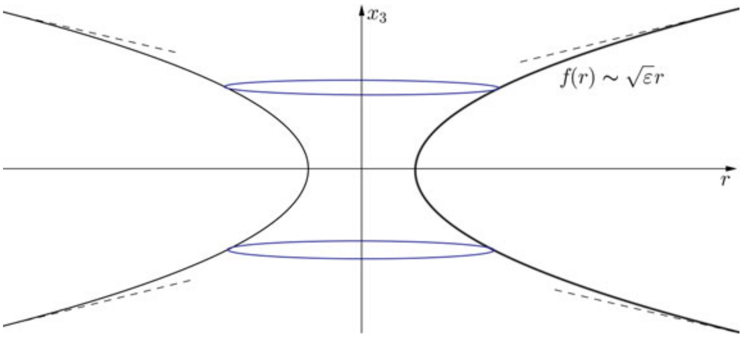


Fig. 5.8 A nonlocal catenoid

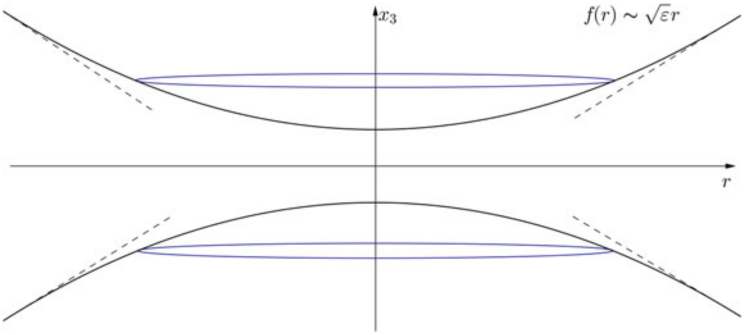


Fig. 5.9 A two-sheet surface with vanishing fractional mean curvature

is small, f is of the order of $1 + (\frac{1}{2} - s)r^2$, but for large r it becomes of the order of $\sqrt{\frac{1}{2} - s} \cdot r$. That is, the two sheets "repel each other" and produce a linear growth at infinity. When s approaches $1/2$ the two sheets are locally closer and closer to two parallel hyperplanes, see Fig. 5.9.

The construction above may be extended to build families of surfaces with vanishing nonlocal mean curvature that can be seen as the nonlocal analogue of k parallel hyperplanes, for any $k \in \mathbb{N}$. These k-sheet surfaces can be seen as the bifurcation, as s is close to $1/2$, of the parallel hyperplanes $\{x_n = a_i\}$, for $i \in \{1, \ldots, k\}$, where the parameters a_i satisfy the constraints

$$a_1 > \cdots > a_k, \qquad \sum_{i=1}^{k} a_i = 0 \tag{5.28}$$

and the balancing relation

$$a_i = 2 \sum_{\substack{1 \le j \le n \\ j \ne i}} \frac{(-1)^{i+j+1}}{a_i - a_j}. \tag{5.29}$$

It is actually quite interesting to observe that solutions of (5.29) correspond to (nondegenerate) critical points of the functional

$$E(a_1, \dots, a_k) := \frac{1}{2} \sum_{i=1}^{k} a_i^2 + \sum_{\substack{1 \le j \le n \\ j \ne i}} (-1)^{i+j} \log |a_i - a_j|$$

among all the k-ples $(a_1, \dots, a_k)$ that satisfy (5.28).

These bifurcation techniques rely on a careful expansion of the fractional perimeter functional with respect to normal perturbations. That is, if E is a (smooth) set with vanishing fractional mean curvature, and h is a smooth and compactly supported perturbation, one can define, for any $t \in \mathbb{R}$,

$$E_h(t) := \{x + th(x)\nu(x),\ x \in \partial E\},$$

where $\nu(x)$ is the exterior normal of E at x. Then, the second variation of the perimeter of $E_h(t)$ at $t = 0$ is (up to normalization constants)

$$\begin{aligned} &\int_{\partial E} \frac{h(y) - h(x)}{|x - y|^{n+2s}}\, d\mathscr{H}^{n-1}(y) + h(x) \int_{\partial E} \frac{\big(\nu(x) - \nu(y)\big) \cdot \nu(x)}{|x - y|^{n+2s}}\, d\mathscr{H}^{n-1}(y) \\ = &\int_{\partial E} \frac{h(y) - h(x)}{|x - y|^{n+2s}}\, d\mathscr{H}^{n-1}(y) + h(x) \int_{\partial E} \frac{1 - \nu(x) \cdot \nu(y)}{|x - y|^{n+2s}}\, d\mathscr{H}^{n-1}(y). \end{aligned}$$

Notice that the latter integral is non-negative, since $\nu(x) \cdot \nu(y) \le 1$. The quantity above, in dependence of the perturbation h, is called, in jargon, "Jacobi operator". It encodes an important geometric information, and indeed, as $s \to 1/2$, it approaches the classical operator

$$\Delta_{\partial E} h + |A_{\partial E}|^2\, h,$$

where $\Delta_{\partial E}$ is the Laplace-Beltrami operator along the hypersurface ∂E and $|A_{\partial E}|^2$ is the sum of the squares of the principal curvatures.

Other interesting sets that possess constant nonlocal mean curvature with the structure of onduloids have been recently constructed in [49] and [24]. This type of sets are periodic in a given direction and their construction has perturbative nature (indeed, the sets are close to a slab in the plane).

It is interesting to remark that the planar objects constructed in [24] have no counterpart in the local framework, since hypersurfaces of constant classical mean curvature with an onduloidal structure only exist in $\mathbb{R}^n$ with $n \geq 3$: once again, this is a typical nonlocal effect, in which the nonlocal mean curvature at a point is influenced by the global shape of the set.

While unbounded sets with constant nonlocal mean curvature and interesting geometric features have been constructed in [24, 48], the case of smooth and bounded sets is always geometrically trivial. As a matter of fact, it has been recently proved independently in [24] and [43] that bounded sets with smooth boundary and constant mean curvature are necessarily balls (this is the analogue of a celebrated result by Alexandrov for surfaces of constant classical mean curvature).

5.3 Boundary Regularity

The boundary regularity of the nonlocal minimal surfaces is also a very interesting, and surprising, topic. Indeed, differently from the classical case, nonlocal minimal surfaces do not always attain boundary data in a continuous way (not even in low dimension). A possible boundary behavior is, on the contrary, a combination of stickiness to the boundary and smooth separation from the adjacent portions. Namely, the nonlocal minimal surfaces may have a portion that sticks at the boundary and that separates from it in a $C^{1,\frac{1}{2}+s}$-way. As an example, we can consider, for any $\delta > 0$, the spherical cap

$$K_\delta := \big(B_{1+\delta} \setminus B_1\big) \cap \{x_n < 0\},$$

and obtain the following stickiness result:

Theorem 5.3.1 *There exists $\delta_0 > 0$, depending on n and s, such that for any $\delta \in (0, \delta_0]$, we have that the s-minimal set in B_1 that coincides with K_δ outside B_1 is K_δ itself.*

That is, the s-minimal set with datum K_δ outside B_1 is empty inside B_1.

The stickiness property of Theorem 5.3.1 is depicted in Fig. 5.10.

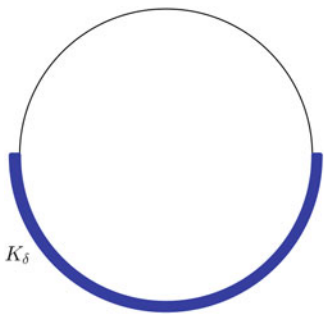


Fig. 5.10 Stickiness properties of Theorem 5.3.1

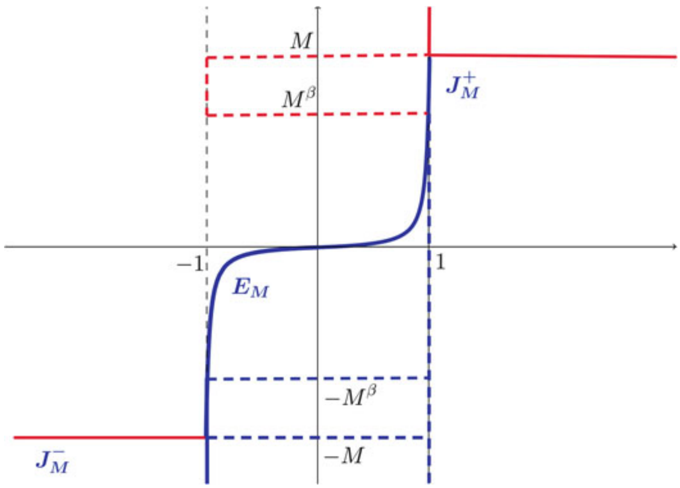


Fig. 5.11 Stickiness properties of Theorem 5.3.2

Other stickiness examples occur at the sides of slabs in the plane. For instance, given $M > 1$, one can consider the s-minimal set E_M in $(-1, 1) \times \mathbb{R}$ with datum outside $(-1, 1) \times \mathbb{R}$ given by the "jump" set $J_M := J_M^- \cup J_M^+$, where

$$J_M^- := (-\infty, -1] \times (-\infty, -M)$$
$$\text{and} \quad J_M^+ := [1, +\infty) \times (-\infty, M).$$

Then, if M is large enough, the minimal set E_M sticks at the boundary of the slab:

Theorem 5.3.2 *There exist $M_o > 0$, $C_o > 0$, depending on s, such that if $M \geq M_o$ then*

$$[-1, 1) \times [C_o M^{\frac{1+2s}{2+2s}}, M] \subseteq E_M^c \tag{5.30}$$

$$\textit{and} \quad (-1, 1] \times [-M, -C_o M^{\frac{1+2s}{2+2s}}] \subseteq E_M. \tag{5.31}$$

The situation of Theorem 5.3.2 is described in Fig. 5.11. We mention that the "strange" exponent $\frac{1+2s}{2+2s}$ in (5.30) and (5.31) is optimal.

For the detailed proof of Theorems 5.3.1 and 5.3.2, and other results on the boundary behavior of nonlocal minimal surfaces, see [63]. Here, we limit ourselves to give some heuristic motivation and a sketch of the proofs.

As a motivation for the (somehow unexpected) stickiness property at the boundary, one may look at Fig. 5.10 and argue like this. In the classical case, corresponding to $s = 1/2$, independently on the width δ, the set of minimal perimeter in B_1 will always be the half-ball $B_1 \cap \{x_n < 0\}$.

Now let us take $s < 1/2$. Then, the half-ball $B_1 \cap \{x_n < 0\}$ cannot be an s-minimal set, since the nonlocal mean curvature, for instance, at the origin cannot vanish. Indeed, the origin "sees" the complement of the set in a larger proportion than the set itself. More precisely, in B_1 (or even in $B_{1+\delta}$) the proportion of the set is the same as the one of the complement, but outside $B_{1+\delta}$ the complement of the set is dominant. Therefore, to "compensate" this lack of balance, the s-minimal set for $s < 1/2$ has to bend a bit. Likely, the s-minimal set in this case will have the tendency to become slightly convex at the origin, so that, at least nearby, it sees a proportion of the set which is larger than the proportion of the complement (we recall that, in any case, the proportion of the complement will be larger at infinity, so the set needs to compensate at least near the origin). But when δ is very small, it turns out that this compensation is not sufficient to obtain the desired balance between the set and its complement: therefore, the set has to "stick" to the half-sphere, in order to drop its constrain to satisfy a vanishing nonlocal mean curvature equation.

Of course some quantitative estimates are needed to make this argument work, so we describe the sketch of the rigorous proof of Theorem 5.3.1 as follows.

Proof (Sketch of the proof of Theorem 5.3.1) First of all, one checks that for any fixed $\eta > 0$, if $\delta > 0$ is small enough, we have that the interaction between B_1 and $B_{1+\delta} \setminus B_1$ is smaller than η. In particular, by comparing with a competitor that is empty in B_1, by minimality we obtain that

$$\operatorname{Per}_s(E_\delta, B_1) \leq \eta, \tag{5.32}$$

where we have denoted by E_δ the s-minimal set in B_1 that coincides with K_δ outside B_1.

Then, one checks that

$$\text{the boundary of } E_\delta \text{ can only lie in a small neighborhood of } \partial B_1 \tag{5.33}$$

if δ is sufficiently small.

Indeed, if, by contradiction, there were points of ∂E_δ at distance larger than ϵ from ∂B_1, then one could find two balls of radius comparable to ϵ, whose centers lie at distance larger than $\epsilon/2$ from ∂B_1 and at mutual distance smaller than ϵ, and such that one ball is entirely contained in $B_1 \cap E_\delta$ and the other ball is entirely contained in $B_1 \setminus E_\delta$ (this is due to a Clean Ball Condition, see Corollary 4.3 in [28]). As a consequence, $\operatorname{Per}_s(E_\delta, B_1)$ is bounded from below by the interaction of these two balls, which is at least of the order of ϵ^{n-2s}. Then, we obtain a contradiction with (5.32) (by choosing η much smaller than ϵ^{n-2s}, and taking δ sufficiently small).

This proves (5.33). From this, it follows that

$$\text{the whole set } E_\delta \text{ must lie in a small neighborhood of } \partial B_1. \tag{5.34}$$

Indeed, if this were not so, by (5.33) the set E_δ must contain a ball of radius, say $1/2$. Hence, $\mathrm{Per}_s(E_\delta, B_1)$ is bounded from below by the interaction of this ball against $\{x_n > 0\} \setminus B_1$, which would produce a contribution of order one, which is in contradiction with (5.32).

Having proved (5.34), one can use it to complete the proof of Theorem 5.3.1 employing a geometric argument. Namely, one considers the ball B_ρ, which is outside E_δ for small $\rho > 0$, in virtue of (5.34), and then enlarges ρ until it touches ∂E_δ. If this contact occurs at some point $p \in B_1$, then the nonlocal mean curvature of E_δ at p must be zero. But this cannot occur (indeed, we know by (5.34) that the contribution of E_δ to the nonlocal mean curvature can only come from a small neighborhood of ∂B_1, and one can check, by estimating integrals, that this is not sufficient to compensate the outer terms in which the complement of E_δ is dominant).

As a consequence, no touching point between B_ρ and ∂E_δ can occur in B_1, which shows that E_δ is void inside B_1 and completes the proof of Theorem 5.3.1.

As for the proof of Theorem 5.3.2, the main arguments are based on sliding a ball of suitably large radius till it touches the set, with careful quantitative estimates. Some of the details are as follows (we refer to [63] for the complete arguments).

Proof (Sketch of the proof of Theorem 5.3.2) The first step is to prove a weaker form of stickiness as the one claimed in Theorem 5.3.2. Namely, one shows that

$$[-1, 1) \times [c_o M\, , M] \subseteq E_M^c \tag{5.35}$$

$$\text{and} \quad (-1, 1] \times [-M,\, -c_o M] \subseteq E_M, \tag{5.36}$$

for some $c_o \in (0, 1)$. Of course, the statements in (5.30) and (5.31) are stronger than the ones in (5.35) and (5.36) when M is large, since $\frac{1+2s}{2+2s} < 1$, but we will then obtain them later in a second step.

To prove (5.35), one takes balls of radius $c_o M$ and centered at $\{x_2 = t\}$, for any $t \in [c_o M, M]$. One slides these balls from left to right, till one touches ∂E_M. When M is large enough (and c_o small enough) this contact point cannot lie in $\{|x_1| < 1\}$. This is due to the fact that at least the sliding ball lies outside E_M, and the whole $\{x_2 > M\}$ lies outside E_M as well. As a consequence, these contact points see a proportion of E_M smaller than the proportion of the complement (it is true that the whole of $\{x_2 < -M\}$ lies inside E_M, but this contribution comes from further away than the ones just mentioned, provided that c_o is small enough). Therefore, contact points cannot satisfy a vanishing mean curvature equation and so they need to lie on the boundary of the domain (of course, careful quantitative estimates are necessary here, see [63], but we hope to have given an intuitive sketch of the computations needed).

In this way, one sees that all the portion $[-1, 1) \times [c_o M\, , M]$ is clean from the set E_M and so (5.35) is established (and (5.36) can be proved similarly).

Once (5.35) and (5.36) are established, one uses them to obtain the strongest form expressed in (5.30) and (5.31). For this, by (5.35) and (5.36), one has only to take care of points in $\{|x_2| \in [C_o M^{\frac{1+2s}{2+2s}}, c_o M]\}$. For these points, one can use again a sliding method, but, instead of balls, one has to use suitable surfaces obtained by appropriate portions of balls and adapt the calculations in order to evaluate all the contributions arising in this way.

The computations are not completely obvious (and once again we refer to [63] for full details), but the idea is, once again, that contact points that are in the set $\{|x_2| \in [C_o M^{\frac{1+2s}{2+2s}}, c_o M]\}$ cannot satisfy the balancing relation prescribed by the vanishing nonlocal mean curvature equation.

The stickiness property discussed above also has an interesting consequence in terms of the "geometric stability" of the flat s-minimal surfaces. For instance, rather surprisingly, the flat lines in the plane are "geometrically unstable" nonlocal minimal surfaces, in the sense that an arbitrarily small and compactly supported perturbation can produce a stickiness phenomenon at the boundary of the domain. Of course, the smaller the perturbation, the smaller the stickiness phenomenon, but it is quite relevant that such a stickiness property can occur for arbitrarily small (and "nice") perturbations. This means that s-minimal flat objects, in presence of a perturbation, may not only "bend" in the center of the domain, but rather "jump" at boundary points as well.

To state this phenomenon in a mathematical framework, one can consider, for fixed $\delta > 0$ the planar sets

$$\begin{aligned} H &:= \mathbb{R} \times (-\infty, 0), \\ F_- &:= (-3, -2) \times [0, \delta) \\ \text{and} \quad F_+ &:= (2, 3) \times [0, \delta). \end{aligned}$$

One also fixes a set F which contains $H \cup F_- \cup F_+$ and denotes by E be the s-minimal set in $(-1, 1) \times \mathbb{R}$ among all the sets that coincide with F outside $(-1, 1) \times \mathbb{R}$. Then, this set E sticks at the boundary of the domain, according to the next result:

Theorem 5.3.3 *Fix $\epsilon_0 > 0$ arbitrarily small. Then, there exists $\delta_0 > 0$, possibly depending on ϵ_0, such that, for any $\delta \in (0, \delta_0]$,*

$$E \supseteq (-1, 1) \times (-\infty, \delta^{\frac{2+\epsilon_0}{1-2s}}].$$

The stickiness/instability property in Theorem 5.3.3 is depicted in Fig. 5.12. We remark that Theorem 5.3.3 gives a rather precise quantification of the size of the stickiness in terms of the size of the perturbation: namely the size of the stickiness in Theorem 5.3.3 is larger than the size of the perturbation to the power $\beta := \frac{2+\epsilon_0}{1-2s}$, for any $\epsilon_0 > 0$ arbitrarily small. Notice that $\beta \to +\infty$ as $s \to 1/2$, consistently with the fact that classical minimal surfaces do not stick at the boundary.

The proof of Theorem 5.3.3 is based on the construction of suitable auxiliary barriers (see Fig. 5.13). These barriers are used to detach a portion of the set in a neighborhood of the origin and their construction relies on some compensations

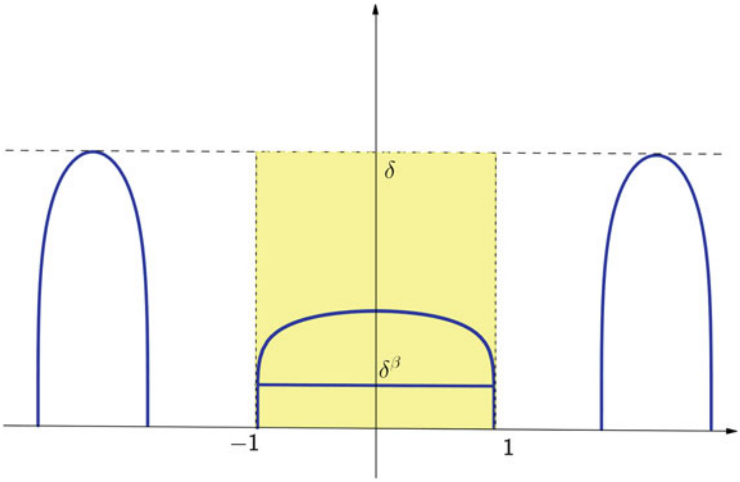


Fig. 5.12 The stickiness/instability property in Theorem 5.3.3, with $\beta := \frac{2+\epsilon_0}{1-2s}$

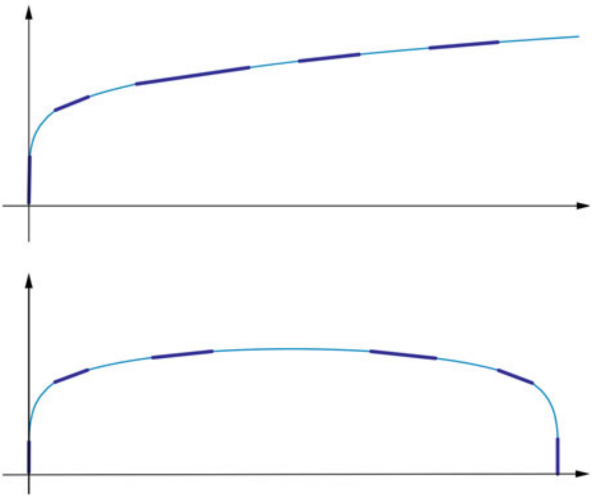


Fig. 5.13 Auxiliary barrier for the proof of Theorem 5.3.3

of nonlocal integral terms. In a sense, the building blocks of these barriers are "self-sustaining solutions" that can be seen as the geometric counterparts of the s-harmonic function x_+^s discussed in Sect. 2.4.

Indeed, roughly speaking, like the function x_+^s, these barriers "see" a proportion of the set in $\{x_1 < 0\}$ larger than what is produced by their tangent plane, but a proportion smaller than that at infinity, due to their sublinear behavior. Once again, the computations needed to check such a balancing conditions are a bit involved, and we refer to [63] for the complete details.

To conclude this chapter, we make a remark on the connection between solutions of the fractional Allen-Cahn equation and s-minimal surfaces. Namely, a suitably scaled version of the functional in (4.9) Γ-converges to either the classical perimeter or the nonlocal perimeter functional, depending on the fractional parameter s. The Γ-convergence is a type of convergence of functionals that is compatible with the minimization of the energy, and turns out to be very useful when dealing with variational problems indexed by a parameter. This notion was introduced by De Giorgi, see e.g. [50] for details.

In the nonlocal case, some care is needed to introduce the "right" scaling of the functional, which comes from the dilation invariance of the space coordinates and possesses a nontrivial energy in the limit. For this, one takes first the rescaled energy functional

$$J_\varepsilon(u,\Omega) := \varepsilon^{2s}\mathscr{K}(u,\Omega) + \int_\Omega W(u)\,dx,$$

where $\mathscr{K}$ is the kinetic energy defined in (4.10). Then, one considers the functional

$$F_\varepsilon(u,\Omega) := \begin{cases} \varepsilon^{-2s}J_\varepsilon(u,\Omega) & \text{if } s\in(0,\ 1/2), \\ |\varepsilon\log\varepsilon|^{-1}J_\varepsilon(u,\Omega) & \text{if } s=1/2, \\ \varepsilon^{-1}J_\varepsilon(u,\Omega) & \text{if } s\in(1/2,\ 1). \end{cases}$$

The limit functional of F_ε as $\varepsilon \to 0$ depends on s. Namely, when $s \in (0,1/2)$, the limit functional is (up to dimensional constants that we neglect) the fractional perimeter, i.e.

$$F(u,\Omega) := \begin{cases} \operatorname{Per}_s(E,\Omega) & \text{if } u|_\Omega = \chi_E - \chi_{E^{\mathscr{C}}}, \text{ for some set } E\subset\Omega \\ +\infty & \text{otherwise.} \end{cases} \tag{5.37}$$

On the other hand, when $s \in [1/2,1)$, the limit functional of F_ε is (again, up to normalizing constants) the classical perimeter, namely

$$F(u,\Omega) := \begin{cases} \operatorname{Per}(E,\Omega) & \text{if } u|_\Omega = \chi_E - \chi_{E^{\mathscr{C}}}, \text{ for some set } E\subset\Omega \\ +\infty & \text{otherwise,} \end{cases} \tag{5.38}$$

That is, the following limit statement holds true:

Theorem 5.3.4 *Let $s \in (0,\ 1)$. Then, F_ε Γ-converges to F, as defined in either (5.37) or (5.38), depending on whether $s\in(0,1/2)$ or $s\in[1/2,1)$.*

For precise statements and further details, see [123]. Additionally, we remark that the level sets of the minimizers of the functional in (4.9), after a homogeneous scaling in the space variables, converge locally uniformly to minimizers either of the fractional perimeter (if $s \in (0, 1/2)$) or of the classical perimeter (if $s \in [1/2, 1)$): that is, the "functional" convergence stated in Theorem 5.3.4 has also a "geometric" counterpart: for this, see Corollary 1.7 in [125].

One can also interpret Theorem 5.3.4 by saying that a nonlocal phase transition possesses two parameters, ε and s. When $\varepsilon \to 0$, the limit interface approaches a minimal surface either in the fractional case (when $s \in (0, 1/2)$) or in the classical case (when $s \in [1/2, 1)$). This bifurcation at $s = 1/2$ somehow states that for lower values of s the nonlocal phase transition possesses a nonlocal interface in the limit, but for larger values of s the limit interface is characterized only by local features (in a sense, when $s \in (0, 1/2)$ the "surface tension effect" is nonlocal, but for $s \in [1/2, 1)$ this effect localizes).

It is also interesting to compare Theorems 5.2 and 5.3.4, since the bifurcation at $s = 1/2$ detected by Theorem 5.3.4 is perfectly compatible with the limit behavior of the fractional perimeter, which reduces to the classical perimeter exactly for this value of s, as stated in Theorem 5.2.

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