Chapter 2
Stability and Stabilization

Abstract This chapter is concerned with the stability and stabilization problems of a class of continuous-time and discrete-time Markov jump linear system (MJLS) with partially unknown transition probabilities (TPs). It will be proved that the system under consideration is more general, which covers the systems with completely known and completely unknown TPs as two special cases, the latter is hereby the switching linear systems under arbitrary switching. Moreover, in contrast with the uncertain TPs, the concept of partially unknown TPs proposed in this chapter does not require any knowledge of the unknown elements. Firstly, the sufficient conditions for stochastic stability and stabilization of the underlying systems are derived via linear matrix inequality (LMI) formulation, and the relationship between the stability criteria currently obtained for the usual MJLS and switching linear systems under arbitrary switching is exposed by the proposed class of hybrid systems. Further, the necessary and sufficient criteria are obtained by fully considering the properties of the transition rates matrices (TRMs) and transition probabilities matrices (TPMs), and the convexity of the uncertain domains. We will show by comparison the less conservatism of the methodologies for obtaining the necessary and sufficient conditions, but note that in the next chapters of Part I, we prefer the ones in the sufficient stability conditions to carry out other studies. The extensions to less conservative results are relatively straightforward and we leave them to readers who are interested.

2.1 Problem Formulation

Fix the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and consider the following continuous-time and discrete-time Markov jump linear systems (MJLSs), respectively:

$$\dot{x}(t) = A(r_t)x(t) + B(r_t)u(t) \tag{2.1}$$

$$x(k+1) = A(r_k)x(k) + B(r_k)u(k) \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$ (respectively, $x(k)$) is the state vector and $u(t) \in \mathbb{R}^l$ (respectively, $u(k)$) is the control input. The jumping process $\{r_t,\ t \geq 0\}$ (respectively,

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$\{r_k,\ k \geq 0\}$), taking values in a finite set $\mathcal{I} \triangleq \{1, \ldots, N\}$, governs the switching among the different system modes. For continuous-time, $\{r_t,\ t \geq 0\}$ is a continuous-time, discrete-state homogeneous Markov process and has the following mode transition rates (TRs)

$$\Pr(r_{t+h} = j | r_t = i) = \begin{cases} \lambda_{ij} h + o(h), & \text{if } j \neq i \\ 1 + \lambda_{ii} h + o(h), & \text{if } j = i \end{cases}$$

where $h > 0$, $\lim_{h \to 0}(o(h)/h) = 0$ and $\lambda_{ij} \geq 0\,(i, j \in \mathcal{I}, j \neq i)$ denotes the switching rate from mode i at time t to mode j at time $t + h$, and $\lambda_{ii} = -\sum_{j=1, j \neq i} \lambda_{ij}$ for all $i \in \mathcal{I}$. Furthermore, the Markov process transition rate matrix (TRM) $\mathbf{\Lambda}$ is defined by:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1N} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{N1} & \lambda_{N2} & \cdots & \lambda_{NN} \end{bmatrix}$$

For discrete-time case, the process $\{r_k,\ k \geq 0\}$ is described by a discrete-time homogeneous Markov chain, which takes values in the finite set $\mathcal{I}$ with mode transition probabilities (TPs)

$$\Pr(r_{k+1} = j | r_k = i) = \pi_{ij}$$

where $\pi_{ij} \geq 0\, \forall\, i, j \in \mathcal{I}$, and $\sum_{j=1}^{N} \pi_{ij} = 1$. Likewise, the transition probability matrix (TPM) $\mathbf{\Pi}$ is defined by:

$$\mathbf{\Pi} = \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1N} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{N1} & \pi_{N2} & \cdots & \pi_{NN} \end{bmatrix}$$

The set $\mathcal{I}$ contains N modes of system (2.1) (or system (2.2)) and for $r_t = i \in \mathcal{I}$ (respectively, $r_k = i$), the system matrices of the ith mode are denoted by $(A_i,\ B_i)$, which are real known with appropriate dimensions.

In addition, the TRs or TPs of the jumping process in this chapter are considered to be partially accessed, i.e., some elements in matrix $\mathbf{\Lambda}$ or $\mathbf{\Pi}$ are unknown. For instance, for system (2.1) or system (2.2) with 4 operation modes, the TRM $\mathbf{\Lambda}$ or TPM $\mathbf{\Pi}$ may be as:

$$\begin{bmatrix} \lambda_{11} & ? & \lambda_{13} & ? \\ ? & ? & ? & \lambda_{24} \\ ? & \lambda_{32} & \lambda_{33} & ? \\ ? & ? & \lambda_{43} & \lambda_{44} \end{bmatrix}, \begin{bmatrix} \pi_{11} & ? & \pi_{13} & ? \\ ? & ? & ? & \pi_{24} \\ \pi_{31} & ? & \pi_{33} & ? \\ ? & ? & \pi_{43} & \pi_{44} \end{bmatrix}$$

where "?" represents the inaccessible elements. For notation clarity, $\forall i \in \mathcal{I}$, we denote $\mathcal{I} = \mathcal{I}_{\mathcal{K}}^{(i)} + \mathcal{I}_{\mathcal{UK}}^{(i)}$ with

$$\begin{aligned} \mathcal{I}_{\mathcal{K}}^{(i)} &\triangleq \{j : \lambda_{ij} (\text{or } \pi_{ij}) \text{ is known}\}, \\ \mathcal{I}_{\mathcal{UK}}^{(i)} &\triangleq \{j : \hat{\lambda}_{ij} \ (\text{or } \hat{\pi}_{ij}) \text{ is unknown}\} \end{aligned} \tag{2.3}$$

where each unknown element is labeled with the tide " ^ ". Moreover, if $\mathcal{I}_{\mathcal{K}}^{(i)} \neq \emptyset$, it is further described as

$$\mathcal{I}_{\mathcal{K}}^{(i)} = (\mathcal{K}_1^{(i)}, \ldots, \mathcal{K}_m^{(i)}), \ \forall 1 \leq m \leq N \tag{2.4}$$

where $\mathcal{K}_m^{(i)} \in \mathbb{N}^+$ represents the mth known element with the index $\mathcal{K}_m^{(i)}$ in the ith row of matrix $\mathbf{\Lambda}$ or $\mathbf{\Pi}$. Also, throughout the chapter, we denote

$$\lambda_{\mathcal{K}}^{(i)} \triangleq \sum\nolimits_{j \in \mathcal{I}_{\mathcal{K}}^{(i)}} \lambda_{ij}, \qquad \pi_{\mathcal{K}}^{(i)} \triangleq \sum\nolimits_{j \in \mathcal{I}_{\mathcal{K}}^{(i)}} \pi_{ij}$$

In the continuous-time case, when $\hat{\lambda}_{ii}$ is unknown, it is necessary to provide a lower bound $\lambda_d^{(i)}$ for it and we have $\lambda_d^{(i)} \leq -\lambda_{\mathcal{K}}^{(i)}$.

Remark 2.1 The accessibility of the jumping process $\{r_t, t \geq 0\}$ (or $\{r_k, k \geq 0\}$) in the existing literature is commonly assumed to be completely accessible ($\mathcal{I}_{\mathcal{UK}}^{(i)} = \emptyset$, $\mathcal{I}_{\mathcal{K}}^{(i)} = \mathcal{I}$) or completely inaccessible ($\mathcal{I}_{\mathcal{K}}^{(i)} = \emptyset, \mathcal{I}_{\mathcal{UK}}^{(i)} = \mathcal{I}$). Moreover, the TRs or TPs with polytopic or norm-bounded uncertainties require the knowledge of bounds or structure of uncertainties, which can still be viewed as accessible. Therefore, our TRM or TPM considered in the sequel is a more natural assumption to the MJSs and hence covers the existing ones.

Remark 2.2 For strictly partially unknown TRM or TPM, the cases $m = N$ (no unknown element) and $m = N - 1$ (only one unknown element) are excluded in (2.4) due to the properties of TRMs and TPMs.

For the underlying systems, the following definitions will be adopted in the rest of this chapter. More details can be referred to [1, 2] and the references therein.

Definition 2.3 System (2.1) is said to be stochastically stable if for $u(t) \equiv 0$ and every initial condition $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathcal{I}$, the following holds,

$$E\left\{\int_0^\infty \|x(t)\|^2 \, |x_0, r_0\right\} < \infty$$

Definition 2.4 System (2.2) is said to be stochastically stable if for $u(k) \equiv 0$ and every initial condition $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathcal{I}$, the following holds,

$$E\left\{\sum\nolimits_{k=0}^\infty \|x(k)\|^2 \, |x_0, r_0\right\} < \infty)$$

The purposes of this chapter are to derive the stochastic stability criteria for system (2.1) and system (2.2) when the TRs or TPs are partially unknown, and to design a state-feedback stabilizing controller such that the resulting closed-loop systems are stochastically stable. The mode-dependent controller is considered here with the form:

$$u(t) = K(r_t)x(t) \text{ (respectively, } u(k) = K(r_k)x(k)) \tag{2.5}$$

where K_i ($\forall r_t = i \in \mathcal{I}$, or $r_k = i \in \mathcal{I}$) is the controller gain to be determined.

To this end, the following lemmas on the stochastic stability of systems (2.1) and (2.2) are firstly recalled and their proofs can be found in the cited references.

Lemma 2.5 ([1]) *The unforced system (2.1) is stochastically stable if and only if there exists a set of symmetric and positive-definite matrices P_i, $i \in \mathcal{I}$ satisfying*

$$A_i' P_i + P_i A_i + \mathcal{P}^{(i)} < 0 \tag{2.6}$$

where $\mathcal{P}^{(i)} \triangleq \sum_{j \in \mathcal{I}} \lambda_{ij} P_j$.

Lemma 2.6 ([2]) *The unforced system (2.2) is stochastically stable if and only if there exists a set of symmetric and positive-definite matrices P_i, $i \in \mathcal{I}$ satisfying*

$$A_i' \mathcal{P}^{(i)} A_i - P_i < 0 \tag{2.7}$$

where $\mathcal{P}^{(i)} \triangleq \sum_{j \in \mathcal{I}} \pi_{ij} P_j$.

2.2 Stability

2.2.1 Continuous-Time Systems

Let us first give stability analysis for the unforced system (2.1) with $u(t) \equiv 0$. The following theorem presents a sufficient condition on the stochastic stability of the considered system with partially unknown TRs (2.3).

Theorem 2.7 *Consider unforced system (2.1) with partially unknown TRs (2.3). The corresponding system is stochastically stable if there exist a set of matrices $P_i > 0$, $i \in \mathcal{I}$ such that*

$$(1 + \lambda_{\mathcal{K}}^{(i)})(A_i' P_i + P_i A_i) + \mathcal{P}_{\mathcal{K}}^{(i)} < 0, \tag{2.8}$$

$$A_i' P_i + P_i A_i + P_j \geq 0, \ \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, j = i, \tag{2.9}$$

$$A_i' P_i + P_i A_i + P_j \leq 0, \ \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, j \neq i \tag{2.10}$$

where $\mathcal{P}_{\mathcal{K}}^{(i)} \triangleq \sum_{j \in \mathcal{I}_{\mathcal{K}}^{(i)}} \lambda_{ij} P_j$.

Proof Based on Lemma 2.5, we know that the system (2.1) is stochastically stable if (2.6) holds. Since one always has $\sum_{j\in\mathcal{I}}\lambda_{ij}=0$, we can rewrite the left-hand side of (2.6) as:

$$\Theta_i \triangleq A_i'P_i + P_iA_i + \mathcal{P}^{(i)} + \sum\nolimits_{j\in\mathcal{I}}\lambda_{ij}\left(A_i'P_i + P_iA_i\right)$$

Thus, from (2.3), we have

$$\begin{aligned}\Theta_i &= (1+\sum\nolimits_{j\in\mathcal{I}_{\mathcal{K}}^{(i)}}\lambda_{ij})\left(A_i'P_i + P_iA_i\right) + \sum\nolimits_{j\in\mathcal{I}_{\mathcal{K}}^{(i)}}\lambda_{ij}P_j\\ &\quad+\sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^{(i)}}\lambda_{ij}\left(A_i'P_i + P_iA_i\right) + \sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^{(i)}}\lambda_{ij}P_j\\ &= (1+\lambda_{\mathcal{K}}^{(i)})\left(A_i'P_i + P_iA_i\right) + \mathcal{P}_{\mathcal{K}}^{(i)}\\ &\quad+\sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^{(i)}}\lambda_{ij}\left(A_i'P_i + P_iA_i + P_j\right)\end{aligned}$$

Then, $\forall j\in\mathcal{I}_{\mathcal{UK}}^{(i)}$ and if $i\in\mathcal{I}_{\mathcal{K}}^{(i)}$, it is straightforward that $\Theta_i<0$ by (2.8), (2.10) and $\lambda_{ij}\geq 0$ ($\forall i,j\in\mathcal{I}, j\neq i$). On the other hand, $\forall j\in\mathcal{I}_{\mathcal{UK}}^{(i)}$ and if $i\notin\mathcal{I}_{\mathcal{K}}^{(i)}$, one can further obtain

$$\begin{aligned}\Theta_i &= (1+\lambda_{\mathcal{K}}^{(i)})\left(A_i'P_i + P_iA_i\right) + \mathcal{P}_{\mathcal{K}}^{(i)}\\ &\quad+\lambda_{ii}\left(A_i'P_i + P_iA_i + P_i\right)\\ &\quad+\sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^{(i)},j\neq i}\lambda_{ij}\left(A_i'P_i + P_iA_i + P_j\right)\end{aligned}$$

Since we have $\lambda_{ii}=-\sum_{j=1,j\neq i}\lambda_{ij}<0$, then according to (2.8)–(2.10), one can also readily obtain $\Theta_i<0$. Therefore, if (2.8)–(2.10) hold (obviously, no knowledge on λ_{ij}, $\forall j\in\mathcal{I}_{\mathcal{UK}}^{(i)}$ is needed therein), we conclude that the system (2.1) is stochastically stable against the partially unknown TRs (2.3), which completes the proof. □

Remark 2.8 Note that if $\mathcal{I}_{\mathcal{UK}}^{(i)}=\emptyset$, $\forall i\in\mathcal{I}$, the underlying system is the one with completely known TRs, which becomes the MJLS in the usual sense. Consequently, the conditions (2.8)–(2.10) are reduced to (2.8), which is equivalent to (2.6). Also, if $\mathcal{I}_{\mathcal{K}}^{(i)}=\emptyset$, $\forall i\in\mathcal{I}$, i.e., the TRs are completely unknown, then the system can be viewed as a switching linear system under arbitrary switching. Correspondingly, the condition (2.8) becomes $A_i'P_i+P_iA_i<0$, (2.9) becomes $-P_i\leq A_i'P_i+P_iA_i$, $\forall i\in\mathcal{I}$ and (2.10) becomes $A_i'P_i+P_iA_i\leq -P_j$, $\forall i\neq j\in\mathcal{I}$. Then, we have $-P_i\leq -P_j$, $\forall i\neq j\in\mathcal{I}$, by which one can conclude $P_i=P_j$, i.e., $P_i=P$. Therefore, the conditions (2.8)–(2.10) are reduced to $A_i'P+PA_i=-P<0$, namely, a latent quadratic common Lyapunov function will be shared among all the modes. Therefore, in the continuous-time context, the condition is such that the resulting switching linear system is globally uniformly asymptotically stable [187].

To process further, the following theorem presents a necessary and sufficient condition on the stochastic stability of the considered system with partially unknown TRs for the unforced system (2.1) (with $u(t) \equiv 0$).

Theorem 2.9 *Consider unforced system (2.1) with partially unknown TRs (2.3). The corresponding system is stochastically stable if and only if there exists a set of matrices $P_i > 0$, $i \in \mathcal{I}$, such that, $\forall i \in \mathcal{I}$,*

$$A_i'P_i + P_iA_i + \mathcal{P}_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{K}}^{(i)}P_j < 0,\ \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)},\ if\ i \in \mathcal{I}_{\mathcal{K}}^{(i)} \tag{2.11}$$

$$\begin{aligned} A_i'P_i + P_iA_i + \mathcal{P}_{\mathcal{K}}^{(i)} + \lambda_d^{(i)}P_i - \lambda_d^{(i)}P_j - \lambda_{\mathcal{K}}^{(i)}P_j < 0, \\ \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)},\ if\ i \in \mathcal{I}_{\mathcal{UK}}^{(i)} \end{aligned} \tag{2.12}$$

where $\mathcal{P}_{\mathcal{K}}^{(i)} \triangleq \sum_{j\in\mathcal{I}_{\mathcal{K}}^{(i)}} \lambda_{ij}P_j$ and $\lambda_d^{(i)}$ is a given lower bound for the unknown diagonal element.

Proof We shall separate the proof into two cases, $i \in \mathcal{I}_{\mathcal{K}}^{(i)}$ and $i \in \mathcal{I}_{\mathcal{UK}}^{(i)}$, and bear in mind that system (2.1) is stochastically stable if and only if (2.6) holds.

1. Case 1: $i \in \mathcal{I}_{\mathcal{K}}^{(i)}$.
   It should be first noted that in this case one has $\lambda_{\mathcal{K}}^{(i)} \le 0$. We only need to consider $\lambda_{\mathcal{K}}^{(i)} < 0$ here since $\lambda_{\mathcal{K}}^{(i)} = 0$ means the elements in the ith row of the TRM are known.

   Now we rewrite the left-hand side of (2.6) as

   $$\begin{aligned} \Theta_i &\triangleq A_i'P_i + P_iA_i + \mathcal{P}_{\mathcal{K}}^{(i)} + \sum_{j\in\mathcal{I}_{\mathcal{UK}}^{(i)}} \hat{\lambda}_{ij}P_j \\ &= A_i'P_i + P_iA_i + \mathcal{P}_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{K}}^{(i)} \sum_{j\in\mathcal{I}_{\mathcal{UK}}^{(i)}} \frac{\hat{\lambda}_{ij}}{-\lambda_{\mathcal{K}}^{(i)}}P_j \end{aligned}$$

   where the elements $\hat{\lambda}_{ij}$, $\forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}$, are unknown. Since we have $0 \le \frac{\hat{\lambda}_{ij}}{-\lambda_{\mathcal{K}}^{(i)}} \le 1$ and $\sum_{j\in\mathcal{I}_{\mathcal{UK}}^{(i)}} \frac{\hat{\lambda}_{ij}}{-\lambda_{\mathcal{K}}^{(i)}} = 1$, we know that

   $$\Theta_i = \sum_{j\in\mathcal{I}_{\mathcal{UK}}^{(i)}} \frac{\hat{\lambda}_{ij}}{-\lambda_{\mathcal{K}}^{(i)}} \left[A_i'P_i + P_iA_i + \mathcal{P}_{\mathcal{K}}^{i} - \lambda_{\mathcal{K}}^{(i)}P_j\right]$$

   Therefore, for $0 \le \hat{\lambda}_{ij} \le -\lambda_{\mathcal{K}}^{(i)}$, $\Theta_i < 0$ is equivalent to $A_i'P_i + P_iA_i + \mathcal{P}_{\mathcal{K}}^{i} - \lambda_{\mathcal{K}}^{(i)}P_j < 0, \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}$, which implies that, in the presence of unknown elements $\hat{\lambda}_{ij}$, the system stability is ensured if and only if (2.11) holds.
2. Case 2: $i \in \mathcal{I}_{\mathcal{UK}}^{(i)}$.
   In this case, $\hat{\lambda}_{ii}$ is unknown, $\lambda_{\mathcal{K}}^{(i)} \ge 0$ and $\hat{\lambda}_{ii} \le -\lambda_{\mathcal{K}}^{(i)}$. Also, we only consider $\hat{\lambda}_{ii} < -\lambda_{\mathcal{K}}^{(i)}$ here since if $\hat{\lambda}_{ii} = -\lambda_{\mathcal{K}}^{(i)}$, then the ith row of the TRM is completely known.

Now the left-hand side of the stability condition in (2.6) can be rewritten as

$$\begin{aligned}\Theta_i &= A_i'P_i + P_iA_i + \mathcal{P}_{\mathcal{K}}^{(i)} + \hat{\lambda}_{ii}P_i + \sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^{(i)}, j\neq i}\hat{\lambda}_{ij}P_j \\ &= A_i'P_i + P_iA_i + \mathcal{P}_{\mathcal{K}}^{(i)} + \hat{\lambda}_{ii}P_i \\ &\quad + \left(-\hat{\lambda}_{ii} - \lambda_{\mathcal{K}}^{(i)}\right)\sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^{(i)}, j\neq i}\frac{\hat{\lambda}_{ij}}{-\hat{\lambda}_{ii} - \lambda_{\mathcal{K}}^{(i)}}P_j\end{aligned}$$

Likewise, since we have $0 \leq \frac{\hat{\lambda}_{ij}}{-\hat{\lambda}_{ii}-\lambda_{\mathcal{K}}^{(i)}} \leq 1$ and $\sum_{j\in\mathcal{I}_{\mathcal{UK}}^{(i)}, j\neq i}\frac{\hat{\lambda}_{ij}}{-\hat{\lambda}_{ii}-\lambda_{\mathcal{K}}^{(i)}} = 1$, we know that

$$\begin{aligned}\Theta_i = \sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^{(i)}, j\neq i}\frac{\hat{\lambda}_{ij}}{-\hat{\lambda}_{ii} - \lambda_{\mathcal{K}}^{(i)}}\Big[&A_i'P_i + P_iA_i + \mathcal{P}_{\mathcal{K}}^{(i)} \\ &+ \hat{\lambda}_{ii}P_i - \hat{\lambda}_{ii}P_j - \lambda_{\mathcal{K}}^{(i)}P_j\Big]\end{aligned}$$

which means that $\Theta_i < 0$ is equivalent to $\forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, j \neq i$,

$$A_i'P_i + P_iA_i + \mathcal{P}_{\mathcal{K}}^{(i)} + \hat{\lambda}_{ii}P_i - \hat{\lambda}_{ii}P_j - \lambda_{\mathcal{K}}^{(i)}P_j < 0 \tag{2.13}$$

As $\hat{\lambda}_{ii}$ is lower bounded by $\lambda_d^{(i)}$, we have

$$\lambda_d^{(i)} \leq \hat{\lambda}_{ii} < -\lambda_{\mathcal{K}}^{(i)}$$

which implies that $\hat{\lambda}_{ii}$ may take any value between $[\lambda_d^{(i)}, -\lambda_{\mathcal{K}}^{(i)} + \epsilon]$ for some $\epsilon < 0$ arbitrarily small. Then $\hat{\lambda}_{ii}$ can be further written as a convex combination

$$\hat{\lambda}_{ii} = -\alpha\lambda_{\mathcal{K}}^{(i)} + \alpha\epsilon + (1-\alpha)\lambda_d^{(i)}$$

where α takes value arbitrarily in $[0, 1]$. Thus, (2.13) holds if and only if $\forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, j \neq i$,

$$A_i'P_i + P_iA_i + \mathcal{P}_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{K}}^{(i)}P_i + \lambda_{\mathcal{K}}^{(i)}P_j - \lambda_{\mathcal{K}}^{(i)}P_j + \epsilon(P_i - P_j) < 0 \tag{2.14}$$

and

$$A_i'P_i + P_iA_i + \mathcal{P}_{\mathcal{K}}^{(i)} + \lambda_d^{(i)}P_i - \lambda_d^{(i)}P_j - \lambda_{\mathcal{K}}^{(i)}P_j < 0 \tag{2.15}$$

simultaneously hold. Since ϵ is arbitrarily small, (2.14) holds if and only if

$$A_i'P_i + P_iA_i + \mathcal{P}_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{K}}^{(i)}P_i < 0$$

which is the case in (2.15) when $j = i$, $\forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}$. Hence (2.13) is equivalent to (2.12).

Therefore, in the presence of unknown elements in the TRM, one can readily conclude that the system is stable if and only if (2.11) and (2.12) hold for $i \in \mathcal{I}_{\mathcal{K}}^{(i)}$ and $i \in \mathcal{I}_{\mathcal{UK}}^{(i)}$, respectively. □

Remark 2.10 The stability criterion developed in Theorem 2.9 is less conservative than the Theorem 2.7. More specifically, in Theorem 2.7, if $i \in \mathcal{I}_{\mathcal{K}}^{(i)}$, the conditions are

$$\begin{cases} \left(1+\lambda_{\mathcal{K}}^{(i)}\right)\left(A_i'P_i+P_iA_i\right)+\mathcal{P}_{\mathcal{K}}^{(i)}<0 \\ A_i'P_i+P_iA_i+P_j\le 0 \end{cases}$$

which, since $\lambda_{\mathcal{K}}^{(i)}<0$, ensure

$$\left(1+\lambda_{\mathcal{K}}^{(i)}\right)\left(A_i'P_i+P_iA_i\right)+\mathcal{P}_{\mathcal{K}}^{(i)}-\lambda_{\mathcal{K}}^{(i)}\left(A_i'P_i+P_iA_i+P_j\right)<0$$

which is (2.11). Also, if $i \in \mathcal{I}_{\mathcal{UK}}^{(i)}$, the criteria in Theorem 2.7 are

$$\begin{cases} \left(1+\lambda_{\mathcal{K}}^{(i)}\right)\left(A_i'P_i+P_iA_i\right)+\mathcal{P}_{\mathcal{K}}^{(i)}<0 \\ A_i'P_i+P_iA_i+P_j\ge 0,\ \forall j\in\mathcal{I}_{\mathcal{UK}}^{(i)}, j=i \\ A_i'P_i+P_iA_i+P_j\le 0,\ \forall j\in\mathcal{I}_{\mathcal{UK}}^{(i)}, j\neq i \end{cases} \tag{2.16}$$

In this case, since $\lambda_d^{(i)}<0$ and $-\lambda_d^{(i)}-\lambda_{\mathcal{K}}^{(i)}>0$, we have

$$\begin{aligned}\left(1+\lambda_{\mathcal{K}}^{(i)}\right)\left(A_i'P_i+P_iA_i\right)+\mathcal{P}_{\mathcal{K}}^{(i)}+\lambda_d^{(i)}\left(A_i'P_i+P_iA_i+P_i\right)\\ +\left(-\lambda_d^{(i)}-\lambda_{\mathcal{K}}^{(i)}\right)\left(A_i'P_i+P_iA_i+P_j\right)<0\end{aligned}$$

which guarantees

$$A_i'P_i+P_iA_i+\mathcal{P}_{\mathcal{K}}^{(i)}+\lambda_d^{(i)}P_i-\lambda_d^{(i)}P_j-\lambda_{\mathcal{K}}^{(i)}P_j<0,\ \ \forall j\in\mathcal{I}_{\mathcal{UK}}^{(i)}$$

Therefore, the conditions (2.11)–(2.12) are less conservative than (2.16). Note that the obtained conditions are without loss of generality since the lower bound, $\lambda_d^{(i)}$, of $\hat{\lambda}_{ii}$ is allowed to be arbitrarily negative.

2.2.2 *Discrete-Time Systems*

The following theorem presents a sufficient condition on the stochastic stability of the unforced system (2.2) with partially unknown TPs (2.3).

Theorem 2.11 *Consider the unforced system (2.2) with partially unknown TPs (2.3). The corresponding system is stochastically stable if there exists a set of matrices $P_i > 0$, $i \in \mathcal{I}$ such that*

$$A_i' \mathcal{P}_{\mathcal{K}}^{(i)} A_i - \pi_{\mathcal{K}}^{(i)} P_i < 0, \tag{2.17}$$

$$A_i' P_j A_i - P_i < 0,\ \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)} \tag{2.18}$$

where $\mathcal{P}_{\mathcal{K}}^{(i)} \triangleq \sum_{j \in \mathcal{I}_{\mathcal{K}}^{(i)}} \pi_{ij} P_j$.

Proof Based on Lemma 2.6, we know that the system (2.2) is stochastically stable if (2.7) holds. Now due to $\sum_{j \in \mathcal{I}} \pi_{ij} = 1$, we rewrite the left-hand side of (2.7) as

$$\Psi_i \triangleq A_i' \left(\sum\nolimits_{j \in \mathcal{I}} \pi_{ij} P_j \right) A_i - \left(\sum\nolimits_{j \in \mathcal{I}} \pi_{ij} \right) P_i$$

Thus, from (2.3), we have

$$\begin{aligned} \Psi_i &= A_i' \left(\sum\nolimits_{j \in \mathcal{I}_{\mathcal{K}}^{(i)}} \pi_{ij} P_j \right) A_i - \left(\sum\nolimits_{j \in \mathcal{I}_{\mathcal{K}}^{(i)}} \pi_{ij} \right) P_i \\ &\quad + A_i' \left(\sum\nolimits_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}} \pi_{ij} P_j \right) A_i - \left(\sum\nolimits_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}} \pi_{ij} \right) P_i \\ &= A_i' \mathcal{P}_{\mathcal{K}}^{(i)} A_i - \pi_{\mathcal{K}}^{(i)} P_i + \sum\nolimits_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}} \pi_{ij} \left(A_i' P_j A_i - P_i \right) \end{aligned}$$

Then, since one always has $\pi_{ij} \geq 0$, $\forall j \in \mathcal{I}$, it is straightforward that $\Psi_i < 0$ if (2.17) and (2.18) hold. Obviously, no knowledge on π_{ij}, $\forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}$ is needed in (2.17) and (2.18), we can hereby conclude that the system (2.2) is stochastically stable against the partially unknown TPs (2.3), which completes the proof. □

Remark 2.12 Analogous to Remark 2.8 for continuous-time case, if $\mathcal{I}_{\mathcal{UK}}^{(i)} = \emptyset$, $\forall i \in \mathcal{I}$, the conditions are reduced to (2.7), the classical criterion to check the stochastic stability for the usual discrete-time MJLS. Also, if $\mathcal{I}_{\mathcal{K}}^{(i)} = \emptyset$, $\forall i \in \mathcal{I}$, the system becomes a discrete-time switching linear system under arbitrary switching. The conditions (2.17) and (2.18) are reduced to $A_i' P_j A_i - P_i < 0$, which is the criterion obtained in [188] by a switched Lyapunov function approach to guarantee the system is globally uniformly asymptotically stable in discrete-time context.

Remark 2.13 It is seen from the above theorems that the stochastic stability for the underlying system is actually guaranteed by the two aspects, i.e., efficiently utilizing the partially known TRs or TPs (see (2.8) and (2.17)), together with some requirements on the latent quadratic Lyapunov function $V_i(x_t, t) = x_t' P_i x_t$, $\forall i \in \mathcal{I}$ (respectively, $V_i(x_k, k) = x_k' P_i x_k$, $\forall i \in \mathcal{I}$). For continuous-time case, the requirements are $V_j(x_t, t) \leq -\dot{V}_i(x_t, t)$, $\forall j \in \mathcal{I}_{\mathcal{UK}}^i$, $j \neq i$ and $-\dot{V}_i(x_t, t) \leq V_i(x_t, t)$, $\forall i \in \mathcal{I}_{\mathcal{UK}}^i$ (from (2.8) and (2.9), respectively), which implies $\dot{V}_i(x_t, t) < 0$ and $V_j(x_t, t) \leq V_i(x_t, t)$. For discrete-time case, the requirements are $\triangle V_i(x_k, k) \triangleq V_i(x_{k+1}, k+1) - V_i(x_k, k) <$

$0, \forall i \in \mathcal{I}_{\mathcal{UK}}^{i}$ and $V_j(x_{k+1}, k+1) - V_i(x_k, k) < 0, \forall j \in \mathcal{I}_{\mathcal{UK}}^{i}, \ j \neq i,$ which can be easily deduced by (2.18).

The following theorem presents a necessary and sufficient condition on the stochastic stability of the unforced system (2.2) with partially unknown TPs.

Theorem 2.14 *Consider the unforced system (2.2) with partially unknown TPs (2.3). The corresponding system is stochastically stable if and only if there exists a set of matrices $P_i > 0$, $i \in \mathcal{I}$ such that*

$$A_i' \left(\mathcal{P}_{\mathcal{K}}^{(i)} + \left(1 - \pi_{\mathcal{K}}^{(i)}\right) P_j\right) A_i - P_i < 0, \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)} \tag{2.19}$$

where $\mathcal{P}_{\mathcal{K}}^{(i)} \triangleq \sum_{j \in \mathcal{I}_{\mathcal{K}}^{(i)}} \pi_{ij} P_j$.

Proof It should be first noted that $\pi_{\mathcal{K}}^{(i)} \leq 1$ in the discrete-time case, and we exclude $\pi_{\mathcal{K}}^{(i)} = 1$ here since it means that all the elements in the ith row are known.

Now the left-hand side of stability condition (2.7) in Lemma 2.6 can be rewritten as

$$\begin{aligned}
\Psi_i &\triangleq A_i' \left(\mathcal{P}_{\mathcal{K}}^{(i)} + \sum\nolimits_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}} \hat{\pi}_{ij} P_j\right) A_i - P_i \\
&= A_i' \left(\mathcal{P}_{\mathcal{K}}^{(i)} + \left(1 - \pi_{\mathcal{K}}^{(i)}\right) \sum\nolimits_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}} \frac{\hat{\pi}_{ij}}{1 - \pi_{\mathcal{K}}^{(i)}} P_j\right) A_i - P_i
\end{aligned}$$

where the elements $\hat{\pi}_{ij}, j \in \mathcal{I}_{\mathcal{UK}}^{(i)}$, are unknown. Since $0 \leq \frac{\hat{\pi}_{ij}}{1-\pi_{\mathcal{K}}^{(i)}} \leq 1, \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}$ and $\sum_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}} \frac{\hat{\pi}_{ij}}{1-\pi_{\mathcal{K}}^{(i)}} = 1$, we know that

$$\Psi_i = \sum\nolimits_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}} \frac{\hat{\pi}_{ij}}{1 - \pi_{\mathcal{K}}^{(i)}} \left[A_i' \left(\mathcal{P}_{\mathcal{K}}^{(i)} + \left(1 - \pi_{\mathcal{K}}^{(i)}\right) P_j\right) A_i - P_i\right]$$

Therefore, for $0 \leq \hat{\pi}_{ij} \leq 1 - \pi_{\mathcal{K}}^{(i)}$, $\Psi_i < 0$ is equivalent to $A_i'(\mathcal{P}_{\mathcal{K}}^{(i)} + (1 - \pi_{\mathcal{K}}^{(i)}) P_j) A_i - P_i < 0, \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}$, which implies that, in the presence of unknown elements $\hat{\pi}_{ij}$, the system stability is ensured if and only if (2.19) holds. □

Remark 2.15 Analogous to Remark 2.10 for the continuous-time case, the necessary and sufficient criterion developed in Theorem 2.9 is also less conservative when compared with Theorem 2.7, where the stability conditions are given by

$$\begin{aligned}
A_i' \mathcal{P}_{\mathcal{K}}^{(i)} A_i - \pi_{\mathcal{K}}^{(i)} P_i &< 0 \\
A_i' P_j A_i - P_i &< 0, \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}
\end{aligned}$$

The inequalities yield

$$A_i'\mathcal{P}_{\mathcal{K}}^{(i)}A_i - \pi_{\mathcal{K}}^{(i)}P_i + \left(1 - \pi_{\mathcal{K}}^{(i)}\right)\left(A_i'P_jA_i - P_i\right) < 0, \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}$$

which is (2.19). Therefore, combined with Remark 2.2, it is seen that the approach adopted in Theorems 2.9 and 2.14 in this section, which uses the TRM or TPM property (the sum of all the elements in each row is zero or one), gives the necessary and sufficient criteria and are less conservative than Theorems 2.7 and 2.11.

2.3 Stabilization

2.3.1 Continuous-Time Systems

Now let us consider the stabilization problem of system (2.1) with control input $u(t)$. The following theorem presents sufficient conditions for the existence of a mode-dependent stabilizing controller with the form (2.5).

Theorem 2.16 *Consider system (2.1) with partially unknown TRs (2.3). If there exist matrices $X_i > 0$ and Y_i, $\forall i \in \mathcal{I}$ such that*

$$\begin{bmatrix} \left(1 + \lambda_{\mathcal{K}}^{(i)}\right) sym(A_iX_i + B_iY_i) + \lambda_{ii}X_i & \mathcal{S}_{\mathcal{K}}^{(i)} \\ * & -\mathcal{X}_{\mathcal{K}}^{(i)} \end{bmatrix} < 0, \forall i \in \mathcal{I}_{\mathcal{K}}^{(i)} \tag{2.20}$$

$$\begin{bmatrix} \left(1 + \lambda_{\mathcal{K}}^{(i)}\right) sym(A_iX_i + B_iY_i) & \mathcal{S}_{\mathcal{K}}^{(i)} \\ * & -\mathcal{X}_{\mathcal{K}}^{(i)} \end{bmatrix} < 0, \ \forall i \notin \mathcal{I}_{\mathcal{K}}^{(i)} \tag{2.21}$$

$$sym(A_iX_i + B_iY_i) + X_j \geq 0, \ \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, \ j = i \tag{2.22}$$

$$\begin{bmatrix} sym(A_iX_i + B_iY_i) & X_i \\ * & -X_j \end{bmatrix} \leq 0, \ \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, \ j \neq i \tag{2.23}$$

where

$$\mathcal{S}_{\mathcal{K}}^{(i)} \triangleq \left[\sqrt{\lambda_{i\mathcal{K}_1^{(i)}}}X_i, \ldots, \sqrt{\lambda_{i\mathcal{K}_m^{(i)}}}X_i\right] \tag{2.24}$$

$$\mathcal{X}_{\mathcal{K}}^{(i)} \triangleq diag\{X_{\mathcal{K}_1^{(i)}}, \ldots, X_{\mathcal{K}_m^{(i)}}\} \tag{2.25}$$

with $\mathcal{K}_1^{(i)}, \ldots, \mathcal{K}_m^{(i)}$ described in (2.4), then there exists a mode-dependent stabilizing controller of the form (2.5) such that the resulting system is stochastically stable. Moreover, if the LMIs (2.20)–(2.23) have a solution, an admissible controller gain is given by

$$K_i = Y_iX_i^{-1} \tag{2.26}$$

Proof Consider system (2.1) with the control input (2.5), replace A_i by $A_i + B_iK_i$ and set $X_i \triangleq P_i^{-1}$, $Y_i \triangleq K_iX_i$ in (2.8)–(2.10), respectively. Firstly, performing a congruence transformation to (2.8) by P_i^{-1}, we can obtain

$$\left(1 + \lambda_{\mathcal{K}}^{(i)}\right) sym\left((A_i + B_iK_i)P_i^{-1}\right) + P_i^{-1}\mathcal{P}_{\mathcal{K}}^{(i)}P_i^{-1} < 0$$

Considering (2.24) and (2.25), by Schur complement (Lemma 1.5), one can obtain that the above inequality is equivalent to (2.20) for $i \in \mathcal{I}_{\mathcal{K}}^{(i)}$, and (2.21) for $i \notin \mathcal{I}_{\mathcal{K}}^{(i)}$, respectively. Similarly, by performing a congruence to (2.9) by X_i, one can get (2.22). Also, by Schur complement, (2.9) is equivalent to

$$\begin{bmatrix} P_i\left(A_i + B_iK_i\right) + \left(A_i + B_iK_i\right)' P_i & I \\ * & -P_j^{-1} \end{bmatrix} \leq 0$$

Performing a congruence to the above by $diag\{X_i, I\}$, one can obtain (2.23). Therefore, if (2.20)–(2.23) hold, (2.8)–(2.10) will be satisfied in Theorem 2.7 such that the underlying system is stochastically stable. Moreover, the desired controller gain is given by (2.26). This completes the proof. □

Remark 2.17 It is worth noting that (2.20) and (2.22) in Theorem 2.16 will not be checked simultaneously due to the fact $\mathcal{I}_{\mathcal{K}}^i \cap \mathcal{I}_{\mathcal{UK}}^i = \emptyset$.

The following theorem presents a necessary and sufficient criterion for the existence of a mode-dependent stabilizing controller of the form in (2.5).

Theorem 2.18 *Consider system (2.1) with partially unknown TRs (2.3). If there exist matrices $X_i > 0$ and Y_i, $\forall i \in \mathcal{I}$ such that*

$$\begin{bmatrix} \Lambda_i + \lambda_{ii}X_i & \mathcal{T}_{\mathcal{K}}^{(i)} & \sqrt{-\lambda_{\mathcal{K}}^{(i)}}X_i \\ * & -\mathcal{X}_{\mathcal{K}}^{(i)} & 0 \\ * & * & -X_j \end{bmatrix} < 0, \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, \mathit{if}\ i \in \mathcal{I}_{\mathcal{K}}^{(i)} \tag{2.27}$$

$$\begin{bmatrix} \Lambda_i + \lambda_d^{(i)}X_i & \mathcal{T}_{\mathcal{K}}^{(i)} & \sqrt{-\lambda_d^{(i)} - \lambda_{\mathcal{K}}^{(i)}}X_i \\ * & -\mathcal{X}_{\mathcal{K}}^{(i)} & 0 \\ * & * & -X_j \end{bmatrix} < 0, \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, \mathit{if}\ i \in \mathcal{I}_{\mathcal{UK}}^{(i)} \tag{2.28}$$

where $\Lambda_i \triangleq A_iX_i + X_iA_i' + B_iY_i + Y_i'B_i'$ and

$$\begin{aligned} \mathcal{X}_{\mathcal{K}}^{(i)} &\triangleq \mathrm{diag}\left[X_{\mathcal{K}_1}, \ldots, X_{\mathcal{K}_{m_i}}\right], \\ \mathcal{T}_{\mathcal{K}}^{(i)} &\triangleq \left[\sqrt{\lambda_{i\mathcal{K}_1}}X_i, \ldots, \sqrt{\lambda_{i\mathcal{K}_{m_i}}}X_i\right] \end{aligned} \tag{2.29}$$

and $\forall s \in \{1, 2, \ldots, m_i\}$, $\mathcal{K}_s$ is described in (2.4), $\mathcal{K}_s \neq i$, then there exists a mode-dependent stabilizing controller of the form in (2.5) such that the closed-loop system *is stochastically stable. Moreover, if the LMIs in (2.27)–(2.28) have a solution, an admissible controller gain is given by (2.26).*

Proof Consider system (2.1) with the control input (2.5) and replace A_i by $A_i + B_iK_i$ in (2.11)–(2.12), respectively. Then, if $i \in \mathcal{I}_{\mathcal{K}}^{(i)}$, performing a congruence transformation to (2.11) by P_i^{-1}, we can obtain

$$\begin{aligned}(A_i + B_iK_i)\, P_i^{-1} + P_i^{-1}\,(A_i + B_iK_i)' \\ +P_i^{-1}\mathcal{P}_{\mathcal{K}}^{(i)}P_i^{-1} - P_i^{-1}\lambda_{\mathcal{K}}^{(i)}P_jP_i^{-1} < 0\end{aligned} \tag{2.30}$$

Setting $X_i \triangleq P_i^{-1}$, $Y_i \triangleq K_iX_i$ and considering (2.29), by Schur complement, one can obtain that (2.30) is equivalent to (2.27). In a similar way, if $i \in \mathcal{I}_{\mathcal{UK}}^{(i)}$, (2.28) can be worked out from (2.12). Meanwhile, due to $Y_i = K_iX_i$, the desired controller gain is given by (2.26). □

Remark 2.19 It is noted from (2.28) that if the diagonal elements in the TRM contain unknown ones, the system stability, the existence of the admissible controller and the controller gains solution will be dependent on $\lambda_d^{(i)}$. This dependency, therefore, will reduce the conservatism existed in the previous "$\lambda_d^{(i)}$-independent" results obtained in Theorem 2.16.

2.3.2 Discrete-Time Systems

Now consider the system (2.2) with control input $u(k)$, the following theorem presents sufficient conditions for the existence of a mode-dependent stabilizing controller with the form (2.5).

Theorem 2.20 *Consider system (2.2) with the partially unknown TPs (2.3). If there exist matrices $X_i > 0$ and Y_i, $\forall i \in \mathcal{I}$ such that*

$$\begin{bmatrix} -\mathcal{X}_{\mathcal{K}}^{(i)} & \mathcal{L}_{\mathcal{K}}^{(i)}\,(A_iX_i + B_iY_i) \\ * & -\pi_{\mathcal{K}}^{(i)}X_i \end{bmatrix} < 0, \tag{2.31}$$

$$\begin{bmatrix} -X_j & A_iX_i + B_iY_i \\ * & -X_i \end{bmatrix} < 0,\ \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, \tag{2.32}$$

where

$$\mathcal{L}_{\mathcal{K}}^{(i)} \triangleq \left[\sqrt{\pi_{i\mathcal{K}_1^{(i)}}}I, \ldots, \sqrt{\pi_{i\mathcal{K}_m^{(i)}}}I\right]', \tag{2.33}$$

$$\mathcal{X}_{\mathcal{K}}^{(i)} \triangleq diag\{X_{\mathcal{K}_1^{(i)}}, \ldots, X_{\mathcal{K}_m^{(i)}}\}, \quad \forall j \in \mathcal{I}_{\mathcal{K}}^{(i)} \tag{2.34}$$

with $\mathcal{K}_1^{(i)}$,...,$\mathcal{K}_m^{(i)}$ described in (2.4), then there exists a mode-dependent stabilizing controller of the form (2.5) such that the resulting system is stochastically stable.

Moreover, if the LMIs (2.31)–(2.32) have a solution, an admissible controller gain is given by (2.26).

Proof First of all, by Theorem 2.16, we know that system (2.2) is stochastically stable with the partially unknown TPs (2.3) if the inequalities (2.17) and (2.18) hold. By Schur complement, (2.17) and (2.18) are respectively equivalent to:

$$\begin{bmatrix} -P_{\mathcal{K}_1^{(i)}} & 0 & \cdots & 0 & \sqrt{\pi_{i\mathcal{K}_1^{(i)}}}P_{\mathcal{K}_1^{(i)}}A_i \\ * & -P_{\mathcal{K}_2^{(i)}} & & \vdots & \sqrt{\pi_{i\mathcal{K}_2^{(i)}}}P_{\mathcal{K}_2^{(i)}}A_i \\ * & * & \ddots & 0 & \vdots \\ * & * & * & -P_{\mathcal{K}_m^{(i)}} & \sqrt{\pi_{i\mathcal{K}_m^{(i)}}}P_{\mathcal{K}_m^{(i)}}A_i \\ * & * & * & * & -\pi_{\mathcal{K}}^{(i)}P_i \end{bmatrix} < 0, \tag{2.35}$$

$$\begin{bmatrix} -P_j & P_jA_i \\ * & -P_i \end{bmatrix} < 0. \tag{2.36}$$

Now, consider the system with the control input (2.5) and replace A_i by $A_i + B_iK_i$ in (2.35) and (2.36), respectively. Setting $X_i \triangleq P_i^{-1}$, performing a congruence transformation to (2.35) by $diag\{\mathcal{X}_{\mathcal{K}}^{(i)}, X_i\}$ and applying the change of variable $Y_i \triangleq K_iX_i$, we can readily obtain (2.31). Also, by $X_i = P_i^{-1}$, $Y_i = K_iX_i$ and performing a congruence transformation to (2.36) by $diag\{X_j, X_i\}$, one can obtain (2.32). Therefore, if (2.31) and (2.32) hold, (2.17) and (2.18) will be satisfied in Theorem 2.11, i.e. the underlying system is stochastically stable. Moreover, the desired controller gain is given by (2.26). This completes the proof. □

From the development in the above theorems, one can clearly see that our obtained stability and stabilization conditions actually cover the results for the usual MJLS and the switching linear systems under arbitrary switching (all the TPs are unknown). Therefore, the systems considered and corresponding criteria explored in the section are more general in hybrid systems field.

Furthermore, the following theorem presents a necessary and sufficient condition for the existence of a mode-dependent stabilizing controller with the form in (2.5).

Theorem 2.21 *Consider system (2.2) with partially unknown TPs (2.3). If there exist matrices $X_i > 0$ and Y_i, $\forall i \in \mathcal{I}$ such that*

$$\begin{bmatrix} -X_i & [\mathcal{L}_{\mathcal{K}}^{(i)}(A_iX_i + B_iY_i)]' \\ * & -\mathcal{X}_{\mathcal{K}}^{(i)} \end{bmatrix} < 0 \tag{2.37}$$

where

$$\mathcal{L}_{\mathcal{K}}^{(i)} \triangleq \left[\sqrt{\pi_{i\mathcal{K}_1}}I, \ldots, \sqrt{\pi_{i\mathcal{K}_{m_i}}}I, \sqrt{1-\pi_{\mathcal{K}}^{(i)}}I\right]' \tag{2.38}$$

$$\mathcal{X}_{\mathcal{K}}^{(i)} \triangleq \mathrm{diag}\left[X_{\mathcal{K}_1}, \ldots, X_{\mathcal{K}_{m_i}}, X_j\right], \; j \in \mathcal{I}_{\mathcal{UK}}^{(i)} \tag{2.39}$$

and $\forall s \in \{1, 2, \dots, m_i\}$, $\mathcal{K}_s$ *is described in (2.4), then there exists a mode-dependent stabilizing controller of the form in (2.5) such that the closed-loop system is stochastically stable. Moreover, if the LMIs in (2.37) have a solution, an admissible controller gain is given by (2.26).*

Proof First of all, by Theorem 2.14, we know that system (2.2) is stochastically stable with partially unknown TPs if the inequality (2.19) holds. By Schur complement, (2.19) is equivalent to

$$\begin{bmatrix} -P_i & * & * & * & * & * \\ \sqrt{\pi_{i\mathcal{K}_1}} P_{\mathcal{K}_1} A_i & -P_{\mathcal{K}_1} & * & * & * & * \\ \sqrt{\pi_{i\mathcal{K}_2}} P_{\mathcal{K}_2} A_i & 0 & -P_{\mathcal{K}_2} & * & * & * \\ \vdots & \vdots & \vdots & \ddots & * & * \\ \sqrt{\pi_{i\mathcal{K}_{m_i}}} P_{\mathcal{K}_{m_i}} A_i & 0 & 0 & \cdots & -P_{\mathcal{K}_{m_i}} & * \\ \sqrt{1-\pi_{\mathcal{K}}^{(i)}} P_j A_i & 0 & 0 & \cdots & 0 & -P_j \end{bmatrix} < 0 \tag{2.40}$$

Now, consider the system with the control input (2.5) and replace A_i by $A_i + B_i K_i$ in (2.40). Setting $X_i \triangleq P_i^{-1}$, performing a congruence transformation to (2.40) by $\mathrm{diag}\{X_i, \mathcal{X}_{\mathcal{K}}^{(i)}\}$ and applying the change of variable $Y_i \triangleq K_i X_i$, we can readily obtain (2.37). Therefore, if (2.37) holds, (2.19) will be satisfied in Theorem 2.14, that is, the underlying system is stochastically stable. Meanwhile, due to $Y_i = K_i X_i$, the desired controller gain is given by (2.26). □

Remark 2.22 In contrast with the continuous-time case, the discrete-time case is relatively simpler since all the elements in the TPM are nonnegative and we need not distinguish the cases of diagonal elements known or unknown.

Remark 2.23 It is noted that an interesting conclusion can be directly drawn from Theorems 2.9 and 2.14. That is, when all the elements in the TRM or TPM are unknown, the underlying systems are subject to switchings without known statistics. This leads to the so-called deterministic switching systems under arbitrary switchings (see [187], [188] for continuous-time and discrete-time case, respectively). We can therefore obtain the necessary and sufficient stability criterion of such switching systems in continuous-time and discrete-time cases, respectively. More specifically, in the discrete-time case, we have the stability condition is $A_i' P_j A_i - P_i < 0, \forall i \times j \in \mathcal{I} \times \mathcal{I}$, which is reduced from (2.19) when all the elements in the TPM are unknown. Likewise, for the continuous-time case, if all the elements in the TRM are unknown, the conditions lead to (2.12) only and it reduces to

$$A_i' P_i + P_i A_i + \lambda_d^{(i)} (P_i - P_j) < 0 \tag{2.41}$$

Since $\lambda_d^{(i)}$ can be arbitrarily negative, inequality (2.41) requires $P_i = P_j \equiv P$ which leads to the condition

$$A_i' P + P A_i < 0, \forall i \times j \in \mathcal{I} \times \mathcal{I}.$$

2.4 Numerical Examples

In this section, several numerical examples will be given to show the validity and potential of our developed theoretical results, respectively, in the continuous-time and discrete-time cases.

Example 2.24 Consider the MJLS (2.1) with four operation modes and the following data:

$$A_1 = \begin{bmatrix} -0.75 & -0.75 \\ 1.50 & -1.50 \end{bmatrix}, A_2 = \begin{bmatrix} -0.15 & -0.49 \\ 1.50 & -2.10 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -0.30 & -0.15 \\ 1.50 & -1.80 \end{bmatrix}, A_4 = \begin{bmatrix} -0.90 & -0.34 \\ 1.50 & -1.65 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 5 \\ -1 \end{bmatrix}, B_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_4 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

The two cases for the TRM are considered in Table 2.1.

Our purpose here is to design a mode-dependent stabilizing controller of the form of (2.5) such that the resulting closed-loop system is stochastically stable with the partially unknown TRs (2.3). By solving (2.20)–(2.26) in Theorem 2.16, the controller gains are solved as:

$$\begin{aligned} \text{Case I: } K_1 &= [\,-0.11\ -0.25\,], K_2 = [\,0.02\ -1.31\,], \\ K_3 &= [\,-0.81\ -0.70\,], K_4 = [\,-0.09\ -0.38\,], \\ \text{Case II: } K_1 &= [\,0.20\ -0.33\,], K_2 = [\,0.12\ -1.10\,], \\ K_3 &= [\,-0.62\ -0.38\,], K_4 = [\,-0.04\ -0.03\,]. \end{aligned}$$

Furthermore, applying the above controllers and giving two possible system modes evolution, the state response of the closed-loop system are shown in Figs. 2.1 and 2.2 under the given initial condition $x_0 = [-1.2\ 0.6]'$.

Now, the following example gives the verification on the results for the discrete-time counterpart.

Table 2.1 Different transition rate matrices (TRMs)

(a) Case I					(b) Case II				
	1	2	3	4		1	2	3	4
1	−1.3	0.2	?	?	1	?	?	0.8	0.3
2	?	?	0.3	0.3	2	0.3	?	0.3	?
3	0.6	?	−1.5	?	3	?	0.1	−1.5	?
4	0.4	?	?	?	4	?	0.2	?	?

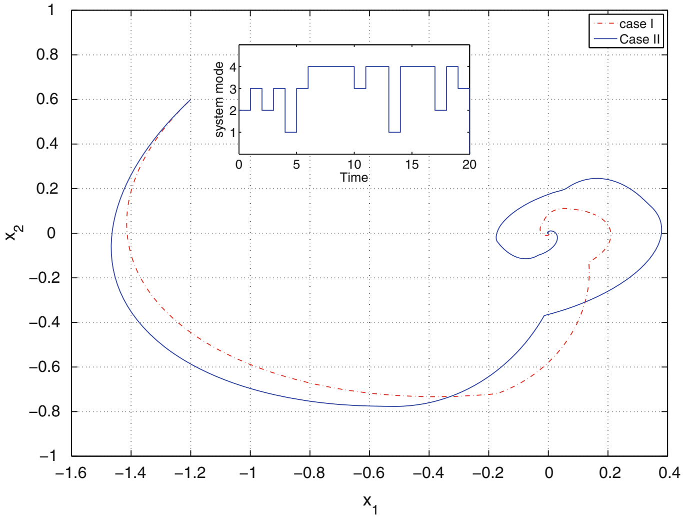

Fig. 2.1 State response of the closed-loop system under mode evolution r_t^1

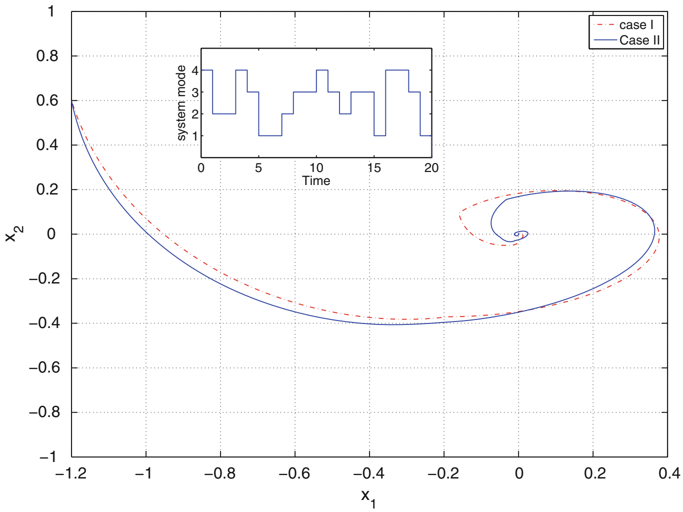

Fig. 2.2 State response of the closed-loop system under mode evolution r_t^2

Table 2.2 Different transition probability matrices (TPMs)

(a) Case I					(b) Case II				
	1	2	3	4		1	2	3	4
1	0.3	?	0.1	?	1	0.3	?	?	0.4
2	?	?	0.3	0.2	2	?	0.2	0.3	?
3	?	0.1	?	0.3	3	?	?	0.5	0.3
4	0.2	?	?	?	4	?	?	0.1	?

Example 2.25 Consider the MJLS (2.2) with four operation modes and the following data:

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 0.32 & -0.40 \\ 0.8 & -0.80 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.08 & -0.26 \\ 0.80 & -1.12 \end{bmatrix}, \\
A_3 &= \begin{bmatrix} 0.16 & -0.08 \\ 0.80 & -0.96 \end{bmatrix}, \; A_4 = \begin{bmatrix} 0.48 & -0.18 \\ 0.80 & -0.88 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, B_4 = \begin{bmatrix} 0.8 \\ -1 \end{bmatrix}.
\end{aligned}
$$

The two cases of the TPMs are considered as in Table 2.2.

Analogous to the continuous-time case, an admissible controller can be solved by (2.31)–(2.26) in Theorem 2.20 with the following control gains:

$$
\begin{aligned}
\text{Case I: } K_1 &= [\,-0.28 \;\, 0.32\,], K_2 = [\,0.36 \;\, -0.42\,], \\
K_3 &= [\,-0.48 \;\, 0.52\,], K_4 = [\,0.25 \;\, -0.45\,], \\
\text{Case II: } K_1 &= [\,-0.21 \;\, 0.24\,], K_2 = [\,0.35 \;\, -0.41\,], \\
K_3 &= [\,-0.47 \;\, 0.51\,], K_4 = [\,0.22 \;\, -0.42\,]
\end{aligned}
$$

Figures 2.3 and 2.4 show the state response of the corresponding closed-loop system for the given initial condition $x_0 = [-0.3 \; 0.4]'$ under two different modes evolution.

It is seen from the curves in Figs. 2.1–2.4 that, despite the partially unknown TPs, the designed controllers are feasible and effective ensuring the resulting closed-loop systems are stable, in the continuous-time or in discrete-time cases, respectively.

The validity and the reduction of conservatism of the results obtained above are verified by the following numerical examples.

Example 2.26 Consider MJLS (2.1) with three operation modes and the following system matrices:

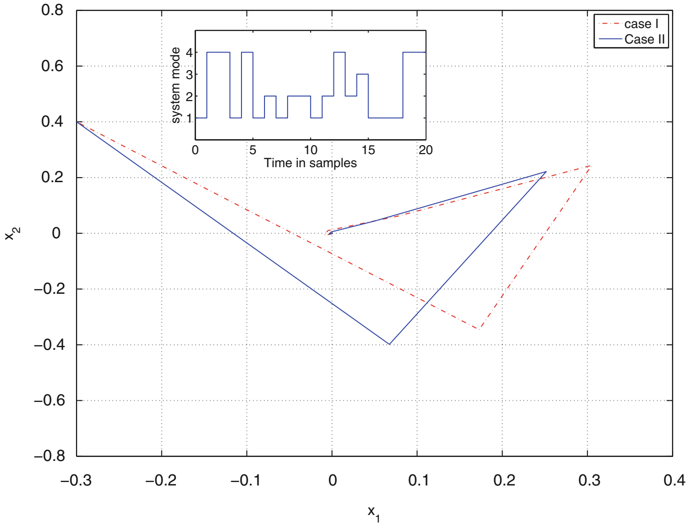

Fig. 2.3 State response of the closed-loop system under mode evolution r_k^1

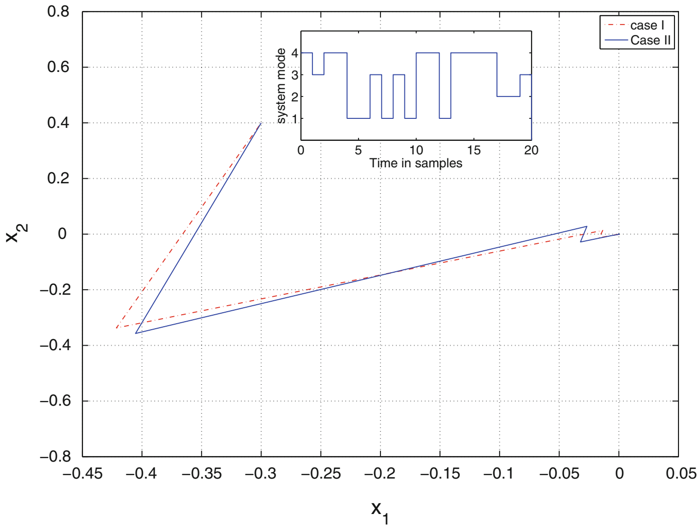

Fig. 2.4 State response of the closed-loop system under mode evolution r_k^2

Table 2.3 Controllers for transition rates matrix (TRM) (2.42)

Results	Controller gains
Theorem 2.18	$K_1 = [-0.47 - 0.53]$
	$K_2 = [-0.170.64]$
	$K_3 = [-0.33 - 0.14]$
Theorem 2.16	Infeasible

$$A_1 = \begin{bmatrix} -0.50 & -0.75 \\ 1 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} -2.4 & -0.33 \\ 1 & -1.4 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -0.20 & 0.1 \\ 1 & -1 \end{bmatrix}, B_1 = \begin{bmatrix} 5 \\ 0 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Assume the TRM is given by

$$TRM = \begin{bmatrix} -1.3 & \hat{\lambda}_{12} & \hat{\lambda}_{13} \\ 0.7 & -1.2 & 0.5 \\ \hat{\lambda}_{31} & \hat{\lambda}_{32} & -0.5 \end{bmatrix} \tag{2.42}$$

where $\hat{\lambda}_{ij}, \forall i \times j \in \mathcal{I} \times \mathcal{I}_{\mathcal{UK}}^{(i)}$ denote the unknown elements.

The purpose of this example is to verify the reduced conservatism of the obtained results in the continuous-time case. First, one can check that the open loop system is unstable by both Theorems 2.7 and 2.9. Then, based on Theorem 2.18, we obtain the controller gains for the system as shown in Table 2.3. However, it is verified that the stabilization criterion developed previously cannot yield a feasible solution of the controller, which shows that the developed approach is less conservative.

Notice that in Example 2.26, all the diagonal elements of TRM (2.42) are known. Now we further provide another example with unknown diagonal elements in the TRM to illustrate the dependency of controller design on the lower bound $\lambda_d^{(i)}$ of the corresponding unknown diagonal element.

Example 2.27 Consider MJLS (2.1) with four operation modes and the following system matrices:

$$A_1 = \begin{bmatrix} -15 & -7.5 \\ 10 & 10 \end{bmatrix}, \ A_2 = \begin{bmatrix} 2.4 & -3.3 \\ 10 & 14 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -2 & 1 \\ 10 & 10 \end{bmatrix}, \ A_4 = \begin{bmatrix} 10 & -2.3 \\ 10 & -11 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, B_4 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Table 2.4 Controllers for transition rates matrix (TRM) (2.43)

Results	Solutions of controller gains
Theorem 2.18 ($\lambda_d^{(2)} = -1$)	$K_1 = [-60.69 - 119.67]$
	$K_2 = [-1.020.07] \times 10^3$
	$K_3 = [96.3681.97]$
	$K_4 = [72.1625.86]$
Theorem 2.18 ($\lambda_d^{(2)} = -1.5$)	$K_1 = [-132.19 - 219.41]$
	$K_2 = [-1.780.13] \times 10^3$
	$K_3 = [169.73145.87]$
	$K_4 = [115.4736.99]$
Theorem 2.18 ($\lambda_d^{(2)} = -2.5$)	Infeasible
Theorem 2.16	Infeasible

The TRM is given by

$$TRM = \begin{bmatrix} -1.3 & 0.2 & \hat{\lambda}_{13} & \hat{\lambda}_{14} \\ \hat{\lambda}_{21} & \hat{\lambda}_{22} & 0.5 & 0.5 \\ 0.1 & \hat{\lambda}_{32} & -2.5 & \hat{\lambda}_{34} \\ 0.4 & 0.2 & 0.6 & -1.2 \end{bmatrix} \tag{2.43}$$

In the 2nd row of TRM (2.43), the diagonal element $\hat{\lambda}_{22}$ is unknown, we assign its lower bound $\lambda_d^{(2)}$ a priori with different values. It can be checked that the open-loop system is unstable based on Theorems 2.7 and 2.9 for any $\lambda_d^{(2)} \in (-\infty, -1]$. Then, by Theorems 2.16 and 2.18 with different $\lambda_d^{(2)}$, we obtain the controller gains as shown in Table 2.4.

It is seen from Table 2.4 that the obtained controller gains are dependent on $\lambda_d^{(2)}$. By applying the bisection method with the conditions in Theorem 2.18, one can further obtain the minimal value of $\lambda_d^{(2)}$, below which the stabilizing controller will not exist (here we get $\underline{\lambda}_d^{(2)} = -2.2758$ by some standard numerical software). It is also worth mentioning here that, for some systems, one may obtain that the controller solution is independent on the bound of diagonal elements, as the system in Example 2.24 shows that the controller exists despite that $\hat{\lambda}_{22}$ is unknown and has no given lower bound.

Example 2.28 Consider MJLS (2.2) with four operation modes and the following system matrices

$$A_1 = \begin{bmatrix} 1 & -1.25 \\ 2.5 & -2.5 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.25 & -0.83 \\ 2.5 & -3.5 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0.5 & -0.25 \\ 2.5 & -3.0 \end{bmatrix}, \; A_4 = \begin{bmatrix} 1.5 & -0.56 \\ 2.5 & -2.75 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, B_4 = \begin{bmatrix} 0.8 \\ -1 \end{bmatrix}.$$

Table 2.5 Controllers for transition probabilities matrix (TPM) (2.44)

Results	Controller gains
Theorem 2.21	$K_1 = [1.87 - 1.60]$
	$K_2 = [1.28 - 1.56]$
	$K_3 = [-5.427.02]$
	$K_4 = [0.76 - 1.37]$
Theorem 2.20	Infeasible

Moreover, the TPM is given by

$$TPM = \begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.4 \\ \hat{\pi}_{21} & 0.2 & 0.3 & \hat{\pi}_{24} \\ \hat{\pi}_{31} & \hat{\pi}_{32} & 0.5 & 0.3 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix} \tag{2.44}$$

The comparison of Theorem 2.20 with Theorem 2.21 is summarized in Table 2.5, where the reduction of conservatism of the new criterion is demonstrated.

2.5 Summary

In this chapter, we have investigated the stability and stabilization problems for a class of continuous-time and discrete-time MJLSs with partially unknown TRs and TPs, respectively. The considered systems are more general than the systems with completely known or completely unknown TRs (or TPs), which can be viewed as two special cases of the ones we tackled here. The LMI-based sufficient stochastic stability and stabilization conditions for the underlying systems are derived for both continuous-time and discrete-time context. Then, necessary and sufficient criteria, which reduces the conservatism of the previously derived sufficient conditions, are obtained by fully exploiting the properties of the TRM and TPM. Numerical examples are presented to show the validity and applicability of the developed results.

Although it has been shown that the methodologies used in obtaining the necessary and sufficient conditions are less conservative, we prefer the ones employed in the sufficient stability conditions to carry out other studies in the chapters of Part I. The corresponding extensions can be carried out as future works of this book.

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