Chapter 2
Stability and Stabilization

Abstract This chapter is devoted to the stability analysis for the four types of time-dependent switched systems. We first summarize the mainly used multiple Lyapunov-like functions (MLFs) approaches, sort out several typical MLFs and review their characteristics. Then, the corresponding methodologies by different MLFs such as switched Lyapunov function, MLFs with μ-times increase at switching instants are developed to study the stability conditions for the underlying system with arbitrary switching, dwell time (DT) switching, average dwell time (ADT) switching, persistent dwell time (PDT) switching and their mode-dependent forms, respectively. The generally analytic results in discrete-time context will be concretely formulated in terms of linear matrix equalities (LMIs). Numerical examples are provided to verify the obtained criteria and to compare the four kinds of time-dependent switched systems. The results obtained in this chapter will lay a foundation for future developments in this book.

2.1 Multiple Lyapunov-Like Functions

In this chapter, consider the unforced discrete-time switched linear systems given by

$$x(k+1) = A_{\sigma(k)}x(k) \tag{2.1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $\sigma(k)$ is the switching signal, which is a piecewise constant function of time and takes its values in the finite set $\mathcal{I} = \{1, \ldots, M\}$, $M > 1$ is the number of subsystems. The switching sequences $k_0, k_1, k_2, \ldots, k_s, \ldots$ are unknown a priori, but are known instantly, in which the switching instant is denoted as k_s, $s \in \mathbb{Z}_+$. When $k \in [k_s, k_{s+1})$, the $\sigma(k_s)$th subsystem (or system mode) is said to be *activated* and the length of the current running time of the subsystem is $k_{s+1} - k_s$. Other descriptions about system (2.1) can be referred to Sect. 1.3 of Chap. 1.

Our aim is to find the stability criteria for the time-dependent switched system (2.1). The following definitions are needed to precisely state what is "stable" in the context of switched systems.

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Definition 2.1 ([1]) The switched system (2.1) is globally uniformly asymptotically stable (GUAS) if there exists a class $\mathcal{KL}$ function $\beta(\|\cdot\|, \cdot)$ such that for all switching signals σ and all initial conditions $x(k_0)$, the solutions of (2.1) satisfy the inequality $\|x(k)\| \leq \beta(\|x(k_0)\|, k - k_0), \forall k \geq k_0$.

Definition 2.2 ([1]) The switched system (2.1) is globally uniformly exponentially stable (GUES) if for constants $c > 0$, $0 < \lambda < 1$, all switching signals σ and all initial conditions $x(k_0)$, the solutions of (2.1) satisfy the inequality $\|x(k)\| \leq c\lambda^{(k-k_0)} \|x(k_0)\|, \forall k \geq k_0$.

Note that for switched systems with above sets of switching signals, the uniformity in above definitions means the uniformity over all switching signals and the set of switching signals with the DT, ADT, PDT properties and their mode-dependent forms, respectively. This differs from the general time-varying system, where the uniformity is only with respect to the initial conditions.

In the stability analysis of switched systems, a common situation is that a global Lyapunov function (GLF) for all subsystems may not exist, or although it does exist, it may be hard to construct and the techniques based on GLF approach are also conservative. Therefore, the so-called MLFs approach is proposed and gradually improved. The key point of MLFs approach is to construct individual Lyapunov (or energy) function for each subsystem and appropriately concatenate these functions at switching instants, aiming to offer more possibilities to demonstrate the stability. Here, the "Lyapunov-like" means that the function associated with each subsystem is nonincreasing within that subsystem, but may increase during the running time of other subsystems. It can be seen that a distinct characteristic of MLFs is that the function values are allowed to jump unlike the continuity in the setting of GLF.

In this section, both the general MLF approach and several special MLFs approaches will be reviewed which are useful to analyze the systems in the discrete-time context. The conservatism and applicability of each MLF will be shown such that the appropriate MLFs can be selected for the time-dependent switched systems with specific switching signals.

Lemma 2.3 *([2]) For the discrete-time nonlinear system $\Sigma_{\sigma(k)} : x(k+1) = f_{\sigma(k)}(x(k))$, $\forall \sigma(k) = i \in \mathcal{I}$, $\mathcal{I} = (1, \ldots N)$. Suppose that the equilibrium point is at the origin. For the Lyapunov functions V_i, $\forall i \in \mathcal{I}$, if at the switching instants k_s and k_v, we have*

$$\sigma(k_s) = \sigma(k_v), \quad \forall s < v, \tag{2.2}$$

where $\sigma(k_s)$ is the switching signal, and the following conditions hold

(a) $\triangle V \triangleq V_i(x(k+1)) - V_i(x(k)) \leq 0, \forall i \in \mathcal{I}$, *when subsystem Σ_i is active;*

(b) $V_{\sigma(k_v)}(x(k_v)) - V_{\sigma(k_s)}(x(k_s)) < 0$;
then the switched system composed by $\Sigma_1, \ldots, \Sigma_N$ is Lyapunov-stable under a certain switching signal.

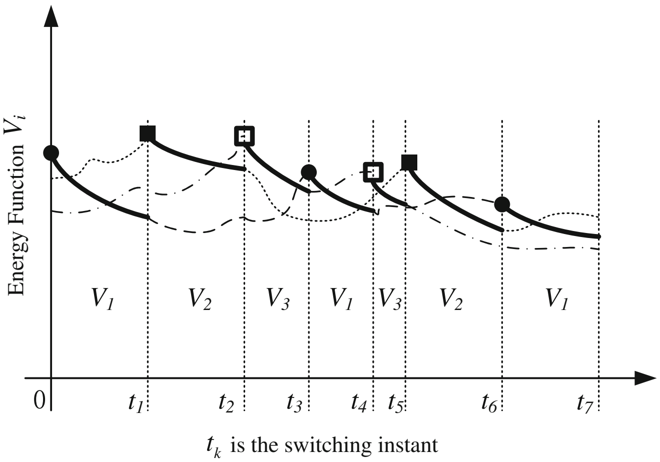

Fig. 2.1 Multiple Lyapunov function ($N = 3$)

The general idea of MLFs approach stated in Lemma 2.3 can be illustrated in Fig. 2.1, taking $N = 3$ for an example. As shown there, it does not only require the Lyapunov stability of each subsystem, but also the monotonically decreasing Lyapunov function values at two consecutive switching instants to a single system. A solid line in the figure denotes that the corresponding subsystem is active.

It can be seen that Lemma 2.3 gives a Lyapunov stability criterion for switched systems from a general perspective. However, owing to the requirement of comparing the Lyapunov function values at two consecutive switching instants to a signal subsystem, Lemma 2.3 is merely able to be applied in the qualitative description of the stability instead of giving the specific and easily-checked conditions. In particular, it is hardly applicable to the cases when the systems are involved with complex behaviors, e.g., uncertainty, nonlinearity and time delays.

Based on the requirements in the two conditions of Lemma 2.3, the evolutions of the general MLFs never ceased during the past decades, targeting the easily-checked stability criteria with less conservatism for switched systems with different regularities. Figure 2.2 shows four typical forms evolved from the general MLFs in Fig. 2.1. We would introduce them one by one by comparing their advantages and disadvantages, respectively.

(1) As shown in Fig. 2.2a, during the running time of each subsystem, the prescribed MLF relaxes condition (a) in Lemma 2.3 into the following condition

$$V_i\left(x(k)\right) \leq h\left(V_i\left(x(k_s)\right)\right), k \in (k_s, k_{s+1}) \tag{2.3}$$

The corresponding MLF is called weak Lyapunov function. From (2.3), allowing the value of the weak Lyapunov function to increase during the running time of each subsystem, the stability criterion is less conservative. However, the rising amplitude h of Lyapunov function value during the running time of each subsystem is hard to be determined, then it is difficult to derive an easily-checked stability criterion, even though the switched system without any complex dynamics, such as uncertainties, time delays.

(2) As shown in Fig. 2.2b, the Lyapunov function is monotonically decreasing during the running time of each subsystem, which is consistent with condition (a) in Lemma 2.3. However, the Lyapunov function value at the switching instant is required to be not higher than that at last switching instant, i.e., condition (b) in Lemma 2.3 can be rewritten as

$$V_j\left(x(k_{s+1})\right) - V_i\left(x(k_s)\right) < 0 \tag{2.4}$$

Therefore, the stability criterion is more conservative, in that this kind of MLF demands a further requirement of general MLF at the switching instant. Such an approach would be very applicable to the stability analysis for switched systems with DT and PDT switching. It could establish a good tradeoff between the less conservative stability criteria (compared with the MLF in Fig. 2.2c) and the easily-checked conditions. However, the investigations based on this approach remain largely open.

(3) As shown in Fig. 2.2c, the Lyapunov function value at instant $k_s + 1$ is non-increasing compared with that at instant k_s, then the corresponding condition (2) in Lemma 2.3 can be rewritten as

$$V_j\left(x(k_s + 1)\right) - V_i\left(x(k_s)\right) \le 0, \quad i \ne j \tag{2.5}$$

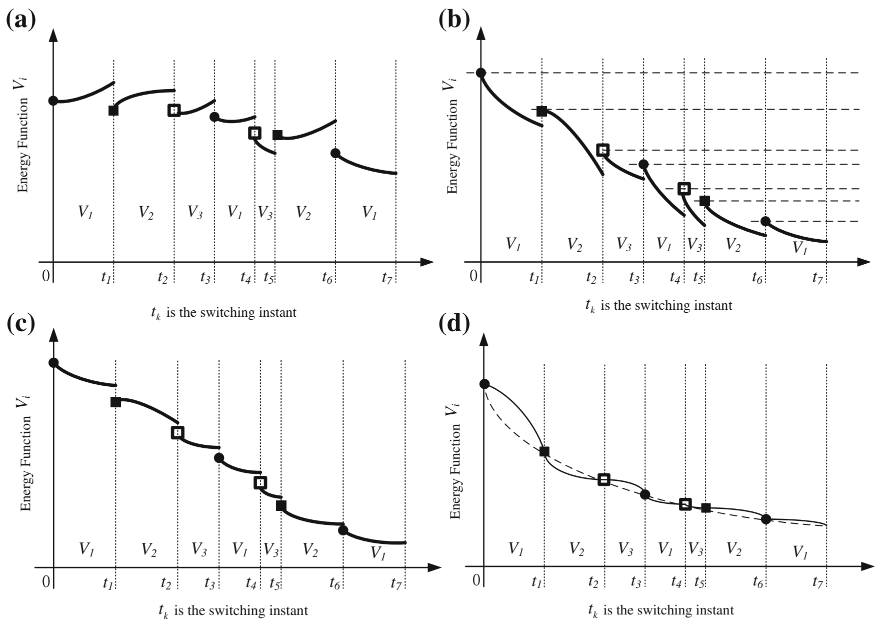

Fig. 2.2 Several MLFs with different forms

Note that k_{s+1} is the next switching instant and $k_s + 1$ is the next sample time to the switching instant.

The MLF with this form is called switched Lyapunov function (SLF) that is coined in [3]. The corresponding stability criterion is much more conservative, since the SLF gives a further requirement between the two function values at the switching instant. However, in the case of discrete time, each subsystem must satisfy $\triangle V \triangleq V_i\left(x(k+1)\right) - V_i\left(x(k)\right) \leq 0$. Then the requirement at the switching instant in (2.5) and condition a) in Lemma 2.3 can be uniformly rewritten as

$$\triangle V \triangleq V_j\left(x(k+1)\right) - V_i\left(x(k)\right) \leq 0 \tag{2.6}$$

where $i = j$ denotes that the switched system is during the running time of ith subsystem, and $i \neq j$ means that the switched system is at the switching instant from ith subsystem to jth subsystem. Therefore, the characteristic of unifying the requirement for Lyapunov functions at the switching instant with that during the running time of each subsystem reduces the difficulty in deriving the stability criterion of discrete-time switched linear system. Then the obtained criterion is quite easy to be checked. As a result, based on the SLF method, other analysis and synthesis problems for the switched linear system under arbitrary switching in the discrete-time domain can be solved effectively, e.g., controller design, filter design, model reduction and so forth.

(4) The MLF shown in Fig. 2.2d, requires constructing the Lyapunov function on each subsystem. Meanwhile, the Lyapunov function values are required to be continuous at the switching instant, which is similar to that in the common Lyapunov function. However, the rates of decay may vary among different running time of each subsystem and the required Lyapunov function is still multiple, which is generally defined as piecewise Lyapunov function in some literature. This kind of form facilitates the analysis for state-dependent switched linear systems in the continuous-time domain according to the continuous feature of function value at the switching instant.

From the above analysis, it can be observed that although the analysis approach via weak Lyapunov function is less conservative, it is inconvenient to derive the stability conditions. The general MLF in Fig. 2.1 requires the comparison between the Lyapunov function values at two consecutive switching instants to a single subsystem, and the MLF in Fig. 2.2b requires the comparison between the Lyapunov function values at two consecutive switching instants. However, in the discrete-time domain, for the nominal linear systems, the system state at each sample time can be given by iterating the state space expression. Thus it is possible to obtain the specific stability criterion for the discrete-time switched linear system. However, when the result is extended to the uncertain switched systems or the stabilization issue, the power of system matrices may appear in the iterative process, and it becomes hard to eliminate the power only based on the MLF introduced above (a further quasi-time-dependent Lyapunov function can be resorted to for a solution). In addition, for the general MLF, note that in the situation that the switched system is under arbitrary switching and the number of subsystems $N = 2$, or under a certain switching rule or a periodic switching sequence, the DT can be employed to derive the stability criterion. Finally, for the SLF, although the requirement of the Lyapunov function values at the switching instant is relatively strict, the method unifies the requirement for Lyapunov function at the switching instant with that during the running time of subsystems, since it is unnecessary to use system state at each sample time, which is used to compare the Lyapunov function at the switching instant. In short, the easily-checked stability criterion can be obtained by the SLF approach. In the following sections, the stability analysis of the approaches based on the aforementioned MLFs will be studied and the comparison of their advantages in concern of the conservatism, easily-checked feature, extensibility, etc., will be addressed.

2.2 Arbitrary Switching

This section will utilize the SLF in Fig. 2.2c to conduct the stability analysis for a class of discrete-time switched nominal linear systems. Consider a class of discrete-time switched linear system (2.1), our objective here is to achieve the stability criterion by constructing the SLF with the requirement at the switching instants in Fig. 2.2c as well as the following expression

$$V(x(k)) = x(k)^T P_i x(k) \tag{2.7}$$

Firstly, the Schur complement lemma is recalled, which will be used in the proof of the main results.

Lemma 2.4 ([4]) *The linear matrix inequality*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} < 0$$

where $S_{11} = S_{11}^T$ *and* $S_{22} = S_{22}^T$ *is equivalent to*

$$S_{11} < 0,\, S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0;\, S_{22} < 0,\, S_{11} - S_{12}^T S_{22}^{-1} S_{12} < 0.$$

By Lemma 2.4, the following theorem presents the asymptotic stability conditions for system (2.1).

Theorem 2.5 *Under arbitrary switching, the discrete-time switched linear system (2.1) is asymptotically stable if there exist matrices* $P_i > 0, \forall i \in \mathcal{I}$ *such that*

$$\begin{bmatrix} -P_j & P_j A_i \\ \star & -P_i \end{bmatrix} < 0, \quad \forall (i, j) \in \mathcal{I} \times \mathcal{I} \tag{2.8}$$

Proof Consider positive constants $\alpha > 0$ and $\beta > 0$. Then the SLF in (2.7) satisfies,

$$\alpha \parallel x(k) \parallel \leq V(x(k)) \leq \beta \parallel x(k) \parallel \tag{2.9}$$

Note that, at the sampling instant $k+1$, $k \in [0, \infty)$, the system may switch into another subsystem. Thus, the following equation can be obtained, $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$

$$\Delta V = V(x(k+1)) - V(x(k)) = x^T(k)(A_i^T P_j A_i - P_i)x(k) \tag{2.10}$$

where, $i = j$ denotes the switched system is during ith subsystem and $i \neq j$ means that the switched system is at the switching instant from the ith subsystem to jth subsystem. Thus, if

$$A_i^T P_j A_i - P_i < 0, \quad \forall (i, j) \in \mathcal{I} \times \mathcal{I} \tag{2.11}$$

we have $\Delta V < 0$, which means that system (2.1) is asymptotically stable. By Lemma 2.4, (2.8) is equivalent to (2.11). □

The stability criteria in Theorem 2.5 are obtained via the SLF approach. If some constraints are placed on the matrices, e.g., letting $P_i \equiv P, \forall i \in \mathcal{I}$, the corresponding stability conditions based on the GLF approach can be obtained as follows.

Proposition 2.6 *If there exists matrix $P > 0$ such that*

$$\begin{bmatrix} -P & PA_i \\ \star & -P \end{bmatrix} < 0, \quad \forall i \in \mathcal{I}$$

then there exist GLFs in the form of $V(k, x(k)) = x(k)^T P x(k)$, which indicates that the system (2.1) is asymptotically stable.

Remark 2.7 Theorem 2.5 presents the stability condition for discrete-time switched systems under arbitrary switching, which is in the form of LMIs. By using the SLF approach, the corresponding results can be easily derived for discrete-time switched systems with complex dynamics such as uncertainty, time delays and so forth, which we will show part of extensions in the following chapters.

For discrete-time switched system (2.1), it should be pointed out that the reason why the switched system is not stable under arbitrary switching is due to the fact that some (at least one) of the subsystems satisfy $\|A_i\| \geq 1, i \in \mathcal{I}$. Actually, the following facts hold for system (2.1)

$$\|x(k+1)\| = \|A_i x(k)\| \leq \|A_i\| \, \|x(k)\|$$

If $\|A_i\| < 1, \forall i \in \mathcal{I}$, we have $\|x(k+1)\| < \|x(k)\|$, and it is straightforward that $\|x(k)\| \to 0$ as $k \to \infty$ for arbitrary switching signal. Therefore, our discussion for the stability analysis of switched system under arbitrary switching is based on the assumption that at least one of the subsystems satisfies $\|A_i\| \geq 1, i \in \mathcal{I}$.

Furthermore, for switched system with polytopic uncertainty in (1.5) and (1.6), similarly,

$$\|x(k+1)\| = \|A_i(\lambda)x(k)\| = \left\|\sum_{m=1}^{s} \lambda_m A_{i,m} x(k)\right\|$$
$$\leq \sum_{m=1}^{s} \lambda_m \left\|A_{i,m}\right\| \|x(k)\|$$

If $\left\|A_{i,m}\right\| < 1$, $\forall i \in \mathcal{I}$, $\forall m \in S$, we have $\|x(k+1)\| < \|x(k)\|$, which indicates the asymptotic stability of uncertain switched system. Thus it can be concluded that at least one vertex matrix satisfies $\left\|A_{i,m}\right\| > 1$, $i \in \mathcal{I}$, $m \in S$, for the discrete-time switched system with polytopic uncertainties.

Considering the discrete time-delay switched system (1.9), if $\|A_i\| + \|A_{di}\| \leq 1$, $\forall\sigma(k) = i \in \mathcal{I}$, we have

$$\begin{aligned}
\|x(1)\| &= \left\|A_{\sigma(k)}x(0) + A_{d\sigma(k)}x(0)\right\| \leq \|x(0)\| \\
\|x(2)\| &= \left\|A_{\sigma(k)}x(1) + A_{d\sigma(k)}x(1 - d(1))\right\| \leq \|x(0)\| \\
&\vdots \\
\|x(d_M)\| &= \left\|A_{\sigma(k)}x(d_M - 1) + A_{d\sigma(k)}x(d_M - 1 - d(d_M - 1))\right\| \leq \|x(0)\| \\
\|x(d_M + 1)\| &= \left\|A_{\sigma(k)}x(d_M) + A_{d\sigma(k)}x(d_M - d(d_M))\right\| \leq \|x(0)\|
\end{aligned}$$

Therefore, for a time-delay switched system, if the condition $\|A_i\| + \|A_{di}\| \leq 1$ is satisfied, the stability can be guaranteed straightforwardly.

More details on stability of uncertain switched systems and switched systems with time delays will be elaborated in later chapters. The discussions on the norm of A can be used for a simple judgement or pre-check on the stability conditions.

Example 2.8 Consider the discrete-time switched linear system (2.1) comprising of two subsystems

$$A_1 = \begin{bmatrix} 0.4 & -0.8 \\ 0.5 & 1.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.1 & -0.3 \\ 0.7 & 0.4 \end{bmatrix}$$

First we can check that $\|A_1\| > 1$, $\|A_2\| > 1$, which indicates that the stability cannot be obtained directly. Our objective here is to test the stability of the system under arbitrary switching. According to Theorem 2.5, by using LMI Toolbox, we can get that there exist feasible solutions to (2.8). The obtained Lyapunov matrices are shown as follows.

$$P_1 = \begin{bmatrix} 124.8798 & 95.2966 \\ 95.2966 & 199.9269 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 126.1364 & 70.7540 \\ 70.7540 & 114.1096 \end{bmatrix}$$

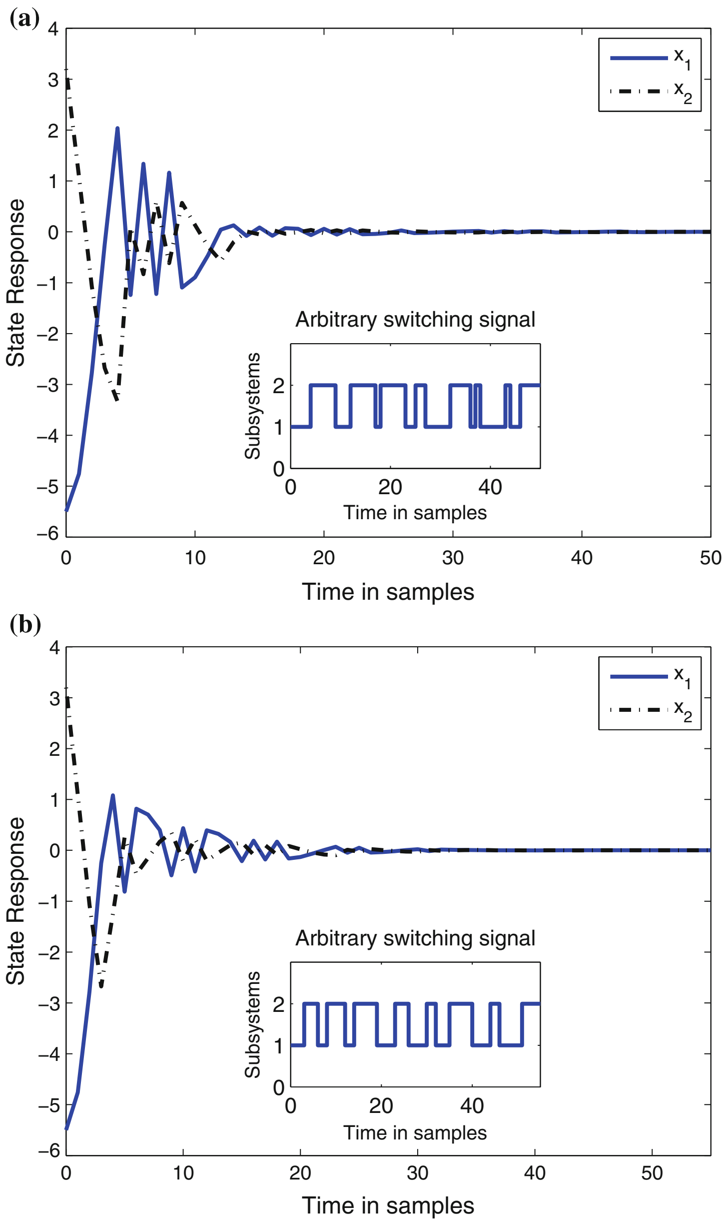

Fig. 2.3 State responses under two arbitrary switching signals

Then it can be concluded that this switched system is stable for any switching signals. In order to test this conclusion, two random switching sequences are generated by the following algorithm

Algorithm 2.1 (*Switching signal generation*) **for** $sample\_T = 1$ **to** $Time\_Length$

$switching\_value =$ **rand**

if $swithing\_value \geq Con$;

$switching\_signal\ (sample\_T) = 2$;

else

$switching\_signal(sample\_T) = 1$;

end

end

where, the **rand** function generates random numbers which are uniformly distributed in the interval (0, 1). For given initial condition $x_0 = [\,-5.5\ 3.2\,]^T$, Fig. 2.3 illustrates the state responses of the switched system under the two arbitrary switching signals, respectively. It can be seen that all the trajectories converge to zero, which proves the validity of Theorem 2.5.

2.3 Dwell Time (DT) Switching

This section will employ the MLFs in Figs. 2.1 and 2.2b, respectively, to investigate the stability problem for system (2.1) with dwell time (DT) constraint on the switching. The corresponding stability criteria of such systems are derived via LMI formulation.

To begin with, based on the general MLF in Fig. 2.1, the same Lyapunov function as shown in (2.7) can be constructed. By Lemma 2.3, one feature of the general MLF is that the values of the Lyapunov function at two consecutive switching instants should be compared as a requirement. However, if $N > 2$, it is almost impossible to determine, under arbitrary switching, which subsystem among $\{A_1, \ldots A_N\}$ the real system will switch into. Thus, although the DT of each subsystem is available, it is still difficult to formulate the state expression of one subsystem at two consecutive switching instants through iteration. Correspondingly, the Lyapunov function in (2.7) can not be obtained, let alone the stability analysis based on Lemma 2.3. On the other hand, if $N = 2$, namely, there exist only two subsystems, in which M_1 and M_2, respectively, representing the DT of each subsystem, and k is one of the switching instant, one can obtain that $x(k + M_1 + M_2) = A_2^{M_2} A_1^{M_1} x(k)$ (or $A_1^{M_2} A_2^{M_1} x(k)$) and consequently, both the state expression at two consecutive switching instants and the corresponding Lyapunov function in (2.7) are available. In addition, if the switching signal of the system (2.1) is under a certain switching rule and the DT is available, it is straightforward to obtain the state values at the corresponding switching instants and the values of the Lyapunov function through iteration. Then, if the switching signals are regulated by the following switching rule:

$$i \xrightarrow{i+1,\ldots,N,1,\ldots,i-1} i, \quad \forall i \in \mathcal{I} \tag{2.12}$$

i.e., one rule of the cyclic switching, we have

$$
\begin{aligned}
& x(k + M_i + M_{i+1} + \cdots + M_N + M_1 + \cdots + M_{i-1}) \\
& = A_{i-1}^{M_{i-1}} \ldots A_1^{M_1} A_N^{M_N} \ldots A_{i+1}^{M_{i+1}} A_i^{M_i} x(k)
\end{aligned}
$$

where k is assumed to be the instant of the system switching into subsystem A_i. Furthermore, the stability analysis for the corresponding switched system can also be conducted. The two-mode system under arbitrary switching (namely, $N = 2$) will be covered as a special case of the rule in (2.12). In what follows, the switched system under cyclic switching will be coped with and a stability criterion by using the general MLF will be presented.

Theorem 2.9 *Suppose that the switched system (2.1) switches into subsystem A_i at the switching instant k_s, then after switching into subsystem $A_i, A_{i+1}, \ldots, A_N, A_1, \ldots, A_{i-1}$ consecutively under cyclic switching (2.12), and finally, switches into subsystem A_i again at the switching instant k_v. Let $M_i, \forall i \in \mathcal{I}$ denote the DT of each subsystem. Then the system (2.1) is asymptotically stable under cyclic switching (2.12) if there exist matrices $P_i > 0, \forall i \in \mathcal{I}$ such that*

$$
A_i^T P_i A_i - P_i \leq 0 \tag{2.13}
$$

$$
\Upsilon_i^T P_i \Upsilon_i - P_i \leq 0 \tag{2.14}
$$

where $\Upsilon_i \triangleq A_{i-1}^{M_{i-1}} \ldots A_1^{M_1} A_N^{M_N} \ldots A_{i+1}^{M_{i+1}} A_i^{M_i}$.

Proof $\forall i \in \mathcal{I}$, system (2.1) can be described by the model of subsystem A_i. Thus if (2.13) holds, one obtains

$$
\Delta V_i \triangleq V_i(x(k+1)) - V_i(x(k)) = x^T(k)(A_i^T P_i A_i - P_i)x(k) < 0
$$

which means that subsystem A_i is stable and satisfies condition (a) in Lemma 2.3. Meanwhile, in Fig. 2.1, we have $V_i(x(k_s)) = x^T(k_s)P_i x(k_s)$ at the switching instant k_s when the system switches into subsystem A_i. Then at the instant k_v when the system switches into subsystem A_i again, system (2.1) has passed through subsystems $A_i, A_{i+1}, \ldots, A_N, A_1, \ldots, A_{i-1}$ consecutively under cyclic switching (2.12). Thus, at the switching instant k_v,

$$
x(k_v) = A_{i-1}^{M_{i-1}} \ldots A_1^{M_1} A_N^{M_N} \ldots A_{i+1}^{M_{i+1}} A_i^{M_i} x(k_s) = \Upsilon_i x(k_s)
$$

and $V_i(x(k_v)) = x^T(k_v)P_i x(k_v) = (\Upsilon_i x(k_s))^T P_i \Upsilon_i x(k_s)$. Therefore,

$$
V_i(x(k_v)) - V_i(x(k_s)) = x^T(k_s)(\Upsilon_i^T P_i \Upsilon_i - P_i)x(k_s)
$$

It is obvious to see that if (2.14) holds, $V_i(x(k_v)) - V_i(x(k_s)) < 0$ and condition (2) in Lemma 2.3 is satisfied. Thus, according to Lemma 2.3, system (2.1) is Lyapunov-stable under cyclic switching (2.12). □

By (2.14) in Theorem 2.9, it is clear that when the switching signals are arbitrary, Υ_i is unavailable, which indicates that the general MLF approach is inapplicable to conduct the stability analysis for the switched system under arbitrary switching. According to (2.4), it can be seen that for the underlying MLF, the approach does not require the comparison between the Lyapunov function values at two consecutive switching instants, and instead, requires the comparison between the Lyapunov function values at two neighboring switching instants. Suppose that the system switches into subsystem A_i at the switching instant k_s, then switches into subsystem A_j $(k_{s+1} - k_s = M_i)$, $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$ at the switching instant k_{s+1}. Thus, with the availability of the DT M_i, $i \in \mathcal{I}$, we obtain that $x(k_s + M_i) = A_i^{M_i} x(k_s)$, which is the state expression between two neighboring switching instants. Furthermore, the expression of the corresponding Lyapunov function can be obtained by (2.7). Through the above analysis, the stability criterion of switched system (2.1) with the MLF in Fig. 2.2b can be derived as follows.

Theorem 2.10 *Suppose that the M_i, $\forall i \in \mathcal{I}$ denote the DT of each subsystem. Then system (2.1) is asymptotically stable if there exist matrices $P_i > 0$, $\forall i \in \mathcal{I}$ such that*

$$A_i^T P_i A_i - P_i \le 0, \quad \forall i \in \mathcal{I} \tag{2.15}$$

$$(A_i^{M_i})^T P_j A_i^{M_i} - P_i < 0, \quad \forall (i, j) \in \mathcal{I} \times \mathcal{I}, i \neq j \tag{2.16}$$

Proof $\forall i \in \mathcal{I}$, system (2.1) can be described by A_i. Thus if the inequality (2.15) holds, one obtains

$$\Delta V_i \triangleq V_i(x(k+1)) - V_i(x(k)) = x^T(k)(A_i^T P_i A_i - P_i)x(k) \le 0$$

which means that subsystem A_i is stable and satisfies condition (a) in Lemma 2.3. Meanwhile, in Fig. 2.2b, it is assumed that the system switches into subsystem A_i at the switching instant k_s, then switches into subsystem A_j $(k_{s+1} - k_s = M_i)$ at the switching instant k_{s+1}. Thus, $V_i(x(k_s)) = x^T(k_s)P_i x(k_s)$, $V_j(x(k_{s+1})) = x^T(k_{s+1})P_j x(k_{s+1})$. If (2.16) holds, one obtains

$$V_j(x(k_{s+1})) - V_i(x(k_s)) = x^T(k_s)((A_i^{M_i})^T P_j A_i^{M_i} - P_i)x(k_s) \le 0 \tag{2.17}$$

Now suppose that system (2.1) switches into subsystem A_i at the switching instant k_{s+n}, and through iteration of (2.17), we have $V_i(x(k_{s+n})) \le V_i(x(k_s))$, which satisfies condition (b) in Lemma 2.3 obviously. Thus, according to Lemma 2.3, system (2.1) is stable. □

Remark 2.11 Note that, one mode-independent case of Theorem 2.10 can be found in [5].

As shown in Theorems 2.9 and 2.10, with the availability of the DT of the subsystems, whether or not the solutions of the corresponding stability conditions in (2.13)–(2.14) and (2.15)–(2.16) are feasible can be verified by the corresponding functions of LMIs toolbox in MATLAB (also, other toolboxes like sedumi and Yalmip). It is worth mentioning that Theorems 2.9 and 2.10 merely give the sufficient conditions. In other words, for certain given values of the DT, the system has the potential to be stable even without the feasible solutions of the stability conditions.

So far, the stability analysis for the discrete-time switched linear systems with the general MLF and the MLF in Fig. 2.2b has been discussed. From Theorems 2.9 and 2.10, it can be seen that the derivations of the stability criteria are based on the assumption of viewing the DT of each subsystem as the known condition. However, when there exist uncertainties in the system matrices, the corresponding form of $A_i^{M_i}$ will appear, which inevitably brings the computational complexity of the matrices. Correspondingly, it will be difficult to conduct the stability analysis for the uncertain switched system, let alone dealing with other control and filtering issues. Thus, it is difficult to extend the approaches of the stability analysis for the system with such two sorts of MLFs if without additional tricks to the uncertain switched system. In Sect. 2.5, the so-called quasi-time-dependent (QTD) Lyapunov function will be constructed to overcome this difficulty that also exists in the switched systems with PDT switching. The readers can refer to [6] for more discussions how the QTD technique evolves and how the conservatism in non-QTD techniques can be reduced.

Example 2.12 Consider the discrete-time switched linear system (2.1), the system matrices of which are shown as follows

$$A_1 = \begin{bmatrix} 1.00 & 0.01 \\ -0.05 & 0.99 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.84 & 0.30 \\ -1.00 & 0.82 \end{bmatrix} \tag{2.18}$$

First, suppose that the DT of the subsystem $M_1 = 1 \sim 20$, $M_2 = 1 \sim 20$. Based on Theorem 2.9 derived from the general MLF and Theorem 2.10 derived from the MLF in Fig. 2.2b, the corresponding DT pairs guaranteeing the stability of system (2.18) are illustrated in Fig. 2.4a, b, respectively, ("•" represents the feasible DT). Obviously, the stability "region" of Fig. 2.4a is larger than that of Fig. 2.4b, which indicates that the general MLF approach is less conservative than the approach of the MLF in Fig. 2.2b when the number of subsystems $N = 2$. In addition, the stability criterion in Theorem 2.5 shows that the switched system (2.18) is unstable. Similarly, in Fig. 2.4, the case with the DT $M_1 = M_2 = 1$ also shows that the system is unstable. Note that in this case, Theorem 2.10 is equivalent to Theorem 2.9.

As shown from the above verification, although the stability analysis based on the SLF shows that the system is unstable, switched system (2.18) under arbitrary switching, i.e., the switching instant and the switching sequence of the subsystems are arbitrary, possesses the potential to become stable as the running time of the DT increases. This demonstrates that the SLF approach is more strict and conservative. However, as known in the conditions (2.14), (2.16) and Fig. 2.4, the DT of each subsystem has to be treated as the known condition in both Theorems 2.9 and 2.10, which means, the switching instant is not completely arbitrary. Therefore, the stability criteria for the systems based on the general MLF and the MLF in Fig. 2.2b suffer the computational complexity and application limitations (i.e., constrained switching) to a certain degree.

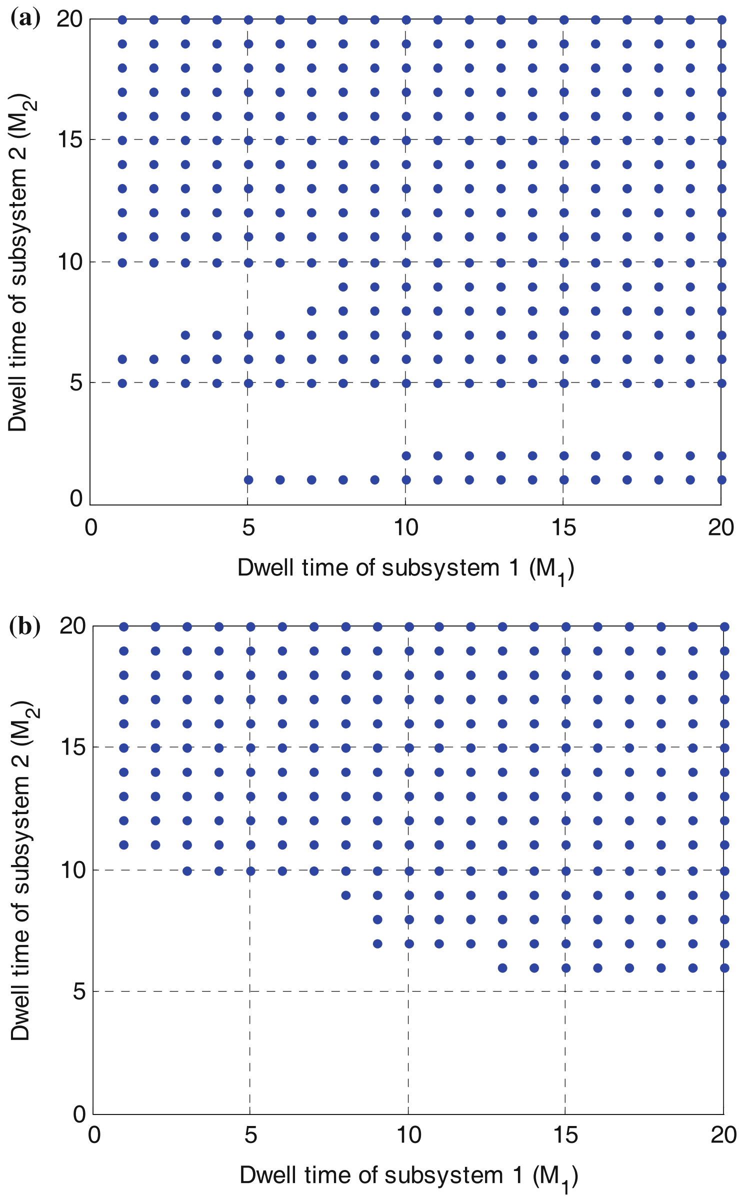

Fig. 2.4 The pairs of MDT such that the switched system is stable

2.4 Average Dwell Time (ADT) Switching

In what follows, the stability conditions for switched systems with average dwell time (ADT) switching will be given. For the convenience of a comparison, we also present the result in continuous-time domain that is first arrived at in [7]. The used Lyapunov function in [7] is sort of MLFs with μ-times increase at switching instants, as can be clearly seen from the derivations in the criteria and thus not listed in Fig. 2.2.

Theorem 2.13 ([7]) *Consider the continuous-time switched system $\dot{x}(t) = f_{\sigma(t)}(x(t))$, $\sigma(t) \in \mathcal{I}$, and let $\lambda > 0$, $\mu > 1$ be given constants. Suppose that there exist $\mathbb{C}^1$ functions $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}$, and two class $\mathcal{K}_\infty$ functions κ_1, κ_2 such that,*

$$\kappa_1(x(t)) \leq V_i(x(t)) \leq \kappa_2(x(t)) \tag{2.19}$$

$$\dot{V}_i(x(t)) \leq -\lambda V_i(x(t)) \tag{2.20}$$

and $\forall(\sigma(t_i) = i,\ \sigma(t_i^-) = j) \in \mathcal{I} \times \mathcal{I},\ i \neq j,$

$$V_i(x(t_i)) \leq \mu V_j(x(t_i)) \tag{2.21}$$

then the system is GUAS for any switching signal with ADT

$$\tau_a \geq \tau_a^* = \frac{\ln \mu}{\lambda} \tag{2.22}$$

Proof The proof of this theorem can be referred to [7] and is omitted here. □

Similar to the stability conditions for continuous-time switched systems, the corresponding results for the discrete-time case are given in the following theorem.

Theorem 2.14 ([8]) *Consider the discrete-time switched system $x(k+1) = f_{\sigma(k)}(x(k))$, $\sigma(k) \in \mathcal{I}$ and let $0 < \lambda < 1$ and $\mu > 0$, $\forall i \in \mathcal{I}$ be given constants. Suppose that there exist positive definite $\mathbb{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$ and two class $\mathcal{K}_\infty$ functions κ_1, κ_2 such that,*

$$\kappa_1(\|x(k)\|) \leq V_i(x(k)) \leq \kappa_2(\|x(k)\|) \tag{2.23}$$

$$\Delta V_i(x(k)) \leq -\lambda V_i(x(k)) \tag{2.24}$$

and $\forall(\sigma(k_i) = i,\ \sigma(k_{i-1}) = j) \in \mathcal{I} \times \mathcal{I},\ i \neq j,$

$$V_i(x(k_i)) \leq \mu V_j(x(k_i)) \tag{2.25}$$

then the system is GUES for any switching signal with ADT

$$\tau_a > \tau_a^* = -\frac{\ln \mu}{\ln(1-\lambda)} \tag{2.26}$$

Proof For $k \in \left[k_l, k_{l+1}\right)$, it follows from (2.24) that

$$V_{\sigma(k)}\left(x(k)\right) \leqslant (1-\lambda)^{k-k_l} V_{\sigma(k_l)}\left(x(k_l)\right) \tag{2.27}$$

Then, according to (2.25) and (2.27), one can obtain

$$\begin{aligned} V_{\sigma(k)}\left(x(k)\right) &\leqslant (1-\lambda)^{(k-k_l)} \mu V_{\sigma(k_{l-1})}\left(x(k_l)\right) \\ &\leqslant \cdots \leqslant (1-\lambda)^{(k-k_0)} \mu^{N_\sigma(k,k_0)} V_{\sigma(k_0)}\left(x(k_0)\right) \end{aligned}$$

From

$$N_\sigma(k, k_0) \leq N_0 + \frac{k-k_0}{\tau_a}$$

it is straightforward to get

$$V_{\sigma(k)}\left(x(k)\right) \leq \mu^{N_0}\left((1-\lambda)\,\mu^{1/\tau_\alpha}\right)^{(k-k_0)} V_{\sigma(k_0)}\left(x(k_0)\right)$$

In addition, for the considered Lyapunov function, it is trivial to know that

$$a_\sigma \left\|x(k)\right\| \leqslant V_\sigma\left(x\right) \leqslant b_\sigma \left\|x(k)\right\|, \sigma \in \mathcal{I}$$

for some $a_\sigma > 0$ and $b_\sigma > 0$. Then we have

$$a \left\|x(k)\right\| \leqslant V\left(x\right) \leqslant b \left\|x(k)\right\|$$

where $a \triangleq \inf\left(a_\sigma\right)$ and $b \triangleq \sup\left(b_\sigma\right)$.

Therefore, if the ADT satisfies (2.26), one can readily obtain

$$(1-\lambda)\,\mu^{1/\tau_\alpha} \leqslant (1-\lambda)\,\mu^{-\ln(1-\gamma)/\ln\mu} \leqslant \frac{(1-\lambda)}{(1-\lambda)} = 1$$

Denoting $\beta \triangleq \sqrt{(1-\lambda)\,\mu^{1/\tau_\alpha}}$, the system state satisfies

$$\left\|x(k)\right\|^2 \leqslant \frac{1}{a} V_{\sigma(k)}\left(x(k)\right) \leqslant \frac{b}{a}\mu^{N_0}\beta^{2(k-k_0)}\left\|x(k_0)\right\|^2$$

which means

$$\|x(k)\| \leqslant \sqrt{\frac{b}{a}\mu^{N_0}\beta^{(k-k_0)}}\,\|x(k_0)\|$$

thus the considered system is GUES, which completes the proof. □

Remark 2.15 It can be seen from Theorem 2.14 that when we increase the value of μ, the existence likelihood of the multiple Lyapunov function for the system stability will be increased, which means the stability of system can be ensured at the expense of increasing μ. In other words, for a given λ, the system stability will be directly dependent on μ. Note that the stability will also depend on decay rate of Laypunov function λ, however, it is not regarded as a design parameter in the section for simplicity.

In the following, the above results will be extended to the case of discrete-time switched systems with modal (MADT) switching.

Theorem 2.16 *Consider the discrete-time switched nonlinear system*

$$x(k+1) = f_{\sigma(k)}(x(k)), \quad \sigma(k) \in \mathcal{I} \tag{2.28}$$

and let $0 < \lambda_i < 1$ *and* $\mu_i \geq 1$, $i \in \mathcal{I}$ *be given constants. Suppose that there exist* $\mathbb{C}^1$ *functions* $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, *and class* $\mathcal{K}_\infty$ *functions* κ_{1i} *and* κ_{2i}, $i \in \mathcal{I}$, *such that* $\forall \sigma(k) = i \in \mathcal{I}$

$$\kappa_{1i}(\|x(k)\|) \leq V_i(x(k)) \leq \kappa_{2i}(\|x(k)\|) \tag{2.29}$$

$$\Delta V_i(x(k)) \leq -\lambda_i V_i(x(k)) \tag{2.30}$$

and $\forall(\sigma(k_p) = i, \sigma(k_{p-1}) = j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$,

$$V_i(x(k_i)) \leq \mu_i V_j(x(k_i)) \tag{2.31}$$

then the system is GUAS for any switching signal with MADT

$$\tau_{ai} > \tau_{ai}^* = -\frac{\ln \mu_i}{\ln(1-\lambda_i)} \tag{2.32}$$

Proof For any $K > 0$, let $k_0 = 0$ and denote $k_1, k_2, \ldots, k_p, k_{p+1}, \ldots, k_{N_\sigma(K,0)}$ the switching times on the interval $[0, K]$, where $N_\sigma(K, 0) = \sum_{i=1}^{N} N_{\sigma i}(K, 0)$.

By (2.30), $\forall i \in \mathcal{I}$,

$$V_i(x(k+1)) - V_i(x(k)) < 0 \tag{2.33}$$

and

$$V_i(x(k+1)) \leq (1-\lambda_i)V_i(x(k)) \tag{2.34}$$

(2.34) together with (2.31) imply

$$\begin{aligned}
V_{\sigma(k_{p+1})}(x(k_{p+1})) &\leq \mu_{\sigma(k_{p+1})} V_{\sigma(k_{p+1}-1)}(x(k_{p+1})) \\
&\leq \mu_{\sigma(k_{p+1})} V_{\sigma(k_{p+1}-1)}(x(k_{p+1}-1))(1-\lambda_{\sigma(k_{p+1}-1)}) \\
&= \mu_{\sigma(k_{p+1})}(1-\lambda_{\sigma(k_p)}) V_{\sigma(k_p)}(x(k_{p+1}-1)) \\
&\leq \mu_{\sigma(k_{p+1})}(1-\lambda_{\sigma(k_p)})^{k_{p+1}-k_p} V_{\sigma(k_p)}(x(k_p)) \\
&\quad \cdots \\
&\leq \prod_{q=0}^{p} \mu_{\sigma(k_{q+1})} \prod_{q=0}^{p} (1-\lambda_{\sigma(k_q)})^{k_{q+1}-k_q} V_{\sigma(k_0)}(x(k_0))
\end{aligned}$$

Then, by (2.34), one obtains

$$\begin{aligned}
&V_{\sigma(K)}(x(K)) \\
&\quad \leq (1-\lambda_{\sigma(k_{N_\sigma})})^{K-k_{N_\sigma}} V_{\sigma(k_{N_\sigma})}(x(k_{N_\sigma})) \\
&\quad \leq (1-\lambda_{\sigma(k_{N_\sigma})})^{K-k_{N_\sigma}} \prod_{j=0}^{N_\sigma-1} \mu_{\sigma(k_{j+1})} \prod_{j=0}^{N_\sigma-1} (1-\lambda_{\sigma(k_j)})^{k_{j+1}-k_j} V_{\sigma(0)}(x(0)) \\
&\quad = \prod_{i=1}^{N} \mu_i^{N_{\sigma i}} \prod_{i=1}^{N} (1-\lambda_i)^{T_i} V_{\sigma(0)}(x(0)) \\
&\quad = \prod_{i=1}^{N} \mu_i^{N_{\sigma i}} \exp\left\{\sum_{i=1}^{N} [T_i \ln(1-\lambda_i)]\right\} V_{\sigma(0)}(x(0)) \\
&\quad \leq \exp\left\{\sum_{i=1}^{N} N_{0i} \ln \mu_i\right\} \exp\left\{\sum_{i=1}^{N} \frac{T_i}{\tau_{ai}} \ln \mu_i + \sum_{i=1}^{N} \ln(1-\lambda_i) T_i\right\} V_{\sigma(0)}(x(0))
\end{aligned}$$

Thus, if there exist constants τ_{ai}, $i \in \mathcal{I}$ satisfying (2.32), the following holds

$$\begin{aligned}
&V_{\sigma(K)}(x(K)) \\
&\quad \leq \exp\left\{\sum_{i=1}^{N} N_{0i} \ln \mu_i\right\} \exp\left\{\max_{i \in \mathcal{I}} \left[\frac{\ln \mu_i}{\tau_{ai}} + \ln(1-\lambda_i)\right] K\right\} V_{\sigma(0)}(x(0))
\end{aligned}$$

Then, it can be concluded that $V_{\sigma(K)}(x(K))$ converges to zero as $K \to \infty$ if the MADT satisfies (2.32), and the asymptotic stability can be obtained with the aid of (2.29). □

Remark 2.17 It can be seen from Theorems 2.13 and 2.14 that the parameters λ and μ are the same for all subsystems, i.e., mode-independent. However, the parameters in Theorem 2.16 are mode-dependent. It can be concluded that $\tau_{ai}^* \leq \tau_a^*$, $\forall i \in \mathcal{I}$, and the mode-dependent features would reduce the conservativeness existed in Theorems 2.13 and 2.14. In fact, note that if $\tau_a = \tau_{ai}$, $\forall i \in \mathcal{I}$, one readily knows from

Definition 1.7 that

$$\sum_{i \in \mathcal{I}} N_{\sigma i}(T,t) \leq \sum_{i \in \mathcal{I}} N_{0i} + \sum_{i \in \mathcal{I}} \frac{T_i}{\tau_a}, \quad \forall T \geq t \geq 0$$

Thus, there exist positive numbers $N_0 = \sum_{i \in \mathcal{I}} N_{0i}$ and $\tau_a = \tau_{ai}$ such that

$$N_\sigma(T,t) \leq N_0 + \frac{T-t}{\tau_a}, \ \ \forall T \geq t \geq 0$$

That is, a switching signal with bounded MADT τ_{ai}^* also has bounded ADT $\tau_a^* \equiv \tau_{ai}^*$, $\forall i \in \mathcal{I}$ in the special case of $\lambda \equiv \lambda_i$, $\mu \equiv \mu_i$, $\forall i \in \mathcal{I}$. From this, it can be concluded that the MADT switching has the advantage of flexibility for a switched system where the switching is able to or needs be designed.

For the issue of stabilizing controller design, consider the switched linear system given as

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{2.35}$$

Our objective here is to find an admissible controller in the form of

$$u(k) = K_{\sigma(k)}x(k) \tag{2.36}$$

where $K_{\sigma(k)}$ is to be determined. Then, the resulting closed-loop system is given by

$$x(k+1) = \bar{A}_{\sigma(k)}x(k) \tag{2.37}$$

where $\bar{A}_{\sigma(k)} \triangleq A_{\sigma(k)} + B_{\sigma(k)}K_{\sigma(k)}$.

Next, based on the results obtained above, we first give the stability conditions for switched systems (2.35) with MADT switching.

Theorem 2.18 *Consider the switched linear system (2.35) when $u(k) \equiv 0$ and let $0 < \lambda_i < 1$ and $\mu_i \geq 1$, $\forall i \in \mathcal{I}$ be given constants. If there exist matrices $P_i > 0$, $\forall i \in \mathcal{I}$ such that $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$,*

$$A_i^T P_i A_i + \lambda_i P_i - P_i \leq 0 \tag{2.38}$$

$$P_i - \mu_i P_j \leq 0 \tag{2.39}$$

then the switched linear system (2.35) is GUES with MADT satisfying (2.32).

Proof Construct the Lyapunov function as follows

$$V_i(x(k)) = x^T(k)P_i x(k), \forall \sigma(k) = i \in \mathcal{I} \tag{2.40}$$

where P_i, $\forall i \in \mathcal{I}$, is a positive definite matrix satisfying (2.38) and (2.39). Then, from (2.30), (2.31), (2.35) and (2.40), it is not hard to obtain, $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$,

$$\begin{aligned}
&\Delta V_i(x(k)) + \lambda_i V_i(x(k)) \\
&\quad = \lambda_i x^T(k) P_i x(k) + x^T(k) A_i^T P_i A_i x(k) - x^T(k) P_i x(k) \\
&\quad = x^T(k) \left(A_i^T P_i A_i + \lambda_i P_i - P_i\right) x(k)
\end{aligned}$$

and

$$V_i(x(k_i)) - \mu_i V_j(x(k_i)) = x^T(k_i)\left(P_i - \mu_i P_i\right) x(k_i)$$

Thus, if (2.38) and (2.39) hold, system (2.35) is GUES for any switching signal with MADT (2.32). □

Now, we are in a position to give the existence conditions of a stabilizing controller for system (2.35) with the MADT switching.

Theorem 2.19 *Consider the switched linear system (2.35) and let* $0 < \lambda_i < 1$ *and* $\mu_i \geq 1$, $\forall i \in \mathcal{I}$ *be given constants. If there exist matrices* $U_i > 0$ *and* T_i, $\forall i \in \mathcal{I}$ *such that* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$,

$$\begin{bmatrix} -U_i & A_i U_i + B_i T_i \\ \star & -(1 - \lambda_i U_i) \end{bmatrix} \leq 0 \tag{2.41}$$

$$U_j \leq \mu_i U_i \tag{2.42}$$

then there exists a stabilizing controller such that system (2.35) is GUAS for any switching signal with MADT satisfying (2.32). Moreover, if (2.41) and (2.42) have a solution, the admissible controller can be given by

$$K_i = T_i U_i^{-1} \tag{2.43}$$

Proof Theorem 2.18 implies that if

$$\begin{aligned}
\bar{A}_i^T P_i \bar{A}_i + \lambda_i P_i - P_i &\leq 0 \\
P_i - \mu_i P_j &\leq 0
\end{aligned}$$

system (2.35) is GUAS for any switching signal with MADT satisfying (2.32). Considering (2.36), setting $U_i \triangleq P_i^{-1}$ and $T_i \triangleq K_i P_i^{-1}$, it can be seen that, if (2.41) holds, (2.38) is satisfied. Moreover, if (2.42) holds, one can obtain that $U_j - \mu_i U_i \leq 0$. By Lemma 2.4, $U_j - \mu_i U_i \leq 0$ can be rewritten as

$$\Lambda \triangleq \begin{bmatrix} -\mu_i U_i & I \\ I & -U_j^{-1} \end{bmatrix} \leq 0$$

Furthermore, note that $\Lambda \leq 0$ is equivalent to $-U_j^{-1} - I(\mu_i U_i)^{-1} I \leq 0$ by Lemma 2.4. Additionally, if the inequalities (2.41) and (2.42) have feasible solutions, the admissible controller gains can be given by (2.43) since $T_i = K_i P_i^{-1}$, which ends the proof. □

Remark 2.20 From the above analysis mentioned, the ADT switching can be viewed as a special case of MADT switching. The stabilizing conditions in the situation of ADT switching can be achieved directly from Theorem 2.19 and therefore are omitted here.

In the following, a numerical example in discrete-time domain will be presented to demonstrate the potential and validity of the results obtained in Sect. 2.4.

Example 2.21 Consider the discrete-time switched linear system (2.35) consisting of three subsystems described by

$$A_1 = \begin{bmatrix} 3.9 & 1.5 \\ 2.5 & 2.3 \end{bmatrix}, B_1 = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}, A_2 = \begin{bmatrix} 1.4 & 0.3 \\ 1 & -2.7 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, A_3 = \begin{bmatrix} -2.2 & 0.1 \\ -2 & -0.4 \end{bmatrix}, B_3 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

Our purpose here is to design a mode-dependent stabilizing controller and find the admissible switching signals with MADT such that the resulting closed-loop system is GUAS.

To illustrate the advantages of the proposed MADT switching scheme, the design results of both controllers and switching signals should be presented for the systems with ADT switching for the sake of comparison. By different approaches and setting the relevant parameters appropriately, the computation results for system (2.35) with two different switching schemes are listed in Table 2.1.

It can be seen from Table 2.1 that the minimal MADT are reduced to $\tau_{a1}^* = 1$, $\tau_{a2}^* = 1$, $\tau_{a3}^* = 4$, for given $\mu = \mu_1 = \mu_2 = \mu_3 = 2$, and one special case of MADT switching is $\tau_a^* = \tau_{a1}^* = \tau_{a2}^* = \tau_{a3}^* = 4$ by setting $\lambda = \lambda_1 = \lambda_2 = \lambda_3 = 0.2$, which is the ADT switching, i.e., the designed MADT switching is more general.

To further show the merit of MADT switching, let us now consider the resulting closed-loop system performances. Applying the obtained controller, under the scheme of ADT switching and MADT switching, respectively, the obtained state responses for each closed-loop subsystem are shown in Fig. 2.5. For each closed-loop subsystem $\bar{A}_i$, it is clear to see that the transient behavior for both subsystem 2 and 3 under controllers Γ_1 and Γ_2 are similar, while the response of subsystem 1 under Γ_2 is much better than that under Γ_1.

Table 2.1 Computation results for the system under two different switching schemes

Switching schemes	ADT switching	MADT switching
Controller gains	Γ_1: $K_1 = [36.16\ 18.90]$ $K_2 = [-7.94\ -8.16]$ $K_3 = [21.08\ 1.30]$	Γ_2: $K_1 = [41.67\ 22.69]$ $K_2 = [-8.64\ 6.83]$ $K_3 = [21.18\ 1.06]$
Switching signals	$\tau_a^* = 4$ $(\mu = 2, \lambda = 0.2)$	$\tau_{a1}^* = 1$, $\tau_{a2}^* = 1$, $\tau_{a3}^* = 4$ $(\mu_1 = \mu_2 = \mu_3 = 2$, $\lambda_1 = 0.97, \lambda_2 = 0.8, \lambda_3 = 0.2)$

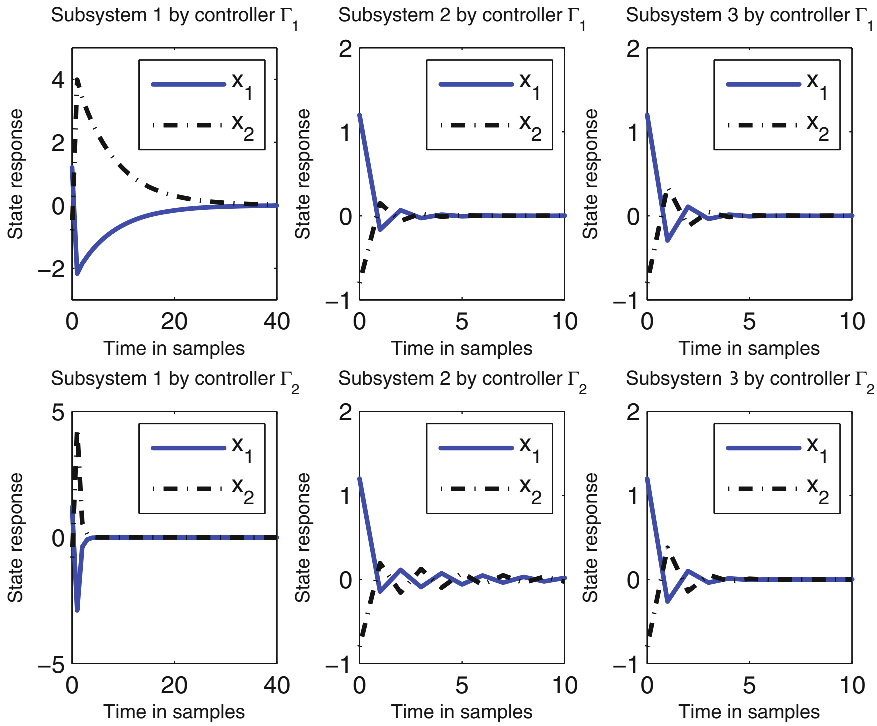

Fig. 2.5 The state response comparisons of the closed-loop subsystems by controllers Γ_1 and Γ_2

Then, generating one possible switching sequence with the ADT property and the MADT property, one can obtain the corresponding state responses of the closed-loop system as shown in Figs. 2.6 and 2.7, respectively, for the same initial state condition. It can be seen from the curves that the state response of closed-loop system is fluctuated under the ADT switching scheme, but can converge to zero in a short time under the MADT switching scheme. To present the reason more clearly, denote the running time of the ith subsystem at the lth working as $t_{i,l}$, $\forall i \in \mathcal{I}, l \in \mathbb{N}^+$, and use $t_{i,l}^A$ and $t_{i,l}^M$ to represent the running time of the subsystem under ADT and MADT switching schemes, respectively. It can be observed that the state responses both begin with subsystem 2 and in Fig. 2.6, $t_{2,1}^A = 2$ and in Fig. 2.7, $t_{2,1}^M = 2$. Then, due to the constraint of ADT switching ($\tau_a = 4$ in Fig. 2.6), we need $t_{1,1}^A \geq 6$. For the case of MADT switching, the constraint on $t_{1,1}^M$ can be removed. The comparison of the switching signals in Figs. 2.6 and 2.7 shows that even for $t_{1,1}^M < t_{1,1}^A$ (we want $t_{1,1}^A$ to be a little shorter), we can attain $t_{1,1}^M < t_{1,1}^A$. This will better the state response because of the shorter running time needed on the subsequent subsystem 1, which demands longer time to converge to zero as shown in Fig. 2.5.

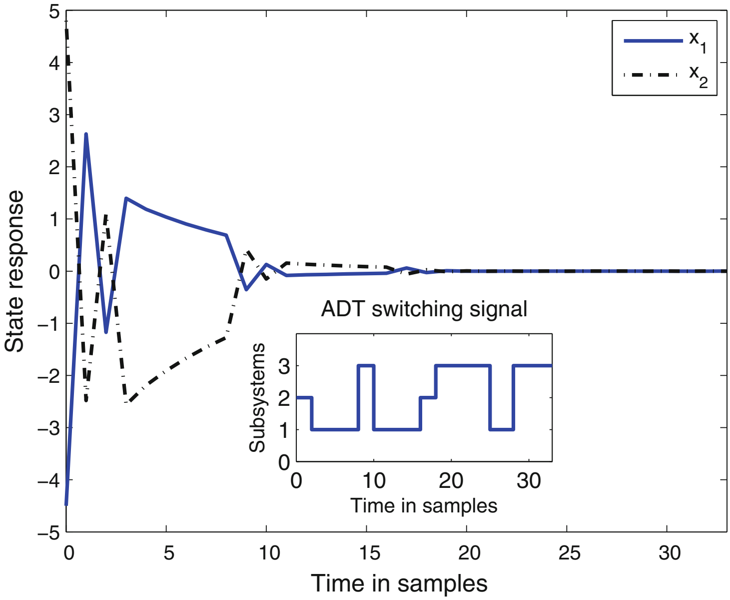

Fig. 2.6 State response of the closed-loop system by controllers Γ_1 under switching signal σ with $\tau_a = 4$

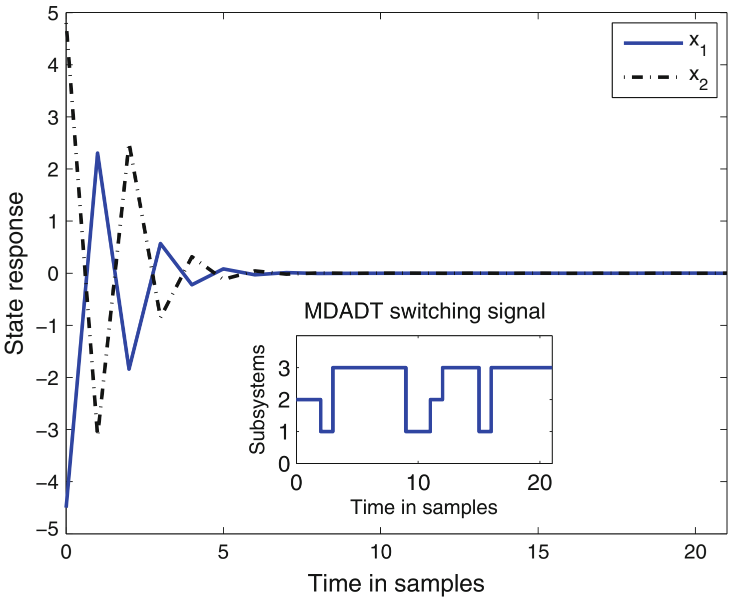

Fig. 2.7 State response of the closed-loop system by controllers Γ_2 under switching signal σ with $\tau_{a1} = 1$, $\tau_{a2} = 1$, $\tau_{a3} = 4$

Thus, from the above discussions, it can be concluded that it will be more flexible in practice to design an MADT switching to perfect or improve the system performances with fewer constraints.

2.5 Persistent Dwell Time (PDT) Switching

Consider the discrete-time switched linear system (2.35), a more general switching signal, "modal persistent dwell-time (MPDT)", is introduced in this section, which not only generalizes the commonly studied DT and ADT switchings, but also further attaches mode-dependency to the PDT switching. The definition of MPDT has been given in Sect. 1.4 of Chap. 1 and is therefore omitted here.

The example below is used to show what an admissible MPDT switchingsequence (we use $\bar{\xi}_{\tau^{[T]}}(k)$ denote inadmissible switching sequences) is. Consider a switched system consisting of three subsystems. The admissible MPDT set, with the period of persistence $\mathbb{T} = 3$, is supposed to be $\tau^{[3]} = \{4, 3, 5\}$. Then $\xi^a_{\tau^{[3]}}(11) = \{1, 1, 1, 1, 2, 1, 3, 2, 2, 2, 2\}$ is an admissible sequence, but both $\bar{\xi}^a_{\tau^{[3]}}(11) = \{1, 1, 1, 3, 2, 1, 3, 3, 3, 3, 3\}$ and $\bar{\xi}^b_{\tau^{[3]}}(11) = \{1, 1, 1, 1, 2, 1, 3, 1, 2, 2, 2\}$ are not since the requirements of $\tau_1 \geq 4$ and $\mathbb{T} \leq 3$ are not satisfied in the former and latter cases, respectively.

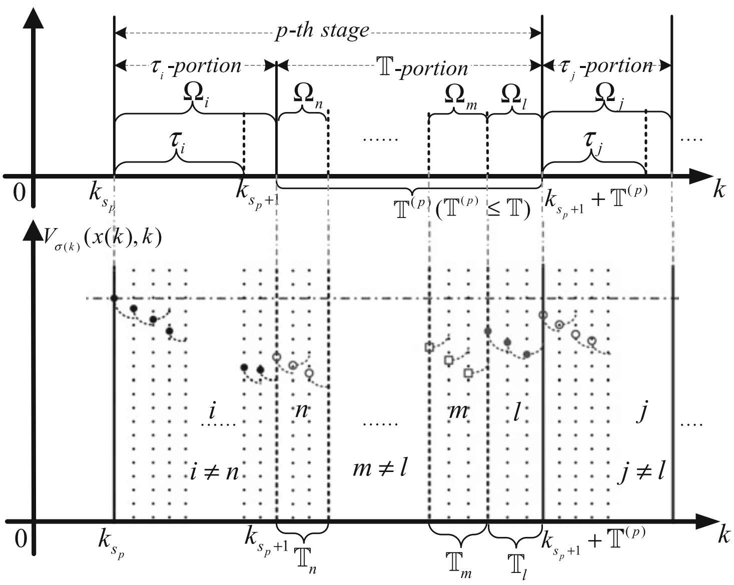

Fig. 2.8 A scenario of MPDT switching (on the *top*), where the period of persistence is $\mathbb{T}$, $i \neq n$, $m \neq l$, $j \neq l$ and $\mathbb{T}^{(p)} \leq \mathbb{T}$. The figure on the *bottom* illustrates the variation of the Lyapunov function used in Theorem 2.22

An illustration on MPDT is given in Fig. 2.8, where the interval consisting of the running time (τ_i-portion) of a certain subsystem and the period of persistence ($\mathbb{T}$-portion) is considered as an *MPDT stage*,[1] and k_{s_p} is denoted as the initial instant of the pth stage, $p \in \mathbb{Z}_{\geq 1}$ with $k_{s_1} \geq k_0$ (here "$\geq$" means a period of persistence may exist before the 1st stage). Let the actual running time of the $\mathbb{T}$-portion at the pth stage be denoted as $\mathbb{T}^{(p)}$, $p \in \mathbb{Z}_{\geq 1}$, it holds that

$$\mathbb{T}^{(p)} \triangleq \sum_{r=1}^{\mathcal{Q}(k_{s_p+1}, k_{s_{p+1}})} \mathbb{T}_{\sigma(k_{s_p+r})} \leq \mathbb{T} \tag{2.44}$$

where $\mathbb{T}_{\sigma(k_{s_p+r})} < \tau_i$ denotes the running time of the subsystem activated at the switching instant $k_{s_p+r} \in \left[k_{s_p+1}, k_{s_{p+1}}\right)$, $r \in \mathbb{Z}_{\geq 1}$, and $\mathcal{Q}(k_{s_p+1}, k_{s_{p+1}})$ stands for the switching times within $\left[k_{s_p+1}, k_{s_{p+1}}\right)$.

In this section, we would like to directly present the stabilization result for the underlying system. The used methodology is similar to the one in Fig. 2.2b, but compares the MLFs at the instants entering into two consecutive stages (can be seen in the derivations of later Theorem 2.18).The specific objectvies are to develop a control policy $\mathcal{F}_{\sigma(k)}(\cdot)$, and find a set of switching signals with admissible MPDT. Here, we are interested in developing a fundamental stabilizing state-feedback policy, but with the *quasi-time-dependent* (QTD) form below, as adopted in [6]

$$\mathcal{F}_{\sigma(k)}(x(k)) \triangleq F_{\sigma(k)}(\vartheta)x(k) \tag{2.45}$$

where ϑ is a scheduled index for the activated subsystem and can be computed online according to the following rules: $\forall \sigma(k) = i \in \mathcal{I}$,

(i) in the τ_i-portion,

$$\vartheta = \begin{cases} k - k_{s_p}, & k \in \left[k_{s_p}, k_{s_p} + \tau_i\right) \\ \tau_i, & k \in \left[k_{s_p} + \tau_i, k_{s_p+1}\right) \end{cases} \tag{2.46}$$

(ii) in the $\mathbb{T}$-portion,

$$\vartheta = k - H_r, k \in \left[k_{s_p+1}, k_{s_{p+1}}\right) \tag{2.47}$$

where $H_r \triangleq \arg\{\max(k_{s_p+r}, r \in \mathbb{Z}_{\geq 1} | k_{s_p+r} \leq k, k_{s_p+r} \in [k_{s_p+1}, k_{s_{p+1}}))\}$ satisfies $\sigma(H_r) = i$.

It has been demonstrated in [6] for switched systems with DT switching that the QTD state-feedback law outperforms the conventional one with less conservatism in achieving minimal DT ensuring the stability of the underlying system. In order to obtain the stabilization criterion by using (2.45) for system (2.35) under MPDT switching, we consider the corresponding QTD Lyapunov function as $V_{\sigma(k)}(x(k), \vartheta)$,

[1]We will slightly abuse the concept as a stage in this book.

where ϑ has been defined in (2.46) and (2.47). Then the stability conditions for the nominal system in nonlinear case can be first arrived at.

Theorem 2.22 *Consider a discrete-time switched nonlinear system $x(k+1) = f_{\sigma(k)}(x(k))$, and $0 < \alpha_i < 1$, $\mu_i > 0$ to be given constants. For a prescribed period of persistence $\mathbb{T}$, suppose that there exist functions $V_{\sigma(k)} : (\mathbb{R}^{n_x}, \mathbb{Z}_{[0,\tau_{\sigma(k)}]}) \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, and two class $\mathcal{K}_\infty$ functions κ_1 and κ_2 such that $\forall \sigma(k) = i \in \mathcal{I}$ and $r \in \mathbb{Z}_{[2,\mathcal{Q}(k_{s_p+1},k_{s_{p+1}})+1]}$*

(i) $\forall \vartheta \in \mathbb{Z}_{[0,\tau_i]}$,

$$\kappa_1(\|x(k)\|) \leq V_i(x(k), \vartheta) \leq \kappa_2(\|x(k)\|) \tag{2.48}$$

(ii) $\forall k \in \left[k_{s_p}, k_{s_p} + \tau_i\right)$,

$$V_i(x(k+1), k+1-k_{s_p}) \leq \alpha_i V_i(x(k), k-k_{s_p}) \tag{2.49}$$

(iii) $\forall k \in \left[k_{s_p} + \tau_i, k_{s_p+1}\right)$,

$$V_i(x(k+1), \tau_i) \leq \alpha_i V_i(x(k), \tau_i) \tag{2.50}$$

(iv) $\forall k \in [k_{s_p+1}, k_{s_{p+1}})$, $r \in \mathbb{Z}_{[1,\mathcal{Q}(k_{s_p+1},k_{s_{p+1}})]}$

$$V_i(x(k+1), k+1-H_r) \leq \alpha_i V_i(x(k), k-H_r) \tag{2.51}$$

(v) $\forall \sigma(k_{s_p+1}) = i \neq j = \sigma(k_{s_p+1} - 1)$,

$$V_i(x(k_{s_p+1}), 0) \leq \mu_j V_j(x(k_{s_p+1}), \tau_j) \tag{2.52}$$

(vi) $\forall \sigma(k_{s_p+r}) = i \neq j = \sigma(k_{s_p+r} - 1)$,

$$V_i(x(k_{s_p+r}), 0) \leq \mu_j V_j(x(k_{s_p+r}), \mathbb{T}_j) \tag{2.53}$$

where $\mathbb{T}_j \in \left[1, \min(\tau_i - 1, \mathbb{T}^{(p)})\right]$, $\forall i \in \mathcal{I}$, $\mathbb{T}^{(p)} \in \mathbb{Z}_{[1,\mathbb{T}]}$. Then the switched nonlinear system is GUAS for MPDT switching signals satisfying (2.48)–(2.53) and

$$\tau_i \geq \frac{(\mathbb{T}+1)\ln \mu_{\max} + \mathbb{T} \ln \alpha_{\max}}{-\ln \alpha_i} \tag{2.54}$$

where $\mu_{\max} \triangleq \max_{i\in\mathcal{I}} \mu_i$, $\alpha_{\max} \triangleq \max_{i\in\mathcal{I}} \alpha_i$.

Proof First of all, if $\mu_{\max}\alpha_{\max} < 1$, then it is straightforward that a switched system is GUAS with $\tau_i = 1$, i.e., under arbitrarily switching. If (2.54) holds, τ_i is at least 1 in discrete-time domain. Then the proof boils down to the case $\mu_{\max}\alpha_{\max} \geq 1$.

Considering $\sigma(k_{s_p}) = i, \sigma(k_{s_p+1} + \mathbb{T}^{(p)}) = j$ in the pth stage of MPDT switching, and supposing an arbitrary switching occurs within $\mathbb{T}^{(p)}$, it follows from (2.49)–(2.53) that

$$\begin{aligned}
&V_j(x(k_{s_p+1}+\mathbb{T}^{(p)}),0)\\
&\le \mu_l V_l(x(k_{s_p+1}+\mathbb{T}^{(p)}),\mathbb{T}_l)\\
&\le \mu_l {\alpha_l}^{\mathbb{T}_l} V_l(x(k_{s_p+1}+\mathbb{T}^{(p)}-\mathbb{T}_l),0)\\
&\le \mu_m\mu_l {\alpha_l}^{\mathbb{T}_l} V_m(x(k_{s_p+1}+\mathbb{T}^{(p)}-\mathbb{T}_l),\mathbb{T}_m)\\
&\le \mu_i\mu_n\cdots\mu_m\mu_l {\alpha_l}^{\mathbb{T}_l}{\alpha_m}^{\mathbb{T}_m}\cdots{\alpha_n}^{\mathbb{T}_n}{\alpha_i}^{k_{s_p+1}-k_{s_p}} V_i(x(k_{s_p}),0)\\
&\le \mu_{\max}^{\mathcal{Q}(k_{s_p},k_{s_p+1}+\mathbb{T}^{(p)})}{\alpha_{\max}}^{\mathbb{T}_l+\mathbb{T}_m+\cdots+\mathbb{T}_n}{\alpha_i}^{\tau_i}V_i(x(k_{s_p}),0)\\
&\le \mu_{\max}^{\mathbb{T}^{(p)}+1}\alpha_{\max}^{\mathbb{T}^{(p)}}{\alpha_i}^{\tau_i}V_i(x(k_{s_p}),0)
\end{aligned} \tag{2.55}$$

where $l, m, \ldots, n$ denote all the possible indices of subsystems being switched within $\mathbb{T}^{(p)}$.

Thus since $\mu_{\max}\alpha_{\max} \ge 1$ and $\mu_{\max}^{\mathbb{T}^{(p)}+1}\alpha_{\max}^{\mathbb{T}^{(p)}} \le \mu_{\max}^{\mathbb{T}+1}\alpha_{\max}^{\mathbb{T}^{(p)}}$ hold. From (2.55), it follows that

$$V_j(x(k_{s_p+1}+\mathbb{T}^{(p)}),0) \le \mu_{\max}^{\mathbb{T}+1}\alpha_{\max}^{\mathbb{T}}{\alpha_i}^{\tau_i}V_i(x(k_{s_p}),0)$$

Then, if (2.54) is satisfied, $\mu_{\max}^{\mathbb{T}+1}\alpha_{\max}^{\mathbb{T}}{\alpha_i}^{\tau_i} \le 1$ holds. Letting $\lambda_i \triangleq \mu_{\max}^{\mathbb{T}+1}\alpha_{\max}^{\mathbb{T}}{\alpha_i}^{\tau_i}$, $\lambda_{\max} \triangleq \max_{i\in\mathcal{I}}\lambda_i$, $\forall i \in \mathcal{I}$, and considering the fact that a period of persistence may exist before the 1st stage, it follows

$$\begin{aligned}
V_{\sigma(k_{s_p})}(x(k_{s_p}),0) &\le \lambda_{\max}V_{\sigma(k_{s_{p-1}})}(x(k_{s_{p-1}}),0) \le \cdots\\
&\le \lambda_{\max}^{p-1}V_{\sigma(k_{s_1})}(x(k_{s_1}),0) \le \lambda_{\max}^{p-1}\mu_{\max}^{\mathbb{T}}\alpha_{\max}^{\mathbb{T}}V_{\sigma(k_0)}(x(k_0),0).
\end{aligned}$$

From (2.48),

$$||x(k_{s_p})|| \le \kappa_1^{-1}(\lambda_{\max}^{p-1}\mu_{\max}^{\mathbb{T}}\alpha_{\max}^{\mathbb{T}}\kappa_2(||x(k_0)||))$$

holds. Thus, due to (2.48)–(2.53), $||x(k)|| \le \kappa_3(||x(k_0)||)$ holds, $\forall k \in (k_{s_p}, k_{s_{p+1}}]$, where

$$\kappa_3(\cdot) \triangleq \kappa_1^{-1}(\mu_{\max}^{\mathbb{T}}\alpha_{\max}^{\mathbb{T}}\kappa_2(\kappa_1^{-1}(\lambda_{\max}^{p-1}\mu_{\max}^{\mathbb{T}}\alpha_{\max}^{\mathbb{T}}\kappa_2(\cdot)))).$$

Thus the GUAS can be inferred by the denotation of $\lambda_{\max}$ and Definition 2.1. This completes the proof. □

Remark 2.23 It should be noted that since the running time of each activated subsystem during the period of persistence is unknown a priori, the worst case of using $\mu_{\max}$, $\alpha_{\max}$ in the derivation of (2.55) is taken into account, as well as the consideration of $\mathbb{T}$ times of switching during the period of persistence.

In Theorem 2.22, if $V_i(x(k),\vartheta) \equiv V_i(x(k))$, $\mu_i \equiv \mu$, $\alpha_i \equiv \alpha$, and $\mathbb{T} \equiv 0$, i.e., the time-dependent Lyapunov function and the PDT switching are considered, and the period of persistence vanishes, then the corresponding MPDT switching reduces to the DT case in the end, and the corresponding stability criterion is simplified to the following corollary.

Corollary 2.24 *Consider nominal system (2.35) with* $u(k) \equiv 0$*, and let* $0 < \alpha < 1$*,* $\mu > 1$ *are given constants. If there exist matrices* $P_i \in \mathcal{S}^n_{>0}$*,* $\forall i \in \mathcal{I}$*, such that* $\forall \sigma(k) = i \in \mathcal{I}$*,*

$$V_i(x(k+1)) \leq \alpha V_i(x(k))$$

holds and $\forall i \times j \in \mathcal{I} \times \mathcal{I}$*,* $i \neq j$*,*

$$V_i(x(k_{s_p+1})) \leq \mu V_j(x(k_{s_p+1}))$$

holds, then the switched system is GUAS for any switching signal with DT satisfying $\tau \geq -\ln\mu/\ln(1-\alpha)$*.*

Remark 2.25 A noteworthy fact is that the conditions in Corollary 2.24 are also the ones ensuring that the underlying switched systems are GUAS with ADT switching, as have been derived in [9]. Therefore, Theorem 2.22 obtained in this section is more general than the existing stability results on switched systems with either DT or ADT switching.

Remark 2.26 Note that, in the frame of ADT switching, the requirement on the ADT ensuring that the switched system is GUAS is also $\tau \geq -\ln\mu/\ln(1-\alpha)$ (cf. [10]), which holds for any $N_0 \geq 2$. However, the requirement in the DT case reduced from PDT switching when $\mathbb{T} \equiv 0$ only holds for $N_0 = 1$.

Then, by considering the QTD Lyapunov function as $V_{\sigma(k)}(x(k), \vartheta) \triangleq x^T(k)P_i(\vartheta)x(k)$, the stabilization criterion for nominal system (2.35) can be readily established in the following theorem.

Theorem 2.27 *Consider system (2.35) and let* $0 < \alpha_i < 1$*,* $\mu_i > 0$ *be given constants,* $i \in \mathcal{I}$*. Suppose there exist matrices* $S_i(\vartheta) \in \mathbb{S}^{n_x}_{>0}$ *and* $U_i(\vartheta)$*,* $\vartheta = 0, 1, \ldots, \tau_i$*,* $\forall i \in \mathcal{I}$*, such that* $\forall\, \vartheta = 0, 1, \ldots, \tau_i - 1$*,* $\forall i \in \mathcal{I}$

$$\begin{bmatrix} -S_i(\tau_i) & A_i S_i(\tau_i) + B_i U_i(\tau_i) \\ \star & -\alpha_i S_i(\tau_i) \end{bmatrix} \leq 0 \tag{2.56}$$

$$\begin{bmatrix} -S_i(\vartheta+1) & A_i S_i(\vartheta) + B_i U_i(\vartheta) \\ \star & -\alpha_i S_i(\vartheta) \end{bmatrix} \leq 0 \tag{2.57}$$

and $\forall (i \times j) \in \mathcal{I} \times \mathcal{I}$*,* $i \neq j$*,*

$$S_j(\mathbb{T}_j) - \mu_j S_i(0) \leq 0 \tag{2.58}$$
$$S_j(\tau_j) - \mu_j S_i(0) \leq 0 \tag{2.59}$$

hold, where $\mathbb{T}_j \in \mathbb{Z}_{[1,\min(\tau_j-1,\mathbb{T}^{(p)})]}$*,* $\mathbb{T}^{(p)} \in \mathbb{Z}_{[1,\mathbb{T}]}$ *with* $\mathbb{T}$ *be the given period of persistence. Then, the resulting closed-loop system is GUAS for MPDT switching signals* *satisfying (2.54). Moreover, the QTD stabilizing controller gain can be obtained by*

$$F_i(\vartheta) = U_i(\vartheta)S_i^{-1}(\vartheta).$$

Proof Based on Theorem 2.22, the proof can be completed by basic matrix manipulations and Lemma 2.4, cf. [11] and is omitted here.

Remark 2.28 In Theorem 2.27, a small τ_i corresponding to fast switching may not guarantee a feasible solution of the admissible controller, then considering α_i and μ_i to be variables, the MPDT can be minimized by solving the following minimization problem.

Problem 2.1

$$\min_{\mu_i, \alpha_i, S_i, U_i} \tau_i, \; s.t. (2.54), (2.56)\text{–}(7.84) \tag{2.60}$$

The minimum of τ_i can be trivially found by bisection method. Note that, for a fixed $\mathbb{T}$, the minimal MPDT means to be the one with smallest $||\tau^{[\mathbb{T}]}||_1$. Furthermore, if the minimal MPDT obtained in such a way are many, the smallest variance of $\tau^{[\mathbb{T}]}$ can be further used to refine them.

If setting $U_i(\vartheta) \equiv U_i$ and $S_i(\vartheta) \equiv S_i$ in Theorem 2.27, one can obtain the corresponding control policy with "non-QTD" controller gains $F_i = U_i S_i^{-1}$, $i \in \mathcal{I}$. As a result, for a certain switched system, the minimal MPDT obtained by an optimization problem similar to Problem 2.1, denoted by θ_i, will be generally greater than the minimal τ_i derived from the QTD control policy. Nevertheless, such non-QTD F_i can be directly used as the stabilizing state-feedback gains for system (2.35).

Example 2.29 Consider system (2.35) consisting of two subsystems described by

$$A_1 = \begin{bmatrix} 1.00 & -0.70 \\ 0.50 & -0.70 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.89 & 0.38 \\ 1.65 & 1.14 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

Our purpose here is to design a QTD stabilizing controller for the nominal system, find out the admissible MPDT switching such that the corresponding closed-loop system is GUAS. Firstly, it can be checked that the nominal switched system does not admit a stabilizing controller under arbitrary switching. By Theorem 2.27 and solving Problem 2.1 for given $\alpha_i = 0.15$, however, the minimal admissible MPDT τ_i can be solved as shown in Table 2.2 for given different $\mathbb{T}$, as well as the θ_i corresponding

Table 2.2 Minimal MPDT by QTD and non-QTD stabilizing controller for different $\mathbb{T}$

$\mathbb{T}$	2	3
(τ_i, θ_i)	$\tau_1 = 3, \tau_2 = 4$; $\theta_1 = 4, \theta_2 = 5$	$\tau_1 = 4, \tau_2 = 5$; $\theta_1 = 5, \theta_2 = 6$

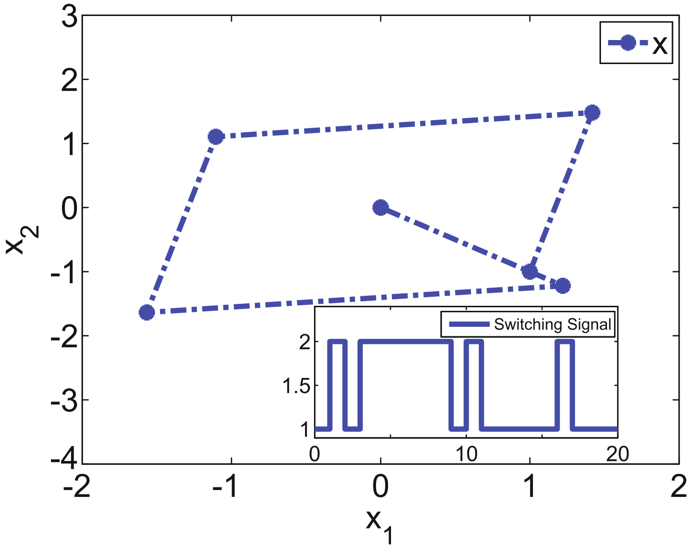

Fig. 2.9 State trajectories of the closed-loop system with MPDT switching ($\mathbb{T} = 3$)

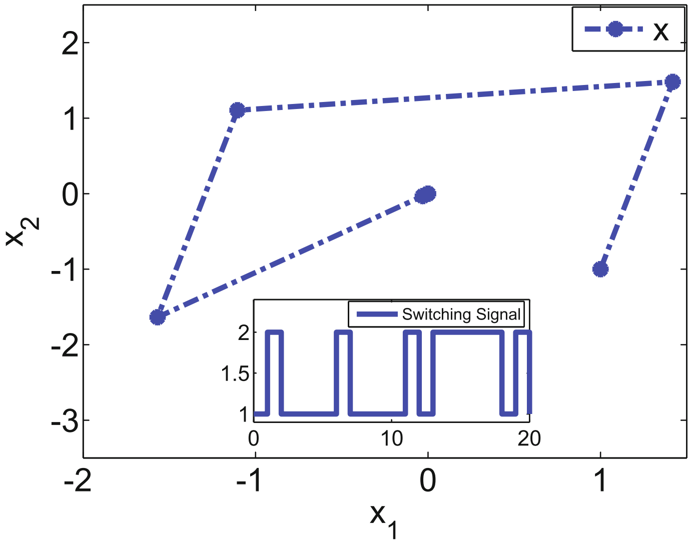

Fig. 2.10 State trajectories of the closed-loop system with MPDT switching ($\mathbb{T} = 2$)

to the non-QTD stabilizing control. It can be seen that the QTD controller has less conservatism in achieving shorter admissible MPDT. The associated controller gains in both cases are omitted here. Given the initial condition $x_0 = [1\ -1]^T$, considering the running time equivalent to the MPDT at each time of switching, and supposing that there exists a period of persistence before the first MPDT stage, the resulting switching signals and the state responses of the corresponding closed-loop system under $\mathbb{T} = 3$ and $\mathbb{T} = 2$ are presented in Figs. 2.9 and 2.10, respectively. It can be seen from Figs. 2.9 and 2.10 that the state trajectory of the resulting closed-loop system converges, verifying the validity of the QTD stabilizing controller.

2.6 Conclusion

In this chapter, we have addressed the stability and stabilization problems of switched systems with several typical time-dependent switching signals. The multiple Lyapunov functions (MLFs) including several evolved forms are introduced to serve as the tools for the stability analysis and stabilizing controller synthesis of switched systems. Specifically, the switched Lyapunov functions are utilized to derive the stability criteria for switched systems under arbitrary switching; the general MLFs and an evolved one (with the comparisons between the Lyapunov function values at two consecutive switching instants) for systems with DT switching; the MLFs with μ-times increase at switching instants for ADT switched systems; and that evolved MLFs but with the comparisons between the MLFs at the instants entering into two consecutive stages for PDT switched systems. Finally, four numerical examples were provided to illustrate the effectiveness of the obtained results.

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