Chapter 2
Satisfactory Interval-Valued Cores of Interval-Valued Cooperative Games

Abstract The aim of this chapter is to develop an effective nonlinear programming method for computing interval-valued cores of interval-valued cooperative games. In this chapter, we define satisfactory degrees (or ranking indexes) of comparing intervals with the features of inclusion and/or overlap relations and discuss their important properties. Hereby we construct satisfactory crisp equivalent forms of interval-valued inequalities. Based on the concept of interval-valued cores, we derive the auxiliary nonlinear programming models for computing interval-valued cores of interval-valued cooperative games and propose corresponding bisection algorithm, which can always provide global optimal solutions. The developed models and method can provide cooperative chances under the situation of inclusion and/or overlap relations between interval-type coalitions' values in which the Moore's interval ranking method (or order relation between intervals) may not assure that an interval-valued core exists. The proposed method is a generalization of that based on the Moore's interval ranking relation. The feasibility and applicability of the models and method proposed in this chapter are illustrated with a numerical example.

Keywords Interval-valued cooperative game • Core • Interval-valued core • Interval ranking • Mathematical programming • Bisection method

2.1 Introduction

Stated as earlier, cooperative games have many successful applications, especially in enterprise management and economics [1, 2]. However, in real situations, player coalitions' values may be imprecise and vague due to the uncertainty of information and the complexity of players' behavior. As a result, interval-valued cooperative games have been studied [3]. In the foregoing Chap. 1, we proposed the concept of the interval-valued least square solutions of interval-valued cooperative games and discussed their important properties. Hereby we developed fast and effective quadratic programming methods for computing such a kind of interval-valued solutions. The interval-valued least square solution is a single-valued solution concept of interval-valued cooperative games. It is well known that the concept of the core [4], which is a set-valued solution, plays an important role in (classical)

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cooperative games [5, 6]. In a very natural way, the core of cooperative games may be extended to the interval-valued core of interval-valued cooperative games. Thereby, Branzei et al. [7] studied the cooperative games under interval uncertainty and the convexity of the interval-valued undominated cores. By introducing the selection of interval-valued cooperative games, Alparslan-Gök et al. [8] and Alparslan-Gök et al. [9] investigated several interval-valued solution concepts of interval-valued cooperative games such as the interval-valued core, the interval-valued dominance core, and stable sets. To study existence of interval-valued cores, they also introduced the notion of Γ-balancedness and extended the Bondareva–Shapley theorem [10] for cooperative games to the interval setting. Han et al. [11] discussed a kind of interval-valued cores through defining a special order relation between intervals. Clearly, all the aforementioned works are conducted on the basis of the traditional interval ranking methods such as the Moore's order relation between intervals [12] (also see Eq. (1.4) in Chap. 1 for a detailed). As stated in the proceeding, these traditional interval ranking methods are relatively strict since they only consider the strict relations including the intersection and being greater whereas they do not consider the inclusion and/or overlap relations between intervals. Additionally, players may accept the inclusion and/or overlap relations between interval-type coalitions' values at some satisfactory degree in practical cooperation. Thus, the main purpose of this chapter is to develop an effective method for computing interval-valued cores of interval-valued cooperative games through introducing the concept of satisfactory degrees (or ranking indexes) of comparing intervals with the feature of the inclusion and/or overlap relations between interval-type coalitions' values.

The rest of this chapter is organized as follows. Section 2.2 gives the concept of satisfactory degrees of comparing intervals, discusses some useful and important properties, and constructs satisfactory crisp equivalent forms of interval-valued inequalities. In Sect. 2.3, we derive the auxiliary nonlinear programming models for computing interval-valued cores of interval-valued cooperative games and propose corresponding bisection method. In Sect. 2.4, a numerical example is used to illustrate the feasibility and applicability of the models and method proposed in this chapter.

2.2 Interval Comparison Satisfactory Degrees and Satisfactory Crisp Equivalent Forms of Interval-Valued Inequalities

The notation of intervals is stated as in Sect. 1.3.1. Namely, $\bar{a} = [a_L, a_R]$ is an interval on the set R of real numbers and $\bar{R}$ is the set of intervals on R.

Alternatively, an interval $\bar{a}$ may be expressed in mean-width (or center-radius) form as $\bar{a} =< m(\bar{a}), w(\bar{a}) >$, where

$$m(\overline{a}) = \frac{a_L + a_R}{2}$$

and

$$w(\overline{a}) = \frac{a_R - a_L}{2}$$

are the mid-point and half-width of the interval $\overline{a} \in \overline{R}$, respectively. Thus, for any intervals $\overline{a} =< m(\overline{a}), w(\overline{a}) >\in \overline{R}$ and $\overline{b} =< m(\overline{b}), w(\overline{b}) >\in \overline{R}$, we can rewrite the addition and the scalar multiplication as follows [13, 14]:

1. $\overline{a} + \overline{b} =< m(\overline{a}) + m(\overline{b}), w(\overline{a}) + w(\overline{b}) >$
2. $\overline{a} - \overline{b} =< m(\overline{a}) - m(\overline{b}), w(\overline{a}) + w(\overline{b}) >$
3. $\gamma\overline{a} =< \gamma m(\overline{a}), |\gamma| w(\overline{a}) >= \begin{cases} < \gamma m(\overline{a}), \gamma w(\overline{a}) > & \text{if } \gamma \geq 0 \\ < \gamma m(\overline{a}), -\gamma w(\overline{a}) > & \text{if } \gamma < 0, \end{cases}$

where $\gamma \in R$ is a real number.

2.2.1 Satisfactory Degrees of Interval Comparison and Properties

The ranking order of intervals is a difficult problem, which has been discussed by some researchers [12, 15, 16]. Moreover, most of the researches about interval-valued cooperative and non cooperative games are conducted on the basis of the interval order relations of Moore [12] and Ishihuchi and Tanaka [15]. Moore [12] believes that $\overline{a} \leq \overline{b}$ if $a_R \leq b_L$, depicted as in the case a of Fig. 1.1. By revising the above Moore's order relation between intervals, Ishihuchi and Tanaka [15] considered that $\overline{a} \leq \overline{b}$ if $a_L \leq b_L$ and $a_R \leq b_R$, depicted as in the cases b and c of Fig. 1.1. The aforementioned two interval ranking methods, which are simply called the Moore's order relation between intervals, are relatively strict in that they only considered the strict relations including the intersection and being greater rather than the inclusion and/or overlap relations between intervals, depicted as in Fig. 1.1. In fact, in terms of the fuzzy set [17, 18], the statement "the interval $\overline{a}$ is not greater than the interval $\overline{b}$" may be regarded as a fuzzy relation between $\overline{a}$ and $\overline{b}$, which is still denoted by $\overline{a} \leq \overline{b}$ for short. Thus, Collins and Hu [19, 20] defined a fuzzy partial order relation between intervals by taking into consideration the inclusion and/or overlap relation between intervals, depicted as in Fig. 2.1.

Comparing Fig. 2.1 with Fig. 1.1, it is obvious that the relations between intervals in the former are more general than those in the latter. In the sequel, we give the concept of satisfactory degrees (or ranking indexes) of intervals' comparison through revising the definition firstly proposed by Collins and Hu [19, 20] (with reference to [21, 22] for a detailed).

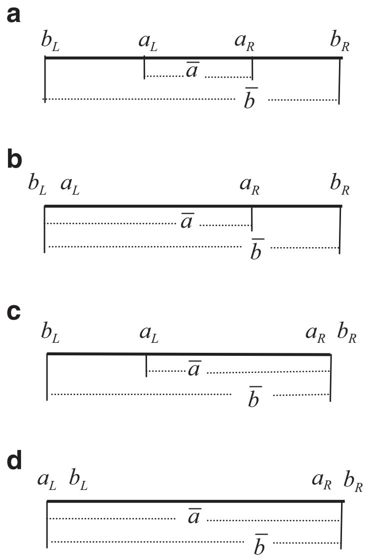


Fig. 2.1 Inclusion and/or overlap relations between two intervals. (**a**) $a_L > b_L$ and $a_R < b_R$, (**b**) $a_L = b_L$ and $a_R < b_R$, (**c**) $a_L > b_L$ and $a_R = b_R$, (**d**) $a_L = b_L$ and $a_R = b_R$

Definition 2.1 Let $\overline{a} = [a_L, a_R] \in \overline{R}$ and $\overline{b} = [b_L, b_R] \in \overline{R}$ be intervals. The premise "$\overline{a} \leq \overline{b}$" is regarded as a fuzzy set, whose membership function is defined as follows:

$$\varphi(\overline{a} \leq \overline{b}) = \begin{cases} 1 & \text{if } a_R < b_L \\ 1^{-} & \text{if } a_L < b_L \leq a_R < b_R \\ \dfrac{b_R - a_R}{2\left(w(\overline{b}) - w(\overline{a})\right)} & \text{if } b_L \leq a_L \leq a_R \leq b_R \text{ and } w(\overline{b}) > w(\overline{a}) \\ 0.5 & \text{if } w(\overline{a}) = w(\overline{b}) \text{ and } a_L = b_L, \end{cases} \tag{2.1}$$

where "1^{-}" is a fuzzy number of "being less than 1," which indicates that the interval $\overline{a}$ is weakly not greater than the interval $\overline{b}$. The fuzzy number "1^{-}" may be adequately chosen according to management situations [1, 18, 23, 24].

"$\overline{a} \leq \overline{b}$" is an interval order relation between $\overline{a}$ and $\overline{b}$, which may be regarded as a generalization of the order relation "$a \leq b$" in the set R of real numbers and has the linguistic interpretation "the interval $\overline{a}$ is essentially not greater than the interval $\overline{b}$." Analogously, we can explain "$\overline{a} \geq \overline{b}$" and "$\overline{a} = \overline{b}$."

Obviously, $0 \leq \varphi(\overline{a} \leq \overline{b}) \leq 1$. Thus, $\varphi(\overline{a} \leq \overline{b})$ may be interpreted as the satisfactory degree (or ranking index) of the premise (or order relation) $\overline{a} \leq \overline{b}$. If $\varphi(\overline{a} \leq \overline{b}) = 0$, then the premise $\overline{a} \leq \overline{b}$ is not accepted by the players. If

$\varphi(\overline{a} \leq \overline{b}) = 1$, then the players are absolutely satisfied with the premise $\overline{a} \leq \overline{b}$. That is to say, the players believe that the premise $\overline{a} \leq \overline{b}$ is absolutely true. If $\varphi(\overline{a} \leq \overline{b}) \in (0, 1)$, then the players accept the premise $\overline{a} \leq \overline{b}$ with different satisfactory degrees between 0 and 1.

In addition, it is obvious from Definition 2.1 that the satisfactory degree of the premise (or order relation) $\overline{a} \leq \overline{b}$ is equal to 0.5 when two intervals $\overline{a}$ and $\overline{b}$ entirely overlap, depicted as in the case d of Fig. 2.1. Moreover, if two intervals degenerate to an identical real number, then the satisfactory degree of the premise $\overline{a} \leq \overline{b}$ is also equal to 0.5. Apparently, the satisfactory degree of the premise $\overline{a} \leq \overline{b}$ is equal to 1 if $a_R < b_L$, depicted as in the case a of Fig. 1.1. Therefore, the interval order relation given by Definition 2.1 includes the Moore's order relation between intervals. If two intervals have the inclusion relation with $w(\overline{b}) > w(\overline{a})$, depicted as in the cases a–c of Fig. 2.1, then we can easily know that the satisfactory degree of the premise $\overline{a} \leq \overline{b}$ is between 0 and 1 according to Eq. (2.1). Due to the condition $w(\overline{b}) > w(\overline{a})$, both the intervals $\overline{a}$ and $\overline{b}$ cannot be reduced to the real numbers at the same time even if $w(\overline{a})$ is equal to 0 or approaches to 0, therefore the satisfactory degree of the premise $\overline{a} \leq \overline{b}$ is also between 0 and 1.

Analogously, we can define the following premise "$\overline{a} \geq \overline{b}$," which indicates the statement "the interval $\overline{a}$ is not smaller than the interval $\overline{b}$."

Definition 2.2 Let $\overline{a} = [a_L, a_R] \in \overline{R}$ and $\overline{b} = [b_L, b_R] \in \overline{R}$ be intervals. The premise "$\overline{a} \geq \overline{b}$" is regarded as a fuzzy set, whose membership function is defined as

$$\varphi(\overline{a} \geq \overline{b}) = 1 - \varphi(\overline{a} \leq \overline{b}),$$

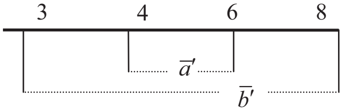
i.e.,

$$\varphi(\overline{a} \geq \overline{b}) = \begin{cases} 0 & \text{if } a_R < b_L \\ 0^+ & \text{if } a_L < b_L \leq a_R < b_R \\ \dfrac{a_L - b_L}{2(w(\overline{b}) - w(\overline{a}))} & \text{if } b_L \leq a_L \leq a_R \leq b_R \text{ and } w(\overline{b}) > w(\overline{a}) \\ 0.5 & \text{if } w(\overline{a}) = w(\overline{b}) \text{ and } a_L = b_L, \end{cases} \tag{2.2}$$

where "0^+" is a fuzzy number of "being greater than 0," which linguistically indicates that the interval $\overline{a}$ is weakly not smaller than the interval $\overline{b}$.

Thus, the interval-valued equality relation "=" can be defined as follows: $\overline{a} = \overline{b}$ if and only if $\overline{a} \geq \overline{b}$ and $\overline{a} \leq \overline{b}$. Alternatively, it is derived from Definitions 2.1 and 2.2 that $\overline{a} = \overline{b}$ is equivalent to both $a_L = b_L$ and $a_R = b_R$. Linguistically, "$\overline{a} = \overline{b}$" may be interpreted as "the interval $\overline{a}$ is equal to the interval $\overline{b}$" in the sense of

Fig. 2.2 Inclusion relation between the intervals $\overline{a}'$ and $\overline{b}'$



Definitions 2.1 and 2.2. Analogously, $\overline{a} > \overline{b}$ if and only if $\overline{a} \geq \overline{b}$ and $\overline{a} \neq \overline{b}$. $\overline{a} < \overline{b}$ if and only if $\overline{a} \leq \overline{b}$ and $\overline{a} \neq \overline{b}$.

In the sequent, the above fuzzy ranking index φ is often called the satisfactory degree. It is easy to prove that φ is continuous except a single special case, i.e., $a_L = b_L$ and $w(\overline{a}) = w(\overline{b})$.

Example 2.1 Let us consider two intervals $\overline{a}' = [4, 6]$ and $\overline{b}' = [3, 8]$, depicted as in Fig. 2.2.

According to the Moore's order relation between intervals (i.e., Eq. (1.4)), we cannot compare the intervals $\overline{a}'$ and $\overline{b}'$ or rank the order of the intervals $\overline{a}'$ and $\overline{b}'$. However, according to Eq. (2.1), we can obtain

$$\begin{aligned}\varphi\left(\overline{a}' \leq \overline{b}'\right) &= \frac{b'_R - a'_R}{2\left(w(\overline{b}') - w(\overline{a}'')\right)} \\ &= \frac{8-6}{(8-3)-(6-4)} \\ &= \frac{2}{3}.\end{aligned}$$

Thus, the satisfactory degree of $\overline{a}' \leq \overline{b}'$ is 2/3 whereas the satisfactory degree of $\overline{a}' \geq \overline{b}'$ is 1/3 according to Definition 2.2. In other words, the statement "$\overline{a}' \leq \overline{b}'$" is true with the possibility 2/3.

Moreover, it is easily derived from Definitions 2.1 and 2.2 that there are some useful and important properties [19], which can be summarized as in Theorem 2.1 as follows.

Theorem 2.1 *For any intervals $\overline{a} \in \overline{R}$, $\overline{b} \in \overline{R}$, and $\overline{c} \in \overline{R}$, then*

1. $0 \leq \varphi\left(\overline{a} \leq \overline{b}\right) \leq 1$
2. $\varphi(\overline{a} \leq \overline{a}) = 0.5$
3. $\varphi\left(\overline{a} \leq \overline{b}\right) + \varphi\left(\overline{a} \geq \overline{b}\right) = 1$
4. *if* $\varphi\left(\overline{a} \leq \overline{b}\right) \geq 0.5$ *and* $\varphi\left(\overline{b} \leq \overline{c}\right) \geq 0.5$, *then* $\varphi(\overline{a} \leq \overline{c}) \geq 0.5$; *or if* $\varphi\left(\overline{a} \leq \overline{b}\right) \leq 0.5$ *and* $\varphi\left(\overline{b} \leq \overline{c}\right) \leq 0.5$, *then* $\varphi(\overline{a} \leq \overline{c}) \leq 0.5$.

Proof. According to Definitions 2.1 and 2.2, we can easily prove that the conclusions of Theorem 2.1 are valid (omitted).

Thus, Definitions 2.1 and 2.2 may provide quantitative methods to determine the exact satisfactory degree for ranking/comparing two intervals. In the proceeding subsection, the satisfactory degree φ is used to define satisfactory crisp equivalent forms of interval-valued inequalities.

2.2.2 Satisfactory Crisp Equivalent Forms of Interval-Valued Inequalities

According to the concept of the satisfactory degrees given above, we can establish the following satisfactory crisp equivalent forms of interval-valued inequality constraints, which will be used to construct auxiliary nonlinear programming models for computing interval-valued cores of interval-valued cooperative games.

Let $\beta \in [0, 1]$ denote the satisfactory degree of the interval-valued inequality constraint $\overline{a} \leq \overline{b}$ which may be satisfied. Then, for the situation in which the intervals $\overline{a}$ and $\overline{b}$ satisfy the following two constraint conditions: $b_L \leq a_L \leq a_R \leq b_R$ and $w(\overline{b}) > w(\overline{a})$, then according to Definition 2.1, the satisfactory crisp equivalent form of the interval-valued inequality $\overline{a} \leq \overline{b}$ is defined as follows:

$$\begin{cases} a_L \geq b_L \\ a_R \leq b_R \\ \varphi(\overline{a} \leq \overline{b}) \geq \beta, \end{cases}$$

which can be further written as the following system of inequalities:

$$\begin{cases} a_L \geq b_L \\ a_R \leq b_R \\ \dfrac{b_R - a_R}{2(w(\overline{b}) - w(\overline{a}))} \geq \beta. \end{cases} \tag{2.3}$$

It is easy to see from Eq. (2.3) that $w(\overline{b}) > w(\overline{a})$ due to $b_R \geq a_R$ and $\beta \in [0, 1]$.

Analogously, for the situation in which the intervals $\overline{a}$ and $\overline{b}$ satisfy the following constraint condition: $a_R < b_L$, then according to Definition 2.1, the satisfactory crisp equivalent form of the interval-valued inequality $\overline{a} \leq \overline{b}$ is defined as follows:

$$a_R < b_L, \tag{2.4}$$

where $\varphi(\overline{a} \leq \overline{b}) = 1$.

For the situation in which the intervals $\bar{a}$ and $\bar{b}$ satisfy the following constraint condition: $a_L < b_L \le a_R < b_R$, then according to Definition 2.1, the satisfactory crisp equivalent form of the interval-valued inequality $\bar{a} \le \bar{b}$ is defined as follows:

$$\begin{cases} a_L < b_L \\ b_L \le a_R \\ a_R < b_R, \end{cases} \tag{2.5}$$

where $\varphi(\bar{a} \le \bar{b}) = 1^-$.

For the situation in which the intervals $\bar{a}$ and $\bar{b}$ satisfy the following two constraint conditions: $a_L = b_L$ and $w(\bar{a}) = w(\bar{b})$, then according to Definition 2.1, the satisfactory crisp equivalent form of the interval-valued inequality $\bar{a} \le \bar{b}$ is defined as follows:

$$\begin{cases} a_L = b_L \\ w(\bar{a}) = w(\bar{b}), \end{cases}$$

where $\varphi(\bar{a} \le \bar{b}) = 0.5$. The above system of equalities can be further written as the following system of equalities:

$$\begin{cases} a_L = b_L \\ a_R - a_L = b_R - b_L. \end{cases} \tag{2.6}$$

In the same way, we can derive the satisfactory crisp equivalent form of the interval-valued inequality $\bar{a} \ge \bar{b}$. More specifically, for the situation in which the intervals $\bar{a}$ and $\bar{b}$ satisfy the following two constraint conditions: $b_L \le a_L \le a_R \le b_R$ and $w(\bar{b}) > w(\bar{a})$, then according to Definition 2.2, the satisfactory crisp equivalent form of the interval-valued inequality $\bar{a} \ge \bar{b}$ is defined as follows:

$$\begin{cases} a_L \ge b_L \\ a_R \le b_R \\ \varphi(\bar{a} \ge \bar{b}) \ge \beta, \end{cases}$$

which can be further written as the following system of inequalities:

$$\begin{cases} a_L \ge b_L \\ a_R \le b_R \\ \dfrac{a_L - b_L}{2\left(w(\bar{b}) - w(\bar{a})\right)} \ge \beta. \end{cases} \tag{2.7}$$

Similarly, for the situation in which the intervals $\overline{a}$ and $\overline{b}$ satisfy the following constraint condition: $a_R < b_L$, then according to Definition 2.2, the satisfactory crisp equivalent form of the interval-valued inequality $\overline{a} \geq \overline{b}$ is defined as follows:

$$a_R < b_L, \tag{2.8}$$

where $\varphi\left(\overline{a} \geq \overline{b}\right) = 0$.

For the situation in which the intervals $\overline{a}$ and $\overline{b}$ satisfy the following constraint condition: $a_L < b_L \leq a_R < b_R$, then according to Definition 2.2, the satisfactory crisp equivalent form of the interval-valued inequality $\overline{a} \geq \overline{b}$ is defined as follows:

$$\begin{cases} a_L < b_L \\ b_L \leq a_R \\ a_R < b_R, \end{cases} \tag{2.9}$$

where $\varphi\left(\overline{a} \geq \overline{b}\right) = 0^+$.

For the situation in which the intervals $\overline{a}$ and $\overline{b}$ satisfy the following two constraint conditions: $a_L = b_L$ and $w(\overline{a}) = w\left(\overline{b}\right)$, then according to Definition 2.2, the satisfactory crisp equivalent form of the interval-valued inequality $\overline{a} \geq \overline{b}$ is defined as follows:

$$\begin{cases} a_L = b_L \\ w(\overline{a}) = w\left(\overline{b}\right), \end{cases}$$

which can be further written as the following system of equalities:

$$\begin{cases} a_L = b_L \\ a_R - a_L = b_R - b_L, \end{cases} \tag{2.10}$$

where $\varphi\left(\overline{a} \geq \overline{b}\right) = 0.5$.

2.3 Nonlinear Programming Models and Method for Interval-Valued Cores of Interval-Valued Cooperative Games

In this section, let us continue to consider how to solve interval-valued cooperative games $\overline{v} \in \overline{G}^n$, which are stated in Sect. 1.3.2.

2.3.1 The Concept of Interval-Valued Cores of Interval-Valued Cooperative Games

Stated as earlier, for any interval-valued cooperative game $\bar{v} \in \overline{G}^n$, its interval-valued imputation set $\bar{I}(\bar{v})$ usually may be very large. As a result, in a parallel way to the concept of cores of cooperative games [1, 2, 4, 25], we may give the concept of interval-valued cores of interval-valued cooperative games. More precisely, the interval-valued core of an arbitrary interval-valued cooperative game $\bar{v} \in \overline{G}^n$, denoted by $\overline{C}(\bar{v})$, is defined as follows:

$$\overline{C}(\bar{v}) = \left\{ \bar{\boldsymbol{x}}(\bar{v}) \in \bar{I}(\bar{v}) \middle| \sum_{i \in S} \bar{x}_i(\bar{v}) \geq \bar{v}(S) \text{ for all } S \subset N \right\}, \tag{2.11}$$

where $\bar{\boldsymbol{x}}(\bar{v}) = (\bar{x}_1(\bar{v}), \bar{x}_2(\bar{v}), \ldots, \bar{x}_n(\bar{v}))^{\mathrm{T}}$ and $\bar{x}_i(\bar{v}) = [x_{Li}(\bar{v}), x_{Ri}(\bar{v})]$ $(i = 1, 2, \ldots, n)$ are stated in Sect. 1.3.2.

Obviously, for an inessential interval-valued cooperative game $\bar{v} \in \overline{G}^n$, whose interval-valued characteristic function $\bar{v}$ is defined as

$$\bar{v}(S) = \sum_{i \in S} \bar{v}(i)$$

for any coalition $S \subseteq N$, i.e., the inessential interval-valued cooperative game $\bar{v}$ is additive, hence its interval-valued core $\overline{C}(\bar{v})$ has a unique element, i.e.,

$$\overline{C}(\bar{v}) = \left\{ (\bar{v}(1), \bar{v}(2), \ldots, \bar{v}(n))^{\mathrm{T}} \right\} = \bar{I}(\bar{v}).$$

Clearly, an inessential interval-valued cooperative game is trivial from a game-theoretic point of view. That is to say, if every player $i \in N$ demands at least $\bar{v}(i)$, then the allocation (or distribution) of the grand coalition $\bar{v}(N)$ can be uniquely determined.

Conversely, for an essential interval-valued cooperative game $\bar{v} \in \overline{G}^n$, which is not additive, then its interval-valued core $\overline{C}(\bar{v})$ may have lots of elements if $\overline{C}(\bar{v})$ is not empty, i.e., $\left|\overline{C}(\bar{v})\right| \geq 1$.

It is obvious from Eq. (2.11) that the interval-valued core $\overline{C}(\bar{v})$ of an interval-valued cooperative game $\bar{v} \in \overline{G}^n$ can be obtained through solving the system of linear interval-valued inequalities as follows:

$$\begin{cases} \sum_{i \in S} \bar{x}_i(\bar{v}) \geq \bar{v}(S) \text{ for all } S \subset N \\ \sum_{i=1}^{n} \bar{x}_i(\bar{v}) = \bar{v}(N), \end{cases} \tag{2.12}$$

where $\bar{x}_i(\bar{v}) = [x_{Li}(\bar{v}), x_{Ri}(\bar{v})]$ $(i = 1, 2, \ldots, n)$ are interval-valued variables.

Example 2.2 There are an investor and two IT technologists, numbered by 1, 2, and 3, respectively. The investor 1 has a fund and looks for a technology patent to invest production. The IT technologists 2 and 3 have a similar IT patent and look for a fund to invest production. Due to the uncertainty of the market demand, it seems to be suitable for expressing the profits with intervals. Thus, this problem may be regarded as a three-person interval-valued cooperative game $\overline{v}'' \in \overline{G}^3$, where the investor 1 and the IT technologists 2 and 3 are regarded as the players 1, 2, and 3, respectively; the grand coalition is $N'' = \{1,2,3\}$; its interval-valued characteristic function (i.e., profit function) $\overline{v}''$ is defined as follows: $\overline{v}''(1,2) = \overline{v}''(1,3) = \overline{v}''\left(N''\right) = [291, 306]$ and $\overline{v}''(S) = 0$ for any other coalitions $S \subset N''$.

Obviously, according to Eq. (2.12), we can easily obtain the interval-valued core of the above interval-valued cooperative game $\overline{v}''$ as follows:

$$\overline{C}\left(\overline{v}''\right) = \left\{([291, 306], 0, 0)^{\mathrm{T}}\right\},$$

which means that there is a unique element (i.e., allocation). In this case, the player 1 (i.e., investor) gets the total profit $[291, 306]$ whereas the players (i.e., technologists) 2 and 3 get nothing from the cooperative production. Clearly, this allocation of the grand coalition $\overline{v}''\left(N''\right)$ seems to be irrational due to the following two aspects. On the one hand, the interval-valued characteristic function $\overline{v}''$ is too simple and special to reflect the real situation. For example, it is obvious that

$$\overline{v}''(1,2,3) = \overline{v}''(1,2) + \overline{v}''(3)$$

and

$$\overline{v}''(1,2,3) = \overline{v}''(1,3) + \overline{v}''(2)$$

due to $\overline{v}''(3) = 0$ and $\overline{v}''(2) = 0$. More importantly, on the other hand, the interval-valued cooperative game $\overline{v}''$ is not convex in that

$$\begin{aligned}\overline{v}''(\{1,2\} \cup \{1,3\}) + \overline{v}''(\{1,2\} \cap \{1,3\}) &= \overline{v}''(1,2,3) + \overline{v}''(1) \\ &= [291, 306] < \overline{v}''(1,2) + \overline{v}''(1,3) \\ &= [582, 612].\end{aligned}$$

Moreover, stated as above, the interval-valued cooperative game $\overline{v}''$ is not strictly superadditive in that

$$\overline{v}''(1,2,3) = \overline{v}''(1,2) + \overline{v}''(3)$$

and

$$\overline{v}''(1,2,3) = \overline{v}''(1,3) + \overline{v}''(2).$$

In fact, the interval-valued cooperative game $\overline{v}''$ is a big boss interval-valued cooperative game. For details regarding the solutions and special properties for big boss interval-valued games, we refer the reader to Branzei et al. [26] and Alparslan-Gök et al. [27].

From the above discussion, the interval-valued core of the interval-valued cooperative game $\overline{v}''$ given in Example 2.2 is very easily obtained by simply observation. In many economic management situations, however, it is very difficult to compute interval-valued cores of interval-valued cooperative games. Particularly, if the interval-valued inequality constraints

$$\sum_{i \in S} \overline{x}_i(\overline{v}) \geq \overline{v}(S) \quad (S \subset N)$$

in Eq. (2.12) are made in the sense of the Moore's order relation between intervals (i.e., Eq. (1.4)), then it is rather possible that there exists no feasible solution to Eq. (2.12) and hereby the interval-valued core $\overline{C}(\overline{v})$ of the interval-valued cooperative game $\overline{v} \in \overline{G}^n$ is empty. Although Alparslan-Gök et al. [8] proved that the interval-valued core of an interval-valued cooperative game is non-empty if and only if the interval-valued cooperative game is the Γ-balanced. But, as stated earlier, the Moore's order relation between intervals is very strict and hereby only a special kind of interval-valued cooperative games has non-empty interval-valued cores.

In the next subsection, we focus on developing a satisfactory-degree-based nonlinear programming method for computing interval-valued cores of interval-valued cooperative games.

2.3.2 Nonlinear Programming Models for Interval-Valued Cores of Interval-Valued Cooperative Games

In this section, we mainly apply Definitions 2.1 and 2.2 (i.e., Eqs. (2.7)–(2.10)) to establish the auxiliary nonlinear programming models for Eq. (2.12).

More specifically, for any coalition $S \subset N$, let

$$\beta_S(\overline{v}) = \varphi\left(\sum_{i \in S} \overline{x}_i(\overline{v}) \geq \overline{v}(S)\right)$$

denote the satisfactory degree of the interval-valued inequality $\sum_{i \in S} \overline{x}_i(\overline{v}) \geq \overline{v}(S)$ which may be satisfied.

For the situation in which the intervals $\sum_{i\in S}\overline{x}_i(\overline{v}) = \left[\sum_{i\in S}x_{Li}(\overline{v}), \sum_{i\in S}x_{Ri}(\overline{v})\right]$ and $\overline{v}(S) = [v_L(S), v_R(S)]$ $(S \subset N)$ satisfy the constraints as follows:

$$v_L(S) \le \sum_{i\in S} x_{Li}(\overline{v}) \le \sum_{i\in S} x_{Ri}(\overline{v}) \le v_R(S)$$

and

$$w(\overline{v}(S)) > w\left(\sum_{i\in S}\overline{x}_i\right),$$

then according to Eq. (2.7), the satisfactory crisp equivalent mathematical programming model for Eq. (2.12) can be constructed as follows:

$$\max\left\{\min_{S\subset N}\{\beta_S(\overline{v})\}\right\}$$
$$\text{s.t.}\begin{cases}\sum_{i\in S} x_{Li}(\overline{v}) \ge v_L(S) \quad (S\subset N)\\ \sum_{i\in S} x_{Ri}(\overline{v}) \le v_R(S) \quad (S\subset N)\\ \beta_S(\overline{v}) = \varphi\left(\sum_{i\in S}\overline{x}_i \ge \overline{v}(S)\right) \quad (S\subset N)\\ \sum_{i\in N} x_{Ri}(\overline{v}) = v_R(N)\\ \sum_{i\in N} x_{Li}(\overline{v}) = v_L(N)\\ x_{Ri}(\overline{v}) \ge x_{Li}(\overline{v}) \quad (i=1,2,\ldots,n).\end{cases} \tag{2.13}$$

Let

$$\beta(\overline{v}) = \min_{S\subset N}\{\beta_S(\overline{v})\}.$$

Then, obviously, $0 \le \beta(\overline{v}) \le 1$. Thereby, according to Definition 2.2 (i.e., Eq. (2.2)), Eq. (2.13) can be rewritten as the following nonlinear programming model:

$$\max\{\beta(\overline{\upsilon})\}$$
$$\text{s.t.}\begin{cases}\sum_{i\in S} x_{Li}(\overline{\upsilon}) \geq \upsilon_L(S) \quad (S\subset N)\\ \sum_{i\in S} x_{Ri}(\overline{\upsilon}) \leq \upsilon_R(S) \quad (S\subset N)\\ \dfrac{\sum_{i\in S} x_{Li}(\overline{\upsilon}) - \upsilon_L(S)}{(\upsilon_R(S)-\upsilon_L(S)) - \left(\sum_{i\in S} x_{Ri}(\overline{\upsilon}) - \sum_{i\in S} x_{Li}(\overline{\upsilon})\right)} \geq \beta(\overline{\upsilon}) \quad (S\subset N)\\ \sum_{i\in N} x_{Ri}(\overline{\upsilon}) = \upsilon_R(N)\\ \sum_{i\in N} x_{Li}(\overline{\upsilon}) = \upsilon_L(N)\\ 0\leq \beta(\overline{\upsilon}) \leq 1\\ x_{Ri}(\overline{\upsilon}) \geq x_{Li}(\overline{\upsilon}) \quad (i=1,2,\ldots,n),\end{cases}$$

which can be rewritten as the following nonlinear programming model:

$$\max\{\beta(\overline{\upsilon})\}$$
$$\text{s.t.}\begin{cases}\sum_{i\in S} x_{Li}(\overline{\upsilon}) \geq \upsilon_L(S) \quad (S\subset N)\\ \sum_{i\in S} x_{Ri}(\overline{\upsilon}) \leq \upsilon_R(S) \quad (S\subset N)\\ (1-\beta(\overline{\upsilon}))\sum_{i\in S} x_{Li}(\overline{\upsilon}) + \beta(\overline{\upsilon})\sum_{i\in S} x_{Ri}(\overline{\upsilon}) \geq (1-\beta(\overline{\upsilon}))\upsilon_L(S) + \beta(\overline{\upsilon})\upsilon_R(S) \ \ (S\subset N)\\ \sum_{i\in N} x_{Ri}(\overline{\upsilon}) = \upsilon_R(N)\\ \sum_{i\in N} x_{Li}(\overline{\upsilon}) = \upsilon_L(N)\\ 0\leq \beta(\overline{\upsilon}) \leq 1\\ x_{Ri}(\overline{\upsilon}) \geq x_{Li}(\overline{\upsilon}) \quad (i=1,2,\ldots,n),\end{cases} \tag{2.14}$$

where $x_{Ri}(\overline{\upsilon}), x_{Li}(\overline{\upsilon})$ $(i=1,2,\ldots,n)$, and $\beta(\overline{\upsilon})$ are decision variables, which need to be determined.

Analogously, for the situation in which the intervals $\sum_{i\in S}\overline{x}_i(\overline{\upsilon})$ and $\overline{\upsilon}(S)$ $(S\subset N)$ satisfy the constraint:

$$\sum_{i\in S} x_{Ri}(\overline{\upsilon}) < \upsilon_L(S),$$

then according to Eq. (2.8), the satisfactory crisp equivalent form of Eq. (2.12) can be constructed as follows:

$$\begin{cases} \sum_{i \in S} x_{Ri}(\overline{v}) < v_L(S) \quad (S \subset N) \\ \sum_{i \in N} x_{Ri}(\overline{v}) = v_R(N) \\ \sum_{i \in N} x_{Li}(\overline{v}) = v_L(N) \\ x_{Ri}(\overline{v}) \geq x_{Li}(\overline{v}) \quad (i = 1, 2, \ldots, n), \end{cases} \tag{2.15}$$

where $x_{Ri}(\overline{v})$ and $x_{Li}(\overline{v})$ $(i = 1, 2, \ldots, n)$ are decision variables.

Equation (2.15) is a system of linear inequalities, which can be solved by using the method of the system of inequalities.

For the situation in which the intervals $\sum_{i \in S} \overline{x}_i(\overline{v})$ and $\overline{v}(S)$ $(S \subset N)$ satisfy the constraint as follows:

$$\sum_{i \in S} x_{Li}(\overline{v}) < v_L(S) \leq \sum_{i \in S} x_{Ri}(\overline{v}) < v_R(S)$$

then according to Eq. (2.9), the satisfactory crisp equivalent form of Eq. (2.12) can be constructed as follows:

$$\begin{cases} \sum_{i \in S} x_{Li}(\overline{v}) < v_L(S) \quad (S \subset N) \\ \sum_{i \in S} x_{Ri}(\overline{v}) \geq v_L(S) \quad (S \subset N) \\ \sum_{i \in S} x_{Ri}(\overline{v}) < v_R(S) \quad (S \subset N) \\ \sum_{i \in N} x_{Ri}(\overline{v}) = v_R(N) \\ \sum_{i \in N} x_{Li}(\overline{v}) = v_L(N) \\ x_{Ri}(\overline{v}) \geq x_{Li}(\overline{v}) \quad (i = 1, 2, \ldots, n), \end{cases} \tag{2.16}$$

which is a system of linear inequalities about the decision variables $x_{Ri}(\overline{v})$ and $x_{Li}(\overline{v})$ $(i = 1, 2, \ldots, n)$.

For the situation in which the intervals $\sum_{i \in S} \overline{x}_i(\overline{v})$ and $\overline{v}(S)$ $(S \subset N)$ satisfy the constraints:

$$\sum_{i \in S} x_{Li}(\overline{v}) = v_L(S)$$

and

$$w\left(\sum_{i \in S} \overline{x}_i\right) = w(\overline{v}(S)),$$

then according to Eq. (2.10), the satisfactory crisp equivalent form of Eq. (2.12) can be constructed as follows:

$$\begin{cases} \sum_{i \in S} x_{Li}(\overline{v}) = v_L(S) \quad (S \subset N) \\ \sum_{i \in S} x_{Ri}(\overline{v}) - \sum_{i \in S} x_{Li}(\overline{v}) = v_R(S) - v_L(S) \quad (S \subset N) \\ \sum_{i \in N} x_{Ri}(\overline{v}) = v_R(N) \\ \sum_{i \in N} x_{Li}(\overline{v}) = v_L(N) \\ x_{Ri}(\overline{v}) \geq x_{Li}(\overline{v}) \quad (i = 1, 2, \ldots, n), \end{cases} \tag{2.17}$$

which is a system of linear inequalities about the decision variables $x_{Ri}(\overline{v})$ and $x_{Li}(\overline{v})$ $(i = 1, 2, \ldots, n)$.

2.3.3 Bisection Algorithm for Computing Interval-Valued Cores of Interval-Valued Cooperative Games

Solving Eq. (2.14) (or Eqs. (2.15)–(2.17)), we can obtain its optimal solution, denoted by $\left(\beta^*(\overline{v}), \overline{\boldsymbol{x}}^*(\overline{v})\right)$ (or $\overline{\boldsymbol{x}}^*(\overline{v})$). Then, $\overline{\boldsymbol{x}}^*(\overline{v})$ is an element of the interval-valued core $\overline{C}(\overline{v})$ of the interval-valued cooperative game $\overline{v} \in \overline{G}^n$, where the maximum satisfactory degree is $\beta^*(\overline{v})$.

Obviously, if $\beta^*(\overline{v}) = 1$, then we can obtain the element of the interval-valued core $\overline{C}(\overline{v})$, which means that all the interval-valued inequalities in Eq. (2.14) are absolutely satisfied.

Generally, $\left(\beta^*(\overline{v}), \overline{\boldsymbol{x}}^*(\overline{v})\right)$ is not a global optimal solution due to the fact that Eq. (2.14) is nonlinear programming. In the following, we propose the bisection method and algorithm for solving Eq. (2.14), which can always provide a global optimal solution to Eq. (2.14).

Assume that a precision $\varepsilon \in \left(0, 1\right]$ is given a priori. Then, according to the bisection method [28], we can determine the iteration number of the proposed bisection algorithm, denoted by m_0, where m_0 is a positive integer which is not smaller than $-\ln\varepsilon/\ln 2$. The bisection procedure and algorithm for solving Eq. (2.14) are summarized as follows:

Step 1: Let $t = 0$, and take $\beta_{Rt}(\overline{v}) = 1$. The nonlinear programming model (i.e., Eq. (2.14)) can be transformed into the linear programming. In this case, solving Eq. (2.14) with $\beta_{Rt}(\overline{v}) = 1$ by using the LINGO tool (or the simplex method of linear programming), if Eq. (2.14) has a feasible solution $\overline{\boldsymbol{x}}_t^*(\overline{v})$, then $\beta^*(\overline{v}) = \beta_{Rt}(\overline{v}) = 1$ is the optimal value of the objective function of Eq. (2.14) and $\overline{\boldsymbol{x}}^*(\overline{v}) = \overline{\boldsymbol{x}}_t^*(\overline{v})$ is an element of the interval-valued core $\overline{C}(\overline{v})$ with the maximum satisfactory degree $\beta^*(\overline{v})$. The algorithm stops. On the contrary, if there is not any feasible solution to Eq. (2.14), then go to Step 2.

Step 2: Take $\beta_{Lt}(\overline{v}) = 0$, and solve Eq. (2.14) by using the LINGO tool (or the simplex method of linear programming), if there is not any feasible solution to Eq. (2.14), which means that this linear programming (hereby Eq. (2.14)) has no optimal solutions, then the algorithm stops. On the contrary, if Eq. (2.14) with $\beta_{Lt}(\overline{v}) = 0$ has a feasible solution $\overline{\boldsymbol{x}}_t^*(\overline{v})$, then we can judge that the optimal value of the objective function of Eq. (2.14) is between 0 and 1, i.e., $\beta^*(\overline{v}) \in (0,1)$, go to Step 3.

Step 3: Let $m(\overline{\beta}_t(\overline{v}))$ be the mean of the lower bound $\beta_{Lt}(\overline{v})$ and the upper bound $\beta_{Rt}(\overline{v})$ of the interval $\overline{\beta}_t(\overline{v}) = [\beta_{Lt}(\overline{v}), \beta_{Rt}(\overline{v})]$. Namely,

$$m(\overline{\beta}_t(\overline{v})) = \frac{\beta_{Lt}(\overline{v}) + \beta_{Rt}(\overline{v})}{2} = \frac{0+1}{2} = 0.5.$$

By using the LINGO tool (or the simplex method of linear programming), solving Eq. (2.14) with $m(\overline{\beta}_t(\overline{v}))$, if there is not any feasible solution to Eq. (2.14), then the optimal value of the objective function of Eq. (2.14) falls into the range which is between the lower bound $\beta_{Lt}(\overline{v})$ and the mean $m(\overline{\beta}_t(\overline{v}))$ of the interval $\overline{\beta}_t(\overline{v})$, i.e., $\beta^*(\overline{v}) \in [\beta_{Lt}(\overline{v}), m(\overline{\beta}_t(\overline{v}))] = [0, 0.5]$, thereby the interval $\overline{\beta}_t(\overline{v})$ is narrowed. Let $\beta_{R,t+1}(\overline{v}) = m(\overline{\beta}_t(\overline{v})) = 0.5$ and $\beta_{L,t+1}(\overline{v}) = \beta_{Lt}(\overline{v}) = 0$, then go to Step 4. On the contrary, if Eq. (2.14) has a feasible solution $\overline{\boldsymbol{x}}_t^*(\overline{v})$, then the optimal value of the objective function of Eq. (2.14) falls into the range which is between the mean $m(\overline{\beta}_t(\overline{v}))$ and the upper bound $\beta_{Rt}(\overline{v})$ of the interval $\overline{\beta}_t(\overline{v})$, i.e., $\beta^*(\overline{v}) \in [m(\overline{\beta}_t(\overline{v})), \beta_{Rt}(\overline{v})] = [0.5, 1]$, thereby the interval $\overline{\beta}_t(\overline{v})$ is narrowed also. Let $\beta_{L,t+1}(\overline{v}) = m(\overline{\beta}_t(\overline{v})) = 0.5$ and $\beta_{R,t+1}(\overline{v}) = \beta_{Rt}(\overline{v}) = 1$, then go to Step 4.

Step 4: Let $t := t+1$. Repeat Step 3 in the new smaller interval $\overline{\beta}_t(\overline{v}) = [\beta_{Lt}(\overline{v}), \beta_{Rt}(\overline{v})]$ until the m_0th iteration. Then, go to Step 5.

Step 5: The length of the narrowed interval $\overline{\beta}_{m_0}(\overline{v}) = [\beta_{Lm_0}(\overline{v}), \beta_{Rm_0}(\overline{v})]$ of the m_0th iteration is not greater than the given precision ε. Let

$$\beta^*(\overline{v}) = \frac{\beta_{Lm_0}(\overline{v}) + \beta_{Rm_0}(\overline{v})}{2},$$

which is the mean of the lower and upper bounds of the interval $\overline{\beta}_{m_0}(\overline{\upsilon})$. Accordingly, $\beta^*(\overline{\upsilon})$ is the optimal value of the objective function of Eq. (2.14) at a given precision ε; $\overline{\boldsymbol{x}}^*(\overline{\upsilon}) = \overline{\boldsymbol{x}}^*_{m_0}(\overline{\upsilon})$ is the element of the interval-valued core $\overline{C}(\overline{\upsilon})$ with the maximum satisfactory degree $\beta^*(\overline{\upsilon})$.

2.4 Real Example Analysis

Suppose that there are three companies p_1, p_2, and p_3 in the electronic product supply chain. They cooperate to develop a new type of electronic products. Each company has different superior resources. Due to a lack of information and/or imprecision of the available information, the managers of the three companies usually are not able to exactly forecast the profit amount of the companies' product under cooperation. Usually, the companies can predict the optimistic profit and the pessimistic profit of product. Hence, intervals are suitable to represent the profit amount of the product from the three companies' perspectives. If the three companies work together for product cooperative innovation, then the optimistic profit of the product may be 44 while the pessimistic profit of the product may be 40, which can be described as the interval-valued profit $\overline{\upsilon}'(1,2,3) = [40,44]$, where the numbers 1, 2, and 3 represent the companies p_1, p_2, and p_3 for short, respectively. Similarly, if the companies p_1 and p_2 cooperate for product innovation, then the interval-valued profit may be $\overline{\upsilon}'(1,2) = [22,30]$. If the companies p_1 and p_3 cooperate for product innovation, then the interval-valued profit may be $\overline{\upsilon}'(1,3) = [24,28]$. If the companies p_2 and p_3 cooperate for product innovation, then the interval-valued profit may be $\overline{\upsilon}'(2,3) = [20,32]$. Because of the limitation of resources, the three companies cannot develop and produce the product alone. Therefore, the profit of the product for each company is 0, i.e., $\overline{\upsilon}'(1) = \overline{\upsilon}'(2) = \overline{\upsilon}'(3) = 0$. Thus, the above problem may be regarded as an interval-valued cooperative game $\overline{\upsilon}' \in \overline{G}^3$. Namely, the three companies p_1, p_2, and p_3 in the electronic product supply chain may be regarded as the players 1, 2, and 3, respectively. The interval-valued characteristic function is $\overline{\upsilon}'$, which is defined on the grand coalition $N' = \{1,2,3\}$ so that $\overline{\upsilon}'(1,2,3) = [40,44]$, $\overline{\upsilon}'(1,2) = [22,30]$, $\overline{\upsilon}'(1,3) = [24,28]$, $\overline{\upsilon}'(2,3) = [20,32]$, and $\overline{\upsilon}'(1) = \overline{\upsilon}'(2) = \overline{\upsilon}'(3) = 0$.

2.4.1 *Computational Results Obtained by the Nonlinear Programming Method*

According to Eq. (2.14), the nonlinear programming model can be constructed as follows:

$$\max\{\beta(\overline{v}')\}$$
$$\text{s.t.}\begin{cases} x_{L1}(\overline{v}') + x_{L2}(\overline{v}') \geq 22 \\ x_{R1}(\overline{v}') + x_{R2}(\overline{v}') \leq 30 \\ x_{L1}(\overline{v}') + x_{L3}(\overline{v}') \geq 24 \\ x_{R1}(\overline{v}') + x_{R3}(\overline{v}') \leq 28 \\ x_{L2}(\overline{v}') + x_{L3}(\overline{v}') \geq 20 \\ x_{R2}(\overline{v}') + x_{R3}(\overline{v}') \leq 32 \\ (1-\beta(\overline{v}'))(x_{L1}(\overline{v}') + x_{L2}(\overline{v}')) + \beta(\overline{v}')(x_{R1}(\overline{v}') + x_{R2}(\overline{v}')) \geq 22(1-\beta(\overline{v}')) + 30\beta(\overline{v}') \\ (1-\beta(\overline{v}'))(x_{L1}(\overline{v}') + x_{L3}(\overline{v}')) + \beta(\overline{v}')(x_{R1}(\overline{v}') + x_{R3}(\overline{v}')) \geq 24(1-\beta(\overline{v}')) + 28\beta(\overline{v}') \\ (1-\beta(\overline{v}'))(x_{L2}(\overline{v}') + x_{L3}(\overline{v}')) + \beta(\overline{v}')(x_{R2}(\overline{v}') + x_{R3}(\overline{v}')) \geq 20(1-\beta(\overline{v}')) + 32\beta(\overline{v}') \\ x_{R1}(\overline{v}') + x_{R2}(\overline{v}') + x_{R3}(\overline{v}') = 44 \\ x_{L1}(\overline{v}') + x_{L2}(\overline{v}') + x_{L3}(\overline{v}') = 40 \\ x_{Ri}(\overline{v}') \geq x_{Li}(\overline{v}') \quad (i = 1, 2, 3) \\ 0 \leq \beta(\overline{v}') \leq 1, \end{cases} \tag{2.18}$$

where $x_{Ri}(\overline{v}')$, $x_{Li}(\overline{v}')$ $(i = 1, 2, 3)$, and $\beta(\overline{v}')$ are decision variables.

Solving Eq. (2.18) by the bisection algorithm given in Sect. 2.3.3, we can narrow the interval $[0, 1]$ in which $\beta(\overline{v}')$ belongs to and infer that the optimal value $\beta^*(\overline{v}') \in [0.875, 0.87506]$. Thus, we can obtain the global optimal solution $(\beta^*(\overline{v}'), \overline{\boldsymbol{x}}^*(\overline{v}'))$ of Eq. (2.18) at a given precision, where $\beta^*(\overline{v}') = 0.875$, $\overline{x}_1^*(\overline{v}') = [9.5, 13.5]$, $\overline{x}_2^*(\overline{v}') = [16, 16]$, and $\overline{x}_3^*(\overline{v}') = [14.5, 14.5]$. Therefore, for the interval-valued cooperative game $\overline{v}' \in \overline{G}^3$, $\overline{\boldsymbol{x}}^*(\overline{v}')$ is an element of its interval-valued core $\overline{C}(\overline{v}')$ with the maximum satisfactory degree $\beta^*(\overline{v}') = 0.875$. In other words, if the maximum satisfactory degree of $\sum_{i\in S}\overline{x}_i(\overline{v}') \geq \overline{v}'(S)$ for the three companies p_1, p_2, and p_3 in the electronic product supply chain is not greater than 0.875, then the interval-valued core of the interval-valued cooperative game $\overline{v}' \in \overline{G}^3$ exists and hereby the three companies may choose product cooperative innovation.

Analogously, according to Eq. (2.15), the system of linear inequalities can be constructed as follows:

$$\begin{cases} x_{R1}\left(\overline{v}'\right) + x_{R2}\left(\overline{v}'\right) < 22 \\ x_{R1}\left(\overline{v}'\right) + x_{R3}\left(\overline{v}'\right) < 24 \\ x_{R2}\left(\overline{v}'\right) + x_{R3}\left(\overline{v}'\right) < 20 \\ x_{R1}\left(\overline{v}'\right) + x_{R2}\left(\overline{v}'\right) + x_{R3}\left(\overline{v}'\right) = 44 \\ x_{L1}\left(\overline{v}'\right) + x_{L2}\left(\overline{v}'\right) + x_{L3}\left(\overline{v}'\right) = 40 \\ x_{Ri}\left(\overline{v}'\right) \geq x_{Li}\left(\overline{v}'\right) \quad (i = 1, 2, 3), \end{cases} \tag{2.19}$$

where $x_{Ri}\left(\overline{v}'\right)$ and $x_{Li}\left(\overline{v}'\right)$ $(i = 1, 2, 3)$ are decision variables.

Solving Eq. (2.19) by using the LINGO tool, we find that there is no feasible solution of Eq. (2.19) and hereby the three companies may have not any cooperative desire for this situation.

According to Eqs. (2.16) and (2.17), the systems of linear inequalities can be constructed as follows:

$$\begin{cases} x_{L1}\left(\overline{v}'\right) + x_{L2}\left(\overline{v}'\right) < 22 \\ x_{L1}\left(\overline{v}'\right) + x_{L3}\left(\overline{v}'\right) < 24 \\ x_{L2}\left(\overline{v}'\right) + x_{L3}\left(\overline{v}'\right) < 20 \\ x_{R1}\left(\overline{v}'\right) + x_{R2}\left(\overline{v}'\right) \geq 22 \\ x_{R1}\left(\overline{v}'\right) + x_{R3}\left(\overline{v}'\right) \geq 24 \\ x_{R2}\left(\overline{v}'\right) + x_{R3}\left(\overline{v}'\right) \geq 20 \\ x_{R1}\left(\overline{v}'\right) + x_{R2}\left(\overline{v}'\right) < 30 \\ x_{R1}\left(\overline{v}'\right) + x_{R3}\left(\overline{v}'\right) < 28 \\ x_{R2}\left(\overline{v}'\right) + x_{R3}\left(\overline{v}'\right) < 32 \\ x_{R1}\left(\overline{v}'\right) + x_{R2}\left(\overline{v}'\right) + x_{R3}\left(\overline{v}'\right) = 44 \\ x_{L1}\left(\overline{v}'\right) + x_{L2}\left(\overline{v}'\right) + x_{L3}\left(\overline{v}'\right) = 40 \\ x_{Ri}\left(\overline{v}'\right) \geq x_{Li}\left(\overline{v}'\right) \quad (i = 1, 2, 3) \end{cases} \tag{2.20}$$

and

$$
\begin{cases}
x_{L1}\left(\overline{v}'\right) + x_{L2}\left(\overline{v}'\right) = 22 \\
x_{L1}\left(\overline{v}'\right) + x_{L3}\left(\overline{v}'\right) = 24 \\
x_{L2}\left(\overline{v}'\right) + x_{L3}\left(\overline{v}'\right) = 20 \\
x_{R1}\left(\overline{v}'\right) + x_{R2}\left(\overline{v}'\right) - \left(x_{L1}\left(\overline{v}'\right) + x_{L2}\left(\overline{v}'\right)\right) = 30 - 22 \\
x_{R1}\left(\overline{v}'\right) + x_{R3}\left(\overline{v}'\right) - \left(x_{L1}\left(\overline{v}'\right) + x_{L3}\left(\overline{v}'\right)\right) = 28 - 24 \\
x_{R2}\left(\overline{v}'\right) + x_{R3}\left(\overline{v}'\right) - \left(x_{L2}\left(\overline{v}'\right) + x_{L3}\left(\overline{v}'\right)\right) = 32 - 20 \\
x_{R1}\left(\overline{v}'\right) + x_{R2}\left(\overline{v}'\right) + x_{R3}\left(\overline{v}'\right) = 44 \\
x_{L1}\left(\overline{v}'\right) + x_{L2}\left(\overline{v}'\right) + x_{L3}\left(\overline{v}'\right) = 40 \\
x_{Ri}\left(\overline{v}'\right) \geq x_{Li}\left(\overline{v}'\right) \quad (i = 1, 2, 3),
\end{cases}
\tag{2.21}
$$

respectively, where $x_{Ri}\left(\overline{v}'\right)$ and $x_{Li}\left(\overline{v}'\right)$ $(i = 1, 2, 3)$ are decision variables.

Solving Eqs. (2.20) and (2.21) by using the LINGO tool, respectively, we find that there are no feasible solutions of Eqs. (2.20) and (2.21) and hereby the three companies may have not any cooperative desire for these situations.

2.4.2 Computational Results Obtained by the Moore's Order Relation Between Intervals

According to Eq. (2.12), we construct the system of linear inequalities as follows:

$$
\begin{cases}
\overline{x}_1\left(\overline{v}'\right) + \overline{x}_2\left(\overline{v}'\right) \geq [22, 30] \\
\overline{x}_1\left(\overline{v}'\right) + \overline{x}_3\left(\overline{v}'\right) \geq [24, 28] \\
\overline{x}_2\left(\overline{v}'\right) + \overline{x}_3\left(\overline{v}'\right) \geq [20, 32] \\
\overline{x}_1\left(\overline{v}'\right) + \overline{x}_2\left(\overline{v}'\right) + \overline{x}_3\left(\overline{v}'\right) = [40, 44] \\
x_{Ri}\left(\overline{v}'\right) \geq x_{Li}\left(\overline{v}'\right) \quad (i = 1, 2, 3),
\end{cases}
\tag{2.22}
$$

where $x_{Ri}\left(\overline{v}'\right)$ and $x_{Li}\left(\overline{v}'\right)$ $(i = 1, 2, 3)$ are decision variables.

Using the Moore's order relation between intervals, i.e., Eq. (1.4), Eq. (2.22) can be rewritten as the following system of inequalities:

$$\begin{cases} x_{R1}\left(\overline{v}'\right) + x_{R2}\left(\overline{v}'\right) \geq 30 \\ x_{L1}\left(\overline{v}'\right) + x_{L2}\left(\overline{v}'\right) \geq 22 \\ x_{R1}\left(\overline{v}'\right) + x_{R3}\left(\overline{v}'\right) \geq 28 \\ x_{L1}\left(\overline{v}'\right) + x_{L3}\left(\overline{v}'\right) \geq 24 \\ x_{R2}\left(\overline{v}'\right) + x_{R3}\left(\overline{v}'\right) \geq 32 \\ x_{L2}\left(\overline{v}'\right) + x_{L3}\left(\overline{v}'\right) \geq 20 \\ x_{R1}\left(\overline{v}'\right) + x_{R2}\left(\overline{v}'\right) + x_{R3}\left(\overline{v}'\right) = 44 \\ x_{L1}\left(\overline{v}'\right) + x_{L2}\left(\overline{v}'\right) + x_{L3}\left(\overline{v}'\right) = 40 \\ x_{Ri}\left(\overline{v}'\right) \geq x_{Li}\left(\overline{v}'\right) \quad (i = 1, 2, 3). \end{cases} \tag{2.23}$$

Solving Eq. (2.23) by using the LINGO tool, we find that there is no feasible solution of Eq. (2.23) and hereby the three companies may have not any desire for product cooperative innovation.

Therefore, the interval-valued core of the interval-valued cooperative game $\overline{v}' \in \overline{G}^3$ does not exist if the Moore's order relation between intervals is used. On the contrary, we can obtain an element of the interval-valued core of the interval-valued cooperative game $\overline{v}' \in \overline{G}^3$ by introducing the satisfactory degrees of comparing intervals. That is to say, the interval-valued core $\overline{C}\left(\overline{v}'\right)$ of the interval-valued cooperative game $\overline{v}' \in \overline{G}^3$ exists, i.e., $\overline{C}\left(\overline{v}'\right) \neq \varnothing$. This result may give more management suggestions for the players (or managers).

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