Chapter 2
Measurability

Central to the discussion of measurability is the notion of a measurable space.

Definition 2.1. A *measurable space* is a set X together with a σ-algebra $\mathbf{S}$ of subsets of X.

Thus, strictly speaking, a measurable space is a pair $(X, \mathbf{S})$. Nevertheless, we frequently denote this space by the single symbol X and refer to X as measurable *with respect to* the σ-algebra $\mathbf{S}$. The elements of $\mathbf{S}$ are called the *measurable subsets* of X. (The requirement that the σ-ring $\mathbf{S}$ be complemented is equivalent to requiring the entire space X to be a measurable set.) Thus, as mentioned in Remark 1.3, in some texts a measurable space is defined to be a pair $(X, \mathbf{S})$ where $\mathbf{S}$ is a σ-ring of subsets of X.) In the event that any possibility of confusion exists, for instance, if X is simultaneously measurable with respect to some other σ-algebra in addition to $\mathbf{S}$, the elements of $\mathbf{S}$ will be said to be *measurable* [$\mathbf{S}$].

Example 2.2. If X is a metric space or, more generally, a topological space then X is a measurable space with respect to the σ-algebra $\mathbf{B}_X$ of Borel sets in X. In the absence of any stipulation to the contrary, whenever, in the sequel, a metric space is regarded as a measurable space, it is the algebra $\mathbf{B}_X$ that is understood to be the σ-algebra of measurable sets.

Example 2.3. In the extended real number system $\mathbb{R}^\natural$ ([**I**, Chapter 2]) the closed rays $[a, +\infty] = \{t \in \mathbb{R}^\natural : t \geq a\}$, $a \in \mathbb{R}^\natural$, generate a σ-ring $\mathbf{B}^\natural$ containing the whole space $\mathbb{R}^\natural$, so $(\mathbb{R}^\natural, \mathbf{B}^\natural)$ is a measurable space. Whenever, in this book, we have occasion to regard $\mathbb{R}^\natural$ as a measurable space, it is this algebra $\mathbf{B}^\natural$ of "extended" Borel sets that we have in mind. (Since $\mathbf{B}^\natural$ contains the half-open interval $[a, b)$ for all finite real numbers a and b, it is clear that the trace of $\mathbf{B}^\natural$ on $\mathbb{R}$ coincides with the algebra $\mathbf{B}_{\mathbb{R}}$ of ordinary real Borel sets. Thus $\mathbf{B}^\natural$ is obtained from $\mathbf{B}_{\mathbb{R}}$ by adjoining the ideal numbers $\pm\infty$—either, neither or both—to the various Borel subsets of $\mathbb{R}$.)

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Example 2.4. If $\mathbf{S}$ is a σ-ring of subsets of a set X, and if $\mathbf{S}$ is *not* complemented, then there are various methods of turning X into a measurable space in such a way that the sets in $\mathbf{S}$ become measurable. The most economical of these procedures is, of course, the one set forth in Example 1.13; see also Problem 1T.

Example 2.5. If $(X, \mathbf{S})$ is a measurable space in which all of the singletons in X are measurable sets, then $\mathbf{S}$ must contain the collection $\mathbf{S}_0$ comprising all the countable subsets of X and their complements. But $\mathbf{S}_0$ is itself a σ-algebra. Thus the singletons in X are all measurable [$\mathbf{S}$] if and only if $\mathbf{S}$ refines $\mathbf{S}_0$.

Example 2.6. If $\{X_\gamma\}_{\gamma \in \Gamma}$ is an indexed partition of a set X, and if $\mathbf{S}_\gamma$ is a σ-algebra in X_γ for each index γ, then the full direct sum $\bigoplus_\gamma \mathbf{S}_\gamma$ of the family $\{\mathbf{S}_\gamma\}$ (Example 1.16) is also complemented in X, and therefore turns X into a measurable space, called the *full direct sum* of the family $\{(X_\gamma, \mathbf{S}_\gamma)\}$ and denoted by

$$\bigoplus_{\gamma \in \Gamma} (X_\gamma, \mathbf{S}_\gamma).$$

In particular, if $(X_1, \mathbf{S}_1)$ and $(X_2, \mathbf{S}_2)$ are measurable spaces with X_1 and X_2 disjoint, then the measurable space $(X_1 \cup X_2, \mathbf{S}_1 \oplus \mathbf{S}_2)$ can also be written as $(X_1, \mathbf{S}_1) \oplus (X_2, \mathbf{S}_2)$; see Example 1.15.

Let X be a measurable space with respect to a σ-algebra $\mathbf{S}$, and let A be an arbitrary subset of X. Then the trace $\mathbf{S}_A = \{E \cap A\colon\ E \in \mathbf{S}\}$ of $\mathbf{S}$ on A is a σ-algebra in A (Problem 1Q), and $(A, \mathbf{S}_A)$ is again a measurable space. Observe that in the event that A is itself measurable, the σ-algebra $\mathbf{S}_A$ simply consists of the measurable subsets of A: $\mathbf{S}_A = \{E \in \mathbf{S}\colon\ E \subset A\}$. In this case, and in this case only, we refer to $(A, \mathbf{S}_A)$ as a *subspace* of $(X, \mathbf{S})$.

Definition 2.7. If X is a measurable space with respect to a σ-algebra $\mathbf{S}$, then a *subspace* of X is a measurable set E (with respect to $\mathbf{S}$) equipped with the σ-algebra $\mathbf{S}_E$.

Example 2.8. If X is a metric space and A an arbitrary subset of X, then A itself becomes a metric subspace of X when equipped with its relative metric ([**I**, Chapter 6]), and the collection of all those subsets of A that are open with respect to this relative metric is precisely $\mathcal{G}_A$, the trace on A of the collection $\mathcal{G}$ of open sets in X ([**I,** Proposition 6.15]). The trace of the σ-algebra $\mathbf{B}_X$ of Borel sets in X is precisely the σ-algebra $\mathbf{B}_A$. Thus there is an unambiguous sense in which A is to be regarded as a measurable space in its own right. Nevertheless, according to the foregoing definition, $(A, \mathbf{B}_A)$ is not a subspace of the measurable space $(X, \mathbf{B}_X)$ unless A is a Borel set in X. Thus, for example, the measurable space $(\mathbb{R}, \mathbf{B}_{\mathbb{R}})$ of real numbers is a subspace of the measurable space $(\mathbb{R}^\natural, \mathbf{B}^\natural)$ of extended real numbers.

Given measurable spaces X and Y, there is a natural way of distinguishing a special class of mappings of X into Y, called the *measurable* mappings.

Definition 2.9. Let $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ be measurable spaces. A mapping $\varphi : X \to Y$ is *measurable* (when necessary, *measurable* $[\mathbf{S}, \mathbf{T}]$) if the inverse image under φ of every set E in $\mathbf{T}$ is measurable $[\mathbf{S}]$, that is, if $\varphi^{-1}(E) \in \mathbf{S}$ for every set E in $\mathbf{T}$. More generally, if φ is defined only on some subset of X that contains the set A, then φ is said to be *measurable on* A if $\varphi|A$ is measurable on the measurable space $(A, \mathbf{S}_A)$. In the special case that Y is a metric space, in keeping with the general convention enunciated in Example 2.2, a mapping φ of X into Y is said simply to be *measurable* (or *measurable* $[\mathbf{S}]$) if it is measurable $[\mathbf{S}, \mathbf{B}_Y]$. Moreover, if both X and Y are metric spaces, the mapping φ is said to be *Borel measurable* if it is measurable $[\mathbf{B}_X, \mathbf{B}_Y]$.

Example 2.10. Any constant mapping of a measurable space $(X, \mathbf{S})$ into a measurable space $(Y, \mathbf{T})$ is measurable, since $\mathbf{S}$ always contains the whole space X. On the other hand, if $\mathbf{S} = \{\varnothing, X\}$, then it may happen that *only* the constant mappings of X into Y are measurable. Dually, if $\mathbf{S} = 2^X$, then *every* mapping of X into Y is measurable $[\mathbf{S}]$.

Example 2.11. Let X and Y be measurable spaces, let $\varphi : X \to Y$ be measurable, and suppose Y has the property that the singletons $\{y_0\}$ in Y are all measurable sets (see Example 2.5). Then all the level sets

$$\{x \in X : \varphi(x) = y_0\} = \varphi^{-1}(\{y_0\})$$

of φ are measurable sets in X. In particular, this is the case if Y is a metric space or, more generally, a Hausdorff topological space.

Example 2.12. Let $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ be measurable spaces, let A and B be subsets of X such that $A \subset B$, and suppose $\varphi : X \to Y$ is measurable on B. Then for each set F in $\mathbf{T}$ there is a set E in $\mathbf{S}$ such that $(\varphi|B)^{-1}(F) = E \cap B$. But then

$$(\varphi|A)^{-1}(F) = \{x \in A : \varphi(x) \in F\} = E \cap A.$$

Thus φ is automatically measurable on A as well, so the collection $\mathcal{A}$ of all those subsets of X on which φ is measurable is closed with respect to the formation of subsets. But $\mathcal{A}$ is not closed with respect to unions in general. Indeed, if A is any *non*measurable subset of X, then the characteristic function χ_A is measurable on both A and $X \setminus A$—being constant on each set—but the real-valued function χ_A is not measurable on X.

On the other hand, the collection $\mathbf{M}$ of all those *measurable* subsets of X on which φ is measurable is clearly a σ-ring contained in $\mathbf{S}$. In particular, a mapping of X into Y that is measurable on each of a countable collection of measurable sets that covers X is itself measurable.

Proposition 2.13. *Let* $(X, \mathbf{S})$ *and* $(Y, \mathbf{T})$ *be measurable spaces, and let* $\varphi : X \to Y$ *be a map. Suppose given a collection* $\mathcal{C}$ *of subsets of* Y *such that* $\mathbf{T} = \mathbf{S}(\mathcal{C})$. *Then* φ *is measurable* $[\mathbf{S}, \mathbf{T}]$ *if (and only if)* $\varphi^{-1}(C) \in \mathbf{S}$ *for every* C *in* $\mathcal{C}$.

Proof. The collection $\{A \subset Y : \varphi^{-1}(A) \in \mathbf{S}\}$ is a σ-ring in Y and contains $\mathcal{C}$, so it must contain $\mathbf{T}$. □

Corollary 2.14. *Let X be a measurable space and let Y be a metric space. Then a mapping $\varphi\colon X \to Y$ is measurable if (and only if) the inverse image under φ of every open [closed] set in Y is measurable.*

In connection with these ideas, and in other contexts as well, the following notation is frequently useful.

Notation

For every real-valued function f on a set X and every real number a we write $E(f < a)$ and $E(f \leq a)$ for the sets $\{x \in X : f(x) < a\}$ and $\{x \in X : f(x) \leq a\}$, respectively. Similarly, $E(f > a)$ and $E(f \geq a)$ denote the sets $\{x \in X : f(x) > a\}$ and $\{x \in X : f(x) \geq a\}$, respectively. In the same vein, if a and b are two real numbers, we write $E(a < f < b)$ for the set

$$\{x \in X : a < f(x) < b\} = E(f > a) \cap E(f < b),$$

and similarly for symbols such as

$$E(a < f \leq b), \quad E(a \leq f \leq b) \quad \text{and} \quad E(f = a).$$

Example 2.15. All the sets mentioned above, associated with a measurable real-valued function f on a measurable space X, are clearly measurable. Conversely, if all of the sets $E(f < t), t \in \mathbb{R}$, are measurable, then f is measurable. Similar statements hold for the families $\{E(f \leq t), t \in \mathbb{R}\}$, $\{E(f > t) : t \in \mathbb{R}\}$, and $\{E(f \geq t) : t \in \mathbb{R}\}$. Exactly the same criteria ensure measurability when f is extended real-valued. See Problems 2A and 2B.

Example 2.16. Every semicontinuous extended real-valued function on a metric space is Borel measurable; see [**I**, Proposition 7.17].

Example 2.17. If X and Y are metric spaces, then every continuous mapping of X into Y is Borel measurable ([**I**, Theorem7.4]). If, in particular, X is a discrete metric space (so every mapping of X into Y is continuous), then the continuous mappings of X into Y exhaust the Borel measurable mappings of X into Y. This case is exceptional; ordinarily one expects to find many Borel measurable mappings that are not continuous.

Example 2.18. A complex-valued function u on a measurable space X is measurable if and only if the real-valued functions $\Re u$ and $\Im u$ are both measurable. Indeed, if a, b, c, and d are any four real numbers such that $a \leq b$ and $c \leq d$, and if we write S_1 and S_2 for the closed strips

$$S_1 = \{\lambda \in \mathbb{C} : a \leq \Re\lambda \leq b\}, \quad S_2 = \{\lambda \in \mathbb{C} : c \leq \Im\lambda \leq d\},$$

then

$$u^{-1}(S_1) = (\Re u)^{-1}([a, b]) \quad \text{and} \quad u^{-1}(S_2) = (\Im u)^{-1}([c, d]).$$

This shows that if u is measurable, then $\Re u$ and $\Im u$ are also measurable. On the other hand, the rectangle $R = [a, b] \times [c, d]$ coincides with $S_1 \cap S_2$. If $\Re u$ and $\Im u$ are both measurable, then $u^{-1}(R)$ is a measurable set, and since the closed rectangles R generate the Borel sets in $\mathbb{C}$ as a σ-ring (Problem 1M), this shows that the complex-valued function u is measurable as well. (Note that this implies that u is measurable if and only if its complex conjugate $\bar{u}$ is measurable.) More generally, a mapping $x \mapsto (f_1(x), \ldots, f_d(x))$ of X into $\mathbb{R}^d$ is measurable if and only if the coordinate functions $f_i, i = 1, \ldots, n$, are all measurable, and similarly for $\mathbb{C}^d$ in place of $\mathbb{R}^d$. (See also Problem 2G.)

Example 2.19. Every monotone real-valued function f on $\mathbb{R}$ is Borel measurable. Indeed, if I is an interval of any type in $\mathbb{R}$, then $f^{-1}(I)$ is likewise an interval of some type. Similarly, every monotone extended real-valued function on $\mathbb{R}$ is Borel measurable. More generally, every monotone extended real-valued function defined on an arbitrary subset A of $\mathbb{R}$ is Borel measurable on A.

We turn next to the consideration of various ways of combining measurable mappings. Our first result is very simple but also very useful.

Proposition 2.20. *Let $(X, \mathbf{S}), (Y, \mathbf{T})$ and $(Z, \mathbf{U})$ be measurable spaces, and let $\varphi \colon X \to Y$ and $\psi \colon Y \to Z$ be measurable mappings. Then the composition $\psi \circ \varphi$ is measurable. In particular, if Y and Z are both metric spaces, and if ψ is Borel measurable, then $\psi \circ \varphi$ is measurable $[\mathbf{S}]$ whenever φ is.*

Proof. If E belongs to $\mathbf{U}$, then $\psi^{-1}(E)$ belongs to $\mathbf{T}$, and thus $\varphi^{-1}(\psi^{-1}(E))$ belongs to $\mathbf{S}$. But $\varphi^{-1}(\psi^{-1}(E)) = (\psi \circ \varphi)^{-1}(E)$. $\square$

Example 2.21. If f is a measurable complex-valued function on a measurable space X, then the powers f^n of $f, n \in \mathbb{N}$, are all measurable as well. Similarly, the functions $|f|^r, r > 0$, are also all measurable. Moreover, the same is true of the functions f^{-n} and $|f|^{-r}$ on the (measurable) subspace $\{x \in X \colon f(x) \neq 0\}$.

Proposition 2.22. *If $f_1, \ldots, f_n$ are arbitrary measurable scalar-valued functions on a measurable space X, then $f_1 + f_2 + \cdots + f_n$ is also measurable.*

Proof. It suffices (by mathematical induction) to prove that the sum $f + g$ of two measurable functions is measurable. To this end, consider the mapping T of X into $\mathbb{C}^2$ defined by $T(x) = (f(x), g(x))$. If U and V are open sets in $\mathbb{C}$, then

$$T^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V)$$

is a measurable set. But every open set in $\mathbb{C}^2$ is a countable union of sets of the form $U \times V$, since $\mathbb{C}$ is separable. Hence T is measurable by Proposition 2.13. Moreover, addition is continuous regarded as a mapping $s : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, whence it follows by Proposition 2.20 that $s \circ T$ is also measurable. Since $(s \circ T)(x) = f(x) + g(x)$, the proof is complete. □

The modification in the proof of Proposition 2.22 required to establish the following result is obvious (see [**I**, Problem 7H]).

Proposition 2.23. *If $f_1, \ldots, f_n$ are measurable real-valued functions on a measurable space X, then the functions $f_1 \vee \cdots \vee f_n$ and $f_1 \wedge \cdots \wedge f_n$ are also measurable.*

Example 2.24. A real-valued function f on a measurable space X is measurable if and only if the positive and negative parts $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$ are both measurable on X.

Proposition 2.25. *If $f_1, \ldots, f_n$ are measurable scalar-valued functions on a measurable space X, and if $\alpha_1, \ldots, \alpha_n$ are scalars, then the linear combination*

$$\alpha_1 f_1 + \cdots + \alpha_n f_n \tag{2.1}$$

and the product

$$f_1 \ldots f_n \tag{2.2}$$

are also measurable.

Proof. The measurability of (2.1) is an immediate consequence of Proposition 2.22 and the second half of this result, in view of the fact that constant functions are measurable. As for the measurability of (2.2), it suffices as before to verify the measurability of the product fg of two measurable functions, and this may either be settled directly via another obvious modification of the proof of Proposition 2.22, or it may be derived from the identity $2fg = (f+g)^2 - (f^2 + g^2)$. □

Corollary 2.26. *If f and g are measurable scalar-valued functions on a measurable space X, then $\{x \in X : f(x) = g(x)\}$ is a measurable set.*

Proof. This is the zero level set of the measurable function $f - g$. □

Example 2.27. If $p(t)$ is a real polynomial and f is a measurable real-valued function on a measurable space X, then $p(f(x))$ is also measurable on X. Similarly, if $p(\lambda)$ is a complex polynomial and f is a measurable complex-valued function on X, then $p(f(x))$ is measurable. These facts may be viewed as consequences of Proposition 2.25, or may be derived directly from Proposition 2.20, since $p(f(x))$ coincides with the composition $p \circ f$, and the function $p(\lambda)$ is continuous.

A significant advantage that measurability enjoys over continuity is that it is readily preserved in passage to a limit. We begin our discussion of these matters with a detailed treatment of the important special case of a sequence of real-valued functions.

Proposition 2.28. *Let $\{f_n\}$ be a sequence of measurable, extended real-valued functions on a measurable space X. Then the functions*

$$\sup_n f_n \quad \textit{and} \quad \inf_n f_n$$

are also measurable. In particular, the set on which $\{f_n\}$ is pointwise bounded above [below] in $\mathbb{R}$ is measurable.

Proof. It suffices to treat upper bounds because

$$\inf_n f_n = -\sup_n(-f_n).$$

Let A denote the set of points $x \in X$ for which the sequence $\{f_n(x)\}$ is bounded above in $\mathbb{R}$. Then $\sup_n f_n$ is constant $(= +\infty)$ on $X \setminus A$, so it is enough to verify that A is measurable and that $\sup_n f_n$ is measurable on A (see Problem 2B).

For each real number M set

$$E_M = \bigcap_{n=1}^{\infty} E(f_n \leq M).$$

Then E_M is clearly measurable, and we also have $E_M = E(\sup_n f_n \leq M)$. Thus, on the one hand,

$$A = \bigcup_{N=1}^{\infty} E_N$$

is measurable, while, on the other hand, $\sup_n f_n$ is measurable on A. □

Corollary 2.29. *If $\{f_n\}$ is a monotone sequence of measurable, extended real-valued functions on a measurable space X, then the pointwise limit $\lim_n f_n$ is also measurable.*

Proof. If $\{f_n\}$ is increasing [decreasing], $\lim_n f_n = \sup_n f_n$ $[= \inf_n f_n]$. □

Corollary 2.30. *For any sequence $\{f_n\}$ of measurable, extended real-valued functions on a measurable space X, the functions $\limsup_n f_n$ and $\liminf_n f_n$ are measurable.*

Proof. It suffices to treat the upper limit. For each positive integer m set

$$g_m = \sup_{n \geq m} f_n.$$

so that $\limsup_n f_n = \inf_m g_m$. The conclusion follows by two applications of Proposition 2.28. □

Theorem 2.31. *For an arbitrary sequence $\{f_n\}$ of measurable (finite) real-valued functions on a measurable space X, the set E of those points x for which the sequence $\{f_n(x)\}$ is convergent in $\mathbb{R}$ is measurable, and the pointwise limit*

$$\lim_n f_n(x), \quad x \in E,$$

is a measurable function on E.

Proof. This follows immediately from Corollary 2.30. □

The result of Theorem 2.31 is true for complex-valued functions as well. (The proof of the following theorem can readily be reduced to the real case, but the argument given extends to more general situations; see Problem 2I.)

Theorem 2.32. *Let X be a measurable space, and let $\{f_n\}$ be a sequence of measurable complex-valued functions on X. Denote by E the set of all those points x of X for which the numerical sequence $\{f_n(x)\}$ is convergent in $\mathbb{C}$. Then E is a measurable set, and the function $f(x) = \lim_n f_n(x)$ is measurable on the subspace E.*

Proof. For each triple (k, m, n) of positive integers write

$$E_{k,m,n} = E(|f_k - f_m| < 1/n).$$

Since $|f_k - f_m|$ is a measurable function, each of the sets $E_{k,m,n}$ is measurable, and so therefore is the set

$$F_n = \bigcup_{p=1}^{\infty} \bigcap_{k=p}^{\infty} \bigcap_{m=p}^{\infty} E_{k,m,n}.$$

Now F_n is precisely the set of points x in X such that $|f_k(x) - f_m(x)| < 1/n$ eventually. The measurability of E follows from the equality $E = \bigcap_{n=1}^{\infty} F_n$, and this holds because a sequence of scalars converges if and only if it satisfies the Cauchy criterion.

In order to see that the function f is measurable on E, let U be an open set in the complex plane and for each $n \in \mathbb{N}$ set $G_n = E \cap f_n^{-1}(U)$. Then G_n is measurable, and so therefore is

$$H_U = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} G_n.$$

The set H consists precisely of those points x of E with the property that $\{f_n(x)\}$ is eventually in U. Next let V be an open set in $\mathbb{C}$ different from $\mathbb{C}$ itself, and let F denote the complement of V. For each positive integer j let U_j denote the set of points in $\mathbb{C}$ whose distance from the closed set F is greater than $1/j$:

$$U_j = \{\lambda \in \mathbb{C} \colon d(\lambda, F) > 1/j\}.$$

The set U_j is an open subset ([**I**, Problem 6J]) and we conclude that $f^{-1}(V) = \bigcup_{j=1}^{\infty} H_{U_j}$ is measurable, thus concluding the proof. □

The following is a convenient summary of Propositions 2.23 and 2.25, Theorems 2.31 and 2.32, and Example 2.27. A linear manifold of scalar-valued functions that is closed with respect to multiplication is called a *function algebra*.

Theorem 2.33. *The collection $\mathcal{M}_{\mathbb{C}}$ of all measurable complex-valued functions on a measurable space X is a complex function algebra that contains all constant functions and is closed with respect to complex conjugation and the formation of limits of pointwise convergent sequences. The collection $\mathcal{M}_{\mathbb{R}}$ of all the real-valued functions in $\mathcal{M}_{\mathbb{C}}$ constitutes a real function algebra that is also a function lattice* ([**I**, *Problem 2M*]).

It is an interesting and useful fact that Theorem 2.33 has a valid converse. Before stating it, however, we introduce a concept that will play a major role in all that follows.

Definition 2.34. A scalar-valued function on a set X is said to be *simple* if it assumes only a finite number of distinct values.

The main facts about simple functions are summarized for convenience of reference in the following proposition, for which no proof need to be given.

Proposition 2.35. *A scalar-valued function on a set X is simple if and only if it can be expressed as a linear combination of characteristic functions of subsets of X. Among such representations of a given simple function s there is precisely one,*

$$s = \sum_{i=1}^{m} \alpha_i \chi_{E_i}, \tag{2.3}$$

in which the sets E_i are disjoint and nonempty and the coefficients α_i are distinct from one another and from zero. (*If $s = 0$, the sum in* (2.3) *is empty.*) *If X is a measurable space, then a simple function s on X is measurable if and only if it can be expressed as a linear combination of characteristic functions of measurable subsets of X. Alternatively, s is measurable if and only if it assumes each of its values on a measurable set or, equivalently, if and only if the sets E_i in* (2.3) *are all measurable.*

Simple functions provide an important link between measurable sets and measurable functions. Their usefulness stems largely from the following fact.

Proposition 2.36. *Let X be a measurable space and consider a function $f : X \to \mathbb{R}$. Then the following three conditions are equivalent.*

(1) *The function f is measurable.*
(2) *There exists a sequence $\{s_n\}$ of measurable simple real-valued functions on X that converges pointwise to f.*

(3) *There exists a sequence* $\{s_n\}$ *of measurable simple real-valued functions on* X *converging pointwise to* f *and satisfying the following additional conditions*:

(a) *at each point* x *of* X *either* $0 \leq s_1(x) \leq s_2(x) \leq \cdots$, *or* $0 \geq s_1(x) \geq s_2(x) \geq \cdots$, *and*

(b) *if* $M \in \mathbb{N}$ *and if* $|f(x)| \leq M$, *then* $|f(x) - s_n(x)| \leq 1/2^n$ *for every* $n \geq M$.

Before giving the proof, we note that property (3a) implies that $|s_n(x)| \leq |f(x)|$ for all x and all n. In particular, if $f(x) = 0$, then $s_n(x) = 0$ for all n. Note also that (3b) implies that $\{s_n\}$ converges uniformly to f on any set on which f is bounded.

Proof. It is obvious that (3) is a stronger condition than (2), while (2) implies (1) by Theorem 2.31. Hence it suffices to show that (1) implies (3). For each positive integer n consider the points

$$t_n = k/2^n, \qquad k = -n2^n, -n2^n + 1, \ldots, n2^n.$$

We define the simple function s_n as follows:

(α) If $f(x) \geq n$ or $f(x) < -n$ set $s_n(x) = 0$,

(β) If $-n \leq f(x) < n$, find k such that $t_k \leq f(x) < t_{k+1}$, and set $s_n(x) = t_k$ if $t_k \geq 0$ and $s_n(x) = t_{k+1}$ if $t_k < 0$.

A moment's reflection shows that the set on which the function s_n assumes each nonzero value t_k is the inverse image under f of a half-open interval, and since all intervals are Borel sets in $\mathbb{R}$, it follows that s_n is measurable. Moreover (3b) follows because $|f(x) - s_n(x)| \leq 1/2^n$ whenever $|f(x)| \leq n$. The verification of (3a) is left to the reader.

□

Remark 2.37. If $\{s_n\}$ is a sequence of measurable simple real-valued functions tending pointwise to a limit f as in (3a) of Proposition 2.36, then all three of the sequences $\{s_n^+\}, \{s_n^-\}, \{|s_n|\}$ are monotone increasing on X and tend pointwise to the limits f^+, f^- and $|f|$, respectively.

The analog of Proposition 2.36 for complex-valued functions is proved in a similar manner by partitioning the closed disk of radius $n2^n$ into parts of diameter $1/2^n$ and selecting for each part a point of minimum absolute value. We record the statement below.

Proposition 2.38. *Let* X *be a measurable space and consider a measurable function* $f : X \to \mathbb{C}$. *There exists a sequence* $\{s_n\}$ *of measurable simple complex-valued functions on* X *converging pointwise to* f *and satisfying the following additional conditions*:

(a) $0 \le |s_1(x)| \le |s_2(x)| \le \cdots \le |f(x)|$, $x \in X$, *and*
(b) *if* $M \in \mathbb{N}$ *and if* $|f(x)| \le M$, *then* $|f(x) - s_n(x)| \le 1/2^n$ *for every* $n \ge M$.

We can now prove the converse of Theorem 2.33.

Theorem 2.39. *Let X be a set and let $\mathcal{M}$ be a function algebra of real-valued functions on X that contains the constant functions and is closed with respect to the formation of limits of pointwise convergent sequences. Then there exists a unique σ-algebra $\mathbf{S}$ in X such that $\mathcal{M}$ is precisely the collection of all real-valued functions measurable $[\mathbf{S}]$. Similarly, a function algebra $\mathcal{M}$ of complex-valued functions on X that contains the constant functions and is closed with respect to complex conjugation and the formation of limits of pointwise convergent sequences is the collection of all complex-valued measurable functions with respect to a unique σ-algebra $\mathbf{S}$ in X.*

We only prove the theorem in the real case, leaving the complex case as an exercise (see Problem 2N). We need a lemma.

Lemma 2.40. *Let U be an open subset of the real line $\mathbb{R}$. Then there exists a sequence $\{p_n\}$ of real polynomials that converges pointwise to the characteristic function χ_U.*

Proof. First, there exists a sequence $\{f_n\}$ of continuous functions on $\mathbb{R}$ that converges pointwise to χ_U (see [**I**, Problem 7F]). By the Weierstrass approximation theorem, there exists, for each index n, a real polynomial p_n such that $|f_n(t) - p_n(t)| \le 1/n$ for all $|t| \le n$, and it is readily verified that the sequence $\{p_n\}$ satisfies the conclusion of the lemma. □

The Weierstrass approximation theorem alluded to here, historically the simplest version of the theorem, may be stated as follows.

Theorem 2.41. (Weierstrass) *For any real-valued function f defined and continuous on a closed interval $[a, b]$ of real numbers, there exists a sequence $\{p_n\}$ of real polynomials converging uniformly to f on $[a, b]$.*

It suffices to prove the theorem when $[a, b] = [0, 1]$. The proof below is due to Bernstein and Korovkin.

Proof. Let $f : [0, 1] \to \mathbb{R}$ be continuous, and define for each $n \in \mathbb{N}$ the Bernstein polynomial

$$B_n(f, t) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} t^k (1-t)^{n-k},$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is the usual binomial coefficient. It suffices to show that B_n converges uniformly to f on $[0, 1]$. In order to do this, we need to calculate B_n explicitly when f is a polynomial of degree at most two. Setting $p_j(t) = t^j$ for $j = 0, 1, 2$, we use the equations

$$B_n(p_0, t) = p_0(t), \quad B_n(p_1, t) = p_1(t), \quad B_n(p_2, t) = p_2(t) + \frac{t(1-t)}{n}$$

for $n \in \mathbb{N}$ and $t \in [0, 1]$. The relevant identities to be verified are

$$\sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} = 1, \tag{2.4}$$

which is simply the binomial theorem, and

$$\sum_{k=0}^{n} (k - nt)^2 \binom{n}{k} t^k (1-t)^{n-k} = nt(1-t) \tag{2.5}$$

(see Problem 2K). The fact that f is bounded and uniformly continuous on $[0, 1]$ ([**I**, Theorem 8.34 and Corollary 8.36]) easily implies that, given $\varepsilon > 0$, there exists a constant $M_\varepsilon > 0$ such that

$$|f(t) - f(s)| \le \varepsilon + M_\varepsilon (t - s)^2, \quad s, t \in [0, 1].$$

This can be written as

$$-\varepsilon - M_\varepsilon (p_2(t) - 2sp_1(t) + s^2) \le f(t) - f(s) \le \varepsilon + M_\varepsilon (p_2(t) - 2sp_1(t) + s^2).$$

Using the fact that $B_n(g, t)$ is nonnegative when g is a nonnegative function, we deduce that

$$\begin{aligned} &- \varepsilon - M_\varepsilon (B_n(p_2, t) - 2sB(p_1, t) + s^2) \le B_n(f, t) - f(s) \\ &\quad \le \varepsilon + M_\varepsilon (B_n(p_2, t) - 2sB_n(p_1, t) + s^2) \end{aligned}$$

for $n \in \mathbb{N}$ and $t, s \in [0, 1]$. Set $t = s$ in this inequality and use the above formulas for $B_n(p_j, t)$ to obtain

$$|B_n(f, s) - f(s)| \le \varepsilon + M_\varepsilon \frac{s(1-s)}{n} \le \varepsilon + \frac{M_\varepsilon}{4n}, \quad n \in \mathbb{N}, s \in [0, 1].$$

Thus $|B_n(f, s) - f(s)| < 2\varepsilon$ for all $s \in [0, 1]$ provided that $n > M_\varepsilon / 4\varepsilon$, thereby concluding the proof. □

We proceed now with the proof of Theorem 2.39 in the real case.

Proof. It is clear that on any measurable space the characteristic function of a set E is measurable if and only if E is measurable. Hence, if the theorem is to hold, the σ-algebra $\mathbf{S}$ must consist precisely of those subsets of X whose characteristic functions belong to $\mathcal{M}$. (Incidentally, this establishes the uniqueness of $\mathbf{S}$.) Accordingly, we set $\mathbf{S} = \{E \subset X \colon \chi_E \in \mathcal{M}\}$, and proceed to verify:

(1) $\mathbf{S}$ is a σ-algebra.
(2) The functions in $\mathcal{M}$ are all measurable $[\mathbf{S}]$.
(3) Every real-valued function that is measurable $[\mathbf{S}]$ belongs to $\mathcal{M}$.

These arguments go as follows.

(1) Since $\chi_{E\setminus F} = \chi_E - \chi_E\chi_F$ and $\chi_{E\cup F} = \chi_E + \chi_F - \chi_E\chi_F$, it is clear that $\mathbf{S}$ is a ring of sets. To see that it is a σ-ring, let $\{E_n\}$ be a sequence of sets in $\mathbf{S}$ having union E. If $F_n = E_1 \cup \cdots \cup E_n$, then F_n belongs to $\mathbf{S}$ for each n and the sequence $\{\chi_{F_n}\}$ tends pointwise to χ_E. It follows that E also belongs to $\mathbf{S}$. Moreover, it is obvious that $\mathbf{S}$ is complemented since the function identically equal to one is in $\mathcal{M}$.
(2) Consider a function $f \in \mathcal{M}$ and let U be an open subset of the real line $\mathbb{R}$. Since $\mathcal{M}$ is a function algebra containing the constants, it follows that $p(f(x))$ belongs to $\mathcal{M}$ for every real polynomial p. In particular, if $\{p_n\}$ is a sequence of real polynomials converging pointwise to χ_U as in Lemma 2.40, then the sequence of functions $h_n(x) = p_n(f(x))$ belongs to $\mathcal{M}$. But $h_n(x)$ converges pointwise to the characteristic function of $f^{-1}(U)$. Thus $f^{-1}(U)$ belongs to $\mathbf{S}$, whence it follows that f is measurable $[\mathbf{S}]$ (Corollary 2.14).
(3) The simple real-valued functions on X that are measurable $[\mathbf{S}]$ are clearly in $\mathcal{M}$ (according to Proposition 2.35 they are linear combinations of the characteristic functions in $\mathcal{M}$), and every real-valued function on X that is measurable $[\mathbf{S}]$ is the pointwise limit of a sequence of such simple functions by Proposition 2.36.

□

The taking of limits of pointwise convergent sequences of functions is an operation of great importance in the theory of measure and integration. We pay special attention to pointwise limits of sequences of continuous scalar-valued functions. (The concepts discussed below may also be found, in part, in [**I**]; see, in particular, [**I**, Problem 8K].)

Definition 2.42. A scalar-valued function f on a metric space X is of *Baire class one* on X if there exists a sequence of continuous scalar-valued functions on X that converges pointwise to f.

Example 2.43. All semicontinuous real-valued functions on a metric space X are of Baire class one ([**I**, Proposition 7.20]). In particular, the characteristic functions of all open subsets and all closed subsets of X are of Baire class one on X ([**I**, Problem 7F]).

Further Baire classes are defined in the same manner as the first. Thus one says that a scalar-valued function f on a metric space X is of *Baire class* *two* if f is the limit of some pointwise convergent sequence of functions of Baire class one, etc. We turn at once to the formal inductive definition toward which these initial constructions clearly point.

Definition 2.44. Let X be a metric space. Set $\mathcal{C}_0$ equal to the collection of all continuous scalar-valued functions on X, let α denote an arbitrary countable ordinal number, and suppose that $\mathcal{C}_\xi$ has already been defined for all ordinal numbers ξ such that $\xi < \alpha$. Then $\mathcal{C}_\alpha$ is defined to be the collection of all functions f with the property that there exists a sequence in $\bigcup_{\xi<\alpha} \mathcal{C}_\xi$ that converges pointwise to f. In this way we obtain by transfinite definition (see [**I**, Theorem 5.12]) a family $\{\mathcal{C}_\alpha\}_{\alpha<\Omega}$ of collections of scalar-valued functions on X indexed by the entire segment $W(\Omega)$ of countable ordinal numbers. The functions belonging to $\mathcal{C}_\alpha$ are said to be of *Baire class* α on X, a terminology obviously consistent with the earlier definitions of Baire classes one and two. (According to this terminology the continuous functions are exactly the functions of Baire class zero.) The functions belonging to the union

$$\mathcal{C}_\Omega = \bigcup_{\alpha<\Omega} \mathcal{C}_\alpha$$

are known as the *Baire functions* on X.

Theorem 2.45. *The class $\mathcal{C}_\Omega$ of Baire functions on a metric space X is the smallest collection of scalar-valued functions on X that contains the class $\mathcal{C}_0$ of all continuous scalar-valued functions on X and that is closed with respect to the formation of limits of pointwise convergent sequences.*

Proof. Assume first that $\{f_n\}$ is a sequence of Baire functions on X that converges pointwise to a limit f. Then each f_n is of Baire class α for some countable ordinal number α—say f_n is of Baire class α_n. According to [**I**, Example 5L] there exists a countable ordinal number η such that $\alpha_n < \eta$ for every n, and it follows that f is of Baire class η. Thus $\mathcal{C}_\Omega$ is indeed closed with respect to the formation of pointwise limits.

Assume next that $\mathcal{C}$ is a collection of scalar-valued functions on X that contains $\mathcal{C}_0$ and is closed with respect to the formation of pointwise limits. If all of the Baire classes $\mathcal{C}_\xi$ are contained in $\mathcal{C}$ for $\xi < \alpha$, where α denotes some countable ordinal number, then $\mathcal{C}_\alpha \subset \mathcal{C}$ as well. Indeed, every function f in $\mathcal{C}_\alpha$ is, by definition, the limit of some pointwise convergent sequence of functions belonging to $\mathcal{C}$. Thus every Baire class $\mathcal{C}_\alpha, \alpha < \Omega$, is contained in $\mathcal{C}$, and therefore $\mathcal{C}_\Omega \subset \mathcal{C}$. □

The preceding result reveals a shorter route to the notion of a Baire function.

Proposition 2.46. *For any collection $\mathcal{F}$ of scalar-valued functions on a set X there is a smallest collection $\mathcal{B} = \mathcal{B}(\mathcal{F})$ of functions on X that contains $\mathcal{F}$ and is closed with respect to the formation of limits of pointwise convergent sequences. This collection will be called the* Baire class generated *by $\mathcal{F}$.*

Proof. The collection of all scalar-valued functions on X is closed with respect to the formation of limits of pointwise convergent sequences. If $\mathcal{D}$ denotes the intersection of any nonempty family of collections each of which is closed with respect to pointwise limits, then $\mathcal{D}$ is also closed with respect to pointwise limits. Thus $\mathcal{B}(\mathcal{F})$ is simply the intersection of the family of all collections of scalar-valued functions on X that contain $\mathcal{F}$ and are closed with respect to the formation of pointwise limits. □

Using this latter concept, we may paraphrase Theorem 2.45 by saying that the class $\mathcal{C}_\Omega$ of Baire functions on a metric space X is the Baire class generated by the class $\mathcal{C}_0$ of continuous functions on X. This simple characterization of the class of Baire functions stands to its initial transfinite definition exactly as the definition of Borel sets in Chapter 1 (as the σ-algebra $\mathbf{B}_X$ generated by the lattice of open sets) stands to the transfinite construction of $\mathbf{B}_X$ in Problem 1O. One definition has the advantage of brevity and avoids the use of transfinite numbers, but also, by that very token, fails to give any information about the grading of Baire functions into numbered classes. Interestingly enough, either definition can be used to prove theorems, as the following propositions demonstrate.

Proposition 2.47. *If f and g are Baire functions on a metric space X, then $f+g$ is also a Baire function on X.*

Proof. Consider first the collection $\mathcal{G}_0$ of all those scalar-valued functions g on X with the property that if f is an arbitrary continuous scalar-valued function on X, then $f+g$ is a Baire function. If $\{g_n\}$ is a sequence in $\mathcal{G}_0$ that converges pointwise to a limit h, and if f is some continuous scalar-valued function on X, then $\{f+g_n\}$ is a sequence of Baire functions converging pointwise to $f+h$ so that $f+h$ is also a Baire function. This shows that $\mathcal{G}_0$ is closed with respect to the formation of limits of pointwise convergent sequences. Since $\mathcal{G}_0$ obviously contains $\mathcal{C}_0$, it follows that $\mathcal{G}_0$ contains *all* Baire functions. Thus we have shown that if f is a continuous scalar-valued function on X, and g is an arbitrary Baire function on X, then $f+g$ is also a Baire function. To complete the proof, consider next the collection $\mathcal{G}$ of all those scalar-valued functions g on X with the property that if f is a Baire function on X, then $f+g$ is too. A repetition of the same argument shows that $\mathcal{G}$ is closed with respect to the formation of pointwise limits. On the other hand, we have just proved that $\mathcal{G}$ contains $\mathcal{C}_0$, and the proposition follows. □

Other results along the same line can be similarly obtained.

Proposition 2.48. *The class of complex-valued Baire functions on a metric space X is closed with respect to complex conjugation. Hence a complex-valued function f on X is a Baire function if and only if $\Re f$ and $\Im f$ are both Baire functions.*

Proof. Consider the collection $\overline{\mathcal{C}}_\Omega$ of all complex conjugates of complex-valued Baire functions on X. This class contains the continuous complex-valued functions on X (the complex conjugate of a continuous function is itself continuous) and is closed with respect to the formation of pointwise limits (if $\{\bar{f}_n\}$ converges pointwise to a limit f, then $\{f_n\}$ converges pointwise to $\bar{f}$, so $f \in \overline{\mathcal{C}}_\Omega$). Hence $\mathcal{C}_\Omega \subset \overline{\mathcal{C}}_\Omega$, and by symmetry $\mathcal{C}_\Omega = \overline{\mathcal{C}}_\Omega$. □

Continuing in this same vein, one easily establishes the following result, whose proof is omitted.

Theorem 2.49. *The real-valued Baire functions on a metric space X form a function algebra that contains the constant functions and is also a function lattice. The complex-valued Baire functions on X form a function algebra that contains the constant functions and is closed with respect to complex conjugation.*

To obtain these and other results concerning Baire functions directly from the original transfinite definition is rather more laborious, requiring as it does the machinery of transfinite induction, but then the end result is more informative. We begin with a pair of observations that clarify the later arguments to some degree.

Proposition 2.50. *On a metric space X the transfinite sequence $\{\mathcal{C}_\alpha\}_{\alpha<\Omega}$ of Baire classes is monotone increasing. For any countable ordinal number α, a function f is of Baire class $\alpha + 1$ on X if and only if it is the limit of a pointwise convergent sequence of functions of Baire class α on X. On the other hand, if λ is a countable limit number, then f is of Baire class λ on X if and only if it is the limit of a pointwise convergent sequence $\{f_n\}$ where each f_n is of Baire class η_n on X, and $\eta_1 < \eta_2 < \ldots < \eta_n < \ldots$ is a strictly increasing sequence in $W(\lambda)$.*

Proof. The stated conditions are obviously sufficient, and the asserted monotonicity follows from the simple fact that a constant sequence is convergent. Moreover, from monotonicity it is clear that $\bigcup_{\xi<\alpha+1} \mathcal{C}_\xi = \mathcal{C}_\alpha$, and hence that the functions in $\mathcal{C}_{\alpha+1}$ are limits of sequences in $\mathcal{C}_\alpha$. Finally, if $f \in \mathcal{C}_\lambda$ where λ is a limit number, then there is a sequence $\{f_n\}$ of functions in $\bigcup_{\xi<\lambda} \mathcal{C}_\xi$ converging pointwise to f. Thus each f_n belongs to some $\mathcal{C}_{\xi_n}, \xi_n < \lambda$, and we have but to define the strictly increasing sequence $\{\eta_n\}$ inductively, setting $\eta_1 = \xi_1$ and $\eta_{n+1} = (\eta_n + 1) \vee \xi_{n+1}$, for $n > 0$. □

Proposition 2.51. *Let X be a metric space, and let g be a continuous mapping of the scalar field into itself. Then for any countable ordinal number α and any function f of Baire class α on X, the composition $g \circ f$ is also of Baire class α.*

Proof. The result is obvious when $\alpha = 0$ because the composition of continuous mappings is continuous. Assume it holds for all $\xi < \alpha$, and $f \in \mathcal{C}_\alpha$. There is a sequence $\{f_n\}$ of functions converging pointwise to f with the property that each f_n is of some Baire class $\xi_n < \alpha$, and the sequence $\{g \circ f_n\}$ shares this property by the inductive hypothesis. In addition $\{g \circ f_n\}$ converges pointwise to $g \circ f$ because g is continuous, and therefore $g \circ f \in \mathcal{C}_\alpha$. □

The following result clearly contains Theorem 2.49 as a corollary.

Theorem 2.52. *Let X be a metric space, let α be a countable ordinal number, and let us write $\mathcal{C}_{\alpha,\mathbb{R}}$ and $\mathcal{C}_{\alpha,\mathbb{C}}$ for the collections of real and complex Baire functions of class α on X, respectively. Then $\mathcal{C}_{\alpha,\mathbb{C}}$ is a complex function algebra on X that contains the constant functions and is closed with respect to complex conjugation. Similarly, $\mathcal{C}_{\alpha,\mathbb{R}}$ is a real function algebra that is also a function lattice on X.*

We only sketch the proof.

Proof. The parts of the theorem that are not immediate consequences of Proposition 2.51 are all derived in the same way, by transfinite induction. As a typical example of such an argument, let us show that $\mathcal{C}_{\alpha,\mathbb{C}}$ is closed with respect to addition.

The result clearly holds for $\alpha = 0$. Suppose it is valid for all $\xi < \alpha$, and $f, g \in \mathcal{C}_{\alpha,\mathbb{C}}$. Choose sequences $\{f_n\}$ and $\{g_n\}$ of complex-valued functions on X converging pointwise to f and g, respectively, such that, for each index n, $f_n \in \mathcal{C}_{\xi_n,\mathbb{C}}$ and $g_n \in \mathcal{C}_{\eta_n,\mathbb{C}}$ for some $\xi_n < \alpha$ and $\eta_n < \alpha$. Then the functions f_n, g_n belong to $\mathcal{C}_{\zeta_n,\mathbb{C}}$, where $\zeta_n = \xi_n \vee \eta_n < \alpha$. The inductive hypothesis implies that $f_n + g_n \in \mathcal{C}_{\zeta_n,\mathbb{C}}$, and therefore $f + g = \lim_n (f_n + g_n) \in \mathcal{C}_{\alpha,\mathbb{C}}$. $\square$

According to Theorem 2.52 (or Theorem 2.49), the class $\mathcal{C}_\Omega$ of complex-valued Baire functions on a metric space X consists of all *measurable* functions with respect to some σ-algebra of subsets of X (Theorem 2.39). The following identifies the σ-algebra.

Theorem 2.53. *On any metric space X the class of scalar-valued Baire functions coincides with the class of Borel measurable scalar-valued functions.*

Proof. As seen earlier, $\mathcal{C}_\Omega$ consists of the scalar-valued functions which are measurable relative to the σ-field $\mathbf{S} = \{E \subset X : \chi_E \in \mathcal{C}_\Omega\}$. We must show that $\mathbf{S}$ coincides with the σ-algebra $\mathbf{B}_X$, and both parts of the proof are easy. On the one hand, $\mathbf{S}$ contains all open sets in X, so $\mathbf{S} \supset \mathbf{B}_X$. On the other hand, the collection of all Borel measurable real-valued functions on X contains the continuous real-valued functions and is closed with respect to the formation of pointwise limits, so every real-valued Baire function on X is Borel measurable. If $E \in \mathbf{S}$, so that χ_E is a Baire function, then χ_E is Borel measurable, and E is a Borel set. $\square$

Problems

2A. Fix a set M dense in $\mathbb{R}$, for example, the set of all rational numbers, or the set of all dyadic fractions. If f is a real-valued function on a measurable space X, then f is measurable if and only if $E(f \le t)$ is measurable for every t in M. Similarly, f is measurable if and only if $E(f < t)[E(f \ge t), E(f > t)]$ is measurable for every $t \in M$.

2B. An extended real-valued function f on a measurable space X is measurable if and only if the two sets $E_{+\infty} = E(f = +\infty)$ and $E_{-\infty} = E(f = -\infty)$ are measurable and the restriction of f to the complement $X \setminus (E_{+\infty} \cup E_{-\infty})$ is a measurable (finite) real-valued function on that subspace.

2C. Let $\{t_n\}_{n=-\infty}^{+\infty}$ be a monotone increasing sequence of real numbers (indexed by the set $\mathbb{Z}$ of all integers), and set $a = \inf_n t_n, b = \sup_n t_n$ (the cases $a = -\infty$ and/or $b = +\infty$ are not excluded). Let f be a real-valued function defined on the interval (a, b), and suppose that f is monotone on each subinterval $(t_n, t_{n+1}), n \in \mathbb{Z}$. Show that f is a Borel measurable function.

2D. Let $\mathrm{mid}\{a, b, c\}$ denote that one of the three real numbers a, b, c that is bracketed by the other two. Show that if f, g and h are any three measurable real-valued functions on a measurable space X, then

$$\mathrm{mid}\{f(x), g(x), h(x)\}$$

is likewise a measurable function on X.

2E. (i) Let $\mathbf{S}_0$ be a σ-ring of subsets of a set X that is *not* complemented (so that $X \notin \mathbf{S}_0$), and let $\mathbf{S}_1$ denote the complemented σ-ring obtained by adjoining to $\mathbf{S}_0$ the complements of the sets in $\mathbf{S}_0$ (see Example 1.13). Describe the algebra of measurable scalar-valued functions on the measurable space $(X, \mathbf{S}_1)$. (Hint: Consider a countable partition of X into sets measurable $[\mathbf{S}_1]$.)

(ii) Let $X = X_1 \cup X_2$ be a partition of a set X, let $\mathbf{S} = \mathbf{S}_1 \oplus \mathbf{S}_2$, where $\mathbf{S}_i$ is a σ-algebra of subsets of $X_i, i = 1, 2$, and let $\mathcal{A}$ denote the algebra of measurable scalar-valued functions on $(X, \mathbf{S})$. If $\mathcal{J}_1$ denotes the set of those functions in $\mathcal{A}$ that vanish on X_2, then $\mathcal{J}_1$ is an ideal in $\mathcal{A}$ that is, in an obvious fashion, isomorphic as an algebra to the algebra $\mathcal{A}_1$ of measurable scalar-valued functions on $(X_1, \mathbf{S}_1)$. Similarly, the ideal $\mathcal{J}_2$ of those functions in $\mathcal{A}$ that vanish on X_1 is isomorphic to the algebra $\mathcal{A}_2$ of measurable scalar-valued functions on $(X_2, \mathbf{S}_2)$. The ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ are complements as subspaces of (the vector space) $\mathcal{A}$; that is, $\mathcal{J}_1 \cap \mathcal{J}_2 = (0)$ while $\mathcal{J}_1 + \mathcal{J}_2 = \mathcal{A}$. Moreover, if functions f and g in $\mathcal{A}$ are written in the form $f = f_1 + f_2, g = g_1 + g_2$, where $f_i, g_i \in \mathcal{J}_i$, $i = 1, 2$, then $fg = f_1 g_1 + f_2 g_2$. (This situation is sometimes expressed by saying that $\mathcal{A}$ splits *internally* into the *direct sum* $\mathcal{A}_1 \oplus \mathcal{A}_2$.) Is the like true for more general notions of the direct sum of measurable spaces (Example 1.16)?

(iii) Let $\mathcal{P}$ be a countable partition of a set X, and let $\mathbf{S}$ be the σ-ring in X corresponding to $\mathcal{P}$ as in Example 1.7. Describe the measurable mappings of $(X, \mathbf{S})$ into $(Y, \mathbf{T})$, where Y is a set and $\mathbf{T}$ is some σ-algebra of subsets of Y containing all singletons (Example 2.5).

2F. (i) Let Y be a metric space, let $\mathcal{C}$ be a collection of subsets of a set X, and suppose given a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of mappings of X into Y with the property that $\varphi_n^{-1}(U) \in \mathcal{C}$ for all n and all open sets U in Y. Show that if $\{\varphi_n\}$ converges pointwise on X to a limit ψ, then $\psi^{-1}(U) \in \mathcal{C}_{\delta\sigma}$ for every open set U in Y (see Problem 1J). In particular, if $(X, \mathbf{S})$ is a measurable space, then the limit of any pointwise convergent sequence of measurable mappings of X into Y is itself measurable.

(ii) Let X be an infinite measurable space, let Y be a metric space, and suppose given a mapping ψ of X into Y. Let $\mathcal{D}$ be the directed set of all finite subsets of X (directed by set inclusion $\subset$), let y_0 be a fixed point of Y, and for each Δ in $\mathcal{D}$ set

$$\varphi_\Delta(x) = \begin{cases} \psi(x), & x \in \Delta \\ y_0, & x \in X \setminus \Delta. \end{cases}$$

Then $\{\varphi_\Delta\}_{\Delta \in \mathcal{D}}$ is a net of mappings of X into Y. Show that this net converges pointwise on X to the given mapping ψ. Show also that the mappings φ_Δ are all measurable under the hypothesis that all singletons in X are measurable sets (see Example 2.5).

(iii) In the foregoing construction take for Y the real line $\mathbb{R}$, for ψ the characteristic function of some subset A of X, and set $y_0 = 0$. Show that in this case the net $\{\varphi_\Delta\}$ converges (pointwise) monotonically upward to χ_A no matter what A is.

Remark 2.54. These constructions clearly show that preservation of measurability under passage to a limit is essentially restricted to *sequential* convergence, and does not extend, in general, to convergent nets, not even monotone nets. This should come as no surprise; the very concept of measurability is rooted in the notion of a countable set.

2G. (Product metrics; see [**I**, Problem 6H]) Let $(X, \mathbf{S})$ be a measurable space, and let $Y_1, \dots, Y_N$ be metric spaces.

(i) If the product $Y = Y_1 \times \ldots \times Y_N$ is equipped with a product metric, and if $\varphi : X \to Y$ is measurable, then the coordinate mappings $\pi_i \circ \varphi$, $i = 1, \dots, N$, are all measurable as well. (Here π_i denotes, as always, the projection of the product Y onto the ith factor Y_i.)

(ii) Conversely, if the spaces Y_i are all separable, and if $\varphi : X \to Y$ has the property that the coordinate mappings $\pi_i \circ \varphi$ are all measurable, then φ is itself measurable. (Hint: See [**I**, Example 6W].)

(iii) Generalize this discussion to the case of a (countably) infinite product $\prod_{n=1}^{\infty} Y_n$ of metric spaces equipped with a product metric.

2H. Let X be a measurable space, and let φ and ψ be measurable mappings of X into a separable metric space (Y, ρ). Show that the real-valued function $x \to \rho(\varphi(x), \psi(x))$ is measurable on X and that $\{x \in X : \varphi(x) = \psi(x)\}$ is a measurable set (see Corollary 2.26). In particular, if φ is a Borel measurable mapping of Y into itself, then the set $\{y \in Y : \varphi(y) = y\}$ of fixed points of φ is a Borel set.

2I. Let X be a measurable space and let $\{\varphi_n\}$ be a sequence of measurable mappings of X into a complete separable metric space Y. Show that the subset E of X consisting of those points x at which the sequence $\{\varphi_n(x)\}$ is convergent in Y is measurable and that $\varphi(x) = \lim_n \varphi_n(x)$ defines a measurable mapping $\varphi : E \to Y$.

2J. The complex version of (the essential part of) Proposition 2.36 goes as follows. For every measurable complex-valued function f on a measurable space X there exists a sequence $\{s_n\}$ of measurable simple complex-valued functions converging pointwise to f and satisfying the following additional conditions:

(i) at each point $x \in X$ either $0 \le \Re s_1(x) \le \Re s_2(x) \le \cdots$, or $0 \ge \Re s_1(x) \ge \Re s_2(x) \ge \cdots$, and likewise, either $0 \le \Im s_1(x) \le \Im s_2(x) \le \cdots$, or $0 \ge \Im s_1(x) \ge \Im s_2(x) \ge \cdots$, and

(ii) if $M \in \mathbb{N}$ and if $|f(x)| \le M$, then $|f(x) - s_n(x)| \le \sqrt{2}/2^n$ for every $n \ge M$.

Show that this assertion is, in fact, valid. (Hint: The proof given in the text can be copied in the complex plane; alternatively, the complex case can be derived from the real case.)

2K. Use the definition of the combinatorial coefficient $\binom{n}{k}$ to derive the equivalent relations

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}, \quad \binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}, \tag{2.6}$$

for all $k = 1, \ldots, n$ (and all $n \in \mathbb{N}$; recall that $0! = 1$ by definition). Set $n = m - 1$ and $k = j - 1$ in the basic identity (2.4), and use (2.6) to show that

$$\sum_{j=1}^{m} \frac{j}{m}\binom{m}{j-1} x^j (1-x)^{m-j} = 1,$$

and hence that

$$\sum_{j=1}^{m} j\binom{m}{j} x^j (1-x)^{m-j} = mx \tag{2.7}$$

for $m = 2, 3, \ldots$. Then use the same trick over again to verify (2.5). The case $n = 1$ needs to be verified separately.

Remark 2.55. It need scarcely be said that these are not simply fortuitous calculations. In fact, in a Bernoulli process consisting of n independent repetitions of a simple trial with probability of success $p = x$ (and therefore with $q = 1 - x$) $\binom{n}{k}x^k(1-x)^{n-k}$ is the probability of precisely k successes. Thus (2.4) says merely that the sum of these probabilities is one, (2.7) states that the expected number of successes is np, and (2.5) gives the standard deviation of this number about its mean as npq—well-known facts of elementary probability theory.

2L. Prove the complex version of Theorem 2.39 by reducing it to the real case. (A direct proof of Theorem 2.39 in the complex case can be based on Problem 2J, along with a complex version of Lemma 2.40, but this would in turn necessitate the introduction of a complex version of the Weierstrass approximation theorem.)

2M. Let us call a mapping φ of a set X into an arbitrary set Y *simple* if it assumes only a finite number of distinct values in Y. (Thus the simple scalar-valued functions defined above in connection with Proposition 2.34 are just the simple mappings of X into the scalar field.)

(i) If $(X, \mathbf{S})$ is a measurable space and Y is a metric space, then a simple mapping φ of X into Y is measurable if and only if it assumes each of its values on a measurable set.

(ii) A metric space Y is said to be σ-*compact* if it is the union of some countable collection of compact sets. Show that if $(X, \mathbf{S})$ is a measurable space and Y is a σ-compact metric space, then every measurable mapping φ of X into Y is the limit of a pointwise convergent sequence $\{\varphi_n\}$ of measurable simple mappings of X into Y. (Hint: There exists an increasing sequence $\{K_n\}$ of compact subsets of Y such that $Y = \bigcup_n K_n$.)

2N. A mapping φ of a measurable space X into a set Y is said to be *elementary* if it assumes only a *countable* number of distinct values. If Y is a metric space, then an elementary mapping of X into Y is measurable if and only if it assumes each of its values on a measurable set. If Y is a separable metric space, then a mapping of X into Y is measurable if and only if it is the limit of a uniformly convergent sequence of measurable elementary mappings of X into Y.

2O. If $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ are measurable spaces and $\varphi : X \to Y$ is measurable, then (as we have noted, see Example 2.12) φ is automatically measurable on any subset A of X (meaning that $\varphi|A$ is measurable as a mapping of $(A, \mathbf{S}_A)$ into Y).

(i) Show, in the converse direction, that if A is measurable, then any measurable mapping $\psi : A \to Y$ is the restriction to A of some measurable mapping $\varphi : X \to Y$.

(ii) Show also that if Y is a complete σ-compact metric space (Problem 2K), then the assumption in (i) that the set A is measurable can be dropped. In particular, then, this is the case if Y is the scalar field $\mathbb{R}$ or $\mathbb{C}$. (Hint: Consider first the case of a simple measurable mapping ψ; use Problem 2I.)

(iii) Let X be a separable metric space, and suppose A is some subset of X that is *not* a Borel set in X (see Problem 1I). Then the identity mapping $\iota : A \to A$ is Borel measurable on A, but does not extend to any Borel measurable mapping of X into A. (Hint: A Borel measurable mapping of X into A is also Borel measurable as a mapping of X into itself, and therefore admits a Borel set of fixed points; recall Problem 2H.)

2P. Let $\mathcal{A}$ denote the algebra of all measurable scalar-valued functions on a measurable space $(Y, \mathbf{T})$, and suppose given a mapping $\varphi : X \to Y$ of some set X onto Y. Show that $\widetilde{\mathcal{A}} = \{f \circ \varphi : f \in \mathcal{A}\}$ is the algebra of measurable scalar-valued functions on X with respect to a unique σ-algebra $\mathbf{S}$ in X, and find $\mathbf{S}$. What becomes of this proposition if we drop the assumption that the mapping φ is onto?

2Q. (i) A scalar-valued function on a metric space X that differs from a continuous function at only finitely many points is of Baire class one on X. Give an example of a function on the real line differing from a continuous function at countably infinitely many points that is of Baire class one, and of another that is not. (Hint: See [**I**, Example 8Q].)

(ii) Every scalar-valued function on a metric space X differing from a continuous function at only a countable set of points is of Baire class two on X.

2R. If a function $f : X \to \mathbb{R}$ on a metric space X has the property that $a \le f(x) \le b$ on X, where $a \le b$ are real numbers, and if f is of Baire class α on X, then f is the limit of a pointwise convergent sequence $\{f_n\}$ of real-valued functions of Baire class less than α on X such that $a \le f_n(x) \le b$ for all x and all n. In particular, if $|f| \le M$ on X, then f is the pointwise limit of a sequence of functions each of Baire class less than α, each of which is similarly bounded. Show, in the same vein, that if f is a complex-valued function of Baire class α on X such that $|f| \le M$ on X, then f is the pointwise limit of a sequence $\{f_n\}$ of complex-valued functions on X where each f_n is not only of Baire class less than α, but also satisfies the condition $|f_n| \le M$ on X. (Hint: For the complex case construct a retraction of $\mathbb{C}$ onto the closed disc $D_M = \{\lambda \in \mathbb{C}\colon |\lambda| \le M\}$; that is, a continuous mapping of $\mathbb{C}$ onto D_M that agrees with the identity mapping on D_M.)

2S. Let $g : \mathbb{R}^d \to \mathbb{C}$ be a continuous function, and suppose $f_1, \dots, f_n$ are real-valued functions of Baire class α on a metric space X. Show that the function

$$h(x) = g(f_1(x), \dots, f_d(x))$$

is also of Baire class α on X. Show that this result remains valid if g is merely defined and continuous on some closed cell containing the range of the mapping $x \mapsto (f_1(x), \dots, f_d(x))$, and devise complex analogs of these results.

2T. Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers with $\sum_n M_n < +\infty$, and suppose that $\{f_n\}$ is a sequence of scalar-valued functions on a metric space X such that $|f_n| \le M_n$ on $X, n \in \mathbb{N}$. Show that if each of the functions f_n is of Baire class α on X (for some countable ordinal number α), then the sum $f = \sum_n f_n$ is also of Baire class α. (Hint: Each f_n is the limit of a pointwise convergent sequence $\{p_k^{(n)}\}_{k=1}^{\infty}$ of functions of Baire class less than α, where $|p_k^{(n)}| \le M_n$ on X for all k and all n. Set

$$q_m = p_1^{(m)} + \cdots + p_m^{(m)},$$

and let m tend to infinity. For all m greater than or equal to a fixed positive integer k we can write $q_m = q'_m + q''_m$, where $\{q'_m\}$ tends pointwise to $f_1 + \cdots + f_k$, while $|q''_m| \le M_{k+1} + \cdots + M_m$. Hence $\{q_m\}$ converges pointwise to f.) Use the foregoing fact to prove that each of the Baire classes $\mathcal{C}_\alpha$ on $X, \alpha < \Omega$, is closed with respect to the formation of limits of uniformly convergent sequences.

2U. A sequence $\{f_n\}$ of scalar-valued functions on a metric space X is said to be *locally uniformly bounded* if for every point x_0 of X there exist a neighborhood V of x and a constant $M > 0$ such that $|f_n(x)| \le M$ for all $x \in V$ and all indices n. (Each of the functions f_n is, then, in particular, locally bounded; see [**I**, Problem 7D].) Prove that the class of locally bounded Baire functions on X is the smallest collection of scalar-valued functions on X that contains the class $\mathcal{C}_0$ of continuous scalar-valued functions and is closed with respect to the formation of limits of pointwise convergent and uniformly locally bounded sequences.

Remark 2.56. In view of Theorem 2.52, it is natural to look for connections between the classification of the Borel sets in a metric space X into numbered classes and the like classification of the Baire functions on X. Such connections do indeed exist, but they are not as tidy as one might hope. The following problem provides a small sampler of such results.

2V. Let X be a metric space.

(i) For each countable ordinal number α let $\mathcal{E}_\alpha$ denote the collection of subsets E of X such that χ_E belongs to the Baire class $\mathcal{C}_\alpha$ on X. Show that $\mathcal{E}_\alpha$ is a complemented lattice of subsets of X. Define inductively classes $\mathcal{G}^\alpha$ for $\alpha < \Omega$ by setting $\mathcal{G}^0 = \mathcal{G}$ (the collection of open sets), and $\mathcal{G}^\alpha = \left(\bigcup_{\beta<\alpha} \mathcal{G}\right)_\delta$ if α is odd and $\mathcal{G}^\alpha = \left(\bigcup_{\beta<\alpha} \mathcal{G}\right)_\sigma$ if $\alpha > 0$ is even. Show also that $\mathcal{G}^\alpha \subset \mathcal{E}_{\alpha+1}$ for every $\alpha < \Omega$.

(ii) Suppose each function of the sequence $\{f_n\}$ of complex-valued functions on X has the property that the inverse image $f_n^{-1}(U)$ belongs to the lattice $\mathcal{G}^\alpha$ for every open set U in $\mathbb{C}$, and suppose $\{f_n\}$ converges pointwise to a limit f. Verify that $f^{-1}(U)$ belongs to $\mathcal{G}^{\alpha+2}$ for every open set $U \subset \mathbb{C}$. (Thus, for example, a scalar-valued function f of Baire class one on X has the property that $f^{-1}(U)$ is a $G_{\delta\sigma}$ for every open set U of complex numbers; see Problem 2F.)

(iii) Use the foregoing observation to prove that if λ is a countable limit number, and if $f \in \mathcal{C}_{\lambda,\mathbb{C}}$, then $f^{-1}(U) \in \mathcal{G}^{\lambda+2}$ for every open set $U \subset \mathbb{C}$. Conclude by transfinite induction that for each countable ordinal number α there is a positive integer k (depending on α) such that if $f \in \mathcal{C}_{\alpha,\mathbb{C}}$, then $f^{-1}(U) \in \mathcal{G}^{\alpha+k}$ for every open set $U \subset \mathbb{C}$.

Remark 2.57. The basic idea of the construction of various Baire classes of mappings on a space X clearly makes sense in a broader context than that of scalar-valued functions. The following two problems are concerned with the classification of mappings taking values in arbitrary metric spaces.

2W. Let (Y, σ) be a metric space.

(i) Show that for any subset $\mathcal{F}$ of Y^X there is a smallest set $\mathcal{B}(\mathcal{F}) \subset Y^X$ that contains $\mathcal{F}$ and is closed with the respect to the formation of pointwise limits. The collection $\mathcal{B}(\mathcal{F})$ is the *Baire class generated* by $\mathcal{F}$.

(ii) What is $\mathcal{B}(\varnothing)$? That is, what collection of mappings of X into Y is generated in this way by the empty collection of mappings? Let A be a subset of Y, and let $\mathcal{F}_A$ denote the collection of constant mappings of X into A. Find $\mathcal{B}(\mathcal{F}_A)$.

(iii) Let $\mathcal{F}$ be a collection of mappings of X into Y. Set $\mathcal{C}_0 = \mathcal{F}$, and supposing, for a countable ordinal number α, that $\mathcal{C}_\xi$ is already defined for all $\xi < \alpha$, define $\mathcal{C}_\alpha$ to consist of all limits of pointwise convergent sequences $\{\varphi_n\}$ of mappings belonging to $\bigcup_{\xi<\alpha} \mathcal{C}_\xi$. Show that $\{\mathcal{C}_\alpha\}_{\alpha<\Omega}$ is a monotone increasing family of subsets of Y^X and that $\mathcal{B}(\mathcal{F}) = \bigcup_{\alpha<\Omega} \mathcal{C}_\alpha$.

2X. Let (X, ρ) and (Y, σ) be metric spaces, and let $\mathcal{C}_0 = \mathcal{C}(X; Y)$ denote the collection of all continuous mappings of X into Y. The mappings in the Baire class $\mathcal{B}(\mathcal{C}_0)$ generated by $\mathcal{C}_0$ are known simply as *Baire mappings* of X into Y. More particularly, for each countable ordinal number α, the collection $\mathcal{C}_\alpha$ defined in the transfinite procedure indicated in part (iii) of the preceding problem (starting with $\mathcal{F} = \mathcal{C}_0$) constitutes the *Baire class* α of mappings of X into Y.

(i) For every countable ordinal number α, a mapping $\psi : X \to Y$ is of Baire class $\alpha + 1$ if and only if it is the limit of a pointwise convergent sequence $\{\varphi_n\}$ of mappings belonging to $\mathcal{C}_\alpha$. On the other hand, if λ is a countable limit number, then ψ is of Baire class λ if and only if there exist a strictly increasing sequence $\{\eta_n\}$ in $W(\lambda)$ and, for each index n, a mapping φ_n in $\mathcal{C}_{\eta_n}$ such that the sequence $\{\varphi_n\}$ converges pointwise to ψ.

(ii) Let (Z, τ) be a third metric space. Show that if $\varphi : X \to Y$ and $\psi : Y \to Z$ are both Baire mappings, then $\psi \circ \varphi : X \to Z$ is also a Baire mapping. Show, more particularly, that if φ is of Baire class α and ψ is of Baire class β, then $\psi \circ \varphi$ is of Baire class $\alpha + \beta$.

(iii) Suppose Y is of the form $Y = Y_1 \times \ldots \times Y_N$, where $Y_1, \ldots, Y_N$ are metric spaces. Let Y be equipped with a product metric, and let α be a countable ordinal number. Show that a mapping $\varphi : X \to Y$ is of Baire class α if and only if the coordinate mappings $\pi_i \circ \varphi, i = 1, \ldots, N$, are all of Baire class α. (Thus, in particular, φ is a Baire mapping if and only if the mappings $\pi_i \circ \varphi$ are all Baire mappings.)

(iv) Part (iii) goes through in the case of a countably infinite product

$$Y = \prod_{n=1}^{\infty} Y_n$$

as well (assuming, as always, that Y is equipped with a product metric). (Hint: Assume, as one may, that $Y \neq \varnothing$, and select a fixed point $y_{n,0}$ in each factor Y_n.)

(v) The Baire mappings of $\mathbb{R}$ into the Cantor set C are just the constant functions $f : \mathbb{R} \to C$. Thus no counterpart of Theorem 2.39 holds in this generalized context.

2Y. Let $(X, \mathbf{S})$ be a measurable space and let (Y, σ) be a metric space. A mapping $\varphi : X \to Y$ will be said to have *property* (MSR) if φ is measurable and its range $\varphi(X)$ is separable (as a subspace of Y). If Y is separable, then every measurable mapping of X into Y has property (MSR); if X is a separable metric space, then every Baire mapping of X into Y has property (MSR).

(i) Prove that if $\varphi : X \to Y$ has property (MSR) and ε is a positive number, then there exists a measurable elementary mapping φ_ε of X into Y such that $\sigma(\varphi(x), \varphi_\varepsilon(x)) < \varepsilon$ for all $x \in X$ (recall Problem 2N). Show, conversely, that any mapping of X into Y that is thus approximable has property (MSR). (Hint: A subset of Y is separable as a subspace if and only if it is contained in the closure of a countable subset of Y.)

(ii) The collection of all mappings in Y^X with property (MSR) is closed under the formation of pointwise limits. (Hint: The separable subsets of Y form a σ-ideal in 2^Y.)

(iii) If $\varphi : X \to Y$ has property (MSR), and if ψ is a mapping of Y into some third metric space Z such that ψ has property (MSR), or such that ψ is a Baire mapping, then $\psi \circ \varphi$ has property (MSR).

(iv) Suppose Y is of the form $Y = Y_1 \times \ldots \times Y_N$, where (Y_i, σ_i) is a metric space, $i = 1, \ldots, N$, and suppose Y is equipped with a product metric. Show that a mapping φ of X into Y has property (MSR) if and only if the coordinate mappings $\pi_i \circ \varphi$ all have that property. Generalize this result to the case of an infinite product $Y = \prod_{n=1}^{\infty} Y_n$.

(v) If two mappings φ and ψ of X into Y both have property (MSR), then the function $x \mapsto \sigma(\varphi(x), \psi(x))$ is measurable.

2Z. Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Show that the set E of points of continuity of f and the set F of differentiability points of f are Borel sets in $\mathbb{R}$. Moreover, the function $f' : F \to \mathbb{R}$ is Borel measurable.

2AA. (Korovkin) Denote by $\mathcal{C}_{\mathbb{R}}([0,1])$ the linear space of continuous functions $f : [0,1] \to \mathbb{R}$. Let $T_n : \mathcal{C}_{\mathbb{R}}([0,1]) \to \mathcal{C}_{\mathbb{R}}([0,1])$, $n \in \mathbb{N}$, be a sequence of real-linear maps with the following properties.

(i) If $f \in \mathcal{C}_{\mathbb{R}}([0,1])$ is a nonnegative function, then $T_n(f)$ is nonnegative for all $n \in \mathbb{N}$.

(ii) The sequence $\{T_n(f)\}_{n=1}^{\infty}$ converges uniformly to f when $f(t) = t^j$ for $j = 0, 1, 2$.

Show that $\{T_n(f)\}_{n=1}^{\infty}$ converges uniformly to f for every $f \in \mathcal{C}_{\mathbb{R}}([0,1])$.

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