Complete Description of Turbulence in Terms of Hopf Functional and LMN Hierarchy: New Symmetries and Invariant Solutions

Marta Wacławczyk

Abstract This paper deals with two methods for the full statistical description of turbulent field, namely the Lundgren–Monin–Novikov hierarchy (Lundgren, Phys Fluids, 10:969–975 1967, [5]) for the multipoint probability density functions (PDFs) of velocity and Hopf functional equation for turbulence (Hopf, J Ration Mech Anal, 1:87–122 1952, [2]). These equations are invariant under certain transformations of dependent and independent variables, so called symmetry transformation. The importance of these symmetries for the turbulence theory and modelling is discussed.

1 Introduction

Although the phenomenon of turbulence is described by deterministic Navier–Stokes equations, due to its sensitivity to small variations in the initial and boundary condition the turbulent field may be treated as a stochastic field. For its full description, all multipoint statistics of arbitrary order should be known. With respect to turbulence research three complete descriptions of turbulence are known, namely the infinite hierarchy of the multi-point correlation equations (so-called Friedmann–Keller (FK) hierarchy, [3]), the infinite hierarchy of the multipoint PDF equations (Lundgren–Monin–Novikov (LMN) equations, [5]) and finally the Hopf functional approach, [2]. The two latter approaches will be discussed below.

The n-point velocity PDF $f_n = f_n(\mathbf{v}_{(1)}, \ldots, \mathbf{v}_{(n)}; \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(n)}, t)$ carries information about all statistics up to n-point statistics of infinite order which can be calculated from the PDF by integration over the sample space variables $\mathbf{v}_{(1)}, \ldots, \mathbf{v}_{(n)}$, for example

$$\langle U_{i_{(1)}}(\mathbf{x}_{(1)}, t) \cdots U_{i_{(n)}}(\mathbf{x}_{(n)}, t)\rangle = \int v_{i_{(1)}} \ldots v_{i_{(n)}} f_n \mathrm{d}\mathbf{v}_{(1)} \ldots \mathrm{d}\mathbf{v}_{(n)}. \tag{1}$$

M. Wacławczyk (✉)
Chair of Fluid Dynamics, Department of Mechanical Engineering, TU Darmstadt, Otto-Berndt Str. 2, 64287 Darmstadt, Germany
e-mail: marta.waclawczyk@igf.fuw.edu.pl

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E. Hopf introduced another very general approach to the description of turbulence. He considered the case where the number of points in PDF goes to infinity, so that the probability density function becomes a probability density functional $F([\mathbf{v}(\mathbf{x})]; t)$ where instead of the vector of sample space variables $\mathbf{v}_{(1)}, \ldots, \mathbf{v}_{(n)}$ at points $\mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(n)}$ one deals with a continuous set of sample space variables $\mathbf{v}(\mathbf{x})$. It is more convenient to consider a functional Fourier transform of the probability density functional, called the characteristic functional than $F([\mathbf{v}(\mathbf{x})]; t)$ itself

$$\Phi([\mathbf{y}(\mathbf{x})]; t) = \int \mathrm{e}^{i(\mathbf{y},\mathbf{v})} F([\mathbf{v}(\mathbf{x})]; t) D\mathbf{v}(\mathbf{x}) = \langle \mathrm{e}^{i(\mathbf{y},\mathbf{U})} \rangle \tag{2}$$

where the integration is performed with respect to the probability measure $F([\mathbf{v}(\mathbf{x})]; t)$ $D\mathbf{v}(\mathbf{x})$ and $(\mathbf{y}, \mathbf{v}) = \int_G y_i v_i \mathrm{d}\mathbf{x}$ is a scalar product of two vector fields. With this definition moments of the velocity can be calculated as the functional derivatives of the characteristic functional at the origin [2]

$$\left. \frac{\delta^n \Phi([\mathbf{y}(\mathbf{x})], t)}{\delta y_{i_{(1)}}(\mathbf{x}_{(1)}) \cdots \delta y_{i_{(n)}}(\mathbf{x}_{(n)})} \right|_{\mathbf{y}=0} = i^n \langle U_{i_{(1)}}(\mathbf{x}_{(1)}, t) \cdots U_{i_{(n)}}(\mathbf{x}_{(n)}, t) \rangle. \tag{3}$$

Hence, Φ may be treated as a functional analogue of the characteristic function Φ_n for $n \to \infty$, defined, in the probability theory, as the inverse Fourier transform of f_n. E. Hopf derived evolution equation for the characteristic functional, cf. [2]. It is only one equation (not a hierarchy) which embodies the statistical properties of the fluid flow in a very concise form.

The objectives of the present work is to discuss the classical and new statistical Lie symmetries that were first found for the FK hierarchy [6] and are also present in the LMN hierarchy [9] and find corresponding symmetries for the Hopf equation. Lie one-point symmetry transformation is such transformation of the independent and dependent variables, $\mathbf{x}$ and $\mathbf{y}$, respectively, which does not change the functional form of a considered equation [1]

$$F(\mathbf{x}, \mathbf{y}, \underset{1}{\mathbf{y}}, \underset{2}{\mathbf{y}}, \ldots, \underset{p}{\mathbf{y}}) = 0 \quad \Leftrightarrow \quad F(\mathbf{x}^*, \mathbf{y}^*, \underset{1}{\mathbf{y}^*}, \underset{2}{\mathbf{y}^*}, \ldots, \underset{p}{\mathbf{y}^*}) = 0 \tag{4}$$

where the transformed variables $\mathbf{x}^* = g(\mathbf{x}, \mathbf{y}; \varepsilon)$ and $\mathbf{y}^* = h(\mathbf{x}, \mathbf{y}; \varepsilon)$ are functions of $\mathbf{x}$ and $\mathbf{y}$ and depend on a group parameter ε. The transformations can also be written in infinitesimal forms after a Taylor series expansion about $\varepsilon = 0$: $\mathbf{x}^* = \mathbf{x} + \xi(\mathbf{x}, \mathbf{y})\varepsilon + O(\varepsilon^2)$ and $\mathbf{y}^* = \mathbf{y} + \eta(\mathbf{x}, \mathbf{y})\varepsilon + O(\varepsilon^2)$. It follows from the Lie's first theorem that knowing the infinitesimal forms ξ and η uniquely determines the global form of the group transformation $g(\mathbf{x}, \mathbf{y}; \varepsilon)$ and $h(\mathbf{x}, \mathbf{y}; \varepsilon)$. With the use of infinitesimals invariant solutions of the considered Eq. (4) may be derived [1]. In fluid mechanics these solutions often represent attractors of the instantaneous fluctuating solutions of the Navier–Stokes equations, i.e. the characteristic turbulent scaling laws.

From the Lie symmetry analysis of the LMN hierarchy it followed that, surprisingly, the new symmetries are connected with intermittent laminar/turbulent flows [9]. The outcome of the symmetry analysis are invariant solutions for turbulence statistics and new possibilities to improve turbulence closures, such that invariance under the whole set of symmetries is accounted for.

2 Symmetries of the LMN Hierarchy

The LMN hierarchy, derived in Ref. [5] is an infinite set of equations for the multipoint PDFs where in the nth equation the $n+1$-point PDF is present

$$
\begin{aligned}
&\frac{\partial f_n}{\partial t} + \sum_{k=1}^{n} v_{i_{(k)}} \frac{\partial f_n}{\partial x_{i_{(k)}}} \\
&= -\frac{1}{4\pi} \sum_{k=1}^{n} \frac{\partial}{\partial v_{i_{(k)}}} \int\int \left(\frac{\partial}{\partial x_{i_{(k)}}} \frac{1}{\left|\mathbf{x}_{(k)} - \mathbf{x}_{(n+1)}\right|} \right) \left(v_{j_{(n+1)}} \frac{\partial}{\partial x_{j_{(n+1)}}} \right)^2 f_{n+1} \mathrm{d}\mathbf{v}_{(n+1)} \mathrm{d}\mathbf{x}_{(n+1)} \\
&\quad - \sum_{k=1}^{n} \frac{\partial}{\partial v_{i_{(k)}}} \left[\lim_{|\mathbf{x}_{(n+1)} - \mathbf{x}_{(k)}| \to 0} \nu \frac{\partial^2}{\partial x_{j_{(n+1)}} \partial x_{j_{(n+1)}}} \int v_{i_{(n+1)}} f_{n+1} \mathrm{d}\mathbf{v}_{(n+1)} \right].
\end{aligned}
\tag{5}
$$

Symmetries of the LMN hierarchy were investigated in Ref. [9]. Therein, it was shown that the hierarchy is invariant under the classical symmetries of the Navier–Stokes, equations, in particular, time and space translations, Galilean invariance and, for $\nu = 0$ two scaling groups

$$
\bar{T}_2: \quad t^* = t, \mathbf{x}^*_{(l)} = \mathrm{e}^{a_2} \mathbf{x}_{(l)}, \mathbf{v}^*_{(l)} = \mathrm{e}^{a_2} \mathbf{v}_{(l)}, \; f^*_n = \mathrm{e}^{-3na_2} f_n, \tag{6}
$$

$$
\bar{T}_3: \quad t^* = \mathrm{e}^{a_3} t, \mathbf{x}_{(l)} = \mathbf{x}_{(l)}, \mathbf{v}^*_{(l)} = \mathrm{e}^{-a_3} \mathbf{v}_{(l)}, \; f^*_n = \mathrm{e}^{3na_3} f_n. \tag{7}
$$

with $l = 1, \ldots, n$, which for $\nu \neq 0$ reduce to one scaling group. Moreover, it was shown that the LMN hierarchy is invariant under new statistical symmetry groups observed in the Friedrich–Keller hierarchy as the scaling and translations of multipoint velocity correlations cf. Ref. [6]. In the LMN approach the symmetries have the following forms

$$
f^*_n = \delta(\mathbf{v}_{(1)}) \cdots \delta(\mathbf{v}_{(n)}) + \mathrm{e}^{a_s} (f_n - \delta(\mathbf{v}_{(1)}) \cdots \delta(\mathbf{v}_{(n)})). \tag{8}
$$

for the scaling and

$$
f^*_n = f_n + \psi(\mathbf{v}_{(1)}) \delta(\mathbf{v}_{(1)} - \mathbf{v}_{(2)}) \cdots \delta(\mathbf{v}_{(1)} - \mathbf{v}_{(n)}). \tag{9}
$$

for the translation symmetry, where ψ is a function such that $\int \psi(\mathbf{v}) \mathrm{d}\mathbf{v} = 0$ and δ is the Dirac delta function.

In Ref. [9] the particular case of a plane Poiseuille channel flow was considered. Both the scaling (8) and the translation symmetry (9) were taken into account. For the channel flow the symmetries have slightly different form. First, the scaling symmetry (8) transforms a PDF f_n of a turbulent signal into the PDF with delta function at $\mathbf{v} = 0$. Next, the translation symmetry with the function ψ defined as $\psi = (1 - \mathrm{e}^{a_s})[\delta(\mathbf{v}_{(1)} - \mathbf{U}_L(\mathbf{x}_{(1)})) - \delta(\mathbf{v}_{(1)})]$ where $\mathbf{U}_L(\mathbf{x}_{(k)}) = [U_0(1 - x^2_{2_{(k)}}/H^2), 0, 0]$, U_0 is the velocity at the centerline and H is the channel half-width, transforms PDF into

$$f_n^* = \mathrm{e}^{a_s} f_n + (1 - \mathrm{e}^{a_s})\delta(\mathbf{v}_{(1)} - \mathbf{U}_L(\mathbf{x}_{(1)})) \dots \delta(\mathbf{v}_{(n)} - \mathbf{U}_L(\mathbf{x}_{(n)})). \tag{10}$$

To sum up, both symmetries, scaling and translation transform a PDF of a turbulent signal into the PDF of an intermittent laminar-turbulent flow. This would correspond to a situation where the flow in a channel is induced by a certain pressure difference ΔP, such that the resulting Reynolds number $Re = U_b H/\nu$ where U_b is the bulk velocity, is close to the critical value Re_{cr}. For certain range of Re both, laminar or turbulent solutions are possible with certain probability, leading to the PDF of the form given in Eq. (10). As discussed in Ref. [9] such interpretation of the symmetries has important consequences. First, it leads to certain conditions on the group parameter e^{a_s}. As this parameter is present in invariant solutions for turbulent statistics derived in [6] it may provide restrictions on the scaling parameters in these laws, such as e.g. the von Karman constant. Second, to properly describe physics, turbulence models should be invariant under the same set of symmetries as the exact equations for statistics (e.g. the FK hierarchy), hence, new symmetries should be included in these models. It may be expected that this is especially important in the case of models describing laminar-turbulent transition.

With the use of new symmetries series of invariant solutions for turbulence statistics can be derived [6]. In Ref. [9] three particular symmetries were taken into account: classical scaling of the Navier–Stokes equations, cf. (5), $y^* = \mathrm{e}^{k_2} y$, $\langle U \rangle^* = \mathrm{e}^{-k_2} \langle U \rangle$ where k_2 is an arbitrary constant, new scaling and translation symmetries of the mean velocity $\langle U \rangle^* = \mathrm{e}^{a_s} \langle U \rangle + C_1(1 - y^2/H^2)$, which follows from Eq. (10). Invariant solution can be found from the solution of the characteristic equation [6]

$$\frac{\mathrm{d}\langle U \rangle}{(a_s - k_2)\langle U \rangle + C_1(1 - y^2/H^2)} = \frac{\mathrm{d}y}{k_2 y} \tag{11}$$

which for $a_s = k_2$ gives

$$\langle U \rangle = \frac{C_1}{k_2} \ln(y) + \frac{C_1}{2k_2}\left(1 - \frac{y^2}{H^2}\right) + \mathscr{C} \tag{12}$$

where $\mathscr{C}$ is a constant. The formula above is, apparently, a sum of the turbulent and laminar velocity in the plane channel flow.

3 Symmetries of the Hopf Equation

Based on the Navier–Stokes equations, E. Hopf derived evolution equation for the characteristic functional [2]. It is only one equation (not a hierarchy) and all turbulence statistics can formally be calculated from the solution of the Hopf equation. The Hopf equation for velocity in the physical space reads [2]

$$\frac{\partial \Phi}{\partial t} = \int_R \mathrm{d}\mathbf{x}\ \widetilde{y}_k(\mathbf{x}) \left[i \frac{\partial}{\partial x_l} \frac{\delta^2 \Phi}{\delta y_l(\mathbf{x}) \delta y_k(\mathbf{x})} + \nu \nabla_x^2 \frac{\delta \Phi}{\delta y_k(\mathbf{x})} \right]. \tag{13}$$

where, in order to eliminate pressure functional Π from the equation, vector field $\widetilde{\mathbf{y}}$ such that $\mathbf{y}(\mathbf{x}) = \widetilde{\mathbf{y}}(\mathbf{x}) + \nabla\phi$ was introduced. The scalar ϕ is chosen such that $\widetilde{\mathbf{y}} = \mathbf{0}$ at the boundary B and the continuity equation is satisfied $\nabla \cdot \widetilde{\mathbf{y}} = 0$. In order to check the invariance of the Hopf functional equation under the scaling groups we first consider transformations of $n+1$-point characteristic functions. From the relation

$$\Phi^*_{n+1} = \int \mathrm{e}^{i \mathbf{v}^*_{(0)} \cdot \mathbf{y}^*_{(0)}} \ldots \mathrm{e}^{i \mathbf{v}^*_{(n)} \cdot \mathbf{y}^*_{(n)}} f^*_{n+1} \mathrm{d}\mathbf{v}^*_{(0)} \ldots \mathrm{d}\mathbf{v}^*_{(n)} \tag{14}$$

we find that the scaling symmetries (6) and (7) will hold if $\mathbf{y}^*_{(i)} = \mathrm{e}^{-k_2} \mathbf{y}_{(i)}$ and $\mathbf{y}^*_{(i)} = \mathrm{e}^{k_3} \mathbf{y}_{(i)}$ for each i, as in such a case the exponent $\mathbf{v}^*_{(i)} \cdot \mathbf{y}^*_{(i)} = \mathbf{v}_{(i)} \cdot \mathbf{y}_{(i)}$ remain unchanged and using (6) we obtain $\Phi^*_n = \Phi_n$. The same holds for the second scaling group (7), i.e. the n-point characteristic function is not transformed $\Phi^*_n = \Phi_n$. We expect that the same should hold for the limit $n \to \infty$ and the characteristic functional. In this case instead of the discrete kth variable $y_{i_{(k)}}$ we deal with $y_i(\mathbf{x})\mathrm{d}\mathbf{x}$. The sums are replaced by integrals in the continuum limit and hence $y_i \mathrm{d}\mathbf{x}$ should scale as $y_{i_{(k)}}$ in the discrete case, i.e. $y^*_i \mathrm{d}\mathbf{x}^* = \mathrm{e}^{-k_2} y_i \mathrm{d}\mathbf{x}$. Because $\mathrm{d}\mathbf{x}$ scales as $\mathrm{d}\mathbf{x}^* = \mathbf{x}\mathrm{e}^{3k_2}$ it follows that $y^*_i = \mathrm{e}^{-4k_2} y_i$.

To sum up, it can be shown that the Hopf functional equation (13) for $\nu = 0$ is invariant under the following transformation of variables

$$\bar{T}_2 : \ \Phi^* = \Phi, \quad \mathbf{x}^* = \mathrm{e}^{k_2}\mathbf{x}, \quad t^* = t, \quad y^*_i \mathrm{d}\mathbf{x}^* = \mathrm{e}^{-k_2} y_i \mathrm{d}\mathbf{x}, \quad \mathbf{y}^* = \mathrm{e}^{-4k_2}\mathbf{y}, \tag{15}$$

$$\bar{T}_3 : \ \Phi^* = \Phi, \quad \mathbf{x}^* = \mathbf{x}, \quad t^* = \mathrm{e}^{k_3} t, \quad y^*_i \mathrm{d}\mathbf{x}^* = \mathrm{e}^{k_3} y_i \mathrm{d}\mathbf{x}, \quad \mathbf{y}^* = \mathrm{e}^{k_3}\mathbf{y}. \tag{16}$$

For $\nu \neq 0$ instead of two scaling groups we obtain one scaling.

The difficulty is connected with the Galilei invariance which is broken. The functional Φ transforms under the Galilean invariance as follows [4]

$$\Phi^* = \langle \mathrm{e}^{i \int \mathbf{U}^*(\mathbf{x},t) \cdot \mathbf{y}^*(\mathbf{x}) \mathrm{d}\mathbf{x}^*} \rangle = \langle \mathrm{e}^{i \int \mathbf{U}(\mathbf{x},t) \cdot \mathbf{y}(\mathbf{x}) \mathrm{d}\mathbf{x}} \rangle \mathrm{e}^{i \int \mathbf{y}(\mathbf{x}) \cdot \mathbf{U}_0 \mathrm{d}\mathbf{x}}. \tag{17}$$

Hence, the transformation of Φ may be written as $\Phi^* = C([\mathbf{y}(\mathbf{x})])\Phi$, where $C([\mathbf{y}(\mathbf{x})])$ is a functional

$$C([\mathbf{y}(\mathbf{x})], t) = \mathrm{e}^{i \int \mathbf{y}(\mathbf{x}) \cdot \mathbf{U}_0 \mathrm{d}\mathbf{x}} \tag{18}$$

The nth derivative of the transformed functional Φ^* at $\mathbf{y}=0$ gives

$$\left.\frac{\delta^n \Phi^*}{\delta y_{i_{(0)}}(\mathbf{x})\cdots\delta y_{i_{(n-1)}}(\mathbf{x})}\right|_{\mathbf{y}=0} = i^n (U_{i_{(0)}}(\mathbf{x},t)+U_{0i_{(0)}})\cdots(U_{i_{(n-1)}}(\mathbf{x},t)+U_{0i_{(n-1)}}) \tag{19}$$

as expected for Galilean invariance. In the Galilean transformation, the space derivatives in Eq. (13) transform as $\partial/\partial x_i^* = \partial/\partial x_i$ and the integral over infinite space $\int \mathrm{d}\mathbf{x}^* = \int \mathrm{d}\mathbf{x}$. As the variables $\mathbf{y}^* = \mathbf{y}$, also the functional derivative remains unchanged $\delta/\delta(y_i(\mathbf{x}))^* = \delta/\delta y_i(\mathbf{x})$. The derivative $\partial/\partial t$ can be presented as

$$\frac{\partial}{\partial t} = \frac{\partial t^*}{\partial t}\frac{\partial}{\partial t^*} + \frac{\partial x_i^*}{\partial t}\frac{\partial}{\partial x_i^*} + \int \frac{\partial y^* \mathrm{d}\mathbf{x}^*}{\partial t}\frac{\delta}{\delta y(x)^*} = \frac{\partial}{\partial t^*} + U_{0i}\frac{\partial}{\partial x_i}\,, \tag{20}$$

The transformed functional equation (13) reads

$$C([\mathbf{y}(\mathbf{x})])\frac{\partial \Phi}{\partial t} = \int y_k(\mathbf{x})\left[i\frac{\partial}{\partial x_l}\frac{\delta^2 C([\mathbf{y}(\mathbf{x})])\Phi}{\delta y_l(\mathbf{x})\delta y_k(\mathbf{x})} + \nu\nabla_x^2\frac{\delta C([\mathbf{y}(\mathbf{x})])\Phi}{\delta y_k(\mathbf{x})} - \frac{\partial \Pi}{\partial x_k}\right]\mathrm{d}\mathbf{x}\,. \tag{21}$$

The functional derivative of Φ^* in (21) reads $C\delta\Phi/\delta y_k(\mathbf{x}) + \Phi\delta C/\delta y_k(\mathbf{x})$ and we note that Laplacian ∇_x^2 of the second term is zero as this term is not a function of $\mathbf{x}$. Hence, the last RHS term of equation (21) inside the integral reads $C([\mathbf{y}(\mathbf{x})])\nu\nabla_x^2\delta\Phi/\delta y_k(\mathbf{x})$. Further, the second functional derivative of Φ^* reads

$$C\frac{\delta^2\Phi}{\delta y_k(\mathbf{x})\delta y_l(\mathbf{x})} + \frac{\delta\Phi}{\delta y_k(\mathbf{x})}\frac{\delta C}{\delta y_l(\mathbf{x})} + \frac{\delta\Phi}{\delta y_l(\mathbf{x})}\frac{\delta C}{\delta y_k(\mathbf{x})} + \Phi\frac{\delta^2 C}{\delta y_k(\mathbf{x})\delta y_l(\mathbf{x})}. \tag{22}$$

Again, the derivative $\partial/\partial x_l$ of the last term is zero, as it does not depend on $\mathbf{x}$. In addition we also have

$$\frac{\partial}{\partial x_k}\left[\frac{\delta\Phi}{\delta y_l(\mathbf{x})}\frac{\delta C}{\delta y_k(\mathbf{x})}\right] = \frac{\delta C}{\delta y_k(\mathbf{x})}\frac{\partial}{\partial x_l}\frac{\delta\Phi}{\delta y_l(\mathbf{x})} = 0, \tag{23}$$

where the first equality follows from the fact that the derivative of C does not depend explicitly on $\mathbf{x}$ and the second, from the continuity condition. It can be seen from the definition of the Hopf functional (2) that its derivative with respect to $y_l(\mathbf{x})$ reads $\langle iU_l(\mathbf{x},t)\exp[i\int \mathbf{U}(\mathbf{x},t)\cdot\mathbf{y}(\mathbf{x})\mathrm{d}\mathbf{x}]\rangle$, hence differentiating once again with respect to x_l is zero as $\partial U_l/\partial x_l = 0$. Finally, the transformed Hopf equation reads

$$\begin{aligned} C([\mathbf{y}(\mathbf{x})])\frac{\partial\Phi}{\partial t} &= C([\mathbf{y}(\mathbf{x})])\int \widetilde{y}_k(\mathbf{x})\left[i\frac{\partial}{\partial x_l}\frac{\delta^2\Phi}{\delta y_l(\mathbf{x})\delta y_k(\mathbf{x})} + \nu\nabla_x^2\frac{\delta\Phi}{\delta y_k(\mathbf{x})}\right]\mathrm{d}\mathbf{x} \\ &\quad + \frac{\delta C}{\delta y_l(\mathbf{x})}\int y_k(\mathbf{x})\left[i\frac{\partial}{\partial x_l}\frac{\delta\Phi}{\delta y_k(\mathbf{x})}\right]\mathrm{d}\mathbf{x} \end{aligned} \tag{24}$$

As it is seen the last RHS term does not cancel. Hence, in the integral formulation of the Hopf equation the Galilei invariance is broken.

We will next consider the transformation of the Hopf functional under the statistical symmetries, formulated for the PDF's in Eqs. (8) and (9). E. Hopf proposed to present the solution of the Hopf functional as the infinite series expansion [2]

$$\Phi = 1 + C_1 + C_2 + \cdots , \tag{25}$$

where

$$C_n = \int K_{i_{(1)}\ldots i_{(n)}}(\mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(n)}, t) y_{i_{(1)}}(\mathbf{x}_{(1)}) \cdots y_{i_{(n)}}(\mathbf{x}_{(n)}) \mathrm{d}\mathbf{x}_{(1)} \cdots \mathrm{d}\mathbf{x}_{(n)} \tag{26}$$

with functions

$$K_{i_{(1)}\ldots i_{(n)}}(\mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(n)}, t) = \frac{i^n}{n!} \langle U_{i_{(1)}}(\mathbf{x}_{(1)}, t) \cdots U_{i_{(n)}}(\mathbf{x}_{(n)}, t) \rangle. \tag{27}$$

If we substitute the statistical symmetries of moments into the above equations we find that the scaling symmetry of multipoint velocity correlations, cf. Ref. [6] and Eq. (8), transforms the kernel functions as

$$K^*_{i_{(1)}\ldots i_{(n)}} = \mathrm{e}^{k_s} K_{i_{(1)}\ldots i_{(n)}}, \tag{28}$$

hence the series expansion reads

$$\Phi^* = 1 + \mathrm{e}^{k_s} \left(\Phi^1 + \Phi^2 + \cdots \right), \tag{29}$$

or

$$\Phi^* = 1 + \mathrm{e}^{k_s} \left(\Phi - 1 \right). \tag{30}$$

The translation symmetry of multipoint correlations, cf. Ref. [6] and Eq. (9), translates the kernel functions by a constant which leads to the following translation of the nth term in the Taylor series expansion

$$\Phi^*_n = \Phi_n + \int C_{i_{(1)}\ldots i_{(n)}} y_{i_{(1)}}(\mathbf{x}_{(1)}) \cdots y_{i_{(n)}}(\mathbf{x}_{(n)}) \mathrm{d}\mathbf{x}_{(1)} \cdots \mathrm{d}\mathbf{x}_{(n)}. \tag{31}$$

Hence, the translation symmetry of the characteristic functional Φ can be written in the following form

$$\Phi^* = \Phi + \Psi([\mathbf{y}(\mathbf{x})]), \tag{32}$$

where Ψ is a functional such that its nth functional derivative at the origin equals $n!C_{i_{(1)}\ldots i_{(n)}}$ and its functional derivatives does not depend explicitly on x or t which makes the Eq. (13) invariant under the transformation (32).

4 Conclusions

To sum up, it was argued that all methods for the full statistical description of turbulence, namely FK hierarchy, LMN hierarchy for PDFs and the Hopf characteristic functional equations are invariant under classical scaling symmetries of Navier–Stokes equations and additionally under the set of statistical symmetries: translation and scaling. Deriving transformations of the Hopf equation equivalent to the FK symmetries is a new contribution of the present work. Through the analysis of PDF equations the statistical translation and scaling were identified in Ref. [9] as connected with the (external) intermittency. Hence, the statistical symmetries indicate the fact that solutions of Navier–Stokes equations may have different character. Such transformation could only be observed in the statistical approach, hence the statistical symmetries were not found in the Lie group analysis of the Navier–Stokes equations. With the use of statistical symmetries series of invariant solutions for turbulence statistics were obtained in Ref. [6]. It can be expected that similar, new results could be obtained for PDFs based on the symmetries of LMN hierarchy. Moreover, the invariance under new statistical symmetries could be introduced into turbulence models to improve their predictions.

The Lie group analysis of infinite hierarchies of equations, such as LMN or FK cannot be performed with the use of common computer algebra systems. Hence, the symmetries of FK equations were rather guessed than calculated and their set may not be complete. A possibility would be to apply the Lie group method to one, Hopf functional equation in order to find possibly new statistical symmetries and next derive corresponding transformations in the LMN and FK hierarchy. For this purpose, the extended Lie group method, introduced in Refs. [7, 8] could be used. This issue is the subject of the current study.

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