Chapter 2
Dynamics of a Material Point

In this chapter we study the dynamics of a material point, namely the laws governing motion by its causes, which are the forces. We shall then start by defining and discussing the concept of force. The experimental method was introduced by Galileo Galilei at the end of the XVI century. He also discovered part of the laws of mechanics. The complete theory of mechanics was built by Isaac Newton, who published in 1686 the "Philosophiae Naturalis Principia Mathematica", known generally as simply "Principia".

The law of inertia was discovered by Galilei and assumed by Newton as the first law of mechanics. It will be studied in Sect. 2.3. The law states that a body in absence of forces acting on it moves naturally with constant velocity in a straight line, a rectilinear uniform motion. The second law was also discovered by Galilei and precisely formulated by Newton. It states that the rate of change of the momentum, a vector that we shall define, namely its time derivative, is equal to the force acting on the body. In an equivalent manner the acceleration is proportional to the force. This is the subject of Sect. 2.4. In the same section we shall discuss Newton's third law, the action-reaction law.

There are several types of force in Nature, as we shall see in the next chapter. In this one, however, in Sect. 2.5, we shall talk of weight, the force acting on all the bodies near the surface of the earth. A few examples will be discussed in Sects. 2.6 and 2.7.

In Sect. 2.8 we introduce two of the fundamental mechanical quantities (beyond momentum, or quantity of motion, already introduced in Sect. 2.4), the angular momentum and the moment of a force.

In Sect. 2.9 we shall study a simple but very important system, the pendulum and its harmonic motion. We shall also see how two concepts of mass, the inertial and the gravitational mass, are in fact only one.

After having introduced the concept of work made by a force and shown the theorem of energy conservation in Sect. 2.10, we shall describe an interesting experiment by Galilei. It establishes that the work done on a body by the weight force depends only on the difference between initial and final heights, not on the

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particular path followed. In modern language the experiment established that the weight force is conservative. This very important concept will be defined in Sect. 2.13. We then demonstrate the energy conservation theorem. Energy conservation is a fundamental law of all physics. We shall deal in this book only with mechanical energy, in its kinetic and potential forms, but we warn the reader that other important forms of energy exist, in particular thermal energy, as we shall discuss in the second volume of this course when dealing with thermodynamics.

The historical process leading to a precise definition of the concept of energy and to the establishment of the law of energy conservation took more than two centuries. Starting with Galilei, it came to maturity around mid XIX century, with the experiments of Mayer and Joule and enunciation of the energy conservation law by Mayer and Helmholtz. We shall give some hints in Sect. 2.14.

In Sect. 2.15 we shall discuss a particular type of force, the central forces. The gravitational attraction of the sun on a planet is an important example of this category.

In the last paragraph we introduce the concept of power, which is the work done by a force per unit of time.

2.1 Force, Operational Definition

The primitive concept of force is linked to muscular strain. If we lift a weight, push an object, we must exert a force with our hands and arms and we feel strain. Since ancient times humans developed simple mechanical devices to exert forces or amplify the muscular effect. The string of an archer's drawn bow exerts a force on the arrow, throwing it in the air; a lever can be used to lift big weights, etc. However, in physics the concept must be quantitative. For that, we must define force accurately enough to be able to measure it. This means that we must be able to compare two forces and establish when they are equal, when one is twice the other, etc. In other words we must be able to determine the ratio between two different forces.

A direct method to compare two forces is based on the lever rule, which was discovered by Archimedes of Syracuse (287–212 BC) more then two thousand years ago. The rule states that two equal forces balance when applied at equal distances on two sides of the pivot (Fig. 2.1a) and that two different forces F_1 and F_2 balance when applied at distances from pivot (l_1 and l_2 respectively) inversely proportional to the forces (Fig. 2.1b), i.e. such as

$$F_1 l_1 = F_2 l_2. \tag{2.1}$$

The first statement can be proven simply with symmetry arguments. If the two forces are equal and the two arms are equal, the system is symmetric. How could it choose on which side to bend? The second statement on the contrary, namely the validity of Eq. (2.1), must be experimentally verified.

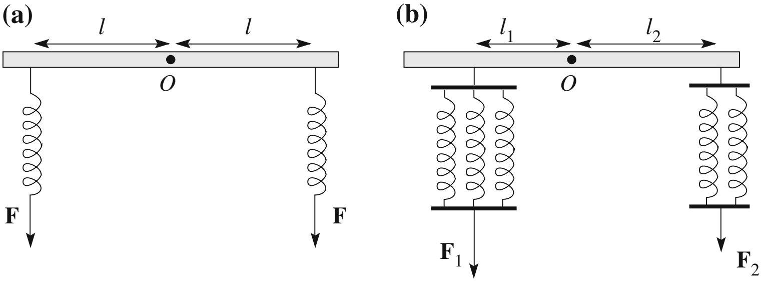


Fig. 2.1 Comparison of two forces

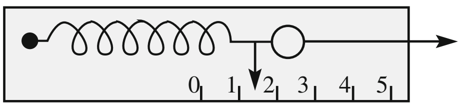
We know that a spring exerts a force when compressed or stretched relative to its natural length; we feel the muscular strain when we compress or pull it. We build a certain number of springs as equal as possible to each other. We can then verify that they exert equal forces when compressed (or stretched) in the same measure by applying those forces at equal distances from the pivot of a lever as in Fig. 2.1a. We can now define as unitary the force expanded of a specific length (N.B.: this is not the official definition).

We can then define the multiples of the unit force. If for example, we want a force of three units, we put three of our springs in parallel. We can experimentally verify the lever rule Eq. (2.1) as shown in Fig. 2.2b with different combinations of unit forces. Once we have stated that, we can use it to measure forces. As a matter of fact the method has been used in steelyards since very ancient times and is still used now in fruit or other goods markets to weigh a wide variety of goods. The weight to be measured is compared with the weight of a standard object seeking for equilibrium by changing the length of the lever arm of the latter.

In the operational definition of the force we have just chosen, we did not make any hypothesis on the relation between the force exerted by the spring and its length. However, this definition is not simple to use in practice. A handier device is the dynamometer (from the Greek *dynami* for force and *metro* for measure).

The dynamometer, shown schematically in Fig. 2.2, is made of a spring fixed at one extreme on a wood, or other material, plate and with a ring at its other extreme. The force to be measured is applied to the ring. A pointer moving on a scale gives a measurement of the dilation of the spring. Once we have built the device we must calibrate it. With the above described procedure we have built a number of springs, multiple and submultiples of the unit. We apply each of them to the ring and mark the position of the pointer on the table. In this way we build a scale on which we will read the values of unknown forces. In practice, we find that the scale is linear,

Fig. 2.2 The dynamometer



namely the stretch is proportional to the applied force, if the stretch is not too large. However, this property is comfortable, but not necessary.

The method we have described is used in practice, but does not allow a precise definition of force. In the SI the unit of force is a derived one, It is the force imparting the unit acceleration (1 m/s^2) to the unit mass (1 kg). It is called newton (N). To have an idea of the order of magnitude, think that the weight of one liter of water, 1 kg, is about 9.8 N. In other words one Newton is about the weight of the water filling a glass.

2.2 Force Is a Vector

In giving the operational definition of force in the previous section we have implicitly assumed, and we did that by definition, that two equal and opposite forces when applied to a point do not cause acceleration. Namely, the two forces are in equilibrium. Clearly, a force not only has a magnitude but also a direction. We can exert a force on a body applying one of our springs and pulling in different directions. We are led to think that force is a vector quantity. However, the conclusion cannot be reached by logic, rather it needs experimental verification. To be a vector, a quantity not only should have a magnitude and a direction, but also satisfy the rule of addition of vectors.

The experience with three forces was originally devised by Pierre Varignon (1654–1722), a contemporary of Newton. Its device is shown in Fig. 2.3. In the

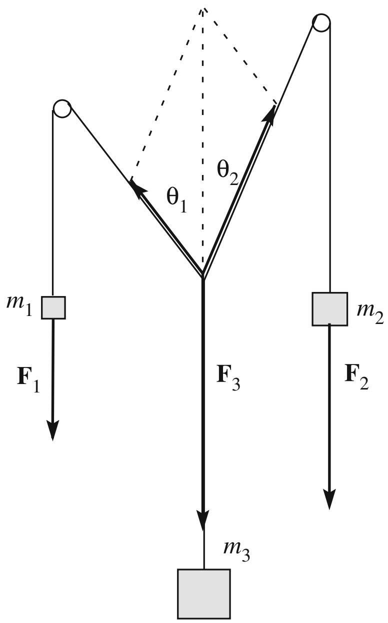


Fig. 2.3 Varignon experiment showing the composition of forces

plane of the figure, which is vertical, three pulleys are fixed. The three weights of masses m_1, m_2 and m_3, act by means of wires, drawn in the figure, joined in the point O. The forces exerted by the wires have magnitudes proportional to the weights and the directions of the wires. Once we have joined the three wires in O and let the system alone, the system moves until it reaches its equilibrium configuration, the one represented in the figure. We know the values of the weights, say F_1, F_2 and F_3, and measure the angles θ_1 and θ_2. We find that the following relations are satisfied:

$$F_1 \sin\theta_1 = F_2 \sin\theta_2, \quad F_1 \cos\theta_1 + F_2 \cos\theta_2 = F_3$$

or

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0.$$

The Varignon experiment and similar ones made afterwards verify the vector character of the force. The most precise tests, however, are indirect and come from the agreement of the experimental data with the predictions made under this hypothesis in the most different conditions.

Once we have established that forces add as vectors, we define as the *resultant* of the set of forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, … and their vector sum

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots. \tag{2.2}$$

Let us now think of some forces that we know from our everyday experience. We can distinguish two types. The just considered forces exerted by a spring, the force a table exerts on an object it supports, the force we exert with our hand pushing an object, are each exerted by contact. A body, the spring, the plane of the table and the hand each apply force to the object touching it. The everyday example of the second type of force is weight. Weight is the force with which earth gravitationally attracts all bodies. It is directed vertically down, towards the center of earth. This force is exerted at a distance i.e., it does not need contact.

2.3 The Law of Inertia

One of the most revolutionary discoveries of Galilei was the establishment of the behavior of a body not subject to forces. The problem lies in the fact that in practice it is impossible to eliminate all the forces. Weight is always present on earth. It cannot be eliminated, but it can be balanced. If we put a body on a horizontal plane, the latter will exert on the body a force equal and opposite to its weight. However, when the body moves, frictional forces due both to the contact between the surfaces of the plane and the body and the air are present. The effect of these “passive” forces is much more difficult to control and was not known before Galilei.

Consider the following experiment. We put a bronze sphere on a horizontal plane. We then give it a push. That is, we apply a force for a brief time interval, giving it a certain initial velocity on the plane. We observe the sphere's motion and see that its velocity gradually decreases and finally stops. To have the sphere moving at constant velocity we need to apply a force continuously. The conclusion seems to be that, when not acted on by forces, a body stands still. If it moves at constant velocity it is acted upon by a force proportional to its velocity. We now know that the conclusion, thought to be true for centuries, is actually false.

Galilei's argument can be summarized as follows. The fact that, when we apply a force to a body and then we cease to apply it, the body slows down and finally stops is obviously true. But the cause is not the absence of acting forces. On the contrary, the cause is the presence of forces that we do not apply, we do not see, yet exist (they are called passive) and we are unable to avoid, like friction and air drag,

Galilei could not prove his statement experimentally by eliminating all the passive resistive forces. He observed however that, when launching a solid polished sphere of brass or ivory on a horizontal guide, the distance travelled by the sphere before coming to rest was longer and longer when the surfaces of the guide and the spheres were smoother and smoother. Mentally going to the limit of infinite smoothness, he concluded that in those conditions the sphere would never stop, but would continue to move forever with the same velocity.

The conclusion is the *law of inertia*. In the words of Newton

> Every body preservs in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by impressed forces.

The law of inertia is not however valid in just any circumstance. Whether it is valid or not depends on the reference frame. Up to now we have made experiments in a reference fixed to earth. We now suppose that we want to build a laboratory on a carriage moving on straight rails at constant velocity, relative to earth. In our laboratory we have a smooth horizontal plane. We lay a bronze sphere on the table and observe that, as expected, it remains still. However, suddenly the sphere moves, accelerates and moves quickly forward, without any visible force acting on it. What did happen? It happened that the carriage suddenly started to slow down till coming to rest. Even if our laboratory is closed with no window to look out, we know that the carriage decelerates because we also experience a mysterious force pushing us forwards.

An observer on earth, namely in the frame we had been considering above, easily interprets the phenomenon. The sphere is free to move horizontally, the table being smooth. A force acted upon by brakes on its reels has slowed the carriage down. This force, however, does not act on the sphere, because the support plane is smooth. The resultant of the forces on the sphere is null. For the law of inertia it will continue in its motion with constant velocity. This is relative to the ground. But the observer on the carriage, which slows down relative to the ground, sees the sphere accelerating to reach the velocity that the carriage had before braking.

A reference frame in which the law of inertia is valid is called an *inertial frame*. We shall see that inertial frames have a privileged role in mechanics, and more generally in physics.

More precisely, the law of inertia can be stated as: *Reference frames do exist in which every body not subject to force indefinitely remains in its state of rest or uniform rectilinear motion.*

One might think that the law of inertia is a consequence of our definition of inertial frame, in other words that the argument is circular. But this is not true. Indeed, we can give arbitrarily any definition we like, but we can never establish by definition a law of nature, namely how she behaves. The existence of inertial frames is a law of nature not a definition by men.

We further observe that we have considered inertial any reference stationary on earth. The conclusion comes from the fact that, while doing experiments in such laboratories, we never observe objects suddenly moving when no force acts on them, nor do we feel as though we are being pushed in one direction or another. However, the conclusion is valid only in a first approximation. Accurate measurements show that frames that are stationary on earth are not exactly inertial. This is due to the fact that earth moves around the sun and rotates on its axis. We shall come back to that in Chap. 4. For the moment it will be enough to know that stationary reference frames on earth are close enough to be inertial for the vast majority of measurements carried out in laboratories and, on the other hand, procedures exist to define inertial reference systems with all the requested precision in case this is needed.

2.4 The Newton Laws of Motion

In the *Principia*, Newton begins by stating, as axioms induced from the experiments, the three fundamental laws from which the description of all the mechanical phenomena, both on earth and in the Universe can be deduced. The first law is the law of inertia we already discussed. The causes of any change of the state of rest or rectilinear uniform motion of a body are to be searched for in the bodies around it. For example the racket that hits it changes the state of motion of a tennis ball, the state of the compass needle is changed by the presence of a magnet, etc. The same hit imparted with a racket to a tennis or ping-pong ball produces different accelerations in the two bodies. By the term *inertial mass* we mean the characteristic of a given object that makes it more or less resistant to changing its state of motion under the action of a given force. Galilei had already proven with his experiments that a body under the action of a constant force, its weight or a component of its weight, moves with a constant acceleration in the direction of the force.

Let us study the phenomenon quantitatively. We have already built springs producing forces of different magnitudes. We have performed an analogous procedure for mass. We have built a number of blocks of the same material making them as equal as possible to each other. We can say that one block has unit inertial mass, two blocks inertial mass equal to two, etc.

We have also prepared a horizontal plane, the function of which is to equilibrate the weights of our blocks. In our experiments we shall put the blocks in motion sliding on the plane and we want to reduce as much as possible the friction forces between the plane and the blocks. We prepare the surface of the plane as smooth as possible. We can also play the following trick. We can build the blocks with a cavity inside and a series of holes between the cavity and the lower face. We fill the cavity with dry ice (frozen CO_2), which will sublimate pushing CO_2 gas through the holes. The thin layer of gas between the block and plane surfaces reduces friction to negligible values.

1. We attach one of our springs to one block, we give it a certain deformation, stretch or compression (Fig. 2.4a). We observe that the body moves with constant acceleration, say a_0, in the direction of the force, as long as we keep constant the force (i.e. the deformation)
2. We attach two springs (Fig. 2.4b) to the block and give them the same deformation as in the first experiment. We observe the body moving again with constant acceleration in the direction of the force. The acceleration is twice as large, 2 a_0.
3. We fix two blocks one on top of the other and attach one spring to which we give once more the same deformation. The acceleration is now one half as in the first experiment, $a_0/2$ (Fig. 2.4c).

Continuing with similar experiments changing the force on a body or the inertial mass, we come to the conclusion that its acceleration a is proportional to the force F and inversely proportional to its inertial mass m_i and we write $F = m_i a$.

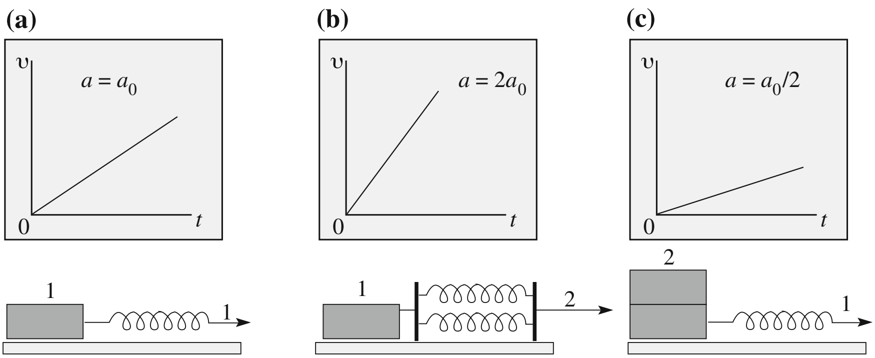


Fig. 2.4 Simple experiments to study the relation between force, acceleration and inertial mass

We can do better, because we have found that acceleration and force, which are two vectors, have the same direction. The second law states that

$$\mathbf{F} = m_i\mathbf{a} = m_i\frac{d\mathbf{v}}{dt} = m_i\frac{d^2\mathbf{r}}{dt^2}. \tag{2.3}$$

This is the form that is more often expressed. However, Newton stated it as

> A change of motion is proportional to the motive force impressed, and takes place in the direction of the right line in which the force is impressed.

The quantity called by Newton "motion" is a fundamental vector quantity, **p**, now called *quantity of motion*, or *momentum* (sometimes *linear momentum*). It is the velocity times the inertial mass

$$\mathbf{p} = m_i\mathbf{v} = m_i\frac{d\mathbf{r}}{dt}. \tag{2.4}$$

Two bodies of different masses can have the same quantity of motion if their velocities are in the inverse proportion of the masses. The second Newton law is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}. \tag{2.5}$$

In words, the rate of change of the momentum of a material point is equal to the force acting on it. Considering that m_i is a constant, and using Eq. (2.4) we have

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m_i\frac{d\mathbf{v}}{dt}. \tag{2.6}$$

As for the law of inertia, the second law is not valid in every reference frame. Recall the example of the sphere in a laboratory on a carriage that starts suddenly to accelerate without any force being acting. Like the first law, the second Newton law is valid only in inertial frames.

Equation (2.3) says that acceleration has the same direction as the relevant force. This may appear to be obvious but it is not true in every circumstance. The equation also says that the acceleration due to a given force acting on a given body is independent of the velocity of the body. Experiments show that both of these, while true at common experience velocities, are not so for velocities close to the speed of light. In these conditions, called relativistic, Eq. (2.3) fails. However, even in these high velocities regimes, Eq. (2.5) remains valid, namely, as Newton stated, the force and the time derivative of momentum are equal. What needs to be changed is the relation between momentum and velocity.

We shall study relativistic mechanics in Chap. 6; we anticipate that in a relativistic regime, the concept of inertial mass remains exactly the same. Mass is a constant, independent of velocity, characteristic of the body. The concept of momentum however must be made more general. Its expression is

$$\mathbf{p} = m_i \gamma(\upsilon)\mathbf{v}, \tag{2.7}$$

where $\gamma(\upsilon)$ is a function of velocity, called the Lorentz factor, after Hendrik Lorentz (1853–1928), one of the fathers of relativistic mechanics. Its value is very close to 1 up to velocities close to that of light, $c \approx 3 \times 10^8$ m/s, but increases very rapidly when υ approaches c.[1]

For comparison, the speed of the earth relative to the sun is about 3×10^4 m/s, 10^{-4} of the speed of light, the speeds of the stars relative to their galaxies, including our sun, are an order of magnitude larger, but still 10^{-3} of the speed of light. For the latter, the Lorentz factor differs from 1 only by 0.5×10^{-6}.

A second limit of validity of the Newton laws is at very small dimensions. Indeed, classical physics ceases to be valid and must be modified in quantum physics, at atomic scales. These however are very small compared to the objects of everyday experience, e.g., atomic radiuses are typically 30–300 pm.

The Newton law gives the acceleration once the forces are known. Consequently, in the analysis of any motion we deal with the position vector, the velocity, which is its first time derivative, and the acceleration, its second time derivative. We do not need higher derivatives. For these reasons we did not go beyond the second derivative of the position vector when we studied kinematics. We recall on purpose that to know the motion of a particle we need to know not only the acting forces, but also the initial position and velocity.

Let us now look at another aspect. The second law can be used in three main ways:

1. If we know the inertial mass of a body and all the forces acting on it, and the initial conditions, we can calculate its motion
2. If we know the motion of a body and its inertial mass, we can infer the forces acting on it.
   Distinguishing the two points of view is not as trivial as it may look. The first point of view is deductive. The laws of mechanics are used to calculate the motion of bodies in all possible circumstances. In this way physicists and engineers design mechanical devices and engines. The second point of view is inductive and is the point of view taken to make progress in physics. The challenge of the physics research is to understand from the study of motion the fundamental nature of the forces that cause it. This is the way followed by Newton to discover universal gravitation from study of the motions of heavenly

[1]The reader is warned that one can still find books and articles calling the product $m_i\gamma(\upsilon)$ "relativistic mass" and m_i "rest mass". The former in a relativistic regime increases with increasing velocity. These concepts were introduced in the last years of the 19th century and the first ones of the 20th when relativity theory was being developed and things were not yet completely clear. They are misleading concepts (what varies with velocity is the Lorentz factor, not the mass, which is invariant) and should be avoided. We shall treat relativity in Chap. 6.

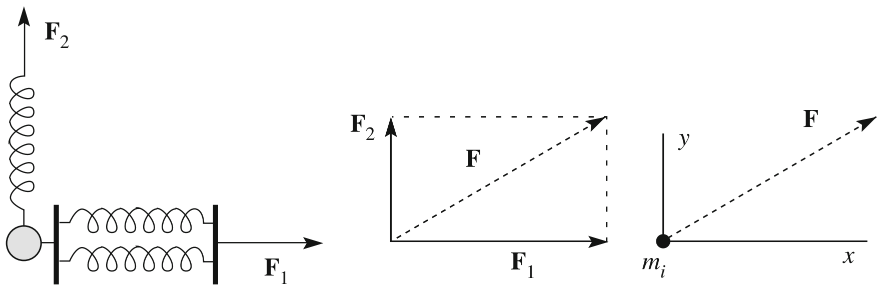


Fig. 2.5 Two forces acting at the same time

bodies. This is the way in which Ernest Rutherford (1871–1937) discovered the atomic nucleus in 1911 when studying the scattering of energetic alpha particle by a thin gold sheet. This is the way followed today to study the properties of atomic nuclei and elementary particles.

We can state that the success of the Newton law is just as follows. It substantially tells us: if you see a body that does not move in a uniform rectilinear motion, a force should act. Search for it and search for the physical agent to which it is due. You will find a force, the mathematical expression of which will be *simple* and, as a consequence, you will be able to lay down a simple theory. From this point of view the Newton law is a research program. We shall see in Chap. 3 that, indeed, the various forces of nature have simple expressions in terms of the co-ordinates and characteristics of the system. The program is successful.

3. A third possibility is that, if we know both forces and motion we can deduce the inertial mass of the body. To know the mass of the proton for example, we can measure how its momentum and energy vary under the action of a known force.

The law of composition of forces. If more than one force act at the same time on the material point we are discussing, their effect is the same as if only one force were acting, equal to the resultant of those forces. Consider for example that two forces are applied as in Fig. 2.5. The first spring exerts the force $\mathbf{F}_1$ in the x direction. When acting alone it produces the acceleration F_1/m_i along x. The second spring exerts the force $\mathbf{F}_2$ in the y direction. When acting alone it produces the acceleration F_2/m_i along y. To know what happens if the two forces act contemporarily is something that cannot be found by logic, rather it has to be found experimentally. Indeed, what experiments show is that the acceleration is just what one calculates assuming that only one force were acting, equal to the resultant $\mathbf{F}$ of $\mathbf{F}_1$ and $\mathbf{F}_2$. In other words, the observed acceleration is $\mathbf{a} = \mathbf{F}/m_i$.

The third Newton law is the *law of action-reaction*.

If a body exerts a force (an action) on a second body, the second always exerts on the first a force (a reaction) that is equal and opposite on the same line of action.

Given its importance, we reproduce how it is stated, in an equivalent manner, by Newton.

> To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Newton gives then a few examples.

> Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse (if I may so say) will be equally drawn back towards the stone; for the distended rope, by the same endeavor to relax or unbend itself will draw the horse as much towards the stone as it does the stone towards the horse and will obstruct the progress of the one as much as it advances that of the other.

We notice that, differently from the first two, the third law deals with two, rather than one, bodies. It tells us that isolated forces (actions) do not exist, only **inter**actions do exist.

Pay attention to the fact that action and reactions are applied in different points, one on one body, the other on the other body. If we push a stone with a finger, the action of the finger is applied in a point of the stone; the reaction of the stone is on the tip of our finger. The force exerted by the horse drawing the stone is exerted on the stone through the rope, the reaction acts, again through the rope, in the point of the horse at the end of the rope. Every object whether it is falling or laying on a support, weighs, meaning that the weight force is applied on it. Weight is the force with which the earth attracts all bodies. As a reaction, each body attracts the earth with an equal and opposite force. The reaction is applied to a point of the earth, its center.

The action-reaction principle, as all physical laws, must be experimentally verified. Direct verifications are based on the fact that in a collision between two bodies the total quantity of motion, namely the vector sum of the two, is conserved, meaning that its values before and after the collision are equal (while each of the two vary).

The vectors we have met so far, position vector, velocity and acceleration depend, as we have seen, on the reference frame. On the contrary, force does not.

2.5 Weight

We know from every day experience that all the bodies on earth are subject to a force, vertically directed downwards, called the weight. We can measure the weight of a body, for example, attaching it to a dynamometer vertically positioned and reading on its scale the position of the pointer, namely the stretch of the spring. If we repeat the measurement in different points of our laboratory we find that it does not vary. However, if we repeat the measurement at much larger distances, for example at the Equator and at 45° latitude, or at different altitudes, for example at the sea level and at 2000 m altitude, we notice small differences (of the order of a few per mille) between them. As we shall discuss in Sect. 5.7, these small variations are due to the rotation of the Erath. Apart from these small corrections, the weight is the gravitational attraction exerted by the earth on the body. This is universal; it is the same force with which the earth attracts the moon. We shall discuss this fundamental force in Chap. 4. We anticipate that the gravitational attraction decreases as the reciprocal of the distance squared. This is one of the reasons (the other is the rotation motion of earth) why the weight of an object is a bit smaller on a mountain than at the sea level.

Different objects, in the same place, may have different weights. This means that the force with which earth attracts a body depends on a characteristic of the body. We state that the gravitational force on a body is proportional to its *gravitational mass*, which we denote with m_g. This is similar to the electric attraction. A charged body A at a certain distance from another body that is also charged, is subject to an electrical force. If in the place of A we put a body B with twice the charge, the force on it is double. Hence, the electric force on a body is proportional to its electric charge. In a similar way two massive bodies, for example two spheres, at a certain distance attract with the gravitational force that is proportional to the gravitational mass of each of them. This force, if between two objects of every day life is quite small, but can be measured with very delicate experiments, as we shall see in Sect. 4.7, but is large between Heavenly bodies. Considering that the gravitational mass is for the gravitational force the analogous of the electric charge for the electric forces, we might call it gravitational charge, but we shall soon see the reason why we call it mass.

The weight force $\mathbf{F}_W$ acting on a body of gravitational mass m_g is then

$$\mathbf{F}_w = m_g \mathbf{g}. \tag{2.8}$$

The vector quantity $\mathbf{g}$ does depend on the location, but in a given site it is equal for all bodies. If $\mathbf{r}$ is the position vector, the vector $\mathbf{g}(\mathbf{r})$ is the gravitational force at $\mathbf{r}$ per unit gravitational mass. It is called *gravity acceleration*. We shall see soon the reason for the name. We notice that the gravitational mass being a characteristic of a body is the same in any point, differently from its weight. If we measure the weights of two bodies in different points on the earth we find that each of them varies a bit, as already mentioned, but the ratio of the two remains rigorously equal. Even if we should do this experiment on the moon.

Operationally, the gravitational mass is the physical quantity measured by a balance. A balance, see Fig. 2.6, consists of a lever with pivot in O and two pans, which we shall consider, to make it simple, exactly at the same distance on the two

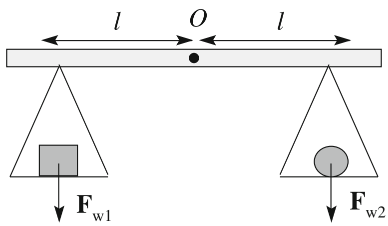


Fig. 2.6 Comparing the weights of two equal masses

sides of O. The balance compares the weights of the two objects on its pans. If they are equal the balance is in equilibrium. We have seen that, by definition, the weights of different objects in the same place are proportional to their gravitational mass. We can then the state that two objects have the same gravitational mass when, put on the pans of the balance, they are in equilibrium.

We now need a body having unit mass by definition. We put it on a pan. Another body has gravitational mass equal to one when, put on the other pan it is in equilibrium. A body has gravitational mass equal to 2, if put on a pan is in equilibrium with two of the unit masses on the other, etc.

Gravitational mass and inertial mass are two different properties of every body. The former is a measure of the strength of the gravitational attraction to which it is subject, the latter of how difficult it is to modify its quantity of motion. However, we know from every day experience, that heavier bodies are also more difficult to accelerate because they are more inert. To search for a mathematical relation, suppose to observe the free fall of two different bodies. Their inertial masses are m_{1i} and m_{2i} and their gravitational masses m_{1g} and m_{2g}. The weight of the first is, $\mathbf{F}_{1w} = m_{1g}\mathbf{g}$, the weight of the second $\mathbf{F}_{2w} = m_{2g}\mathbf{g}$. Calling $\mathbf{a}_1$ and $\mathbf{a}_2$ the two accelerations, we have:

$$m_{1g}\mathbf{g} = m_{1i}\mathbf{a}_1, \quad m_{2g}\mathbf{g} = m_{2i}\mathbf{a}_2,$$

which can be written as

$$\mathbf{a}_1 = \frac{m_{1g}}{m_{1i}}\mathbf{g}, \quad \mathbf{a}_2 = \frac{m_{2g}}{m_{2i}}\mathbf{g}. \tag{2.9}$$

We see that the free fall accelerations of different bodies in the same place are proportional to the ratios of their gravitational and inertial mass. Consequently, if this ratio is equal for all the bodies, light or heavy, all of them fall with the same acceleration. This fundamental property was experimentally shown to be true by G. Galilei.

It is often told that Galilei dropped contemporarily two balls, one made of lead, one of wood, from the Pisa tower and that he observed them reaching ground at the same instant, showing in this way that they fall with the same acceleration. The experiment was absolutely success and spectacularly carried out in 1971 by the NASA Apollo 15 astronaut D. Scott dropping a hammer and a feather on the moon. As a matter of fact Galilei never mentions having made his fundamental experiments in such a way. He new very well that it could not work, both for the perturbing effects of the atmosphere and due to the smallness of the fall times, a fact that did not allow him precise measurements. His very precise experiments were done with reduced, to say so, weight forces, with spheres on inclined planes and with pendulums. We shall discuss this in Sect. 2.9.

We can conclude that the free fall accelerations of all bodies in a given place are equal, action of the atmosphere apart. The ratio between gravitational and inertial mass is a universal constant, the same for all bodies. The value of the constant is arbitrary, because depends on the choice of the two units. Clearly, the most convenient choice is to have the ratio equal to one. With this choice gravitational and inertial mass are not only proportional, they are equal. The unit of both is the kilogram. From now on we shall indicate with the same symbol, for example m without any subscript, both quantities.

2.6 Examples

In this section we study a number of examples of application of the Newton laws. A good way to proceed is the following.

The first step is to identify all the bodies present in the problem. Next we identify for each of them all the forces acting on it. To do that it is convenient to wrap it, ideally in an envelope, in order to identify all the forces acting on the body from its exterior. To this aim it is often useful to draw each object separately, in its ideal envelope, and the acting forces and write down for each of them its type and its agent (for example: weight due to earth, normal force due to the constraint, friction due to the supporting surface). If the problem contains more than one body, we must identify the action and reaction pairs, and the bodies on which they act. Once all the forces are identified we must calculate the resultants on each of the bodies. To do that we choose a reference frame. The choice should be guided by any symmetry the problem might have. We must then calculate the Cartesian components of the resultant by summing the correspondent components of all the forces. The components divided by the mass of the body are the three components of the acceleration of the body. From the acceleration we find the law of motion with the procedures we studied in Sects. 1.15 and 1.16.

Example E 2.1. Place a block on a horizontal frictionless surface horizontally drawn by a rope.

Frictionless means a physical surface that does not exert forces parallel to it. It is an idealization. Friction always exists, but we can reduce it, for example with the dry ice trick of Sect. 2.4. We attach a rope to the block and draw it horizontally with the force $\mathbf{F}_r$. The situation is shown in Fig. 2.7.

Knowing $\mathbf{F}_r$ and the mass m of the block we want to know its motion, considering it as a point. We draw the body in its ideal envelope. We identify the forces

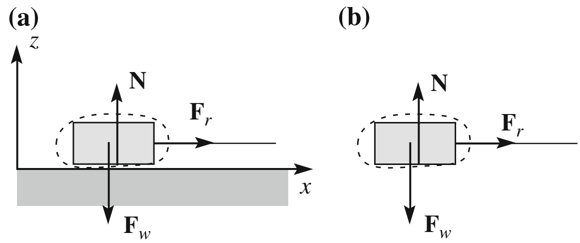


Fig. 2.7 **N** normal constraint force, $\mathbf{F}_r$ force exerted by the rope, $\mathbf{F}_w$ weight, due to earth

acting through the surface: (1) the weight of the block $\mathbf{F}_w$, due to earth, vertically directed downwards, (2) the constraint force exerted by the plane. As we have assumed it to be frictionless the force is normal to the surface, upwards and we call it $\mathbf{N}$, (3) the force (tension) exerted by the rope, $\mathbf{F}_r$. We have drawn all of that in Fig. 2.7b. As we are considering the block as a material point, all the forces are applied in the same point. One of the forces, $\mathbf{N}$, is not given. This is always the case of constraint forces. The body cannot penetrate the support plane because the molecules of the body and the plane repel each other. We know that the body has no vertical acceleration. We infer that the support develops the force that is exactly what is needed to keep it steady. We will find it by solving the equations.

All the forces of the problem lay in the same vertical plane. It is then convenient to choose a reference frame with one axis, say z, vertical upwards and a second one, say x, horizontal to the right in the figure. We do not need the third axis because there are neither forces nor motion in that direction. We now write the second Newton law and its two components

$$\mathbf{F}_r + \mathbf{N} + \mathbf{F}_w = m\mathbf{a}, \quad N - F_w = 0, \quad F_r = ma_x.$$

We conclude that the normal force exerted by the support plane has magnitude equal to the weight. Both forces are vertical and have opposite direction; hence their resultant is zero. The resultant of the forces is the tension of the rope, which causes a uniformly accelerated motion in the x direction.

Example E 2.2 A block moving on a horizontal frictionless surface drawn by a rope at an angle with the horizontal.

The situation is the same as in the previous example, but for the rope now pulling at an angle θ with the horizontal (see Fig. 2.8a). However, we still assume that the motion is on the plane, namely that there is no vertical acceleration. The forces are the same, but $\mathbf{F}_r$ has different components. We have

$$\mathbf{F}_r + \mathbf{N} + \mathbf{F}_w = m\mathbf{a}, \quad N - F_w + F_r \sin\theta = 0, \quad F_r \cos\theta = ma_x.$$

The equation for the z components gives again the normal constraint force, $N = F_w – F_r \sin\theta$. If $\theta > 0$ as in the figure, N is smaller than in the previous example because the rope helps in sustaining the block, the opposite if $\theta < 0$. The second equation gives horizontal acceleration.

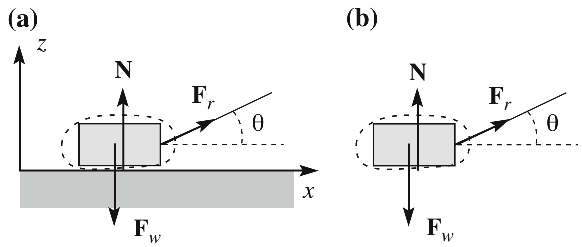


Fig. 2.8 $\mathbf{N}$ normal constraint force, $\mathbf{F}_r$ force exerted by the rope, $\mathbf{F}_w$ weight, due to earth

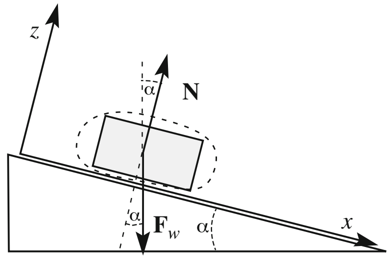


Fig. 2.9 A *block* on a frictionless incline

Notice that a physical limitation of this analysis exists. The normal force cannot be negative, because the support plane cannot attract the body (there is no glue). Hence, if $F_r \sin\theta > F_w$, the assumed conditions cannot be satisfied. Clearly, in this situation the block is lifted up and its acceleration has a vertical component.

Example E 2.3 Block on an inclined frictionless surface.

There are two forces acting on the body (Fig. 2.9), the weight $\mathbf{F}_w$ and the constraint force $\mathbf{N}$ perpendicular to the support plane, which is now inclined. The convenient choice of the axes is to take z perpendicular to the plane and x along the plane, downwards. Clearly, the body will slide accelerating downwards, namely in the x direction we have chosen.

The Newton equation and its components are

$$\mathbf{N} + \mathbf{F}_w = m\mathbf{a}, \quad F_w \sin\alpha = ma_x, \quad N - F_w \cos\alpha = 0.$$

The z component gives us the normal force $N = F_w \cos\alpha$. The x component gives the acceleration ($a = a_x$). Recalling that $F_w = mg$, we have that the notion on an inclined frictionless plane is uniformly accelerated with acceleration

$$a = g\sin\alpha. \tag{2.10}$$

We see that the motion on an incline is completely similar to the motion of free fall, as long as we can neglect the resistive forces. The difference is that the acceleration is smaller on the incline by a factor $\sin\alpha$. We can reduce acceleration by reducing the slope of the plane. If the motion starts from rest from the origin, the law of motion is obtained by integrating twice Eq. (2.10), obtaining

$$x(t) = \frac{1}{2}at^2 = \frac{1}{2}(g\sin\alpha)t^2. \tag{2.11}$$

In words: *the distances travelled are proportional to the squares of the times taken to travel them.*

The incline allows us to slow down the free fall motion and to study its laws over longer times, which can be measured with better precision.

As mentioned in Sect. 2.5 this is one of the great discoveries of Galilei. He did not have a modern chronometer, but invented an ingenious water chronometer, with which he was able to measure the times of the motion, a few seconds long, with a precision better than 0.1 s. He describes his experiments in the book "Dialogues and mathematical demonstrations concerning two new sciences" or "Two new sciences" published in 1638. He writes:

> A piece of wooden molding or scantling, about 12 cubits long, half a cubit wide, and three finger-breadths thick, was taken; on its edge was cut a channel a little more than one finger in breadth; having made this groove very straight, smooth, and polished, and having lined it with parchment, also as smooth and polished as possible, we rolled along it a hard, smooth, and very round bronze ball. Having placed this board in a sloping position, by lifting one end some one or two cubits above the other, we rolled the ball, as I was just saying, along the channel, noting, in a manner presently to be described, the time required to make the descent. We repeated this experiment more than once in order to measure the time with an accuracy such that the deviation between two observations never exceeded one-tenth of a pulse-beat. Having performed this operation and having assured ourselves of its reliability, we now rolled the ball only one-quarter the length of the channel; and having measured the time of its descent, we found it precisely one-half of the former. Next we tried other distances, comparing the time for the whole length with that for the half, or with that for two-thirds, or three-fourths, or indeed for any fraction; in such experiments, repeated a full hundred times, we always found that the spaces traversed were to each other as the squares of the times, and this was true for all inclinations of the plane, i.e., of the channel, along which we rolled the ball. We also observed that the times of descent, for various inclinations of the plane, bore to one another precisely that ratio which, as we shall see later, the Author had predicted and demonstrated for them.
>
> For the measurement of time, we employed a large vessel of water placed in an elevated position; to the bottom of this vessel was soldered a pipe of small diameter giving a thin jet of water, which we collected in a small glass during the time of each descent, whether for the whole length of the channel or for a part of its length; the water thus collected was weighed, after each descent, on a very accurate balance; the differences and ratios of these weights gave us the differences and ratios of the times, and this with such accuracy that although the operation was repeated many, many times, there was no appreciable discrepancy in the results.

Example E 2.4 A block at rest in a lift.

A block of mass m lies in a lift on a horizontal pan of a balance, one of those, for example, that are used to weigh people. What is the apparent weight of the block when the lift accelerates up or down?

As usual we imagine the block in an ideal envelope (Fig. 2.10). Two forces act on it, the weight $\mathbf{F}_w$ vertical down, and the normal constraint of the pan $\mathbf{N}$ upwards. The balance measures the reaction to $\mathbf{N}$, namely the force on it, which is $-\mathbf{N}$. Hence, N is the apparent weight of the block.

If the lift moves with acceleration a upward, the unknown N is given by the Newton law $N - F_w = ma$. Hence, the apparent weight is $N = F_w + ma = m(g + a)$, which is larger than the true weight. If the lift accelerates downwards, the apparent weight is $N = m(g - a)$, smaller than the real one. Notice that if the acceleration downwards is g the apparent weight is null. Indeed, the block is falling with the same acceleration of the lift.

If the lift moves uniformly both upwards and downwards the apparent weight is equal to the real one, as if it were standing. We feel an increase of our weight either

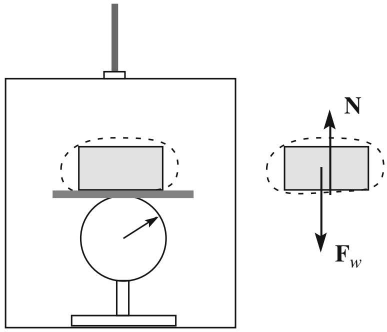


Fig. 2.10 A *block* in an accelerating lift

if the lift accelerates going up or if it decelerates going down. In both cases its acceleration is upwards. Similarly we feel a decrease of our weight when the lift slows down going up or accelerates going down.

Tension of the ropes and wires. In some of the examples we made we have used a stretched rope or wire to apply a force in a point of a body. This force is equal to the tension of the wire. We generally assume the wire to be inextensible, meaning that its length does not vary whichever the tension may be, and perfectly flexible, meaning that the tension is always parallel to the wire, and of negligible mass. Once more, these are idealizations.

Let us clarify the concept of tension. Consider a wire, stretched and steady as in Fig. 2.11a. We mentally isolate a small segment, enlarged in Fig. 2.11b. Two forces act on the segment (neglecting the weight), applied to its extremes and due to the contiguous elements of the wire. These are the tension forces. As the wire is at rest, the two forces are equal and opposite. Consequently, the tension is the same in every section of the wire.

Each of the extremes of the wire is not in contact with another element. As it does not accelerate, a force must act on it from outside equal in magnitude to the tension and directed outwards, as in Fig, 2.11a. The forces on the extremes are equal and opposite and have the magnitude of the tension.

Consider now the case in which the wire moves. As an example, suppose that one extreme is fixed to a block of mass M lying on a horizontal plane of negligible friction. We draw the block applying to the free extreme of the wire a force $\mathbf{F}_1$ obtaining an acceleration $\mathbf{a}$, as shown in Fig, 2.12a. We want to understand under which conditions we really can neglect the mass of the wire. To do that, let us start assuming the mass of the wire to be m.

We are now dealing with two bodies, the block and the wire. We ideally isolate each of them and draw the force diagrams on each of them, in Fig. 2.12b, c.

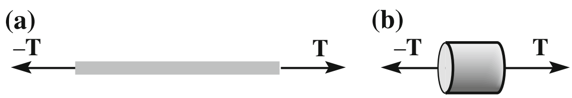


Fig. 2.11 **a** The tension forces on a wire and, **b** on a segment

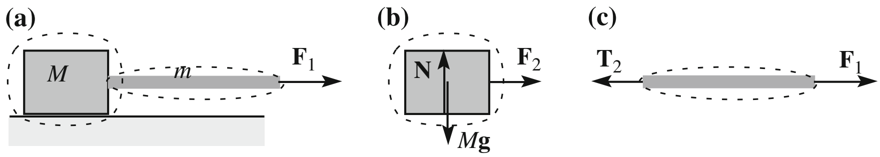


Fig. 2.12 **a** Accelerated motion of a *block* drawn by a rope, **b** **N** normal constraint force, $M\mathbf{g}$ weight due to earth, $\mathbf{F}_2$ force due to the wire, **c** $\mathbf{T}_2$ force on the wire due to the *block*, $\mathbf{F}_1$ force pulling the wire

We next identify the action reaction pairs. There is one such pair, consisting of the forces $\mathbf{F}_2$ applied to the block and $\mathbf{T}_2$ applied to the left extreme of the wire. They are equal and opposite. The force $\mathbf{F}_1$ applied to the right extreme of the wire is its tension and we can call it $\mathbf{T}_1$. The Newton equations for the two bodies are

$$\mathbf{F}_2 = -\mathbf{T}_2 = M\mathbf{a}, \quad \mathbf{T}_1 + \mathbf{T}_2 = m\mathbf{a},$$

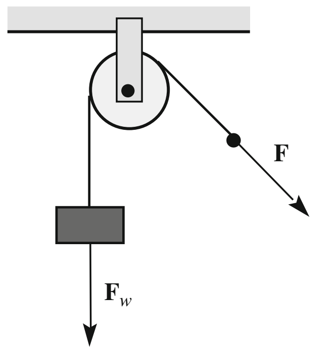
hence, for the magnitudes, $T_1 = (M + m)a$ and $T_2 = Ma$. We see that the tensions at the two extremes are different. Indeed $T_1 > T_2$ because T_1 must accelerate wire and block, T_2 only the block. Let us consider their ratio

$$\frac{T_1}{T_2} = \frac{M + m}{M} = 1 + \frac{m}{M},$$

which becomes unity for $m/M \to 0$. We can then state that the tensions at the extremes can be considered equal if the mass of the wire is negligible compared to the mass of the block. When we speak of massless ropes or wires we mean of negligible mass compared to the masses of the other objects.

Notice that we can arrange a stretched wire, or rope, to have forces at its extremes of equal magnitude but different directions, by using pulleys. We did so already discussing the Varignon experiment (Fig. 2.3). Notice that in these cases, if the motion is accelerated, the magnitudes of the tensions at the extremes can be considered equal only if also the mass of the pulley is negligible and if it can rotate with negligible friction on the pivot (Fig. 2.13).

Fig. 2.13 With a pulley, the direction of the force exerted by a wire can be changed



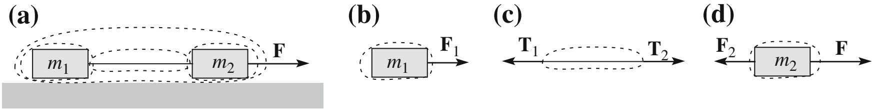


Fig. 2.14 **a** Two *blocks* connected by a wire, **b** force on m_1, **c** forces on the wire, **d** forces on m_2

Example E 2.5 Two blocks linked by a rope of negligible mass.

Figure 2.14a shows two blocks of masses m_1 and m_2 lying on a horizontal frictionless plane, connected by an inextensible wire of negligible mass. To the second block, at the right, a horizontal force **F** is applied. The motion is on the support plane. To know it, we do not need to analyze the vertical forces, which have zero resultants (Fig. 2.14a, b, c, d).

We start by considering the whole system, thinking of it as a unique ideal envelope. The only force acting on this surface is **F**. Hence we have $F = (m_1 + m_2)a$. which gives an acceleration a equal for the two bodies.

We now isolate each of the bodies. The block on the left (Fig. 2.14b) is attached to an extreme of the wire. This exerts on the block the horizontal force $\mathbf{F}_1$. For the action-reaction law the block exerts on the extreme of the wire an equal and opposite force, which is the tension of the wire at that extreme ($\mathbf{F}_1 = -\mathbf{T}_1$). Two other forces act on the block, the external force **F** and the force $\mathbf{F}_2$ due to the right extreme of the wire (Fig. 2.14d). Again, for the action-reaction law the block exerts, on the right extreme of the wire, a force equal and opposite to $\mathbf{F}_2$ that is the tension $\mathbf{T}_2$ at that extreme ($\mathbf{F}_2 = -\mathbf{T}_2$). As we have discussed above, the magnitude of the tension is the same in all points of the wire. Taking into account the directions we have $\mathbf{T}_1 = -\mathbf{T}_2$ (Fig. 2.14c). Calling T the magnitude of the tension we can write the Newton equations as $T = m_1 a,\ F - T = m_2 a$.

The sum of the two equations gives the acceleration of the system $a = F/(m_1 + m_2)$. If we want the value for tension, we substitute a in the first equation obtaining

$$T = \frac{m_1}{m_1 + m_2} F.$$

We see, in particular, that $T < F$, namely the tension is smaller than the force with which we pull.

2.7 Curvilinear Motion

In the previous section we have studied a few examples in which the forces were known, a part of the constraint ones, and the motion that had to be found. In this action we shall consider the inverse problem, namely, the motion of a material point being known, find the resultant of the forces. The singular forces, in case more than one is present, cannot be found, because systems of forces with the same resultant produce the same motion in the case of material points.

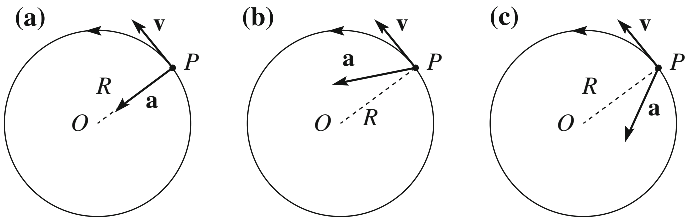


Fig. 2.15 Circular motion, **a** uniform, **b** increasing velocity, **c** decreasing velocity

Circular uniform motion
Consider the motion of a material point P with mass m constrained to move on a circumference of radius R. Suppose the motion to be uniform, namely the magnitude of its speed υ to be constant, as in Fig. 2.15. The motion is however accelerated, because the direction of the velocity varies. As we already found, the acceleration has a constant magnitude (Eq. 1.57) $a = \upsilon^2/R$ and is in every point directed to the center (centripetal acceleration). This acceleration must be given by a force of magnitude

$$F = ma = m\frac{\upsilon^2}{R}. \tag{2.12}$$

The corresponding force has the same direction as the acceleration and is called *centripetal force*. The adjective "centripetal, from the Latin "petere" for "point towards", recalls only its direction but does not specify at all its nature. It may be the tension of a wire, the normal force of a circular guide, the gravitational force of the earth on the moon, etc. We shall discuss a few examples in Sect. 3.4.

Variable speed motion.
If the magnitude of the velocity of a particle moving on a circle varies, its acceleration has two components. One component, a_n, is perpendicular to the trajectory, or, the latter being circular, directed to the center. It is again the variation of the direction of the velocity, namely the just discussed centripetal acceleration of value υ^2/R where υ, we must now specify, is the instantaneous velocity. The second component, a_t, is in the direction of the motion, i.e. tangent to the trajectory and expresses the variation in time of the magnitude of the velocity. We have

$$a_t = \frac{d\upsilon}{dt}, \quad a_n = \frac{\upsilon^2}{R}. \tag{2.13}$$

The acceleration vector, and the force, is directed at an angle with the radius that is forward if the velocity is increasing (Fig. 2.15b), backward if it is decreasing (Fig. 2.15c). The magnitude of the force is $F = ma = m\sqrt{a_n^2 + a_t^2}$.

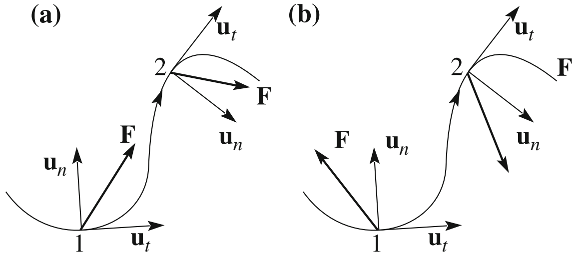


Fig. 2.16 General plane motion. **a** Increasing speed, **b** decreasing speed

As an example, consider a block lying on the platform of a merry go round, which is initially still. When the platform starts moving, gradually increasing its angular velocity, the acceleration of the block has two components, one centripetal and one tangential. The corresponding force, equal to the mass of the ball times this acceleration, is given by the friction on the platform. If the latter is not enough, the block slides towards the periphery of the platform.

As a second example consider the launch of the hammer. The athlete acting on the rope he holds in his hands puts the hammer in rotation with increasing speed. The force on the hammer must be adequate to keep it on a circular orbit (component mv^2/R towards the center) and makes its speed increase (a component in the direction of the motion). The rope must then be directed forward, as in Fig. 2.15b

General plane motion.
We consider now a material point of mass m moving on a plane trajectory of arbitrary shape with velocity not necessarily constant in magnitude. We have already studied the kinematics of the problem in Sect. 1.14. Even in this case, the acceleration has two components, a tangential and a normal one, as in Eq. (1.62). They are given by Eq. (2.13).

The only difference from the circular case is that now R is the local curvature radius, which is not fixed but varies along the trajectory. The second Newton law tells us that the resultant of the forces acting on the point must be its acceleration times its mass.

If we know only the trajectory, but nothing of the velocity, we can still say that in every point of the trajectory in which the curvature is not zero, the resultant of the forces must be directed on the side of the curvature center, pointing forward from it (Fig. 2.16a) or backwards (Fig. 2.16b) depending on whether the motion is accelerated or delayed respectively.

2.8 Angular Momentum and Moment of a Force

Consider a material point P moving in an inertial frame as shown in Fig. 2.17. Let $\mathbf{p} = m\mathbf{v}$ be its momentum and $\mathbf{r}$ its position vector. Consider a generic point Ω, which may be at rest or moving relative to the frame. We shall now introduce the concepts of *angular momentum* and *moment of a force* about the *pole* Ω.

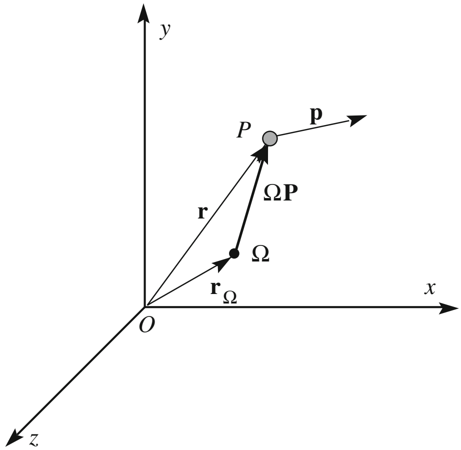


Fig. 2.17 The vectors relevant for angular momentum

We have already defined the moment of a bound vector in Sect. 1.8. The angular momentum is the moment of the linear momentum, considering it, for this purpose, as applied to the material point, as shown in Fig. 2.17.

Hence, the angular momentum of the point P about the pole Ω is the vector product of the vector from Ω to P and its quantity of motion (or momentum).

$$\mathbf{l}_\Omega = \mathbf{\Omega P} \times \mathbf{p}. \tag{2.14}$$

Consider the force $\mathbf{F}$ applied to P. The moment of the force about the pole Ω is the vector product of the vector from Ω to P and $\mathbf{F}$

$$\tau_\Omega = \mathbf{\Omega P} \times \mathbf{F}. \tag{2.15}$$

Remember that the order of the factors matters in cross products. Notice also that the moments change if the reference frame changes.

Let us now see how the angular momentum changes in time. For that, we take the time derivative of Eq. (2.14) using the rule of the derivative of products, paying attention to the order of the factors

$$\frac{d\mathbf{l}_\Omega}{dt} = \frac{d\mathbf{\Omega P}}{dt} \times \mathbf{p} + \mathbf{\Omega P} \times \frac{d\mathbf{p}}{dt}. \tag{2.16}$$

To find the derivative of the vector $\mathbf{\Omega P}$ we notice that it is the difference of two vectors, both varying with time, $\mathbf{\Omega P} = \mathbf{r} - \mathbf{r}_\Omega$. Deriving we have.

$$\frac{d\mathbf{\Omega P}}{dt} = \mathbf{v} - \mathbf{v}_\Omega.$$

The meaning of this expression is clear: the derivative of a vector joining two moving points is the relative velocity of those points. We substitute this expression in Eq. (2.16) and also notice that the derivative of the momentum is equal to the resultant $\mathbf{F}$ of the forces acting on P, because the frame is inertial. We get

$$\frac{d\mathbf{l}_\Omega}{dt} = \mathbf{v} \times \mathbf{p} - \mathbf{v}_\Omega \times \mathbf{p} + \Omega\mathbf{P} \times \mathbf{F}.$$

The first term in the second member is zero, being the cross product of two parallel vectors; the last term is the moment of the resultant about the pole $\boldsymbol{\tau}_\Omega$. In conclusion

$$\frac{d\mathbf{l}_\Omega}{dt} = \boldsymbol{\tau}_\Omega - \mathbf{v}_\Omega \times \mathbf{p}. \tag{2.17}$$

This is a very important equation that we shall use often in the following. It becomes particularly simple if we choose a stationary pole in the reference frame. The equation becomes

$$\boldsymbol{\tau}_\Omega = \frac{d\mathbf{l}_\Omega}{dt}. \tag{2.18}$$

In words the equation is called the angular momentum theorem for a material point: *the time derivative of the angular momentum of a material point about a pole fixed in an inertial reference frame is equal to the moment of the resultant of the forces acting on it about the same pole.*

Notice that if the body is extended, as we shall discuss in the following chapter, the different forces acting on it, say $\mathbf{f}_1$, $\mathbf{f}_2$,…, may be applied in different points and the moment of their resultant $\mathbf{F} = \mathbf{f}_1 + \mathbf{f}_2 + \cdots$, $\boldsymbol{\tau}_\Omega = \Omega\mathbf{P} \times \mathbf{F}$ is in general different from the vector sum of their moments. In the case under study however, all the forces are applied in P and

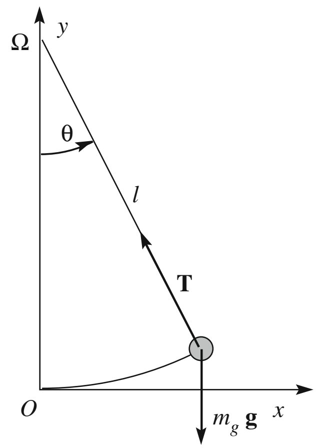
$$\begin{aligned}\boldsymbol{\tau}_\Omega &= \Omega\mathbf{P} \times \mathbf{F} = \Omega\mathbf{P} \times (\mathbf{f}_1 + \mathbf{f}_2 + \cdots) = \Omega\mathbf{P} \times \mathbf{f}_1 + \Omega\mathbf{P} \times \mathbf{f}_2 + \cdots \\ &= \boldsymbol{\tau}_{\Omega_1} + \boldsymbol{\tau}_{\Omega_2} + \cdots.\end{aligned}$$

The resultant of the moments is equal to the moment of the resultant of the forces. We stress that this is true only if all the forces are applied at the same point.

2.9 The Simple Pendulum

The pendulum is a material point constrained to move on an arc of a circumference. It can be simply made by fixing a thin wire to a small sphere on an extreme and to a fixed point on the other, which we call Ω. The length l of the wire, or better the distance between the fixed point and the center of the sphere, is called the *length of the pendulum*. If we take the pendulum away from its equilibrium position O and abandon it with zero velocity, the body moves towards O under the action of two

Fig. 2.18 The simple pendulum



forces, the weight, directed vertically down, and the tension of the wire (**T**), directed as the wire. The acceleration has the direction of the resultant of these two forces. Consequently it is always in the plane defined by the wire and the vertical. If the initial velocity is zero, the motion is on the plane. As the distance from Ω is kept fixed by the wire, which we assume inextensible, the trajectory is an arc of a circle of radius l.

As shown in Fig. 2.18, we take a reference system with the origin O in the rest position of the pendulum, the y-axis vertical upwards, the x-axis horizontally in the plane of motion and z such as to complete the triplet. The z-axis is normal to the figure towards the observer. We call θ the angle between the wire and the vertical, taking it positive if seen anticlockwise by the observer.

Historically, as we have already mentioned, the study of the motion of pendulums, with their periodic motion, made a fundamental contribution to the development of mechanics. Galilei discovered two important properties. The first one is the isochronism of small oscillations; if the amplitude is not too large (we shall be more precise in the following), the oscillation period is independent of the amplitude. This property allowed building of precise clocks. The second property is even more important; the oscillation periods of pendulums of the same lengths and different masses are identical. This proves, as we shall now see, that gravitational mass and inertial mass are equal. The property was later called *equivalence principle* and is at the basis of general relativity.

In our demonstration, we start by assuming that the two masses might be different. We call m_i the inertial mass of the pendulum, namely the proportionality constant between acceleration and force, and m_g its gravitational mass, the constant that appears in the weight, which is then $m_g\mathbf{g}$.

The tension is a constraint force, due to the wire, which we assume to be perfectly flexible and inextensible. The constraint develops a force, in general unknown a priori, automatically adjusted to make the motion happen, in our case, at a fixed distance from Ω. We do not know the intensity of the wire tension $\mathbf{T}$, but we know its direction, which is along the wire.

In our study of the motion we shall use the angular momentum theorem. We choose the pole in the suspension point Ω, for reasons that will become clear soon. We use Eq. (2.18) with

$$\boldsymbol{\tau}_\Omega = \boldsymbol{\Omega}\mathbf{P} \times \left(\mathbf{T} + m_g\mathbf{g}\right) = \boldsymbol{\Omega}\mathbf{P} \times \mathbf{T} + \boldsymbol{\Omega}\mathbf{P} \times m_g\mathbf{g}.$$

We now see the reason for our choice of pole. The first term is always zero, being the vector product of two parallel vectors. Consequently we do not need to know the intensity of the tension. We have

$$\boldsymbol{\tau}_\Omega = \boldsymbol{\Omega}\mathbf{P} \times m_g\mathbf{g}. \tag{2.19}$$

The angular momentum about the same pole is

$$\mathbf{l}_\Omega = \boldsymbol{\Omega}\mathbf{P} \times m_i\mathbf{v}, \tag{2.20}$$

where the mass is the inertial one. Equation (2.18) gives

$$\boldsymbol{\Omega}\mathbf{P} \times m_g\mathbf{g} = \frac{d(\boldsymbol{\Omega}\mathbf{P} \times m_i\mathbf{g})}{dt}. \tag{2.21}$$

All the vectors in these equations, in any position of the pendulum, belong to the plane xy. Both vector products are consequently in z direction. The equation has only the z component. The z component of $\boldsymbol{\Omega}\mathbf{P} \times m_g\mathbf{g}$ is $-lm_g g \sin\theta$. The velocity is always perpendicular to $\boldsymbol{\Omega}\mathbf{P}$. As a consequence the z component of $\boldsymbol{\Omega}\mathbf{P} \times m_i\mathbf{v}$ is simply $lm_i\upsilon$, where $\upsilon = l\frac{d\theta}{dt}$. So, we have

$$-lm_g g \sin\theta = lm_i \frac{d\upsilon}{dt}.$$

And finally we can write

$$\frac{d^2\theta}{dt^2} + \frac{m_g g}{m_i l}\sin\theta = 0. \tag{2.22}$$

This is a differential equation, whose unknown is a function of time $\theta(t)$. Once it is solved, we know the motion of the pendulum, because if we know θ, we know its position. Equation (2.22) cannot be solved analytically. However, if the oscillations are "small", we can approximate the sine with its argument and the equation becomes

$$\frac{d^2\theta}{dt^2} + \frac{m_g g}{m_i l}\theta = 0. \tag{2.23}$$

This is a well-known differential linear equation with constant coefficients, which we shall meet several times. We leave its study to calculus courses and directly give the general solution, which is

$$\theta(t) = \theta_0 \cos(\omega_0 t + \phi), \tag{2.24}$$

where

$$\omega_0 = \sqrt{\frac{m_g g}{m_i l}} \tag{2.25}$$

is called *proper angular frequency*. As one sees, it depends only on the characteristics of the pendulum, including its weight.

The reader can easily verify, with two derivatives, that this expression indeed satisfies Eq. (2.23), for whatever values of the constants θ_0 and ϕ. These constants do not depend on the characteristics of the pendulum but on how the motion has started. They should be found in each case on the basis of two initial conditions. We can use the position and velocity at the starting time that we shall take as $t = 0$. We immediately see that

$$\theta(0) = \theta_0 \cos\phi, \quad \left(\frac{d\theta}{dt}\right)_{t=0} = -\theta_0\omega_0 \sin\phi.$$

The initial velocity being zero, the second equation gives $\phi = 0$ ($\theta_0 = 0$ is also a solution of Eq. (2.24) but identically null). The first condition says that θ_0 is just the initial angle, the angle at which we have let the pendulum go. In conclusion the motion of the pendulum is described by the equation

$$\theta(t) = \theta_0 \cos(\omega_0 t). \tag{2.26}$$

The motion is periodic, meaning that, for any instant of time t we can consider, both the position and the velocity become the same after a certain time interval T, called the *period*, namely at the instant $t + T$. From Eq. (2.26) we immediately see that the period is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m_i l}{m_g g}}, \tag{2.27}$$

where we used Eq. (2.25).

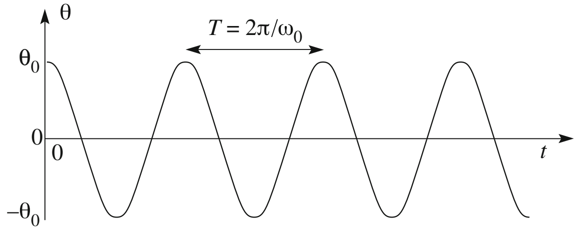


Fig. 2.19 Angular harmonic motion

The motion is represented in Fig. 2.19. This is the most common periodic motion in Nature. It is called harmonic motion. In the next chapter we shall study it in depth.

We now make an important observation on the expressions of proper angular frequency and period. Angular frequency and period depend on the length of the pendulum, but not on the oscillation amplitude: two pendulums of the same length (and in the same place, hence at the same g) are isochronous. On the other hand, angular frequency and period depend on the *ratio* between gravitational and inertial masses. If this ratio is the same for all bodies, independently of the substance they are made of and of their position, that ratio is a constant and angular frequency will be independent of the mass of the pendulum. If we want to experimentally test if gravitational and inertial masses are proportional or not, we can test whether pendulums of the same length, and different masses or made of different substances, do oscillate or not with the same period.

Galilei noticed that this method is much more accurate than others he knew. In principle, one could think to drop two spheres, e.g. one of wood and one of lead, from the top of a tower and check if they the reach the ground simultaneously. However, Galilei never mentions having done such an experiment, from the leaning tower of Pisa. This is a legend without any historical support. Indeed, Galilei observed, and wrote, that that method is not accurate enough, because is too fast and at high speeds the air resistance noticeably perturbs observations. Galilei used inclined slopes, as we have discussed, to slow down the motion, reduce the air drag, and increase the relative measurement accuracy due to the longer times to be measured. The use of pendulums allows even better accuracy. He used pairs of pendulums made of different materials and of exactly the same lengths, took them out of equilibrium and let them go at the same time. He found that they keep oscillating in phase for hundreds of periods. The air drag did act more effectively on the lighter pendulum gradually reducing its amplitude more than that of the heavier one. However this did not matter because the period is independent of amplitude.

G. Galilei describes accurately his progress toward a more precise experiment, gradually eliminating the spurious effects and the sources of errors in the "Dialogs concerning two new sciences" (1638) (translation from Italian by Henry Crew and Alfonso de Salvio). He established the proportionality of inertial and gravitational mass with an uncertainty of $2–3 \times 10^{-3}$.

> The experiment made to ascertain whether two bodies, differing greatly in weight will fall from a given height with the same speed, offers some difficulty; because, if the height is considerable, the retarding effect of the medium, … will be greater in the case of the small momentum of the very light body than in the case of the great force of the heavy body; so that, in a long distance, the light body will be left behind; if the height be small, one may well doubt whether there is any difference; and if there be a difference it will be inappreciable. It occurred to me therefore to repeat many times the fall through a small height in such a way that I might accumulate all those small intervals of time that elapse between the arrival of the heavy and light bodies respectively at their common terminus, so that this sum makes an interval of time which is not only observable, but easily observable. In order to employ the slowest speeds possible and thus reduce the change which the resisting medium produces upon the simple effect of gravity it occurred to me to allow the bodies to fall along a plane slightly inclined to the horizontal. For in such a plane, just as well as in a vertical plane, one may discover how bodies of different weight behave: and besides this, I also wished to rid myself of the resistance which might arise from contact of the moving body with the aforesaid inclined plane. Accordingly I took two balls, one of lead and one of cork, the former more than a hundred times heavier than the latter, and suspended them by means of two equal fine threads, each four or five cubits long. Pulling each ball aside from the perpendicular, I let them go at the same instant, and they, falling along the circumferences of circles having these equal strings for semi-diameters, passed beyond the perpendicular and returned along the same path. This free vibration repeated a hundred times showed clearly that the heavy body maintains so nearly the period of the light body that neither in a hundred swings nor even in a thousand will the former anticipate the latter by as much as a single moment, so perfectly do they keep step. We can also observe the effect of the medium which, by the resistance which it offers to motion, diminishes the vibration of the cork more than that of the lead, but without altering the frequency of either.

In conclusion, Galilei experimentally demonstrated the equality of inertial and gravitational masses with an accuracy of about one per mille, namely that

$$\frac{m_i}{m_g} - 1 < 10^{-3}. \tag{2.28}$$

Newton repeated this later on the Galilei experiments. He writes in the "Principia":

> It has been, now of a long time, observed by others, that all sorts of heavy bodies (allowance being made for the inequality of retardation which they suffer from a small power of resistance in the air) descend to the earth from equal heights in equal times; and that equality of times we may distinguish to a great accuracy, by the help of pendulums. I tried the thing in gold, silver, lead, glass, sand, common salt, wood, water, and wheat. I provided two wooden boxes, round and equal: I filled the one with wood, and suspended an equal weight of gold (as exactly as I could) in the center of oscillation of the other.

He concluded that:

> By these experiments, in bodies of the same weight, I could manifestly have discovered a difference of matter (i.e. *inertial mass*) less than the thousandth part of the whole, had any such been.

Hence Newton confirmed what Galilei had discovered with a similar precision of 1×10^{-3}. After having found an expression of the gravitational force, Newton did also a check of the equivalence principle, on a solar system scale. He did that, in particular, on the system of Jupiter and its satellites. We shall see his argument in Sect. 4.4. Here we just say that the precision was, once more, of one per mille.

Having established the proportionality of the two types of mass, we can make them equal by choosing their units. With this choice Eqs. (2.25) and (2.27) become

$$\omega_0 = \sqrt{\frac{g}{l}}, \quad T = 2\pi\sqrt{\frac{l}{g}}. \tag{2.29}$$

To have a feeling of the orders of magnitude, we can easily calculate that a 1-m long pendulum has a period of about 2 s.

We now recall having approximated the sine of the angle with the angle (in radiants) itself. Let us verify when the approximation is good. For example, if $\theta = 30°$, or 0.52 rad, its sine is sin 30° = 0.50. The relative error is (0.52–0.50)/0.50 = 4 %, which is quite small. Even for $\theta = 60°$, or 1.05 rad, the error is not enormous, but already noticeable. Indeed, sin 60° = 0.87 and the corresponding error is 20 %. These are the relative errors making the sine equal to the angle, but the corresponding ones on the period are even smaller, as we now shall see.

The exact Eq. (2.22), as we said, cannot be solved analytically. However, it can be solved by successive approximations. In fact, the approximation we made is a series expansion stopped at the first term ($\sin\theta = \theta$); the next approximation we stop at the second term ($\sin\theta = \theta - \theta^3/6$). The resulting expression for the period with amplitude θ_0, calling T_o the period given by Eq. (2.28), is

$$T(\theta_0) = T_0\left[1 + \frac{1}{4}\sin^2\frac{\theta_0}{2}\right],$$

which, as it is seen, depends on the amplitude θ_0. The relative error made using the usual expression of the period is $\frac{1}{4}\sin^2\frac{\theta_0}{2}$. Going back to the above examples, we find that the relative error for $\theta = 30°$ is 1.6 %, the one for $\theta = 60°$ is 6.3 %. They are not large.

We make a last observation. If the oscillations are small, the pendulum moves substantially on the horizontal, namely on the x-axis in Fig. 2.18. Now $x = l\tan\theta$, that we can approximate with $x = l\theta$. We can then conclude that the motion, as represented by the x coordinate, has the equation

$$x(t) = x_0\cos(\omega t). \tag{2.30}$$

As expected it is a harmonic motion, of amplitude x_0.

2.10 The Work of a Force. The Kinetic Energy Theorem

In this section we introduce the concepts of work, done by a force, and kinetic energy, of a body. The meaning of "work" in physics is rather different from its meanings in everyday language and consequently from what intuition might suggest. For example, holding in one hand a heavy object even if we do not move it we still need to apply a force with our muscles and make some effort. However, we do not perform any work, in the language of physics. In physics, a force makes work only if its application point moves. In the example, the work done by the force we exert on the body is positive if we raise, negative if we lower it, but zero if we do not move it.

Consider the material point P moving in a reference frame with position vector $\mathbf{r}$, along a certain trajectory, the curve Γ. As shown in Fig. 2.20, consider the position vector in the instants t, $\mathbf{r}(t)$, and immediately after $t + dt$, $\mathbf{r}(t + dt)$. The displacement of P in the interval dt is the infinitesimal vector

$$d\mathbf{s} = \mathbf{r}(t + dt) - \mathbf{r}(t). \tag{2.31}$$

If $\mathbf{F}$ is a force acting on the point, its work for the infinitesimal displacement (2.31) is defined as

$$dW \equiv \mathbf{F} \cdot d\mathbf{s}. \tag{2.32}$$

The finite work having been done by the force, a finite displacement of the point, say from A to B along the trajectory Γ, is the line integral along the curve Γ from A to B

$$W_{AB;\Gamma} = \int_{A;\Gamma}^{B} dW = \int_{A;\Gamma}^{B} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{s}, \tag{2.33}$$

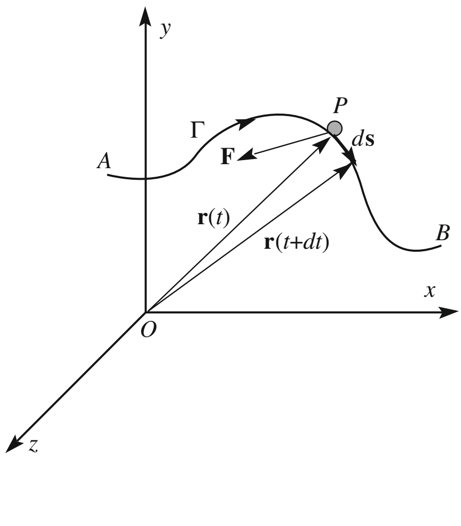


Fig. 2.20 The elements to define the work of force **F**

where $\mathbf{F}(\mathbf{r})$ is the force in the point of position $\mathbf{r}$. The line integral is the sum of all the elementary dot products $\mathbf{F}(\mathbf{r}) \cdot d\mathbf{s}$ on all the elements of the curve. Clearly, the integral does not depend only on the initial and final points A and B, but also on the specific path taken to go from the former to the latter. Indeed, if the path changes, also the force in the new points may change. To make this explicit in the notation we have included both A and B and Γ in the subscripts of W. The case in which the integral depends on the origin and the end but not on the path is however important and will be studied in Sect. 2.13.

Notice that more forces, call them $\mathbf{F}_i$, may act contemporarily on the point P, for example weight, friction, air resistance, etc. In this case, the total work made by all the forces is equal to the sum of the works each force would do if acting separately

$$W_{AB;\Gamma} = \sum_i \int_{A;\Gamma}^{B} \mathbf{F}_i(\mathbf{r}) \cdot d\mathbf{s} \tag{2.34}$$

Clearly, the elementary (meaning “infinitesimal”) displacement of the application points of the forces $d\mathbf{s}$ is the same for all them. Considering that the sum of integrals is equal to the integral of the sum, which is in our case the resultant of the forces $\mathbf{R} = \sum_i \mathbf{F}_i$, we have

$$W_{AB;\Gamma} = \int_{A;\Gamma}^{B} \sum_i \mathbf{F}_i(\mathbf{r}) \cdot d\mathbf{s} = \int_{A;\Gamma}^{B} \mathbf{R} \cdot d\mathbf{s}. \tag{2.35}$$

Namely, the total work made by the acting forces is equal to the work made by their resultant. Notice, again, that this is true <u>only</u> if all forces are applied in the same point.

The physical dimension of the work is those of a force times a displacement. Its unit is the jule, with symbol J, which is the work done by the unit force, 1 N, when its application point moves one unit of length, 1 m, in the direction of the force. To appreciate the order of magnitude, a jule is roughly the work you do when you raise a glass of water by 1 m.

We now prove the work-kinetic energy theorem. Being a consequence of the second Newton law it is valid in inertial frames. Consider a material point and the resultant $\mathbf{R}$ of the forces acting on it. The Newton law says

$$\mathbf{R} = m\frac{d\mathbf{v}}{dt}.$$

We take the scalar product with the elementary displacement $d\mathbf{s} = \mathbf{v}\, dt$ of the two members

$$\mathbf{R} \cdot d\mathbf{s} = m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}dt = m\mathbf{v} \cdot d\mathbf{v}.$$

Now consider the dot product $\mathbf{v} \cdot d\mathbf{v}$. We recall that the square of a vector is the dot product of the vector by itself, in this case $\upsilon^2 = \mathbf{v} \cdot \mathbf{v}$. Differentiating this expression we have

$$d\left(\upsilon^2\right) = d(\mathbf{v} \cdot \mathbf{v}) = d\mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot d\mathbf{v} = 2\mathbf{v} \cdot d\mathbf{v},$$

hence $\mathbf{R} \cdot d\mathbf{s} = \frac{1}{2}m(d\upsilon^2)$.

The work done by $\mathbf{R}$ when the point moves from A to B on the given trajectory is then

$$W_{AB;\Gamma} = \frac{1}{2}m\int_A^B d\left(\upsilon^2\right) = \frac{1}{2}m\upsilon_B^2 - \frac{1}{2}m\upsilon_A^2. \tag{2.36}$$

We then define the *kinetic energy* of the material point of mass m and velocity υ as

$$U_K = \frac{1}{2}m\upsilon^2, \tag{2.37}$$

which is independent of the position. The kinetic energy has the same physical dimension as the work and is measured in jule. We finally can write Eq. (2.36) as

$$W_{AB;\Gamma} = U_K(B) - U_K(A), \tag{2.38}$$

which is the work-kinetic energy theorem. In words: *when a material point moves on a certain trajectory from* A *to* B*, the work done by the forces acting on it is equal to the difference between the kinetic energy of the point has in* B *and that it had in* A.

It is sometimes useful to express kinetic energy in terms of momentum rather than velocity, namely

$$U_K = \frac{p^2}{2m}. \tag{2.39}$$

2.11 Calculating Work

In this section we shall see two examples of calculation of works, made respectively by weight and friction, when the application point P moves on its trajectory from the initial position A to the final one B. We shall see that in the former case the work

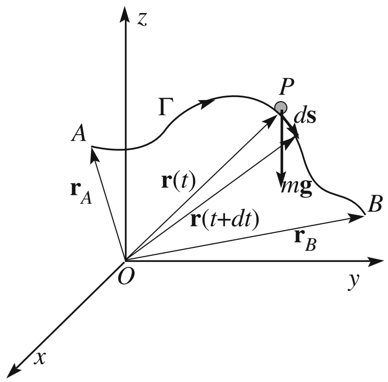


Fig. 2.21 Trajectory of the material point and its weight

depends only on the initial and final position, and not on the path taken between them, in the latter it depends on the path too.

Starting with weight, Fig. 2.21 shows the reference frame (not necessarily inertial) where we have chosen the z-axis to be vertical. Point P moves on the trajectory from the position A, with the position vector $\mathbf{r}_A = (x_A, y_A, z_A)$ to the position B, with the position vector $\mathbf{r}_B = (x_B, y_B, z_B)$. The figure shows also the position vector at the generic instant t and in the immediately following instant $t + dt$. The force acting on the point is its weight $m\mathbf{g}$, which is equal in all points. The elementary work done by the weight, which is vertically directed downwards is $dW = m\mathbf{g} \cdot d\mathbf{s} = -mgdz$. The total work is given by the integral

$$W_{AB;\Gamma} = \int_{A;\Gamma}^{B} m\mathbf{g} \cdot d\mathbf{s} = -\int_{A;\Gamma}^{B} mgdz = mgz(A) - mgz(B). \qquad (2.40)$$

We see that in this relevant case the work is independent of the path, depending only on the final and initial position, even better, on their heights only. This conclusion was experimentally proven by Galilei with a simple experiment that we shall describe in the next section.

This is not the case of the second example, the friction force, which we shall study in Sect. 3.5.

Suppose we have an object, say a book or a brick, lying on a table. In real cases, the constraint does not apply to the body only the normal force, but also a friction that is tangent to the contact surface. If we want to move the body on the trajectory Γ in Fig. 2.22 at a constant speed, as we know from every day experience, we need to pull it, apply a force, parallel to the plane in the direction of the displacement. This means that the plane exerts on the body a force equal and opposite to our pull, because the velocity is constant in magnitude and then the resultant of the forces in the direction of the motion must be zero. Indeed, as we shall see in Sect. 3.5, the friction force, $\mathbf{F}_a$, is always parallel and opposite to the elementary displacement $d\mathbf{s}$. We now calculate the work of $\mathbf{F}_a$.

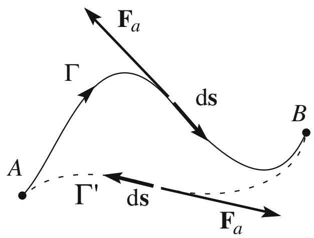


Fig. 2.22 Calculating the work of the friction force

The elementary work is $dW = \mathbf{F}_a \cdot d\mathbf{s} = -F_a ds$ which is always negative. The total work is given by the line integral on the trajectory

$$W_{AB;\Gamma} = \int_{A;\Gamma}^{B} \mathbf{F}_a \cdot d\mathbf{s} = -\int_{A;\Gamma}^{B} F_a ds = -F_a s_{AB}(\Gamma), \tag{2.41}$$

where $s_{AB}(\Gamma)$ is the length of the trajectory Γ between A and B. The work is proportional to the length of the path, a quantity obviously depending on the path.

We conclude with an observation that we shall generalize in Sect. 2.13. We have seen that the work of the weight force for displacement A to B is $W_{AB} = -mg(z_B - z_A)$. Suppose now that the point goes back to A. The work of weight is $W_{BA} = -mg(z_A - z_B) = -W_{AB}$. Namely the total work of the weight on a closed path is zero. On the other hand, the work of the friction force to go from A to B on the curve Γ is $W_{AB,\Gamma} = -F_a s_{AB}(\Gamma)$. If we now go back on another curve, say Γ' in the Fig. 2.22, the work of the friction is $W_{BA,\Gamma} = -F_a s_{BA}(\Gamma)$, which is again negative. Consequently the work of the friction on a closed path is not zero, it is negative.

2.12 An Experiment of Galilei on Energy Conservation

One of the discoveries of G. Galilei was the fact, as we have mentioned, that the velocity of body descending under the action of is weight only, starting from rest, depends on the difference between the initial and final levels, and not on the followed path.

In the "Dialogue on Two new sciences" he states that the velocities of bodies descending on inclines of different slopes and the same height are equal. In his words (translations by the author):

> All contrasts and impediments removed… a heavy and perfectly round ball, descending through the lines CA, CD, CB would reach the final points A, D, C with the same moments

with reference to Fig. 2.23a reproduced from the book. Notice that, at the time, Galilei was searching for and developing the laws of mechanics and that several

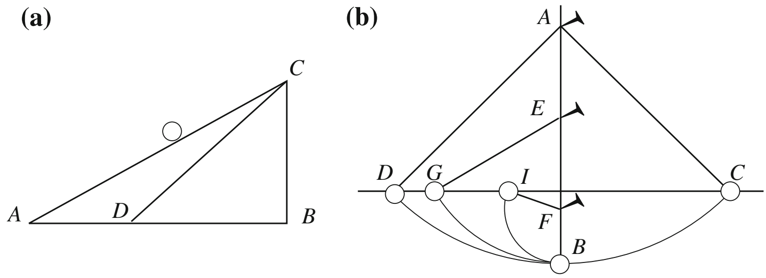


Fig. 2.23 **a** Ball falling on inclines of different slopes; **b** the pendulum and nail experiment

concepts had not yet been completely defined. In particular, impetus, momentum, kinetic energy were not well-separated concepts.

However, accurate measurements of those velocities were impossible to do. To prove the statement, he invented a simple and genial experiment, using a pendulum and a nail. Figure 2.23b is also reproduced from his book.

Salviati, the person who in the Dialogues represents Galilei, starts with the description:

> Suppose this sheet to be a vertical wall and to have a lead ball of one or two ounces hanging from a nail fixed in the wall, suspended to a thin wire AB, two or three arms long, perpendicular to the horizon… and about two finger far from the wall.

Then draw the vertical line AB and, perpendicular to it DC. Move the wire with the ball in AC and let it go. We shall see the ball

> descending first through the arc BCD, and going beyond point B as much as, sliding on the arc BD, almost reaching the drawn horizontal CD, failing to reach it by a very small gap, which has been taken away by the impediments of the air and the wire; from which we can likely conclude that the momentum (impetus) gained by the ball in B, in the descent on the arc CB, was so much to pull it back through the similar arc BD to the same height.

He continues with the request to repeat the experiments several times to check the result. Then

> I want we fix in the wall, grazing the vertical AB, a nail, like in E or in F, which should protrude out five or six fingers.

As before, the wire with the ball is moved to *AC* and let go. The ball will again move on the arc *CB*. But, when it is in *B*, the wire hits the nail, forcing the ball to move on the arc *BG*, having center in *E*.

> Now, my Lords, you will see with enjoyment the ball reaching the horizontal line in the point *G*, and the same to happen if the obstacle would be lower, as in *F*, where the ball would go through the arc *BI*, always finishing its ascent on the line *CD*.

Salviati concludes that the momentum acquired by a body descending from a certain height is just what is needed to bring it back to the same height, through whatever path. He observes that the momentum acquired in the descent on a given arc is equal to the momentum needed to rise through the same arc. He concludes that the momentum, and we can say also the velocity and kinetic energy in *B,* is the same whether it descends through *CB*, or *GB* or *IB* or any arc beginning on the horizontal *DC* and having its lowest point in *B*. On the other hand, the fall along an arc can be thought of as the fall on an "incline" of varying slope, proving the assumption.

The importance of the result of this experiment became clear in the following evolution of mechanics. In his experiment the kinetic energy of the ball in *B* is the same whatever the path starting from stillness from the same level. We now know that this energy is equal to the work done by the weight force. We conclude that the work done by the weight depends only on the difference of level and not on the particular path followed. We have already discussed this property in the previous section. Indeed, it is a fundamental one; it shows that there is a quantity, the energy, which is conserved, does not change in the motion under the action of weight. Weight is a conservative force, as we shall now see.

2.13 Conservative Forces

In general the work of a force on a point depends on the trajectory of the point. However, we have seen a case, the case of the weight force, in which the work depends only on the origin *A* and end *B* and not on the trajectory between them. Forces having these properties are said to be *conservative*. In the opposite case, as for the friction, they are said to be *non-conservative* or *dissipative*.

Let $\mathbf{r}$ be the position vector in the chosen reference frame and $\mathbf{F}(\mathbf{r})$ be a conservative force, a function of the position. The definition of conservative force states that, for whatever curve Γ with origin in *A* and end in *B*,

$$W_{AB;\Gamma} = \int_{A;\Gamma}^{B} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{r}_A, \mathbf{r}_B), \tag{2.42}$$

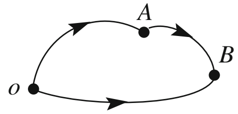
where f is a function of the co-ordinates of *A* and of *B*. It is easy to show that in this case it is always possible to find a function of the co-ordinates, which we shall indicate with $U_p(\mathbf{r})$, such as

$$W_{AB} = U_p(\mathbf{r}_A) - U_p(\mathbf{r}_B). \tag{2.43}$$

To show that, consider an arbitrarily chosen point o, as in Fig. 2.24. The work from A to o on whatever path is

$$W_{oA} = f(\mathbf{r}_o, \mathbf{r}_A) \tag{2.44}$$

Fig. 2.24 Different paths



and similarly the work from o to B is

$$W_{oB} = f(\mathbf{r}_o, \mathbf{r}_B). \tag{2.45}$$

But, we can go from o to B also going from o to A and then from A to B. Considering that work is an additive quantity we can write $W_{oB} = W_{oA} + W_{AB}$. Hence

$$W_{oA} + W_{AB} = f(\mathbf{r}_o, \mathbf{r}_B). \tag{2.46}$$

By subtracting Eq. (2.43) from this expression we have

$$W_{AB} = f(\mathbf{r}_o, \mathbf{r}_B) - f(\mathbf{r}_o, \mathbf{r}_A). \tag{2.47}$$

We then reach the result by putting $U_p(\mathbf{r}) = f(\mathbf{r}_o, \mathbf{r})$. The function $U_p(\mathbf{r})$ is the *potential energy* of the force $\mathbf{F}(\mathbf{r})$ and is a function of the co-ordinates only. In conclusion the potential energy, or better its difference, is defined by the relation

$$U_p(\mathbf{r}_B) - U_p(\mathbf{r}_A) = -\int_A^B \mathbf{F} \cdot d\mathbf{s}. \tag{2.48}$$

In words: *the difference of potential energy of the force* **F** *in the point* B *and in the point* A *is equal to the opposite of the work done by the force when its application point moves from* A *to* B, *following any trajectory.*

The reason of the—sign, or the word "opposite", is the following. To be concrete, consider the weight. If we move a body of mass m from the level z_A to the higher level z_B, the displacement is opposite to the force and the work $-mg(z_B - z_A)$ is negative. The potential energy of the body is then larger when its level is higher. The work done by the weight force is equal and opposite to the gain of potential energy of the body. This energy can be given back as work by the body, taking it down to the original level. The higher the body, the greater is its potential to produce work.

We can conclude, and this is true in complete generality, by stating that *the potential energy difference between two states of a body is equal to the work we need to do against the force acting on the body to change it from the first to the second state.*

Notice again that a potential energy can be defined for a force only if its work is independent of the path. No potential energy exists, for example, for the friction forces.

Notice also that only differences of potential energy can be defined, not its absolute value. In other words, potential energy is defined up to an arbitrary

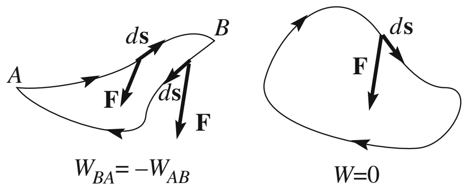


Fig. 2.25 The paths discussed in the text

additive constant. In practice, we fix the constant choosing a reference position, say o, in which we define the potential energy to be zero ($U_p(o) = 0$), The potential energy in the arbitrary point P is then

$$U_p(P) = U_p(o) - \int_o^P \mathbf{F} \cdot d\mathbf{s} = - \int_o^P \mathbf{F} \cdot d\mathbf{s}.$$

For example for the weight, we arbitrarily fix a reference level at which the potential energy is zero by definition. This may be the ground level but some other level too. We take that level as the origin of the vertical upward directed z-axis and the potential energy is

$$U_p(z) = mgz. \tag{2.49}$$

We have stated that a force **F** is conservative if the work it does on a point when it moves from position A to B is independent of the path. There are two equivalent ways to state the same, which may be useful in certain circumstances.

1. A force is conservative if the work it does moving from A to B on any path is equal and opposite to the work done moving from B to A on any path (Fig. 2.25). This follows immediately from (2.48).
2. The work of a conservative force on any closed path is zero.
   In summary we can briefly say that the (equivalent) properties of conservative forces are: (1) its work does not depend on the path, (2) admits a potential energy, (3) the works going and going back are equal and opposite, (4) the work on a closed path is zero.

2.14 Energy Conservation

Consider a material point P of mass m moving from the position A to the position B on the trajectory Γ under the action of the (only) force **F**. Whether the force is conservative or not its work is equal to the change of the kinetic energy of the point. Denoting with U_k the kinetic energy, we write

$$W_{AB,\Gamma} = U_k(B) - U_k(A). \tag{2.50}$$

If, and only if, **F** is conservative, the same work is also the opposite of the change of potential energy of the force

$$W_{AB,\Gamma} = U_p(A) - U_p(B). \tag{2.51}$$

It immediately follows that

$$U_p(B) + U_k(B) = U_p(A) + U_k(A). \tag{2.52}$$

Considering that the positions A and B are arbitrary, we conclude that the sum of the kinetic and potential energies is the same, i.e., is constant, in every position of the motion. The sum is the *total mechanical energy*, say U_{tot} of the material point. The conclusion is so important that it is often called a "principle". The principle, or law, of energy conservation states that

$$U_{\text{tot}} = U_p + U_k = \text{constant}. \tag{2.53}$$

If more than one force is acting on the point P and all of them are conservative, Eq. (2.53) is still valid, provided that U_p is the sum of the potential energies of all the acting forces, or, in an equivalent manner, if it is the potential energy of the resultant of those forces. Notice however, that the law is no longer valid even if only one of the forces is dissipative.

In words, the law of energy conservation states that *if a point moves under the action of conservative forces only, its total mechanical energy is conserved during its motion.*

Consider now the case, which is what happens in practice, that also dissipative forces are present. Consider for example the motion on an incline under the actions of weight and friction. The kinetic energy theorem is still valid. The work done by the forces for the displacement from A to B on the curve Γ, can be written as the sum of the work W_{AB}^C of the conservative forces and that $W_{AB,\Gamma}^D$ of the dissipative ones and we have

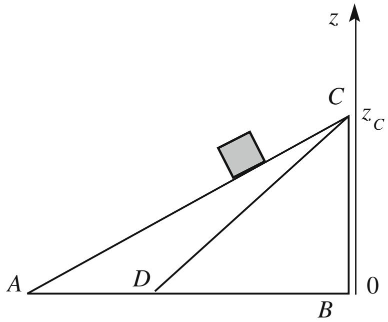
$$W_{AB}^C + W_{AB,\Gamma}^D = U_k(B) - U_k(A)$$

but $W_{AB}^C = U_p(A) - U_p(B)$, and in conclusion

$$U_{\text{tot}}(B) - U_{\text{tot}}(A) = W_{AB,\Gamma}^D. \tag{2.54}$$

We see that, if non-conservative forces are active, the total mechanical energy varies and its variation is equal to the work of the non-conservative forces. The work of these forces is negative, as we saw for friction. Hence the energy diminishes. This is the reason of the *dissipative* term.

Fig. 2.26 Fall on inclines or vertical



The physical dimension of kinetic, potential and total energies are the same as of the work. The measurement unit is consequently the jule.

Example E 2.1 Let us go back to the discussion made in Sect. 2.12 on the experiments by Galilei on inclined planes. Figure 2.26 shows a body of mass m, which can fall, starting from rest from point C, on inclines of different slopes CA or CD or vertically on CB. Take a vertical upwards axis z, and denote by z_C the height of C (that is the height of the inclined plane).

Consider the motion on CA. If friction is negligible the force exerted by the constraint is normal and does not make work. The other acting force is the weight $m\mathbf{g}$.

The energy conservation principle applied to the displacement CA from C, where the velocity is zero, to A, where $z = 0$, gives

$$mgz_C = \frac{1}{2}m\upsilon_A^2 \tag{2.55}$$

or

$$\upsilon_A = \sqrt{mgz_C}. \tag{2.56}$$

We see that the final velocity depends only on the difference in level not on the inclination.

If the friction is not negligible, the final energy is less than we have just calculated. We can obtain it with Eq. (2.54) calculating the work of friction. The latter does depend on the inclination for two reasons: the lengths of the paths are different and the body pushes with different forces on the plane. To do the calculation, however, we need to know something more on friction. We shall do that in the next chapter.

We finally observe that the above arguments are valid if the body can be considered a material point. If the body also rotates, like balls do, there is also kinetic energy associated to the latter that should be considered. We shall discuss this point in Sect. 8.16.

As we have just seen, in the presence of dissipative forces, the total mechanical energy, namely the sum of kinetic and potential energy, is not conserved. However, these are only two of many forms of energy. As a matter of fact the law of energy conservation is one of the basic laws of physics. The law is universally valid, without any exception, provided all the forms of energy are included in the balance. Other forms of energy are chemical energy, thermal energy, electric energy, nuclear energy, etc. Every time energy seems not to be conserved, it is because we have failed to include one of its forms. The issue is one of the main objects of thermodynamics, which will be discussed in the second volume of this course. The historic process that led to clarification of the concept of energy and to the establishment of the universal law of energy conservation was very long. Starting, as we have seen, already with Galilei, the process came to maturity only in the middle of the XIX century. It was then established with the first law of thermodynamics, mainly by Julius von Mayer (1814–1878) and James Prescott Joule (1818–1889). Energy is conserved also in the presence of dissipative forces if internal thermal energy is included in addition to macroscopic mechanical energy.

2.15 A Theorem Concerning Central Forces

A region of space in which a force that is a function only of the point, and possibly of time, is called a *force field*. If the force does not depend on time, the field is said to be *stationary*; if it does not depend on the position, it is said to be *uniform*.

The most common example of a uniform stationary field is weight, which is constant in time and space (at least within the limits of a laboratory). On the contrary, the viscous drag, the resistance of air to the motion, say, of a car or an airplane, is an (increasing) function of speed and consequently is not a force field.

A force field is said to be *central* if in every point P the force is directed as the line between P and a fixed point, called the *center of the forces*. The situation is sketched in Fig. 2.27, where C is the center of the forces.

It is clearly convenient to choose the center of the forces as the origin of the reference frame. We shall employ polar co-ordinates in which $\mathbf{r} = (r,\theta,\phi)$ is the

Fig. 2.27 A central field of forces

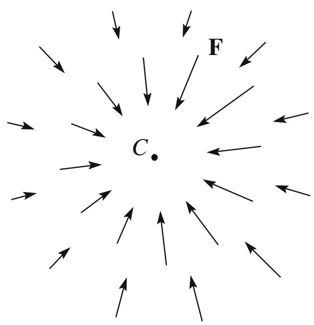
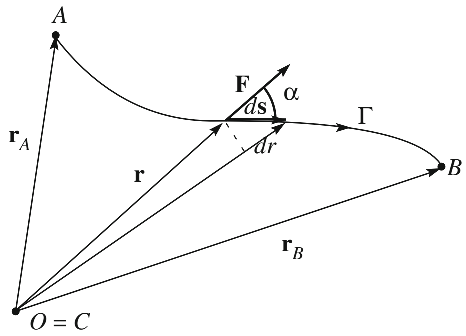


Fig. 2.28 Work by a central force



position vector. Let $\mathbf{F}(\mathbf{r})$ be the force under consideration. Saying that the force is central means that the two vectors $\mathbf{F}$ and $\mathbf{r}$ are everywhere parallel. They may have the same or opposite directions. The component of the force on the position vector, the radial component, is its magnitude in the former case, the opposite in the latter. This quantity may depend on the three coordinated, the two angles and the distance r from the center. If the force depends only on r, the field is said to have a *spherical symmetry*. On the other hand, a central force may be conservative or not. We shall now prove that these two properties are correlated: if a field of central forces has spherical symmetry, the force is conservative and, vice versa, if a central field of force is conservative it possesses spherical symmetry.

We start with the first statement. The radial component of the force, say $F_r(r)$, is by hypothesis a function of the distance from the center r only. Given any two points like A and B in Fig. 2.28, let us calculate the work done by the force on an arbitrary curve Γ, having A as origin and B as end. We shall proof that it is independent of the chosen curve. We indicate with $d\mathbf{s}$ the generic element of the curve. The work corresponding to this elementary displacement is

$$dW = \mathbf{F}(r) \cdot d\mathbf{s} = F_r(r) ds \cos \alpha, \tag{2.57}$$

where α is the angle between $\mathbf{F}$ and $d\mathbf{s}$, which is also the angle between the directions of $\mathbf{r}$ of $d\mathbf{s}$. Hence, $ds \cos \alpha$ is the projection of $d\mathbf{s}$ on the direction of $\mathbf{r}$, namely simply dr, i.e. the elementary variation of the distance from center. N.B. Pay attention! This notation is universally employed, but is ambiguous. The designation dr means the variation of the magnitude of the vector $\mathbf{r}$, namely $d|\mathbf{r}|$, <u>not</u> the magnitude of the vector variation of $\mathbf{r}$, namely $|d\mathbf{r}|$.

Anyway we have

$$dW = F_r(r) dr. \tag{2.58}$$

Notice that this elementary work may be positive or negative depending on F_r and dr having the same or opposite sign. The total work on the curve Γ is

$$W = \int_A^B F_r(r)dr \tag{2.59}$$

which is independent of the chosen curve, proving that the force is conservative.

What we have proven is valid for whatever dependence on r. A particularly important case is the gravitational force exerted by a mass M, which we shall consider to be point-like, on another mass m, point-like too. We shall study the gravitational force in Chap. 5. We anticipate here that such a force acting on m is in any point directed towards the position of M; namely it is central. Its magnitude is proportional to the product of the two masses and inversely to the square of their distance r. Indicating by G_N the proportionality constant, the force is

$$F_r(r) = -G_N \frac{Mm}{r^2}, \tag{2.60}$$

where the minus sign indicates that the force is always in the direction opposite to $\mathbf{r}$, namely is attractive. The work done on a displacement from A to B is

$$W = \int_A^B F_r(r)dr = \int_A^B -G_N \frac{Mm}{r^2} dr = G_N \frac{Mm}{r_B} - G_N \frac{Mm}{r_A}. \tag{2.61}$$

As expected, it is independent of path. We can then define the potential energy of the gravitational force. The potential energy difference between the point of position vector $\mathbf{r}_B$ and the point of position vector $\mathbf{r}_A$ is the opposite of the work Eq. (2.61), namely

$$U_p(\mathbf{r}_B) - U_p(\mathbf{r}_A) = -G_N \frac{Mm}{r_B} + G_N \frac{Mm}{r_A}. \tag{2.62}$$

As always, the potential energy is defined up to an arbitrary additive constant, namely

$$U_p(\mathbf{r}) = -G_N \frac{Mm}{r} + \text{constant}. \tag{2.63}$$

The constant is fixed choosing a point in which the potential energy is zero by definition. In this case it is obviously convenient (but not at all necessary) to choose this point at infinite distance, obtaining

$$U_p(\mathbf{r}) = -G_N \frac{Mm}{r}. \tag{2.64}$$

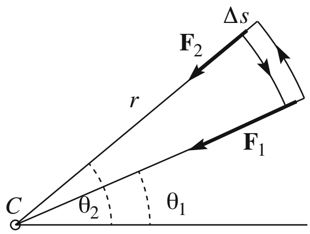


Fig. 2.29 The closed path used in the demonstration

This is the *potential energy* of a point-like mass m (the earth for example) in the gravitational field of the point-like mass M (the sun). Notice that, in fact, this is the energy of the pair of masses m and M (see Chap. 7).

We now prove the second of the above stated properties. We assume the force to be central and conservative and show that its component (magnitude with sign) on the position vector cannot depend on angles.

Let us consider for simplicity displacements on a plane. Consider a closed path, as in Fig. 2.29, composed of two circular arcs centered on the center of forces C, and two radial segments joining their extremes, at the angles θ_1 and θ_2 respectively. Take the radial segments of a very short length Δs. Assume by contradiction that the magnitude of the force F would depend not only on r but also on the angle θ. Under this hypothesis F_r has different values on the two radial sides that are at different angles, say F_{r1} and F_{r2}. Let us calculate the work of the force on this path. The contributions of the arcs are zero because on them the force is perpendicular to displacement. The contributions of the radial segments are $-F_{r1}\Delta s$ and $F_{r2}\Delta s$. The total work is then $W = (F_{r2} - F_{r1})\Delta s \neq 0$, in contradiction with the hypothesis that the force is conservative.

2.16 Power

In physics, power is defined as the work done per unit time. For a given delivered work, the power is larger for shorter delivery times. The simplest case is the work done by a force, say $\mathbf{F}$, on a material point, say P. Consider the elementary displacement $d\mathbf{s}$ of the point, taking place between the instants t and $t + dt$. The work done by the force is $dW = \mathbf{F} \cdot d\mathbf{s}$. The *power* w given by the force the work divided by the corresponding time interval, that is

$$w = \frac{dW}{dt} = \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} = \mathbf{F} \cdot \mathbf{v}. \quad (2.65)$$

In words: the power delivered by the force $\mathbf{F}$ acting on a material point moving at the velocity $\mathbf{v}$ in a given instant is equal to the dot product of the force and the velocity of the point in that instant. If the force is a function of the position, it must be obviously evaluated in the position of the point.

The physical dimensions of the power are those of a work divided by a time. Its unit is the watt, after James Watt (1736–1819) One watt is the power developed by a force delivering the work of one joule in one second (1 W = 1 J/1 s). To have an idea of the order of magnitude, you develop about 1 W if you raise a glass of water by 1 m in one second.

Problems

2.1 A person is sitting on a chair supported by a horizontal ground. Draw the diagrams of the forces for the person, the chair, and the earth. Describe each of the forces, identifying the body that produces them and the body on which they act. Identify the action reaction pairs.

2.2 A block hangs from the ceiling through a rope. A second rope is attached to the bottom of the block. It hangs vertically and you draw it with your hands downwards. Draw the diagrams of the forces for the block, each of the ropes, your body, the ceiling and the earth. Describe each of the forces, identifying the body that produce them and the body on which they act. Identify the action reaction pairs

2.3 Fig. 2.30 represents two blocks of masses m_1 and m_2 on frictionless planes. The plane of the first block is horizontal; the plane of the second is at an angle θ. The two blocks are tied by a mass less inextensible wire that can slide over a pulley without friction. (a) mentally insulate each block and draw the force diagrams; then write three equations of motion, (b) find the tension of the wire and the acceleration of m_2.

2.4 A body of mass m = 1 kg moves in a circular uniform motion on a circle of radius R = 0.1 m. What is the value of the centripetal force?

2.5 The system represented in Fig. 2.31 is in a vertical plane. $M > m$. Letting it free, M goes down and m goes up. Neglecting the frictions, draw the diagrams of the forces and determine the accelerations of M and of m.

2.6 With a hammer of mass m = 0.1 kg we beat on a nail, which is already partially stuck in a piece of wood, with a speed of v = 1 m/s. The nail advances a distance of s = 1 cm. Find the force exerted by the hammer.

2.7 Two people pull a rope, each on one end, each with a force of magnitude F. What is the tension? F or $2F$? Why?

2.8 Two ropes hang from the ceiling. Two spheres of different masses hang at the two ends. With both your hands you apply to the two spheres the same force

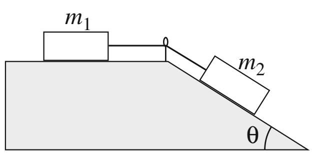


Fig. 2.30 The two blocks of problem 2.3

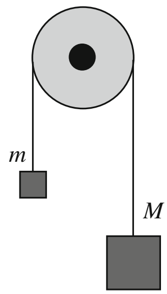


Fig. 2.31 The two blocks of problem 2.5

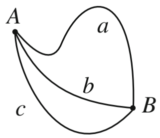


Fig. 2.32 The three guides of problem 2.9 in a vertical plane

F, which is not necessarily in the direction of the rope. What are the forces on each hand?

2.9 The three curves in Fig. 2.32 represent three rigid guides in a vertical plane. Three rings of different masses slide without friction, one on each of them. The three rings start from A at the same time with null velocity. State for each of the following statements if it is true or false. 1. The rings reach B contemporarily. 2. The rings reach B with velocities equal in magnitude.

2.10 A man of mass $m = 80$ kg jumps from a platform at the eight $h = 0.5$ m above ground. Reaching the soil he forgets to fold his legs. Fortunately the ground is quite soft and stops the motion in a distance $s = 2$ cm. What is the average force exerted on his bones during the stoppage?

2.11 Give an approximate evaluation of the height reached by a pole vaulter athlete able to reach in his run the speed of $\upsilon = 10$ m/s.

2.12 Fig. 2.33 shows three blocks of equal weight F_p. The pulley is frictionless. If we gradually increase all the weights, keeping them equal to each other, which rope will break?

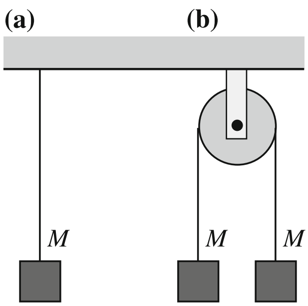


Fig. 2.33 The system of problem 2.12

2.13 Two spheres, one with mass double that of the other, are launched upwards with the same initial momentum p_0. If the resistance of air can be neglected, what is the ratio of the heights they reach.

2.14 A particle of mass $m = 2$ kg oscillates on the x-axis. The equation of its motion is $x = 0.2\sin(5t - \pi/6)$, with x in meters and t in seconds. (a) What is the magnitude of the force acting on the particle at time $t = 0$? What is the maximum value of the force?

2.15 A twine of length l can hold a maximum tension T. It is employed to rotate a mass m on a circle. Find the maximum velocity the body can rotate if (a) the rotation is in a horizontal plane, (b) in a vertical plane. Draw in each case the diagram of the forces.

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