Chapter 2
Dynamics of Bubbles in an Infinite Volume of Liquid

2.1 The Problems of Force Controlled Bubble Evolutions

Let us consider the case when both the heat supply (output) conditions to the bubble and the molecular kinetics of the evaporation (condensation) processes have no substantial effect on its evolution. In this setting the variation of the radius of a single bubble in an infinite volume of still liquid will be completely governed by the hydrodynamics of motion of the ambient liquid. In other words, in this case the evolution of a bubble is governed to a large extent by the relevant forces: the pressure difference, the inertia forces, as well as the viscosity and surface tension forces. We are thus led to consider a large class of problems in *bubble dynamics*.

Form the whole variety of the problems in the dynamics of bubbles, both of scientific and practical interest, in this chapter we consider the problem of *dynamic growth of a bubble* and the *bubble collapse* problem. The present chapter will mainly be concerned with these problems, which, in addition of being precisely stated and having an elegant mathematical formulation, have a significant practical importance.

The study of the majority of dynamic problems solved in this chapter was carried out within the frameworks of the same logic.

In the majority of cases it is possible to obtain an analytic solution as a quadrature, which often does not reduce to standard integrals, and hence should be calculated numerically. In some cases using the symbol algebra computer programs, the analytical solution can be reduced to a cumbersome combination of standard integrals in practice being useless for direct calculations. Next, we studied the limit (degenerate) solutions of the problem under study that have a transparent physical meaning and a fairly simple analytic form. In conclusion, on the basis of the so-obtained asymptotic relations and numerical analysis it proved possible to construct practical formulas capable of providing good accuracy in the entire range of variation of the regime parameters and securing the required passages to the limit.

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Dynamic growth of a bubble Assume that in at time $t = 0$ a spherical cavity (bubble) of radius R_0, filled with vapour, gas, or gas-vapour mixture, originates in a bulk of still incompressible liquid. The pressure inside the cavity is p_v, and the pressure p_∞ in the ambient liquid is constant. Assume that the pressure difference $\Delta p = p_v - p_\infty$ is constant in time.

If $p_v > p_\infty$, then the bubble will start to grow under loading of pressure difference. The rate of its expansion will depend, on the one hand, upon the pressure difference Δp, and on the other hand, on the inertial forces in liquid, the surface tension forces, and the viscosity forces, which oppose the bubble expansion. The surface tension forces are responsible for a pressure surge on the interfacial boundary, which reduces the total pressure drop. For bubbles of fairly small size, the surface tension forces can completely balance the pressure difference Δp, making the bubble growth impossible.

Figure 2.1a qualitatively shows the distribution of pressure and velocity near a growing bubble. It is seen that away from the interfacial boundary the perturbing action of a bubble on the ambient liquid dies down, tending to zero at large distances from it.

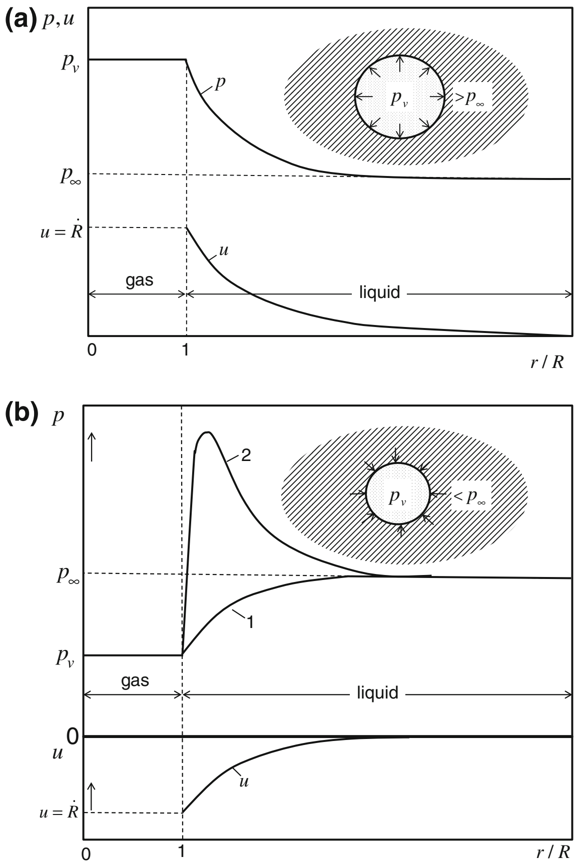

Fig. 2.1 Qualitative form of distributions of pressure and velocity over the radius. **a** Bubble growth; **b** bubble collapse, *1*—the distribution of pressure in the initial period of collapse $(R > 0.63R_0)$, *2*—the distribution in the final period of collapse $(R < 0.63R_0)$

In early stages of bubble growth, the dynamic effects prevail, as a rule, over the thermal ones, and so the bubble grows according to the *dynamically controlled laws being of the main interest for the present chapter*, while for large growth times the effects of heat supply to the interfacial boundary become prevailing, and we thus have the *thermally controlled bubble growth model,* which will be considered in detail in the next chapter.

Bubble collapse If $p_v < p_\infty$, then under pressure difference the bubble radius becomes to decrease, and so the bubble "collapses" (the process opposite to the bubble growth process), Fig. 2.1b.

It becomes an interesting question to inquire on what happens with the pressure distribution near a collapsing bubble. Below we shall see that in the early stages of the process, while the current radius R of a collapsing bubble is still above $0.63R_0$, where R_0 is the initial bubble radius, the distribution of pressure in the liquid, is monotone in nature (as in the bubble growth problem), see curve 1 in Fig. 2.1b. In what follows (for $R < 0.63R_0$), the law of pressure variation in the liquid becomes nonmonotone, the pressure assumes a maximum value, and besides, as the collapse process ensues, the maximum value sharply increases and moves towards the bubble surface direction, see curve 2 in Fig. 2.1b.

A practical interest in the dynamics of gas bubbles was related in the first place with the *cavitation problem*—a disruption of liquid with formation of gas-vapour caverns (cavities) and their subsequent pulsations and collapsing. They are called vapour or gas cavities depending on the concentration of the vapour or gas in the cavity. Under the conventional conditions the intermediate case is usually realized: a cavitation bubble is filled with the gas-vapour mixture.

In the concluding stage of the collapse process, the pressure and temperature in the bubble assume considerable values. After a cavity is collapsed, a spherical shock wave is propagated in the ambient liquid, which is damped in space. In the course of their life cycle the bubbles usually lose their spherical form. The largest deformations are observed in the terminal collapse stage for near-wall bubbles. A collapse of a bubble often results in a "cumulative jet" destroying the surface of a rigid body. This phenomenon is known as the cavitation erosion.

The appearance of cavitation cavities alters dramatically the hydrodynamic characteristics of the flowing parts of many hydraulic machines and devices (pumps, hydraulic turbines, ship propellers, etc.), reducing their performance and life-time. Similar phenomena may also appear under outer flows of bodies moving in a liquid. The correct understanding of the accompanying processes helps in a number of cases not only to minimize the negative effect of cavitation, but also exploit this phenomenon for useful purposes.

It is worth noting that the cavitation triggers various physical and chemical phenomena in liquids: sonoluminescence (fluorescence of liquids); chemical effects (sonochemical reactions); dispergation (fragmentation of solid particles in a liquid); emulsification (mixing and homogenization of immiscible liquids).

The bibliography on the dynamics of bubbles comprises hundreds of titles. By now this part of liquid and gas mechanics has become classical. Nevertheless, it is worth noting that a good deal of aspects of this problem has not received complete treatment, and hence at present time the interest to particular questions in the inertial dynamics of bubbles is not declining.

2.2 The Rayleigh-Lamb Equation

Now we consider a spherical gas bubble with impermeable surface submerged in an bulk of still liquid. Assume that the pressure over the entire volume of the bubble is constant and equals p_v, and the pressure in the liquid is p_∞. If $p_v > p_\infty$, then under loading of pressure drop the bubble will grow, and in the case $p_v < p_\infty$, it will decrease. The last phenomenon is called the *bubble collapsing*. In this stage of analysis we shall not impose any restrictions on the sign of the difference $p_v - p_\infty$. Consequently, the equation obtained in this section will be of fairly general nature and will hold both for the cases of dynamic growth considered below and for the collapsing of bubbles.

The continuity equation for liquid in the spherical polar system is as follows

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 u\right) = 0, \tag{2.1}$$

where u is the liquid velocity.

In view of the impermeability of the interfacial boundary, the velocity of the liquid on the bubble boundary, u_R, will equal the rate of change of its radius, $\dot{R} \equiv dR/dt$. Taking this into account, we find the integral of Eq. (2.1):

$$r^2 u = R^2 u_R = R^2 \dot{R} = const = F(t), \tag{2.2}$$

where $F(t)$ is the known function of time.Consequently, we get the field of radial velocity in the liquid:

$$u = R^2 \dot{R}/r^2. \tag{2.3}$$

By formula (2.3), the velocity of liquid decays as $u \sim 1/r^2$ as we move away from the bubble surface along the radial coordinate. This means that the liquid at infinity from the bubble remains unperturbed.

The equation of conservation of momentum for a liquid is the Navier-Stokes equation in the spherical polar system:

$$-\frac{1}{\rho_l}\frac{\partial p}{\partial r} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \nu_l\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) - \frac{2u}{r^2}\right]. \tag{2.4}$$

Here, ρ_l, ν_l are, respectively, the density and the kinematic viscosity coefficient of liquid.

Substituting the velocity u from (2.3) into Eq. (2.4) we obtain

$$\frac{1}{r^2}\left(2R\dot{R}^2 + R^2\ddot{R}\right) - \frac{2}{r^5}R^4\dot{R}^2 = -\frac{1}{\rho_l}\frac{\partial p}{\partial r}. \tag{2.5}$$

As is seen from Eq. (2.5), for the case of spherical expansion (compression), the viscous term in the square brackets on the right (2.4) vanishes. This implies the conclusion that the bubble dynamics equation of (2.5), which was rigorously derived in the model of a viscous liquid, agrees with the similar equation obtained in the model of a perfect liquid.

To explain this fact we consider a one-dimensional flow of viscous incompressible liquid. In this setting, the viscous term in the Navier-Stokes equation reads as

$$W_0 = \nu_l\frac{\partial^2 u}{\partial x^2}, \tag{2.6}$$

for a plane flow in the Cartesian coordinates (index "0"). For an axially symmetric flow in the cylindrical coordinate system (index "1") the viscous term can be represented as

$$W_1 = \nu_l\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right), \tag{2.7}$$

and, for a spherically symmetric flow, it reads, in the spherical polar system (index "2"),

$$W_2 = \nu_l\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) - \frac{2u}{r^2}\right]. \tag{2.8}$$

Relations (2.6)–(2.8) can be written in a unified way

$$W_n = \nu_l\frac{\partial}{\partial x}\left[\frac{1}{x^n}\frac{\partial(x^n u)}{\partial x}\right], \tag{2.9}$$

where $n = 0, 1, 2$, respectively, for the planar, axially symmetric, and spherically symmetric problems; x is the corresponding axial coordinate.

The volumetric flow rate of liquid for each of the above three cases can be written, up to a constant coefficient, as

$$V_n = x^n u, \tag{2.10}$$

and the continuity equation for the type of flows in question reads as

$$\frac{\partial V_n}{\partial x} = \frac{\partial(x^n u)}{\partial x} = 0. \tag{2.11}$$

Substituting (2.11) in (2.9), we see that for this class of one-dimensional flows the value of viscous stresses at each point inside the liquid is zero. Nevertheless, as we shall see below, the effect of the viscosity forces on the bubble dynamics is manifested in terms of the boundary condition on the interfacial boundary.

Integrating Eq. (2.5) from $r = R$ to $r = \infty$, we obtain the distribution of pressure in the liquid

$$\frac{p(r) - p_\infty}{\rho_l} = \frac{2R\dot{R}^2 + R^2\ddot{R}}{r} - \frac{R^4\dot{R}^2}{2r^4}, \tag{2.12}$$

where $\ddot{R} \equiv d^2R/dt^2$.

Equation (2.12) holds for the entire volume of liquid up to the boundary of the bubble, where at $r = R : p = p_l$. Inside the vapour phase the pressure is constant and equals p_v. To find the relations for the quantities p_v, p_l we apply the *dynamic coupling condition* on the interfacial boundary:

$$p_v - p_l = 4\frac{\mu_l\dot{R}}{R} + 2\frac{\sigma}{R}. \tag{2.13}$$

Here, μ_l is the dynamic viscosity of liquid, σ is the surface tension coefficient.

The first term on the right (2.13) expresses the viscous resistance of liquid on the spherical expansion (compression) of a bubble. The quantity $2\sigma/R$ is the Laplace jump of pressures due to the curvature of the interfacial boundary.

Applying Eq. (2.12) to the bubble surface and using condition (2.13), we see that

$$\frac{p_v - p_\infty}{\rho_l} = \frac{3}{2}\dot{R}^2 + R\ddot{R} + 4\frac{\nu_l\dot{R}}{R} + 2\frac{\sigma}{\rho_l R}. \tag{2.14}$$

Thus, in a general case in the equation of bubble dynamics there appears the viscous term—notwithstanding that in Eq. (2.5) there is no effect of viscous stresses, the viscous effect manifest themselves in terms of the dynamic coupling condition on the interfacial boundary (2.13) (Leighton 1994).

The boundary conditions for Eq. (2.14) are as follows:

$$\text{as } r \to R(t)\text{: } p = p_v + 2\sigma/R(t); \tag{2.15}$$

$$\text{as } r \to \infty\text{: } p \to p_\infty. \tag{2.16}$$

The initial condition is the pressure homogeneity condition on the entire space occupied by the liquid

$$\text{for } t = 0\text{: } r = R_0;\ p = p_\infty. \tag{2.17}$$

In the absence of the viscosity and surface tension effects $(\nu_l = \sigma = 0)$, the dynamic coupling condition (2.13) is reduced to the condition of no pressures jump at the interfacial boundary:

$$p_v - p_l = 0. \tag{2.18}$$

As a result, Eq. (2.14) assumes a simpler form

$$\frac{p_v - p_\infty}{\rho_l} = \frac{3}{2}\dot{R}^2 + R\ddot{R}. \tag{2.19}$$

Equation (2.19) is the classical *Rayleigh's equation*, which was obtained in 1917. A more general form (2.14) of this relation, which accounts for the effect of the viscosity forces and the surface tension forces, is often called the *Rayleigh-Plesset's equation* [with reference to *Plesset's work* (1949)]. However, is worth pointing out that Eq. (2.14) was first obtained by Lamb (1923). Later it was studied in the works of Herring (1941), and of Kirkwood and Bethe (1942). In accordance with the above, Eq. (2.14) will be called the *Rayleigh-Lamb's equation*.

We also recall that in the derivation of the Rayleigh-Lamb's equation the interfacial boundary was assumed to be impermeable, which enabled one to assume that the liquid velocity on the bubble surface is exactly the velocity of surface motion. It is not difficult to take into account the effect of permeability [see, for example, Brennen (1995)], however, as one may easily check, for the class of problems under study it will have an effect only in the direct vicinity of the thermodynamic critical point, when the density of the vapour and liquid phases are close.[1]

The Rayleigh equation relates the law of variation of the bubble radius in time $R(t)$ with the pressure drop $\Delta p(t)$. In other words, from the known law of variation $\Delta p(t)$ it allows one to calculate the evolution of the bubble radius in time: $R = R(t)$. The problem may be turned around: from the known function $R(t)$ it is required to find the law of variation of $\Delta p(t)$.

Rayleigh's equation may also be obtained from the balance of energy. Let us find the kinetic energy of liquid motion in the entire volume:

$$E = \rho_l \int\limits_R^\infty \frac{u^2}{2} 4\pi r^2 dr. \tag{2.20}$$

[1]In the following chapters we shall show that, for the problem of thermally controlled bubble growth problem, the effect of permeability of the interfacial boundary on the intensity of the interfacial heat and mass exchange becomes quite substantial in many cases of practical interest.

Substituting the liquid velocity distribution from formula (2.3) in integral (2.20) and integrating, this gives

$$E = 2\pi\rho_l R^3 \dot{R}^2 \tag{2.21}$$

The increment of the kinetic energy of liquid dE is equal to the work done by the pressure excess $p_v - p_\infty$ when the bubble volume is increased by dV:

$$dE = (p_v - p_\infty)dV, \tag{2.22}$$

where $V = 4/3\pi R^3$ is the volume of the bubble.

Equation (2.22) is the *energy interpretation* of the classical Rayleigh's equation. It is easily shown that Eqs. (2.22) and (2.19) are equivalent. Indeed, by (2.21) and (2.22),

$$dE = 2\pi\rho_l d\left(R^3\dot{R}^2\right) = 4\pi R^2(p_v - p_\infty)dR. \tag{2.23}$$

It follows that

$$\frac{1}{2R^2}\frac{d}{dR}\left(R^3\dot{R}^2\right) = \frac{3}{2}\dot{R}^2 + R\dot{R}\frac{d\dot{R}}{dR} = \frac{p_v - p_\infty}{\rho_l}. \tag{2.24}$$

We write the equality

$$\dot{R}\frac{d\dot{R}}{dR} = \dot{R}\frac{d\dot{R}}{dt}\frac{dt}{dR} = \ddot{R}. \tag{2.25}$$

Now the equivalence of Eqs. (2.22) and (2.19) follows from (2.22) to (2.25). In applications one may use the both forms of the Rayleigh's equation: the *dynamic form* (2.19) and *the energy form* (2.22).

It is worth pointing out that formula (2.22) is not a trivial reformulation of the differential Eq. (2.19), as it might at first appear. On the contrary, expanding dE for possible different realizations of the liquid flow, one may in a number of cases obtain fairly interesting results.

2.3 Collapse of a Vapour Bubble

2.3.1 Change of the Bubble Radius in Time

Consider the process of *collapsing of a vapour bubble* in an unbounded volume of ambient liquid. In addition to the simplifying assumptions adopted in the previous section we assume that both the pressure in the bubble, p_v, and the pressure in the liquid, p_∞, are constant in time.

If one considers a gas cavity, then in the process of its compression the pressure of the gas inside it will increase. Hence, for a gas bubble the model under consideration applies only in the case $p_v \ll p_\infty$, which in essence is equivalent to the Rayleigh assumption.[2]

If, however, one is concerned with a vapour cavity, then it is assumed that the processes of vapour condensation during the collapse phase of a bubble have infinitely large intensity. Besides, the pressure p_v will be constant in time and equal to the saturation pressure, which corresponds to the temperature of the liquid.

The Rayleigh equation does not explicitly contain the time t. This enables one to carry over from the variables $R(t)$ to the variables $\dot{R}(R)$. Differentiating in time $\dot{R} = \dot{R}(R)$ as a composite function, we obtain

$$\ddot{R} = \dot{R}\frac{d\dot{R}}{dR} = \frac{1}{2}\frac{d}{dR}\left(\dot{R}^2\right) = \frac{dz(R)}{dR}, \tag{2.26}$$

where $z = \dot{R}^2$ is the squared bubble expansion velocity.

In view of (2.26) Eq. (2.19) can be transformed into the equation

$$R\frac{dz}{dR} + 3z = 2\frac{p_v - p_\infty}{\rho_l}, \tag{2.27}$$

which is of a first-order in z. The solution of (2.27), which satisfies the initial condition

$$\text{for } R = R_0\text{: } z = 0, \tag{2.28}$$

has the following form

$$z \equiv \dot{R}^2 = \frac{2}{3}\frac{\Delta p}{\rho_l}\frac{R_0^3}{R^3}\left(1 - \frac{R^3}{R_0^3}\right), \tag{2.29}$$

where $\Delta p = p_\infty - p_v$. Here and in what follows, for convenience we shall understand by Δp the absolute value of the pressure drop ($\Delta p > 0$), clarifying the choice of the sign of it for each problem under study.

For the case of a bubble collapse, we have $\dot{R} < 0$. Then, choosing in Eq. (2.29) the negative square root, we write down the expression for the rate of variation of the bubble radius:

$$\dot{R} = -\sqrt{\frac{2}{3}\frac{\Delta p}{\rho_l}}\left(\frac{R_0}{R}\right)^{3/2}\left(1 - \frac{R^3}{R_0^3}\right)^{1/2}. \tag{2.30}$$

[2]In the original paper by *Rayleigh* (1914) it was assumed that the gas bubble is empty; that is, $p_v = 0$.

The initial condition for Eq. (2.30) is as follows:

$$\text{for } t = 0\text{: } R = R_0. \tag{2.31}$$

From Eq. (2.30) it seen that with decreasing the radius of the cavity the absolute value of the velocity of its boundary is increasing, and as $R \to 0$ it tends to infinity: $|\dot{R}| \sim R^{-3/2} \to \infty$.

Integrating Eq. (2.30) with initial condition (2.31) gives the formula relating the dimensionless bubble radius $\xi = R/R_0$ with the dimensionless time t/t_d:

$$\int_{\xi}^{1} \sqrt{\frac{\xi^3}{1-\xi^3}} d\xi = \frac{t}{t_d}. \tag{2.32}$$

Here, $t_d = R_0\sqrt{\frac{3}{2}\frac{\rho_l}{\Delta p}}$ is the dynamic scale of the bubble collapse time.

Let us determine the complete collapse time t_0 of a bubble. To this aim we substitute in (2.32) the lower limit of integration $\xi = 0$, then the integral on the left of (2.32) will become a constant:

$$\int_{0}^{1} \sqrt{\frac{\xi^3}{1-\xi^3}} d\xi = \frac{1}{3}\frac{\Gamma(1/2)\Gamma(5/6)}{\Gamma(4/3)} \approx 0.7468 \tag{2.33}$$

where $\Gamma(n)$ is the gamma-function.

As a result, we have

$$t_0/t_d = 0.7468 \tag{2.34}$$

or, in the dimension form,

$$t_0 = 0.9146 R_0\sqrt{\frac{\rho_l}{\Delta p}}. \tag{2.35}$$

In view of (2.33) Eq. (2.32) can be represented in the following "reduced" form:

$$\int_{\xi}^{1} \sqrt{\frac{\xi^3}{1-\xi^3}} d\xi = 0.7468\frac{t}{t_0}. \tag{2.36}$$

Expression (2.36) is the required law of variation of the bubble radius in time, $\xi \equiv R/R_0 = f(t/t_0)$. The integral on the left is not standard, and hence it should be calculated numerically. The dependence $R/R_0 = f(t/t_0)$, as obtained by calculation, is shown in Fig. 2.2 by a solid line. It is seen that the rate of variation of the

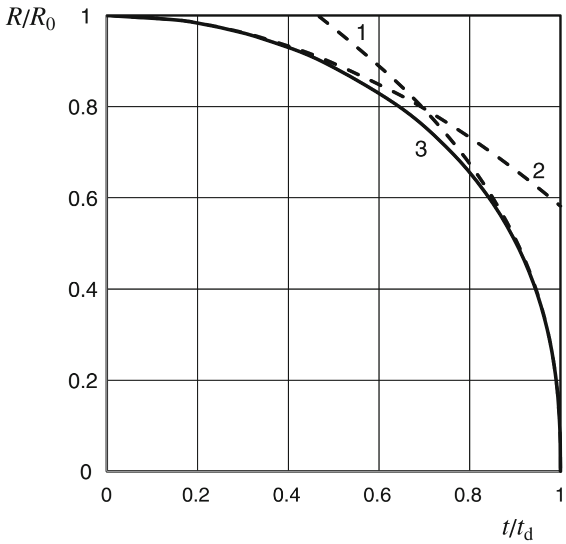

Fig. 2.2 Variation of the bubble radius during collapse. *1*—the asymptotic dependence for the final period (2.43), *2*—the asymptotic dependence for the initial period (2.38), *3*—numerical calculation by formula (2.36)

radius, which is fairly small at the initial time, is found to sharply increase as the process develops, tending to infinity in final stages of the process. Such behaviour of the curve is explained by the fact that the process under study is governed by the inertial dynamics of the liquid: when subjected to a constant pressure drop Δp the liquid around the bubble will gradually build up speed.

For the initial and final stages of bubble collapse, the asymptotic laws of collapse may be obtained analytically.

The initial stage of collapse In the initial stage of bubble collapse $t/t_0 \ll 1$ and $R/R_0 \approx 1$. We set $\xi \approx 1 - \chi, \chi \ll 1$. Now the integral in (2.36) can be written as

$$\int_{\xi}^{1} \sqrt{\frac{\xi^3}{1-\xi^3}} d\xi \approx \int_{0}^{\chi} \frac{d\chi}{\sqrt{3\chi}} \approx 2\sqrt{\frac{\chi}{3}} = 2\sqrt{\frac{1-\xi}{3}} \approx 0.4183\sqrt{1-\frac{R}{R_0}}. \tag{2.37}$$

Consequently, in view formula (2.33) we get the asymptotic dependence:

$$R/R_0 = 1 - 0.4183(t/t_0)^2. \tag{2.38}$$

The final stage of collapse For the final stage of bubble collapse we have $t/t_0 \approx 1$ and $\xi = R/R_0 \ll 1$. Taking this into account, we represent the integral in Eq. (2.32) as

$$\int_{\xi}^{1} \sqrt{\frac{\xi^3}{1-\xi^3}} d\xi = \int_{0}^{1} \sqrt{\frac{\xi^3}{1-\xi^3}} d\xi - \int_{0}^{\xi} \sqrt{\frac{\xi^3}{1-\xi^3}} d\xi. \tag{2.39}$$

The first integral on the right of (2.39) can be found exactly from formula (2.33). The second integral is estimated by expanding in a series for $\xi \ll 1$:

$$\int_0^{\xi} \sqrt{\frac{\xi^3}{1-\xi^3}} d\xi \approx \frac{2}{5}\xi^{5/2}. \tag{2.40}$$

Now expression (2.39) as $\xi \to 0$ is written as

$$\int_{\xi}^{1} \sqrt{\frac{\xi^3}{1-\xi^3}} d\xi = \frac{1}{3}\frac{\Gamma(1/2)\Gamma(5/6)}{\Gamma(4/3)} - \frac{2}{5}\xi^{5/2} \tag{2.41}$$

From Eqs. (2.33), (2.36) and (2.41) we find that

$$\xi^{5/2} = \frac{5}{6}\frac{\Gamma(1/2)\Gamma(5/6)}{\Gamma(4/3)}\left(1 - \frac{t}{t_0}\right)^{2/5} \tag{2.42}$$

or

$$R/R_0 = 1.284(1 - t/t_0)^{2/5}. \tag{2.43}$$

In Fig. 2.3 we show by dashed lines the results of calculation by the asymptotic formulas (2.38) and (2.43). These relations are seen to fairly well describe the initial and finite periods of bubble growth.

The analytic approximation For practical purposes it is convenient to have an analytic dependence for $R(t)$.

It is seen from Fig. 2.3 that the form of the curve obtained numerically is similar to the quarter circle. Hence, in the first approximation we shall seek the approximation of quadrature (2.36) in the form of the dependence

$$R/R_0 \approx \left[1 - (t/t_0)^2\right]^n. \tag{2.44}$$

(for a quarter circle, we have $n = 0.5$). We shall require that this dependence, as $t/t_0 \to 0$, will pass into asymptotics (2.38). Expanding (2.44) in a Taylor series near $t/t_0 = 0$ and using the first term of the expansion, we see that the exponent should be corrected as follows: $n = 0.5 \to 0.4183$. As a result, we arrive at the following simple approximation:

$$\frac{R}{R_0} = \left[1 - \left(\frac{t}{t_0}\right)^2\right]^{0.4183}. \tag{2.45}$$

In spite of the fact that near the point $t/t_0 \to 1$ formula (2.45) does not provide an exact transition into the asymptotic dependence (2.43), this formula is in a very good agreement with the results of numerical integration of Eq. (2.36). In the entire range $0 \le t/t_0 \le 1$ the maximal absolute error in the determination of the bubble radius by formula (2.45) is at most 0.009.

2.3.2 The Pressure Field in the Liquid

Having the dependence $R(t)$ for the bubble collapse at our disposal, let us examine the function $p(r)$ for an extremum.[3]

Equating the left-hand side of Eq. (2.5) to zero, we obtain the value of the radial coordinate $r_{\max}$ corresponding to the extremal pressure:

$$r_{\max} = R\left(\frac{2\dot{R}^2}{2\dot{R}^2 + R\ddot{R}}\right)^{1/3}. \tag{2.46}$$

Let us express the second derivative of the bubble radius in time from the Rayleigh Eq. (2.19):

$$\ddot{R} = -\left(\frac{\Delta p}{\rho_l} + \frac{3}{2}\dot{R}^2\right)\frac{1}{R}. \tag{2.47}$$

From (2.47) we have that $\ddot{R}$ is always negative.

Since the numerator of the fraction on the right of (2.46) is always negative and $\ddot{R} < 0$, it follows that the inequality

$$2\dot{R}^2 + R\ddot{R} > 0 \tag{2.48}$$

should be satisfied in order that $r_{\max}$ be positive.

In order to find the extremal pressure p_* one needs to substitute the value of $r_{\max}$ from formula (2.46) in Eq. (2.12):

$$\frac{p_{\max} - p_\infty}{\rho_l} = \frac{3}{4}\frac{R}{r_{\max}}\left(2\dot{R}^2 + R\ddot{R}\right). \tag{2.49}$$

Hence, using (2.47), we obtain

$$\frac{p(r) - p_\infty}{p_\infty - p_V} = \frac{R}{r}\frac{1 - 4\xi^3}{3\xi^3} - \left(\frac{R}{r}\right)^4\frac{1 - \xi^3}{3\xi^3}. \tag{2.50}$$

We denote the dimensionless coordinate as $\tilde{r} = r/R$ and write the dimensionless pressure drop as

[3] Below we shall show that this extremum is always a maximum.

$$P(r) = \frac{p(r) - p_v}{p_\infty - p_v}. \tag{2.51}$$

In view of this Eq. (2.50) assumes the form

$$P(\tilde{r}) = 1 + \frac{1 - 4\xi^3}{3\xi^2 \tilde{r}} - \frac{\xi\left(1 - \xi^3\right)}{3\tilde{r}^4}. \tag{2.52}$$

From Eqs. (2.49) to (2.52) we find the dimensionless pressure difference at the point of extremum of $P_{\max}$

$$P_{\max} = 1 + \frac{1}{4^{4/3}} \frac{1}{\xi^3} \frac{\left(1 - 4\xi^3\right)^{4/3}}{\left(1 - \xi^3\right)^{1/3}}, \tag{2.53}$$

the radial coordinate of the extremum is as follows:

$$\tilde{r}_{\max} = 4^{1/3} \xi \left(\frac{1 - \xi^3}{1 - 4\xi^3}\right)^{1/3}. \tag{2.54}$$

In a more transparent form relation (2.54) reads as:

$$\frac{r_{\max}}{R} = \left(4 \frac{1 - \xi^3}{1 - 4\xi^3}\right)^{1/3}. \tag{2.55}$$

In view of Rayleigh' Eq. (2.19) condition (2.48) becomes

$$2\dot{R}^2 + R\ddot{R} = \frac{1}{2}\dot{R}^2 + \frac{2}{3}\frac{\Delta p}{\rho_l} > 0. \tag{2.56}$$

Substituting in (2.56) the quantity $\dot{R}^2$ from Eq. (2.29), we find that

$$\xi^3 \equiv (R/R_0)^3 < 1/4. \tag{2.57}$$

From inequality (2.57) it follows that the extremum may not exist if the dimensionless radius is greater than

$$\xi \equiv R/R_0 = 4^{-1/3} \approx 0.630. \tag{2.58}$$

By twice differentiating Eq. (2.52), we find, in view of (2.54), the second derivative of the function $P(\tilde{r})$ with $\tilde{r} = \tilde{r}_{\max}$:

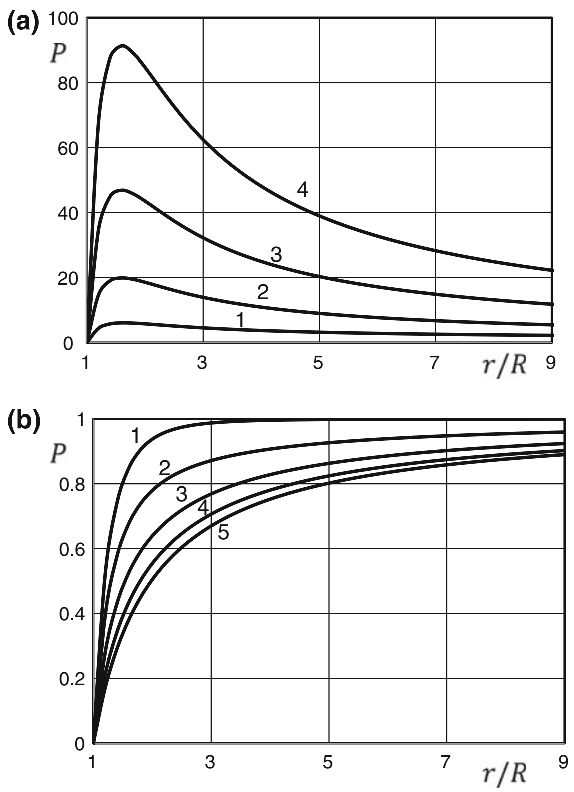

Fig. 2.3 The distribution of pressure in the vicinity of a collapsing bubble. **a** The initial stage of collapse $(R/R_0 \geq 0.63)$, *1*—$R/R_0 = 0.63$, *2*—0.7, *3*—0.8, *4*—0.9, *5*—0.99; **b** the final stage of collapse $(R/R_0 < 0.63)$, *1*—$R/R_0 = 0.3$, *2*—0.2, *3*—0.15, *4*—0.12

$$\left.\frac{\partial^2 P}{\partial \tilde{r}^2}\right|_{r=\tilde{r}_{\max}} = -\frac{1}{4}\frac{\left(1-4\xi^3\right)^2}{\xi^5\left(1-\xi^3\right)}. \tag{2.59}$$

From Eq. (2.59) it follows that for $\xi^3 < 1$ one always gets $\left.\frac{\partial^2 P}{\partial \tilde{r}^2}\right|_{r=\tilde{r}_{\max}} < 0$. Hence, as was conjectured, the extremum is always a maximum.

Let us now consider Eqs. (2.53) and (2.55) for the coordinates of the extremum. For $\xi^3 = 1/4$ it follows from (2.53) to (2.55) that $r_{\max} \to \infty, p_{\max} = p_\infty$. In the region $0.63 < \xi < 1$ these equations have no physical meaning (because of negative values of $r_{\max}$). This is quite clear: according to condition (2.57) an extremum may not exist here. Hence, in the initial stage of the process $(0.63 < \xi < 1)$, the pressure in the liquid should vary monotonically, see Fig. 2.3a.

If ξ drops down to the value 0.63, then a weakly manifested maximum of $P(\tilde{r})$ will appear infinitely far from the bubble. If the bubble radius decreases, then its quantity rapidly increases and its radial coordinate will monotonically move in the direction to the bubble surface, see Fig. 2.4b.

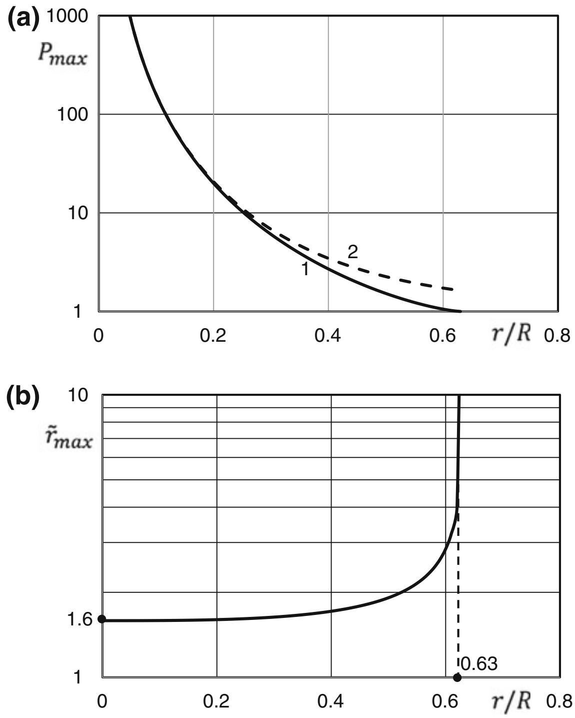

Fig. 2.4 Dependence of the magnitude of pressure maximum and coordinate of pressure maximum on the reduced bubble radius. **a** The value of pressure maximum, 1—calculation by the exact formula (2.53), 2—calculation by the asymptotic formula (2.60); **b** the coordinate of pressure maximum

In the final collapse stage, as $R \to 0$, it follows from (2.53) to (2.55) that

$$P_{\max} = 1 + \frac{1}{4^{3/4}} \frac{1}{\xi^3} \to \infty; \tag{2.60}$$

$$\tilde{r}_{\max} \to 4^{1/3} R \approx 1.587R, \tag{2.61}$$

Figure 2.4 depicts $\tilde{r}_{\max}$ and $P_{\max}$ versus the dimensionless bubble radius r/R. From (2.61) is seen that in the final collapse stage the position of the pressure maximum stabilizes at a distance from the centre at ca. 1.6 times the current bubble radius, and the maximum pressure by formula (2.60) increases inversely with the cube of the bubble radius.

From Fig. 2.4a it is seen that in the final stages of the collapse process the maximal pressure may attain fairly large values. A violation of the spherical symmetry of the flow in the final stage of collapse may lead to the appearance of a "cumulative jet". The evolution of the form of a collapsing cavity was numerically studied in a number of papers; for example, Popinet and Zaleski (2002), Sussman and Smereka (1997) and Curtiss et al. (2013). Calculations showed that small initial deformations of a bubble increase with time. The process starts with a small deviation from sphericity and is terminated with the formation of a high-speed jet directed towards the wall. An impingement on wall gradually causes damage to the wall material. This phenomenon is known as the “cavitation erosion”.

2.3.3 The Influence of Capillary Effects and Viscosity Forces

In the final stages of bubble collapse, as $R \to 0$, the Laplace jump in pressure $\Delta p_\sigma = (2\sigma/R)$ tends to infinity according to the linear law $\Delta p_\sigma \sim (1/R)$. Nevertheless, a commonly accepted point of view is that the effect of the surface tension forces on the process of bubble collapse is completely negligible (Labuntsov and Yagov 1978). This is explained by the fact that in the final stages of bubble collapse the maximum of pressure by formula (2.60) tends to zero according to the law $P_{\max} \sim (R_0/R)^3$; i.e., the dynamic effects for a collapse of a bubble are prevailing. In this section we shall show that this conclusion is correct only in the case when at the initial time (with $R = R_0$) the influence of the capillarity effects can be neglected in comparison with the external pressure drop: $2\sigma/R_0 \ll p_\infty - p_v$. If these two quantities are commensurable, the surface tension forces may have a substantial effect on the process of collapsing.

Let us consider in more detail the effect of the surface tension forces on the collapse of a bubble. To this aim we write the Rayleigh-Lamb’s Eq. (2.14) with an account of the capillary forces:

$$\frac{3}{2}\dot{R}^2 + R\ddot{R} = -\frac{1}{\rho_l}\left(\Delta p + \frac{2\sigma}{R}\right), \tag{2.62}$$

where $\Delta p = p_\infty - p_v$ is the external pressure difference.

In this setting the Laplace jump in pressures $2\sigma/R(t)$ will play the role of an additional pressure drop which accelerates the collapse of a bubble. As distinct from the case considered in the previous section, the resulting pressure drop will increase as the bubble collapses, tending to infinity as $R \to 0$.

At the initial time, the additional pressure drop due to surface tension forces will be equal

$$\Delta p_\sigma = 2\sigma/R_0. \tag{2.63}$$

Let us introduce the dimensionless capillarity parameter $\Re$, which takes into account the relative role of the effects of surface tension,

$$\Re = \frac{\Delta p_\sigma}{\Delta p} = \frac{2\sigma}{R_0 \Delta p} = \frac{R_*}{R_0}, \tag{2.64}$$

where $R_* = 2\sigma/\Delta p$ is the critical bubble radius.

By a similar analysis as in the derivation of Eqs. (2.26)–(2.29) we obtain, instead of Eq. (2.32),

$$\int_{\xi}^{1} \sqrt{\frac{\xi^3}{1-\xi^3+\frac{3}{2}\Re\left(1-\xi^2\right)}}d\xi = \frac{t}{t_d}, \tag{2.65}$$

where $t_d = R_0\sqrt{\frac{3}{2}\frac{\rho_l}{\Delta p}}$ is the dynamic scale of bubble collapse time.

To determine the total bubble collapse time t_0, we substitute in (2.65) the lower limit of integration $\xi = 0$. As a result, the dimensionless bubble collapse time will be a function of the capillarity parameter:

$$t_0/t_d = 0.7468\psi(\Re), \tag{2.66}$$

where $\psi(\Re)$ is the function allowing for the effect of surface tension.

In Fig. 2.5 the solid line (curve 1) shows the correction to the surface tension effect ψ versus the capillarity parameter $\Re$, as obtained from numerical calculation of integral (2.65). It is seen that as $\Re \to 0$ (in practice, for $\Re < 0.01$), the capillarity effect is absent, the parameter ψ becomes unity, and relation (2.66) becomes formula (2.34).

As $\Re$ increases the collapse time is reduced, as should follow from the physical considerations. It is easily shown that as $\Re \to \infty$ the value of ψ tends to zero according to

$$\psi \to 0.6756/\sqrt{\Re}. \tag{2.67}$$

The results of calculation by formula (2.67) are shown in Fig. 2.5 by the dashed line (curve 2). It is seen that already for $\Re > 5$ the exact solution of (2.65) is in good agreement with the asymptotic dependence (2.67).

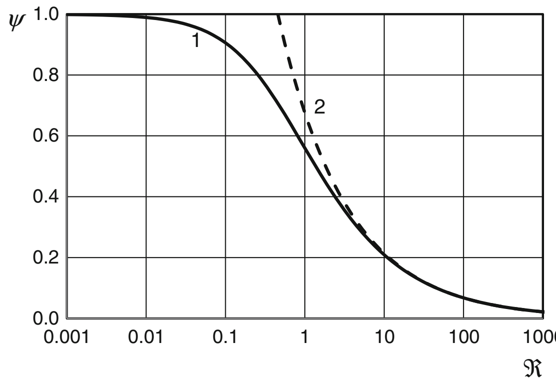

Fig. 2.5 The correction to the surface tension effect versus the capillarity parameter with collapsing bubble. *1*—Numerical calculation of the integral (2.65) (exact solution), *2*—calculation by the asymptotic formula (2.67)

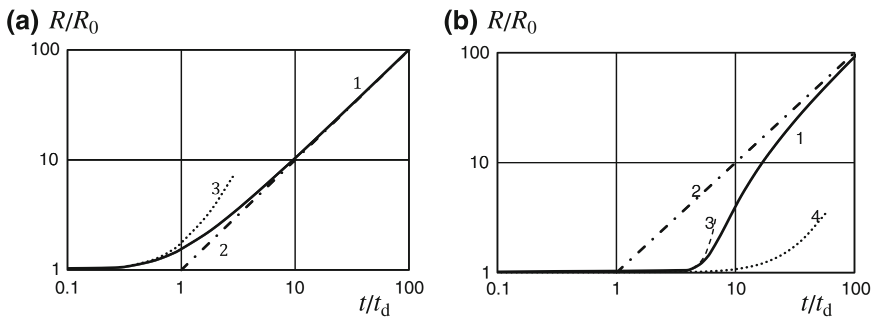

Fig. 2.6 Dynamical growth of a bubble. **a** Growth in the absence of the surface tension forces, *1*—the exact solution (2.75), *2*—the asymptotic growth law for large time (2.79) (the Rayleigh inertial growth law), *3*—the asymptotic growth law for the initial period (2.82); **b** the bubble growth in the presence of the surface tension forces ($\bar{\Re} = 0.001$), *1*—the exact solution (2.85), *2*—the Rayleigh inertial growth law (2.79), *3*—the quadratic asymptotic growth law (2.87), *4*—the linear asymptotic (2.87)

In the general case, the function $\psi(\Re)$ can be approximated with relative error at most 0.2 % by the formula

$$\psi = \frac{1}{\sqrt{1 + 2.19\Re}}, \tag{2.68}$$

which provides the necessary passages to the limit. Now the total collapse time of a bubble may be found from the formula

$$t_{0\sigma} = \frac{0.9146 R_0}{\sqrt{1 + 2.19\Re}} \sqrt{\frac{\rho_l}{\Delta p}}, \tag{2.69}$$

which differs from (2.35) by the presence of the correction function of $\Re$.

From the total collapse time one can find the time dependence of the bubble radius in the reduced form, $R/R_0 = f_\sigma(t/t_{0\sigma})$. The calculations show that the reduced law of variation of the bubble radius in time $f_\sigma(t/t_{0\sigma})$ is practically independent of the parameter $\Re$ and is described, with the maximal relative error not exceeding 3 %, by dependence (2.45), which was obtained earlier for the inertial collapse of bubble

This result is of considerable interest. Physically it means that the main parameter governing the bubble collapse time is the total initial pressure drop

$$\Delta p_0 = p_\infty - p_v + 2\sigma/R_0, \tag{2.70}$$

whereas the detailed law of its variation in time proves unimportant.

It is worth noting that this conclusion is of fairly general nature: *various processes accompanying a collapse of a spherical bubble (condensation of vapour, dissolution of gas, heat exchange on the interfacial boundary, and so on) have a relatively weak effect on the dynamics of this process, while the initial pressure drop Δp_0 is determining*.

To estimate the effect of viscosity forces we write the complete Rayleigh-Lamb Eq. (2.14)

$$\frac{3}{2}\dot{R}^2 + R\ddot{R} = -\frac{1}{\rho_l}\left(\Delta p + \frac{2\sigma}{R} + 4\mu_l\frac{\dot{R}}{R}\right). \tag{2.71}$$

The first term in brackets on the right allows for the initial pressure drop, the second term allows for the effect of the surface tension forces, and the third terms, the effect of viscosity forces.

Unfortunately, in the presence of viscosity forces the attempts to analytically solve the bubble collapse problem proved futile. Nevertheless, from the above considerations it is seen that the role of the viscosity forces in the bubble collapse process is insignificant.

Indeed, in the framework of the problem under study, the initial conditions at $t = 0$ have the form $R = R_0, \dot{R} = 0$; i.e., at the initial moment the liquid around the bubble is at rest. Hence, in the initial stage of the bubble collapse process there will be no effect of the viscosity forces. As a result, it seems logical to suppose that the effects of viscosity will have a higher order of smallness in comparison with the inertial forces, and hence in most cases they will have no significant effect practically on the bubble collapse dynamics.[4] This conjecture is supported by numerical results. According to Hammitt (1980) the effect of viscosity is manifested only in the concluding stages of the bubble collapse process (for $R/R_0 < 10^{-3}$) in high-viscosity liquids like lubricating oil.

2.4 Dynamic Growth of Vapour Bubble

Now we consider the situation when the pressure inside a bubble exceeds that in the ambient liquid, $p_v - p_\infty > 0$. For the analysis we shall always assume that this the pressure drop is constant in time. Due to the pressure drop there will be a growth of a bubble, see Fig. 2.1a. The dynamics of its growth, alongside with the actual pressure drop initiating the bubble expansion, will be determined by the inertial forces of liquid growth that retard the growth, as well as by the surface tension and viscosity forces. In this section we shall successively study the effect of these forces.

[4]For highly viscous liquids these considerations may in principle fail to hold. Criteria for estimation of the effective limits of the viscosity forces will be formulated below in Sect. 2.4.3.

It is worth pointing out that, unlike the bubble collapse problem, the present book seems to be the first to give a systematic exposition of the laws of the dynamically controlled growth of bubbles.

2.4.1 The Effect of Inertial Forces

We rewrite the Rayleigh Eq. (2.19) in the form (2.29):

$$\dot{R}^2 = \frac{2}{3}\frac{\Delta p}{\rho_l}\frac{R_0^3}{R^3}\left(\frac{R^3}{R_0^3} - 1\right). \tag{2.72}$$

For the case of bubble growth we have $\Delta p > 0, \dot{R} > 0$. Choosing in Eq. (2.72) the positive square root, we write the expression for the rate of variation of the radius:

$$\dot{R} = \sqrt{\frac{2}{3}\frac{\Delta p}{\rho_l}}\left(\frac{R_0}{R}\right)^{3/2}\left(\frac{R^3}{R_0^3} - 1\right)^{1/2}. \tag{2.73}$$

Integrating (2.73) with the initial condition

$$\text{at } t = 0\text{: } R = R_0, \tag{2.74}$$

we obtain the following quadrature

$$\int_1^{\xi} \frac{\xi^{3/2} d\xi}{\left(\xi^3 - 1\right)^{1/2}} = \frac{t}{t_d}, \tag{2.75}$$

In the general case the integral on the left of Eq. (2.75) can be written as a fairly cumbersome combination of Euler integrals. By calculating this integral one may obtain the general dependence of the bubble radius on time, as given in Fig. 2.6a in the dimensionless coordinates (curve 1).

For large growth times ($t/t_d \to \infty, \xi \to \infty$), the problem of dynamic bubble growth has a simple analytic solution. In this case the integral in (2.75) can be written as

$$\int_1^{\xi} \sqrt{\frac{\xi^3}{1 - \xi^3}} d\xi \approx \xi = \frac{R}{R_0}. \tag{2.76}$$

Taking this into account we obtain that in the final stage of dynamic bubble growth its dimensionless radius increases in time according to the linear law:

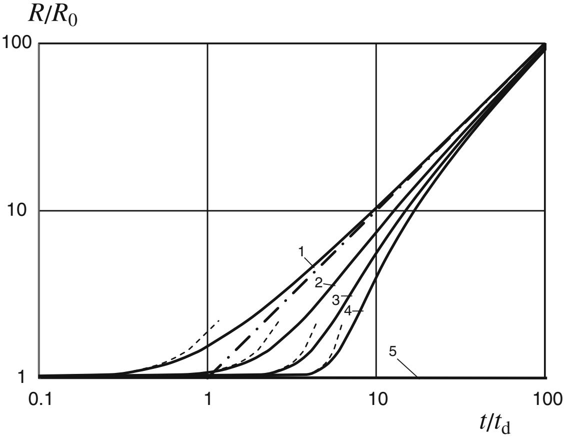

Fig. 2.7 Dynamic bubble growth with various values of the capillarity parameter. *1*—$\bar{\Re} = 1$ (the absence of the effect of the surface tension), *2*—0.1, *3*—0.01, *4*—0.001, *5*—0.0 (*horizontal line*, corresponds to the critical bubble in equilibrium with surrounding liquid)

$$R/R_0 = t/t_d. \tag{2.77}$$

In Fig. 2.6a dependence (2.77) is shown by the dash-dot line (curve 2).

In dimension form formula (2.77) looks like

$$R = t\sqrt{\frac{2}{3}\frac{\Delta p}{\rho_l}}. \tag{2.78}$$

In accordance with this equation, at large times the inertial bubble growth, when subjected to a time constant pressure drop, occurs with a constant rate independent of the bubble radius:

$$\dot{R} = \sqrt{\frac{2}{3}\frac{\Delta p}{\rho_l}} = \text{const.} \tag{2.79}$$

If the difference in the radial motion of the bubble boundary and the liquid velocity at its boundary is taken into account (allowance for the permeability of the interfacial boundary), then Eq. (2.79) can be written as

$$\dot{R} = \frac{\rho_l}{\rho_l - \rho_v}\sqrt{\frac{2}{3}\frac{\Delta p}{\rho_l}} = \text{const.} \tag{2.80}$$

Equation (2.80) is a mathematical expression for the degenerate dynamic inertia-controlled bubble growth low. Within its frameworks it is assumed that the bubble growth under a constant pressure drop Δp is governed only by the inertial dynamic reaction of the liquid against the spherical expansion, while the other forces (viscosity and surface tension forces) are absent.

In the literature the classical formula (2.80) is often interpreted as a universal *inertial-controlled bubble growth law by Rayleigh*. From the above it follows that such an approach is not fully justified: this formula is only an asymptotic growth law, which holds for sufficiently large times (practically for $t/t_d > 5$), when the bubble "forgets" the effect of initial parameters.

Now let us consider the initial bubble growth stage ($\xi \approx 1$). We set $\xi \approx 1 + \chi, \chi \ll 1$. Hence, the integral in (2.75) will assume the form

$$\int_1^{\xi} \frac{\xi^{3/2} d\xi}{\left(\xi^3 - 1\right)^{1/2}} \approx \int_0^{\chi} \frac{d\chi}{(3\chi)^{1/2}} \approx \frac{2}{\sqrt{3}}\chi^{1/2} = \frac{2}{\sqrt{3}}(\xi - 1)^{1/2} = \frac{2}{\sqrt{3}}\left(\frac{R}{R_0} - 1\right)^{1/2}. \tag{2.81}$$

Substituting (2.81) into (2.75) we obtain the final expression for the bubble growth in the initial period of time:

$$\frac{R}{R_0} = 1 + \frac{3}{4}\left(\frac{t}{t_d}\right)^2 = 1 + \frac{1}{2}\frac{\Delta p t^2}{\rho_l R_0^2}. \tag{2.82}$$

Thus, in the initial period the bubble grows according to the quadratic law (2.82). The results of calculation by the asymptotic formula (2.82) are shown in Fig. 2.6a by the dot line (curve 3).

From Eq. (2.82) we have the dimension formula for *the inertia-controlled bubble growth rate during the initial period*:

$$\dot{R} = \frac{\Delta p}{\rho_l R_0} t. \tag{2.83}$$

As distinct from the "Rayleigh's formula for inertial bubble growth" (2.79), in the initial period the bubble growth rate is not constant, but rather increases in time according to a linear law. The dependence of the bubble growth rate on the pressure drop is also different.

A comparison of the exact solution (2.75) with the asymptotic formulas (2.78) and (2.82), as given in Fig. 2.6a, shows that in the initial stage ($0 < t/t_d < 0.2$) the bubble is growing fairly slowly, its growth rate continuously increasing. Finally, at large growth times (practically for $t/t_d > 5$) the bubble "forgets" about the initial conditions and shifts to the asymptotic "inertia-controlled growth law", which is characterized by a constant growth rate.

This pattern has a transparent physical interpretation: Rayleigh's equation, which underlies the current analysis, is an equation of dynamics, within the framework of which the pressure drop in the liquid is balanced by the inertial forces of the liquid under spherically symmetric motion. If a pressure drop is suddenly started in a liquid at rest, then liquid accelerates smoothly, which provides for a gradual transition to the asymptotic dependence (2.78).

Quadrature (2.75) can be approximated by the following simple expression:

$$\frac{R}{R_0} = \frac{1}{3} + \sqrt{\frac{4}{9} + \left(\frac{t}{t_d}\right)^2}. \tag{2.84}$$

Expression (2.84) satisfies both asymptotics (2.78) and (2.82) and describes the exact solution (2.75) with error smaller than 2 %.

2.4.2 The Effect of the Surface Tension Forces

Let us consider the problem of dynamic growth of a vapour bubble which suddenly originates in the volume of a superheated liquid. Assume that the pressure inside the bubble is constant and is equal to the saturation pressure of the liquid for a given temperature $p_v = p_{sat}(T_l) > p_\infty$. Respectively, the pressure drop $\Delta p = p_v - p_\infty$ will be constant in time. The presence of the surface tension forces causes an additional pressure drop $\Delta p_\sigma = 2\sigma/R$. Hence, by the Rayleigh-Lamb Eq. (2.62), under the conditions in question the bubble growth will be triggered by the time-dependent pressure drop, which equals $\Delta p - \Delta p_\sigma$. With increasing R the capillary pressure drop Δp_σ unboundedly decreases. As a result, the effective pressure drop will tend to the constant value Δp. In other words, it follows that the relative role of the surface tension forces should be most sharply manifested in the initial period of bubble growth, and then it will start to decay as the size of the bubble will be increasing.

Similarly to the case of bubble collapse in the presence of surface tension forces (see Sect. 2.3.3), we introduce the capillarity parameter $\Re = R_*/R_0$. In the case of bubble growth, the parameter $\Re$ may vary from zero to one.

The degenerate case $\Re = 0$ corresponds to the condition $R_0 \gg R_*$. Besides, $\Delta p_\sigma \ll \Delta p$, and the influence of capillary effects on the bubble growth will not be felt, even at the initial time, and its dynamics will obey the inertial growth relations from the previous section.

In the other degenerate case $\Re = 1$, the initial bubble radius will be equal to the critical radius, $R_0 = R_*$. Moreover, here $\Delta p = \Delta p_\sigma$, and the bubble will be in equilibrium with the ambient superheated liquid, and its growth will be impossible.

In cases when the capillarity parameter is close to unity the problem under study is reduced in essence to the problem on the initial growth stage of an equilibrium vapour nucleus under some perturbation from equilibrium.

Under the conditions in question, instead of Eq. (2.65) we have

$$\int_1^\xi \sqrt{\frac{\xi^3}{(\xi^3 - 1) + \frac{3}{2}\Re(1 - \xi^2)}}\, d\xi = \frac{t}{t_d}. \tag{2.85}$$

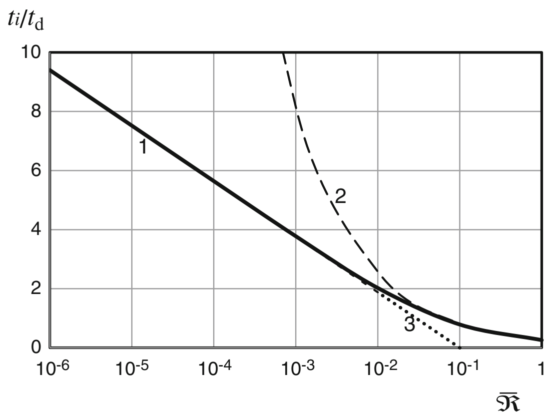

Fig. 2.8 The duration of "incubation period of bubble growth" versus the capillarity parameter. *1*—Calculation by formula (2.88) with $\chi = 0.05$, *2*—calculation by the asymptotic dependence (2.89) (weak manifestation of capillarity), *3*—calculation by the asymptotic formula (2.90) (capillary effects are prevailing)

Formula (2.85) differs from (2.65) by the sign of the difference $\left(\xi^3 - 1\right)$ and by the order in the limits of integration.

The integral on the left of (2.85), which can be obtained analytically, is a fairly cumbersome combination of elliptic integrals. To analyze the results obtained it is useful, in parallel with $\Re$, to utilize the quantity $\bar{\Re} = 1 - \Re$, which is complementary to $\Re$.

The dependences $\xi = R/R_0 = f(t/t_d)$, as calculated by formula (2.85) for various $\bar{\Re}$, are shown in Fig. 2.7.

In the limit case $\bar{\Re} = 1$, the dependence $R/R_0 = f(t/t_d)$ (curve 1) agrees with the numerical results by formula (2.84), as obtained in the case where there are no capillary effects. In the limit case, $\bar{\Re} = 0(\Re = 1)$, which physically corresponds to a critical vapour nucleus in equilibrium with the ambient superheated liquid, the design dependence is shown by the horizontal line $R/R_0 = 1$, labeled 5; i.e., there is no bubble growth.

In the case of large growth times, $t/t_d \to \infty$; $\xi \gg 1$, the quadrature (2.85) is asymptotically goes into Rayleigh's formula (2.78), which is shown in Fig. 2.7 by the dash-dot curve. Besides, physical considerations show that the effect of capillarity decays: the design curves $R/R_0 = f(t/t_d)$, as obtained for various s, are found to approach curve 1, which corresponds to $\bar{\Re} = 1$, the transition time to the asymptotic growth phase increases with decreasing $\bar{\Re}$.

Of special interest are the initial portions of the curves, when the effects of capillarity become determining: within a certain time period, whose length is a function of $\bar{\Re}$, the growth rate of bubbles is insignificant, this is followed by a rapid growth of a bubble. In other words, there is a certain "incubation period", within which the bubble growth is insignificant.

Let us study this effect in more detail. To analyze the laws for the initial growth period of a bubble, we replace ξ in formula (2.85) by $\chi = \xi - 1$:

$$2\int_0^{\chi}\sqrt{\frac{(\chi+1)^3}{4\chi^3+6\chi^2(2-\Re)+12\chi(1-\Re)}}d\chi=\frac{t}{t_d}. \tag{2.86}$$

In the case $\chi \ll 1$, neglecting the first and second terms in the denominator of this expression, it is found that, in the initial period of bubble growth, as $t/t_d \to 0$,

$$\xi=\frac{R}{R_0}=1+\frac{3}{4}(1-\Re)\left(\frac{t}{t_d}\right)^2. \tag{2.87}$$

For $\Re = 0$ relation (2.87) coincides with asymptotics (2.82), which was obtained without consideration of capillary forces.

In the case of practical importance, when the initial deviations from the equilibrium parameters are insignificant, ($\bar{\Re} = (1-\Re) \to 0$), one may not neglect the second term in the denominator of (2.86) in comparison with the third one. Hence, formula (2.87) does not apply for small $\bar{\Re}$. To illustrate this fact, the dot line in Fig. 2.6b shows the results of calculation by formula (2.87) (dot line, curve 4) versus the exact solution (2.85), which was obtained with $\bar{\Re} = 0.001$ and is shown by a solid line (curve 1). The approximation (2.87) is seen to poorly describe the exact solution (2.85).

Let us obtain the following quadratic approximation. To this aim we neglect the first term in the denominator of (2.86). Integrating the resulting expression, this gives the following quadratic approximation:

$$\frac{t}{t_d}=\sqrt{\frac{2}{3}}\ln\left[1+\frac{\chi+\sqrt{\chi^2+2\chi(1-\mathrm{S})}}{1-\mathrm{S}}\right]. \tag{2.88}$$

The results of calculation by formula (2.88) are shown in Fig. 2.6b by a dashed line (curve 3). The initial growth phase of a bubble is seen to be well fitted by Eq. (2.88) in the entire range of variation of capillarity parameter, whereas for small values of capillarity parameter formula (2.87) does not work.

Using (2.88) one may estimate the duration of the "incubation period" of a vapour nucleus. We fix $\chi = 0.05$, which corresponds to $R/R_0 = 1.05$. Then from (2.35) one may get the dependence of the dimensionless duration of the latent period of growth of a vapour nucleus t_i/t_d as a function of the initial perturbation $\bar{\Re}$, as shown in Fig. 2.8 by a solid line (curve 1). Note that t_i never vanishes. Besides, $t_i/t_d \to 0.26$, even when there are no capillarity effects ($\bar{\Re} \to 1$), which is explained by the effect of inertial "runaway" of liquid, which was discussed in the previous section, and then t_i unboundedly increases with decreasing $\bar{\Re}$.

It is easily shown that near $\bar{\Re} \approx 1$

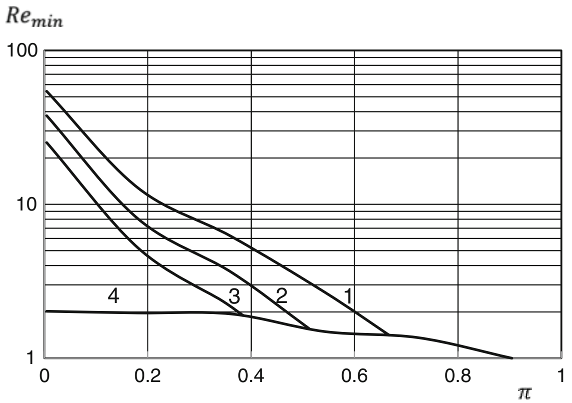

Fig. 2.9 The minimal Reynolds number versus the reduced pressure for water. *1*—Superheat 10.0 K, *2*—20.0, *3*—40.0, *4*—the maximal (spinodal) superheat

$$\frac{t_i}{t_d} = 0.816\sqrt{\frac{0.1}{\bar{\Re}}} \tag{2.89}$$

(weak manifestation of capillary effects), while for $\bar{\Re} \ll 1$

$$t_i/t_d = -0.816 \ln \bar{\Re} - 1.88 \tag{2.90}$$

(the capillary effects are prevailing). The calculation results of the duration of the "incubation period" by formulas (2.89) and (2.90) are shown in Fig. 2.8 (curves 2 and 3, respectively). It is seen that in practice formula (2.89) applies only in the range $0.04 < \bar{\Re} \leq 1$, and the asymptotics (2.90) works only for $\bar{\Re} < 0.01$.

2.4.3 The Effect of Viscosity Forces

To analyze the effect of viscosity forces on the process of dynamic growth of a bubble we shall use the complete Rayleigh-Lamb Eq. (2.14)

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho_l}\left(\Delta p - 2\frac{\sigma}{R} - 4\frac{\mu_l \dot{R}}{R}\right) \tag{2.91}$$

with the initial condition

$$\text{for } t = 0\text{: } R = R_0, \dot{R} = 0. \tag{2.92}$$

Under these conditions the bubble growth is governed by the time-stationary pressure drop, which is considered by the first (positive) term in round brackets on the right of this equation, whereas the varying in time surface tension and viscosity forces, which are accounted for by, respectively, the second and third (negative) terms hinders the growth of a bubble.

Unfortunately, the presence of the viscous term makes impossible to obtain a complete analytic solution of the problem under study. Nevertheless, a qualitative analysis, in addition with the asymptotic analytic solution, being complemented by the results of numerical analysis, enable one to see into the peculiarities of the physical mechanisms behind the effect of the viscosity on the growth process of bubbles and provide quantitative estimates, which are helpful for practical purposes.

Qualitative analysis Following Avdeev (2015), we consider, first of all, the degenerate dynamic viscous-controlled growth low of a bubble (Labuntsov 1974). Within its frameworks, we assume that at any time instant the pressure drop Δp is balanced only by the normal component of the viscous stress tensor at the bubble boundary. Here, there are neither inertial forces on the liquid and nor the surface tension forces. For this idealized case, from (2.91) we obtain the expression for the bubble growth rate

$$\dot{R} = \frac{\Delta p R}{4\mu_l}. \tag{2.93}$$

This equation with initial condition $R = R_0$ with $t = 0$ has the solution

$$R = R_0 \exp\left(\frac{1}{4}\frac{\Delta p}{\mu_l}\right), \tag{2.94}$$

and the bubble growth rate in time obeys the law

$$\dot{R} = \frac{1}{4}\frac{\Delta p R_0}{\mu_l}\exp\left(\frac{1}{4}\frac{\Delta p}{\mu_l}t\right). \tag{2.95}$$

At the initial time, for $t = 0$, the bubble growth rate is as follows

$$\dot{R} = \frac{1}{4}\frac{\Delta p R_0}{\mu_l}. \tag{2.96}$$

From Eq. (2.93) it follows that the growth rate increases with increasing radius. In other words, the role of viscosity effects continuously dies down as the radius increases. It is interesting to compare (2.93) with the degenerate dynamic inertia-controlled growth law (2.80). In accordance with (2.80), the role of dynamic inertial effects will remain constant with increasing bubble radius. Hence, the viscous-controlled growth law is capable of governing the growth rate of a vapour bubble only for small values of its radius, whereas for sufficiently large values of the bubble radius (large growth times) a transition to the inertia-controlled scheme of bubble growth is inevitable. The effects of viscosity will vanish not only for large growth times, but also in the initial growth phase of a bubble. For the problem in question, condition (2.92) implies that $\dot{R} = 0$ at $t = 0$. Now the viscous term in the

Rayleigh-Lamb Eq. (2.91) will also be zero at the start of bubble growth. Hence, the viscosity forces may have effect on the dynamics of a bubble only at some intermediate growth stage. Nevertheless, we shall see below that under certain conditions (in particular, for near-spinodal superheats) the effects of viscosity may have quite a decisive influence on the growth of bubbles.

We put the Rayleigh-Lamb Eq. (2.91) into the dimensionless form. As a linear scale we take, as before, the initial bubble radius R_0, and as the time scale, the dynamic scale

$$t_d = R_0\sqrt{\frac{3}{2}\frac{\rho_l}{\Delta p}}. \tag{2.97}$$

Hence,

$$u_0 = \frac{R_0}{t_d} = \sqrt{\frac{2}{3}\frac{\Delta p}{\rho_l}} \tag{2.98}$$

will serve as the universal scale of the velocity. In view of the above, Eq. (2.91) in the dimensionless form will assume the final form

$$\xi^2\ddot{\xi} + \frac{3}{2}\dot{\xi}^2\xi + \frac{4}{\mathrm{Re}}\dot{\xi} + \frac{3}{2}\Re - \frac{3}{2} = 0. \tag{2.99}$$

For the problem under study the initial conditions (2.92) in the dimensionless form can be rewritten as

$$\text{for } \tilde{t} \equiv t/t_d = 0\text{: } \xi = 1, \dot{\xi} = 0. \tag{2.100}$$

The fourth term on the left of the dimensionless Rayleigh-Lamb Eq. (2.99) allows for the effect of the surface tension forces. It incorporates the capillarity parameter $\Re$, which is the ratio of the critical bubble radius R_* to the initial radius R_0. The third term takes into account the effect of viscosity forces. The Reynolds number, which is contained in the third term, is determined in terms of the initial bubble radius R_0 and the characteristic velocity u_0:

$$\mathrm{Re} = \frac{R_0 u_0}{\nu_l}, \tag{2.101}$$

The physical sense of the Reynolds number is the order of the ratio of the inertial forces to the viscosity forces. Hence, for large Reynolds numbers ($\mathrm{Re} \to \infty$) the effect of viscosity dies out, while for small ones it becomes determining.

Let us estimate the order of the Reynolds number. By (2.101), the values of Re will be small for minimal values of the initial bubble radius R_0. For the bubble growth problem its initial radius may not be smaller than the critical radius,

$$R_0 \geq R_*, \tag{2.102}$$

for otherwise the capillary pressure drop will exceed the available pressure drop Δp, and hence no bubble growth will be possible.

Substituting $R_0 = R_*$ in formula (2.101), and expanding u_0 by formula (2.98), we may estimate the smallest possible Reynolds number

$$\mathrm{Re}_{\min} = \frac{2\sigma}{\mu_l}\sqrt{\frac{2}{3}\frac{\rho_l}{\Delta p}}. \tag{2.103}$$

In the superheated liquid the vapour pressure in a bubble corresponds to the saturation pressure for a given temperature of liquid on the interfacial boundary: $p_v = p_{sat}(T)$. We have $\Delta p = p_v - p_\infty = p_{sat}(T) - p_\infty$, and so the pressure drop Δp turns out to be uniquely related to the liquid superheat. Figure 2.9 depicts the value of $\mathrm{Re}_{\min}$ versus the reduced pressure $\pi = p/p_{cr}$, which are calculated for water with various superheats (curves 1–3). From formula (2.103) it follows that $\mathrm{Re}_{\min}$ will decrease with increasing liquid superheat. The maximally possible superheat of liquid corresponds to the boundary of its thermodynamic stability, spinodal (Skripov 1974; Debenedetti 1996). The value of $\mathrm{Re}_{\min}$, which was determined from the value of spinodal superheat, is shown in Fig. 2.10 (curve 4). It is seen that in the range

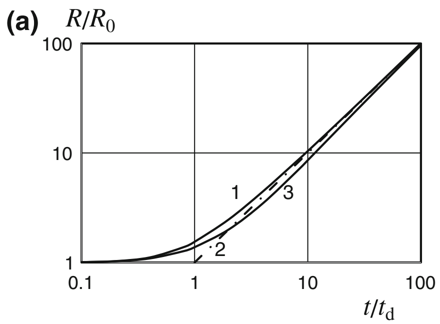

Fig. 2.10 Dynamic bubble growth in the presence of viscosity. **a** in the absence of the surface tension forces ($\Re = 0$)· *1*—the exact solution for the bubble growth in the absence of viscosity forces (2.84), *2*—the inertial Rayleigh growth law, *3*—numerical calculation of the bubble growth with maximal manifestation of viscosity forces (Re = 2); **b**: in the presence of surface tension forces, *1*—the exact solution for the bubble growth in the absence of viscosity and surface tension forces (2.84) ($\Re = 0, \mathrm{Re} \to \infty$), *2*—Rayleigh inertial growth law, *3*—growth in the presence of the surface tension ($\Re = 0.99$) and in the absence of viscosity ($\mathrm{Re} \to \infty$), *4*—growth in the presence of viscosity and surface tension forces ($\Re = 0.99, \mathrm{Re} = 10$), *5*—growth with maximal manifestation of viscosity forces ($\Re = 0.99, \mathrm{Re} = 2$)

$\pi < 0.5$ (for water this corresponds to pressures up to 11 MPa) $\mathrm{Re}_{\min} \approx 2$, and in the range $0.5 < \pi < 0.9$ the values of $\mathrm{Re}_{\min}$ gradually decrease, approaching the unity.

Thus, a conclusion can be made that for the majority of liquids the minimal possible values of Reynolds numbers, which correspond to the maximal manifestation of the viscosity effect, may not decrease below the values of approximately 1–2. The only exclusion is the region of direct vicinity of the thermodynamic critical point.

Another important conclusion is that the maximal manifestation of the viscosity effects will be seen only when studying the growth of near-critical vapour nucleuses with near-spinodal superheats.

The asymptotic solution Let us obtain an asymptotic solution of Eq. (2.99) with initial condition (2.100) for the initial period of bubble growth ($\tilde{t} \ll 1$).

Let us expand the dimensionless bubble radius $\xi = R/R_0$ in a series in powers of $\tilde{t}$ near $\tilde{t} = 0$, retaining only the terms of at most third order of smallness:

$$\xi = 1 + a\tilde{t} + b\tilde{t}^2 + c\tilde{t}^3; \tag{2.104}$$

here, a, b, c are the sought-for expansion coefficients. We have

$$\dot{\xi} = a + 2b\tilde{t} + 3c\tilde{t}^2 \tag{2.105}$$

and

$$\ddot{\xi} = 2b + 6c\tilde{t}. \tag{2.106}$$

By the boundary condition (2.100) for $\tilde{t} = 0$ we have $\dot{\xi} = 0$, and hence $a = 0$.

Taking this into account we get

$$\begin{cases} \xi = 1 + b\tilde{t}^2 + c\tilde{t}^3, \\ \dot{\xi} = 2b\tilde{t} + 3c\tilde{t}^2, \\ \ddot{\xi} = 2b + 6c\tilde{t}. \end{cases} \tag{2.107}$$

Substituting (2.107) in the reduced Rayleigh-Lamb Eq. (2.99), combining the terms of the zero and first order of smallness, we obtain

$$\begin{cases} b = \frac{3}{4}(1 - \Re), \\ c = -\frac{1-\Re}{\mathrm{Re}}. \end{cases} \tag{2.108}$$

Hence, the sought-for asymptotic solution assumes the form

$$\xi = 1 + (1 - \Re)\left(\frac{3}{4}\tilde{t}^2 - \frac{1}{\mathrm{Re}}\tilde{t}^3\right). \tag{2.109}$$

This being so, the effect of viscosity forces in the initial period of bubble growth is manifested only in the cubic in time term of the expansion.

In case there are no viscosity forces ($\mathrm{Re} \to \infty$), formula (2.109) goes over to the previously obtained asymptotic dependence (2.87).

Since the attempts to obtain a general analytic solution of the above problem proved futile, our further analysis will depend on a numerical method. Figure 2.10a gives the dependence $R/R_0 = f(\tilde{t})$, which is calculated for a bubble growth with no surface tension effects ($\Re = 0$), curve 3. In the same figure, for comparison, we show the results by formula (2.84), as obtained for a dynamical inertial bubble growth (curve 1), as well as the asymptotic formula (2.78), curve 2. The calculations were performed with $\mathrm{Re} = 2$, which is close to the minimal possible values. The analysis shows that the effect of viscosity forces is manifested only for the region of intermediate values of $\tilde{t}$ ($0.8 < \tilde{t} < 80$); this effect vanishes both for small and very values of the dimensionless time. The effect of viscosity is maximal near the point $\tilde{t} \approx 4$, at which it can be as high as 30 %.

In the case of presence of the surface tension forces, the effect of viscosity forces on the bubble growth is manifested much stronger. As an example, in Fig. 2.10b we compare the exact solution of the bubble growth problem under no viscosity (2.85) in the case $\Re = 0.99$ (curve 3, the solid line), and for a numerical solution of the bubble growth problem with viscosity in the case $\Re = 0.99$ and $\mathrm{Re} = 2$ (curve 5, the dot line). It is seen that under these conditions the effect of viscosity is highly essential: near point $\tilde{t} \approx 9$ the bubble size decreases to almost 2.5 times under the action of viscosity forces. The effect of viscosity rapidly dies down with increasing the Reynolds number; this is illustrated by the calculation results with $\mathrm{Re} = 10$, which are shown in Fig. 2.10b by dashed line (curve 4). Nevertheless, as in the above case of absence of the surface tension forces, the effect of viscosity forces is insignificant during the initial period of bubble growth ($\tilde{t} < 1.4$), as well as for very large growth times.

Using the asymptotic solution (2.109) one may estimate the initial growth period during which the effect of the viscosity forces is weakly manifested (Avdeev 2015). According to (2.109),

$$\xi_\mu = 1 + \frac{3}{4}(1 - \Re)\tilde{t}^2 - \frac{1 - \Re}{\mathrm{Re}}\tilde{t}^3. \tag{2.110}$$

In the absence of the viscosity effect, this formula goes into dependence (2.87), in accordance with which

$$\xi_\sigma = 1 + \frac{3}{4}(1 - \Re)\tilde{t}^2. \tag{2.111}$$

Specifying $\xi_\sigma - \xi_\mu \approx 0.05$ and rounding off the resulting coefficient, it is seen that the effect of viscosity is weakly manifested in the initial period of time within the interval

$$\tilde{t} < 0.4\sqrt[3]{\frac{\mathrm{Re}}{1 - \Re}}. \tag{2.112}$$

By comparing with numerical results this inequality is seen to properly describe the initial duration of the growth period, within which the bubble growth satisfies the analytic dependences obtained in the model of perfect liquid.

Thus, the analysis carried out in the present section allows one to draw up the conclusion that the effects of viscosity have no influence both during the initial stage of growth of bubbles and for large growth times. Nevertheless, in the intermediate region, the effect of the viscosity forces may manifest itself in a number of cases rather strongly, even for low-viscous liquids. The effect of viscosity occurs most strong for the growth of bubbles of near-critical sizes and for near-spinodal superheatings.

2.5 Conclusions

- The present chapter was concerned with a wide range of problems in *bubble dynamics,* for which the evolution of a bubble is mainly governed by the forces acting in liquid: the pressure difference, the inertial forces, the viscosity and the surface tension forces. From the entire variety of problems of bubble dynamics of scientific and practical interest the following problems are singled out: the *dynamic-controlled bubble growth* problem and the *bubble collapse* problem.
- The Rayleigh-Lamb equation, which is the main equation of bubble dynamics, is obtained by considering the hydrodynamics of motion of the ambient liquid and through the balance of energy. A detailed study is given to the "viscosity paradox", in accordance with which the viscous forces in the Rayleigh-Lamb equation are manifested only in terms of the boundary condition on the interfacial boundary.
- The classical problem on the collapse of a bubble is solved. A detailed analysis is given to the laws of variation of its radius and the distribution of the pressure in the liquid. The asymptotical branches of the solutions are considered. The effect of surface tension forces on the process of collapsing is examined. Contrary to a widespread opinion, the surface tension forces are shown as having a considerable effect on the process of collapsing in terms of the variation of the initial pressure difference. Interpolation design formulas are obtained capable of describing, to a high degree, the process of collapsing and providing for the necessary passages to the limit.
- For the first time, a systematic treatment is given to the process of dynamic bubble growth with account of the effect of the surface tension and viscosity forces. In the absence of viscosity and surface tension forces (the *dynamic inertia-controlled scheme of growth*) in the initial period, the bubble growth rate is shown to increase in time according to the linear law, and only later it

asymptotically goes over to the concluding stage, which is characteristic of constant growth rate.
- A study is given of the initial growth period of vapour nucleus in equilibrium with the ambient superheated liquid, depending on the value of the initial perturbation. A conclusion is made that there exists a kind of an "incubation period" of bubble growth, within which the bubble growth is not very strong. The effect of viscosity is shown to degenerate both in the initial growth stages of a bubble and for large growth times. Nevertheless, in the intermediate growth stages, the effects of viscosity may have a considerable effect on the bubble growth even for relatively low-viscous liquids like water. The effect of viscosity is manifested most noticeable for the growth of vapour nucleuses for liquid superheats close to spinodal ones.

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