

# RM and its Nice Properties

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**Abstract** Dunn–McCall logic **RM** is by far the best understood and the most well-behaved logic in the family of logics developed by the school of Anderson and Belnap. However, it is not considered to be a relevant logic by the relevant logicians, since it fails to have the variable-sharing property. Instead, **RM** is usually characterized as being “semi-relevant,” without explaining what this notion means. In this paper we suggest a plausible definition of semi-relevance, and show that according to it, **RM** is a strongly maximal semi-relevant logic having a conjunction, a disjunction, and an implication. We also review and prove the most important nice properties of **RM**, especially strong completeness results about it (the full proofs of which are difficult to find in the literature).

**Keywords** Degrees of truth · Fuzzy logics · Paraconsistent logics · Relevant logics · Semi-relevance

## 1 Introduction

The central idea behind the design of  $\mathbf{R}_{\rightarrow}$ , the basic, purely implicational fragment of the relevant logic **R**, is that  $\varphi \rightarrow \psi$  should relevantly follow from  $\mathcal{T}$  iff there is a proof of  $\psi$  from  $\mathcal{T}$ ,  $\varphi$  in which  $\varphi$  is actually *used*.<sup>1</sup> But what exactly is meant by ‘ $\mathcal{T}, \varphi$ ’ in this formulation? In textbooks on logics, this is usually just an abbreviation for  $\mathcal{T} \cup \{\varphi\}$ , where  $\mathcal{T}$  is a *set* of formulas. However, this interpretation is problematic from the point of view of  $\mathbf{R}_{\rightarrow}$ . To see why, consider the question whether  $\varphi \rightarrow \varphi$  should relevantly follow from the assumption  $\varphi$ . According to the above criterion, this is the case iff there is a proof of  $\varphi$  from  $\varphi$ ,  $\varphi$  that actually uses  $\varphi$ . By the standard interpretation, this means that there is a proof of  $\varphi$  from  $\{\varphi\} \cup \{\varphi\}$  that uses  $\varphi$ , i.e., there is a proof of  $\varphi$  from  $\{\varphi\}$  that uses  $\varphi$ . This is certainly the case, and so we should

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<sup>1</sup>This principle is practically abandoned in the full system **R**.

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conclude that indeed  $\varphi \rightarrow \varphi$  relevantly follows from  $\varphi$ , implying that  $\varphi \rightarrow (\varphi \rightarrow \varphi)$  should be provable. Unfortunately, this formula is *not* provable in  $\mathbf{R}_{\rightarrow}$ . The reason is that the above criterion leads to  $\mathbf{R}_{\rightarrow}$  only if the term ‘ $\mathcal{T}, \varphi$ ’ in its formulation is understood as the *multiset* which is obtained by adding (a copy of)  $\varphi$  to the *multiset*  $\mathcal{T}$ .

It somewhat looks strange to take relevant entailment as a relation between *multisets* of formulas and formulas, rather than between *sets* of formulas and formulas (as consequence relations are usually and most naturally taken to be). This observation motivated J.M. Dunn and S. McCall in investigating the results of adding to  $\mathbf{R}$  and its fragments the mingle axiom  $\varphi \rightarrow (\varphi \rightarrow \varphi)$  considered above. In the case of  $\mathbf{R}_{\rightarrow}$ , this addition yields  $\mathbf{RM0}_{\rightarrow}$ , which is the minimal system in which the above criterion for relevant entailment is met, with the latter taken as a relation between *sets* of formulas and formulas. In the case of the full system  $\mathbf{R}$ , it yields a very interesting system called  $\mathbf{RM}$  (“ $\mathbf{R}$ -mingle”). As noted in Dunn and Restall (2002), Dunn–McCall logic  $\mathbf{RM}$  “is by far the best understood of the Anderson–Belnap style systems.” However, it is not considered to be a relevant logic by the relevant logicians, since it fails to have the variable-sharing property. Instead,  $\mathbf{RM}$  is usually characterized as being “semi-relevant,” without explaining what this notion means. In this paper we suggest a plausible definition of semi-relevance, and show that according to it,  $\mathbf{RM}$  is a strongly maximal semi-relevant logic having a conjunction, a disjunction, and an implication. We also review and prove known important properties of  $\mathbf{RM}$ , especially *strong* completeness results whose full proofs are difficult to find in the literature.

## 2 Preliminaries

### 2.1 Propositional Logics

In the sequel,  $\mathcal{L}$  denotes a propositional language. The set of well-formed formulas of  $\mathcal{L}$  is denoted by  $\mathcal{W}(\mathcal{L})$ , and  $\varphi, \psi, \sigma$  vary over its elements.  $\mathcal{T}, \mathcal{S}$  vary over theories of  $\mathcal{L}$  (where by a ‘theory’ we simply mean here a subset of  $\mathcal{W}(\mathcal{L})$ ), and  $\Gamma, \Delta$  vary over *finite* sets of formulas. We denote by  $\text{Atoms}(\varphi)$  ( $\text{Atoms}(\mathcal{T})$ ) the set of atomic formulas that appear in  $\varphi$  (in formulas of  $\mathcal{T}$ ).

**Definition 2.1** A (Tarskian) *consequence relation* (tcr) for a language  $\mathcal{L}$  is a binary relation  $\vdash$  between theories in  $\mathcal{W}(\mathcal{L})$  and formulas in  $\mathcal{W}(\mathcal{L})$ , satisfying the following three conditions.

- [R] *Reflexivity*:  $\psi \vdash \psi$  (i.e.,  $\{\psi\} \vdash \psi$ ).
- [M] *Monotonicity*: If  $\mathcal{T} \vdash \psi$  and  $\mathcal{T} \subseteq \mathcal{T}'$ , then  $\mathcal{T}' \vdash \psi$ .
- [C] *Cut (Transitivity)*: If  $\mathcal{T} \vdash \psi$  and  $\mathcal{T}', \psi \vdash \varphi$ , then  $\mathcal{T} \cup \mathcal{T}' \vdash \varphi$

**Definition 2.2** Let  $\vdash$  be a Tarskian consequence relation for  $\mathcal{L}$ .

- $\vdash$  is *structural*, if for every  $\mathcal{L}$ -substitution  $\theta$  and every  $\mathcal{T}$  and  $\psi$ , if  $\mathcal{T} \vdash \psi$ , then  $\theta(\mathcal{T}) \vdash \theta(\psi)$ .
- $\vdash$  is *non-trivial*, if  $p \not\vdash q$  for distinct atoms  $p, q \in \mathbf{Atoms}(\mathcal{L})$ .
- $\vdash$  is *finitary*, if for every theory  $\mathcal{T}$  and every formula  $\psi$  such that  $\mathcal{T} \vdash \psi$ , there is a *finite* theory  $\Gamma \subseteq \mathcal{T}$  such that  $\Gamma \vdash \psi$ .

*Note 2.1* The condition of non-triviality is strictly stronger than the more familiar condition of consistency used by Dunn in Sect. 29.4 of Anderson and Belnap (1975), which says that  $p \not\vdash q$  for  $q \in \mathbf{Atoms}(\mathcal{L})$ . Thus the tcr  $\vdash$  for which  $\mathcal{T} \vdash \varphi$  iff  $\mathcal{T} \neq \emptyset$  is structural, finitary, and consistent, but not non-trivial.

**Definition 2.3**

- A (propositional) *logic* is a pair  $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ , where  $\mathcal{L}$  is a propositional language, and  $\vdash_{\mathbf{L}}$  is a structural and non-trivial tcr for  $\mathcal{L}$ .<sup>2</sup>
- A logic  $\langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$  is *finitary* if  $\vdash_{\mathbf{L}}$  is finitary.

**Definition 2.4** Let  $\mathbf{L}_1 = \langle \mathcal{L}_1, \vdash_{\mathbf{L}_1} \rangle$  and  $\mathbf{L}_2 = \langle \mathcal{L}_2, \vdash_{\mathbf{L}_2} \rangle$  be propositional logics.

- $\mathbf{L}_1$  is an *extension* of  $\mathbf{L}_2$ , if  $\mathcal{L}_2 \subseteq \mathcal{L}_1$  and  $\vdash_{\mathbf{L}_2} \subseteq \vdash_{\mathbf{L}_1}$ .
- $\mathbf{L}_1$  is a *simple extension* of  $\mathbf{L}_2$ , if  $\mathcal{L}_2 = \mathcal{L}_1$  and  $\vdash_{\mathbf{L}_2} \subseteq \vdash_{\mathbf{L}_1}$ .
- $\mathbf{L}_1$  is a *proper extension* of  $\mathbf{L}_2$ , if  $\mathcal{L}_2 \subseteq \mathcal{L}_1$  and  $\vdash_{\mathbf{L}_2} \subsetneq \vdash_{\mathbf{L}_1}$ .
- $\mathbf{L}_1$  is a *strongly proper extension* of  $\mathbf{L}_2$ , if  $\mathcal{L}_2 \subseteq \mathcal{L}_1$ , and there is a sentence  $\varphi$  of  $\mathcal{L}_2$  such that  $\vdash_{\mathbf{L}_1} \varphi$  but  $\not\vdash_{\mathbf{L}_2} \varphi$ .
- $\mathbf{L}_1$  is a *conservative extension* of  $\mathbf{L}_2$ , if  $\mathcal{L}_2 \subseteq \mathcal{L}_1$ , and  $\mathcal{T} \vdash_{\mathbf{L}_1} \psi$  iff  $\mathcal{T} \vdash_{\mathbf{L}_2} \psi$  whenever  $\mathcal{T} \cup \{\psi\} \in 2^{\mathcal{W}(\mathcal{L}_2)}$ .
- $\mathbf{L}_1$  is a *weakly conservative extension* of  $\mathbf{L}_2$ , if  $\mathcal{L}_2 \subseteq \mathcal{L}_1$ , and  $\vdash_{\mathbf{L}_1} \psi$  iff  $\vdash_{\mathbf{L}_2} \psi$  whenever  $\psi \in \mathcal{W}(\mathcal{L}_2)$ .
- $\mathbf{L}_1$  is an *axiomatic extension* of  $\mathbf{L}_2$ , if  $\mathcal{L}_2 \subseteq \mathcal{L}_1$ , and there is a set  $\mathcal{S}$  of sentences in  $\mathcal{L}_1$  such that  $\vdash_{\mathbf{L}_1}$  is the minimal structural tcr  $\vdash$  on  $\mathcal{L}_1$  which satisfies the following conditions:  $\vdash_{\mathbf{L}_2} \subseteq \vdash$ , and  $\vdash \varphi$  for every  $\varphi \in \mathcal{S}$ .

**Definition 2.5** Let  $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$  be a propositional logic.

- A binary connective  $\supset$  of  $\mathcal{L}$  is called an *implication for  $\mathbf{L}$*  if the classical deduction theorem holds for  $\supset$  and  $\vdash_{\mathbf{L}}$ . That is,

$$\mathcal{T}, \varphi \vdash_{\mathbf{L}} \psi \quad \text{iff} \quad \mathcal{T} \vdash_{\mathbf{L}} \varphi \supset \psi.$$

- A binary connective  $\wedge$  of  $\mathcal{L}$  is called a *conjunction for  $\mathbf{L}$*  if it satisfies the following condition:

$$\mathcal{T} \vdash_{\mathbf{L}} \psi \wedge \varphi \quad \text{iff} \quad \mathcal{T} \vdash_{\mathbf{L}} \psi \text{ and } \mathcal{T} \vdash_{\mathbf{L}} \varphi.$$

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<sup>2</sup>The condition of non-triviality is not always explicitly demanded, but we have found it (here and elsewhere) convenient to include it in order to avoid uninteresting pathological cases.

- A binary connective  $\vee$  of  $\mathcal{L}$  is called a *disjunction for  $\mathbf{L}$*  if it satisfies the following condition:

$$\mathcal{T}, \psi \vee \varphi \vdash_{\mathbf{L}} \sigma \quad \text{iff} \quad \mathcal{T}, \psi \vdash_{\mathbf{L}} \sigma \text{ and } \mathcal{T}, \varphi \vdash_{\mathbf{L}} \sigma.$$

**Definition 2.6** We call a logic *normal* if it has all the basic connectives above (conjunction, disjunction, implication).<sup>3</sup>

**Definition 2.7** Let  $\mathcal{L}$  be a propositional language.

- A *matrix* for  $\mathcal{L}$  is a triple  $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ , where
  - (1)  $\mathcal{V}$  is a non-empty set of truth values;
  - (2)  $\mathcal{D}$  is a non-empty proper subset of  $\mathcal{V}$  (the *designated* elements of  $\mathcal{V}$ );
  - (3)  $\mathcal{O}$  is a function that associates an  $n$ -ary function  $\tilde{\diamond}_{\mathcal{M}}: \mathcal{V}^n \rightarrow \mathcal{V}$  with every  $n$ -ary connective  $\diamond$  of  $\mathcal{L}$ .

We say that  $\mathcal{M}$  is *(in)finite*, if so is  $\mathcal{V}$ .

- Let  $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$  be a matrix for  $\mathcal{L}$ . An  $\mathcal{M}$ -*valuation* for  $\mathcal{L}$  is a function  $v: \mathcal{W}(\mathcal{L}) \rightarrow \mathcal{V}$  such that  $v(\diamond(\psi_1, \dots, \psi_n)) = \tilde{\diamond}_{\mathcal{M}}(v(\psi_1), \dots, v(\psi_n))$  for every  $n$ -ary connective  $\diamond$  of  $\mathcal{L}$  and every  $\psi_1, \dots, \psi_n$  in  $\mathcal{W}(\mathcal{L})$ .
- An  $\mathcal{M}$ -valuation  $v$  is an  $\mathcal{M}$ -*model* of a formula  $\psi$ , or  $v$   $\mathcal{M}$ -*satisfies*  $\psi$  (notation:  $v \models_{\mathcal{M}} \psi$ ), if  $v(\psi) \in \mathcal{D}$ . We say that  $v$  is an  $\mathcal{M}$ -*model* of a theory  $\mathcal{T}$  (notation:  $v \models_{\mathcal{M}} \mathcal{T}$ ), if it is an  $\mathcal{M}$ -model of every element of  $\mathcal{T}$ .
- Let  $\mathcal{M}$  be a matrix for  $\mathcal{L}$ .  $\vdash_{\mathcal{M}}$ , the consequence relation that is induced by  $\mathcal{M}$ , is defined by:  $\mathcal{T} \vdash_{\mathcal{M}} \psi$  if every  $\mathcal{M}$ -model of  $\mathcal{T}$  is an  $\mathcal{M}$ -model of  $\psi$ . We shall denote by  $\mathbf{L}_{\mathcal{M}}$  the logic  $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$  which is induced by  $\mathcal{M}$ .

**Definition 2.8** Let  $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$  be a propositional logic, and let  $\mathcal{M}$  be a matrix for  $\mathcal{L}$ .

- If  $\mathbf{L}_{\mathcal{M}}$  is an extension of  $\mathbf{L}$ , we say that  $\mathbf{L}$  is *sound* for  $\mathcal{M}$ .
- If  $\mathbf{L}$  is an extension of  $\mathbf{L}_{\mathcal{M}}$ , we say that  $\mathbf{L}$  is *complete* for  $\mathcal{M}$ .
- $\mathcal{M}$  is a *characteristic* matrix for  $\mathbf{L}$ , if  $\mathbf{L} = \mathbf{L}_{\mathcal{M}}$  (that is, if  $\mathbf{L}$  is both sound and complete for  $\mathbf{L}_{\mathcal{M}}$ ).
- $\mathbf{L}_{\mathcal{M}}$  is *weakly sound* for  $\mathbf{L}$ , if for every  $\psi \in \mathcal{W}(\mathcal{L})$ ,  $\vdash_{\mathcal{M}} \psi$  implies that  $\vdash_{\mathbf{L}} \psi$ .  $\mathbf{L}_{\mathcal{M}}$  is *weakly complete* for  $\mathbf{L}$ , if  $\vdash_{\mathbf{L}} \psi$  implies that  $\vdash_{\mathcal{M}} \psi$ .
- $\mathcal{M}$  is a *weakly characteristic* matrix for  $\mathbf{L}$ , if  $\mathbf{L}$  is both weakly sound and weakly complete for  $\mathbf{L}_{\mathcal{M}}$  (that is,  $\vdash_{\mathcal{M}} \psi$  iff  $\vdash_{\mathbf{L}} \psi$ ).

## 2.2 Some Basic Relevant Logics

In this section, we shortly review some basic relevant logics, together with their properties that will be used later in our study of **RM**. (**RM** itself will be introduced in Sect. 4.) We start with the central relevant logic **R**.

<sup>3</sup>Our notion of normality should not be confused with the notion of normality used in modal logics, or the notion of normal theory used in Anderson and Belnap (1975).

**Definition 2.9** Let  $\mathcal{L}_R = \{ \wedge, \vee, \rightarrow, \neg \}$ .

**Definition 2.10**  $\mathbf{R}$  is the logic in  $\mathcal{L}_R$  which is induced by the system  $HR$  that is presented in Fig. 1.

For our purposes, the most important property of  $\mathbf{R}$  is the following theorem, an (implicit) proof of which can be found, e.g., in Anderson and Belnap (1975, p. 301).

**Theorem 2.11**  $\vee$  is a disjunction for any axiomatic extension of  $\mathbf{R}$ .

A particularly important fragment of  $\mathbf{R}$  is its intensional fragment.

**Definition 2.12** Let  $HR_{\rightarrow}$  be the Hilbert-type systems in  $\{ \neg, \rightarrow \}$  whose axioms and rule are those axioms and rule of  $HR$  which do not mention  $\wedge$  or  $\vee$  (i.e., [Id], [Tr], [Pe], [Ct], [N1], [N2], and [MP]).  $\mathbf{R}_{\rightarrow}$  is the logic in  $\{ \neg, \rightarrow \}$  which is induced by  $HR_{\rightarrow}$ .

The following theorem has been proved by Meyer (see Anderson and Belnap 1975).

**Proposition 2.13**  $\mathbf{R}$  is a conservative extension of  $\mathbf{R}_{\rightarrow}$ . In other words,  $HR_{\rightarrow}$  axiomatizes the  $\{ \neg, \rightarrow \}$ -fragment of  $\mathbf{R}$ .

The most significant property of  $\mathbf{R}_{\rightarrow}$  is that very natural relevant deduction theorems obtain for it. The simplest one is the following proposition from Avron (2014) (originally due to Church).

**Fig. 1** The proof system  $HR$

**Axioms:**

[Id]	$\varphi \rightarrow \varphi$
[Tr]	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \sigma) \rightarrow (\varphi \rightarrow \sigma))$
[Pe]	$(\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \sigma))$
[Ct]	$(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$
[N1]	$(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)$
[N2]	$\neg\neg\varphi \rightarrow \varphi$
[ConE1]	$\varphi \wedge \psi \rightarrow \varphi$
[ConE2]	$\varphi \wedge \psi \rightarrow \psi$
[ConI]	$(\sigma \rightarrow \varphi) \wedge (\sigma \rightarrow \psi) \rightarrow (\sigma \rightarrow \varphi \wedge \psi)$
[DisI1]	$\varphi \rightarrow \varphi \vee \psi$
[DisI2]	$\psi \rightarrow \varphi \vee \psi$
[DisE]	$(\varphi \rightarrow \sigma) \wedge (\psi \rightarrow \sigma) \rightarrow (\varphi \vee \psi \rightarrow \sigma)$
[Dist]	$\varphi \wedge (\psi \vee \sigma) \rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \sigma)$

**Rules:**

[MP]	$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}$	[Ad]	$\frac{\varphi \quad \psi}{\varphi \wedge \psi}$
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**Proposition 2.14** *Let  $\mathbf{L}$  be an axiomatic extension of  $\mathbf{R}_{\rightarrow}$ . Then  $\mathbf{L}$  satisfies the following relevant deduction theorem:*

$$\mathcal{T}, \varphi \vdash_{\mathbf{L}} \psi \text{ iff } \mathcal{T} \vdash_{\mathbf{L}} \psi \text{ or } \mathcal{T} \vdash_{\mathbf{L}} \varphi \rightarrow \psi.$$

Another important property of  $\mathbf{R}_{\rightarrow}$  (see Anderson and Belnap 1975; Dunn and Restall 2002) is the fact that it has a corresponding cut-free Gentzen-type calculus, which can be used for a decision procedure. That system can also be used for an easy proof of the next lemma.

**Definition 2.15**

- $\varphi + \psi =_{Df} \neg\varphi \rightarrow \psi$
- $\varphi \otimes \psi =_{Df} \neg(\varphi \rightarrow \neg\psi)$
- $\varphi \leftrightarrow \psi =_{Df} (\varphi \rightarrow \psi) \otimes (\psi \rightarrow \varphi)$

**Lemma 2.16** *All instances of the following formulas are provable in  $HR_{\rightarrow}$ :*

1.  $(\varphi \leftrightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$  and  $(\varphi \leftrightarrow \psi) \rightarrow (\psi \rightarrow \varphi)$
2.  $(\varphi + \psi) \leftrightarrow (\psi + \varphi)$  and  $((\varphi + \psi) + \sigma) \leftrightarrow (\varphi + (\psi + \sigma))$
3.  $(\varphi + \varphi) \rightarrow \varphi$
4.  $(\varphi_1 \rightarrow \psi_1) \rightarrow ((\varphi_2 \rightarrow \psi_2) \rightarrow ((\varphi_1 + \varphi_2) \rightarrow (\psi_1 + \psi_2)))$
5.  $(\varphi \rightarrow \sigma) \rightarrow ((\psi \rightarrow \sigma) \rightarrow ((\varphi + \psi) \rightarrow \sigma))$
6.  $\neg\varphi + \varphi$
7.  $\neg\neg\varphi \leftrightarrow \varphi$
8.  $(\neg\psi \rightarrow \sigma) \rightarrow ((\psi \rightarrow \sigma) \rightarrow \sigma)$
9.  $(\psi \rightarrow \neg\sigma) \rightarrow ((\psi \rightarrow \sigma) \rightarrow \neg\psi)$
10.  $(\varphi \rightarrow (\varphi + \varphi)) \leftrightarrow (\neg\varphi \rightarrow (\neg\varphi \rightarrow \neg\varphi))$
11.  $((\varphi \rightarrow \varphi) + (\psi \rightarrow \psi)) \leftrightarrow (\varphi \rightarrow (\psi \rightarrow (\varphi + \psi))) \leftrightarrow ((\varphi \rightarrow \psi) + (\psi \rightarrow \varphi))$

With the help of [DisI1] and [DisI2], item 5 of Lemma 2.16 entails

**Lemma 2.17**  $\vdash_{\mathbf{R}} (\varphi + \psi) \rightarrow \varphi \vee \psi$ .

One more important property of  $\mathbf{R}$  and  $\mathbf{R}_{\rightarrow}$  that we will need is given in the next proposition.

**Proposition 2.18** *Every simple extension  $\mathbf{L}$  of either  $\mathbf{R}$  or  $\mathbf{R}_{\rightarrow}$  has the replacement property, that is, if  $\vdash_{\mathbf{L}} \psi \leftrightarrow \varphi$ , then  $\vdash_{\mathbf{L}} \sigma\{\varphi/p\} \leftrightarrow \sigma\{\psi/p\}$  for every sentence  $\sigma$  and atom  $p$ .*

Another central purely intensional relevant logic is the following logic, which can easily be seen to be a simple axiomatic extension of  $\mathbf{R}_{\rightarrow}$ .

**Definition 2.19** Let  $HRMI_{\rightarrow}$  be the Hilbert-type system in  $\{\neg, \rightarrow\}$  that is obtained from  $HR_{\rightarrow}$  by replacing the identity axiom [Id] by the mingle axiom:

$$[Mi] \quad \varphi \rightarrow (\varphi \rightarrow \varphi)$$

$RMI_{\rightarrow}$  is the logic in  $\{\neg, \rightarrow\}$  which is induced by  $HRMI_{\rightarrow}$ .

$RMI_{\rightarrow}$  has been investigated in Avron (1984). It is shown there that it has the following properties.

- the variable-sharing property<sup>4</sup>;
- a weakly characteristic infinite matrix that provides a decision procedure;
- Scroggs’ property (which **RM** has as well—see Theorem 6.9);
- an associated cut-free Gentzen-type system  $GRMI_{\rightarrow}$  which provides a decision procedure too.

$GRMI_{\rightarrow}$  can be used for verifying the next lemma. Alternatively, the lemma can easily be proved with the help of Lemma 2.16, the definition of  $+$ , and the mingle axiom.

**Lemma 2.20** *All instances of the following formulas are theorems of  $RMI_{\rightarrow}$ .*

1.  $\psi + \psi \leftrightarrow \psi$  (and so  $\psi \leftrightarrow (\neg\psi \rightarrow \psi)$ )
2.  $(\psi \rightarrow \sigma) \rightarrow (\sigma \rightarrow (\psi \rightarrow \sigma))$
3.  $(\psi \rightarrow \sigma) \rightarrow (\neg\psi \rightarrow (\psi \rightarrow \sigma))$
4.  $\neg((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \psi)) \rightarrow \psi$
5.  $\neg(\psi \rightarrow \sigma) \rightarrow ((\psi \rightarrow \sigma) \rightarrow \sigma)$
6.  $\neg(\psi \rightarrow \sigma) \rightarrow ((\psi \rightarrow \sigma) \rightarrow \neg\psi)$
7.  $(\neg\psi \rightarrow \sigma) \rightarrow (\neg\psi \rightarrow (\psi \rightarrow \sigma))$
8.  $(\sigma \rightarrow \neg\psi) \rightarrow (\sigma \rightarrow (\psi \rightarrow \sigma))$

### 3 Semi-relevance

In Avron (2014) we have tried to characterize the notion of a relevant logic. A central part in that characterization was the presence of an implication  $\rightarrow$  with certain properties, including the famous variable-sharing property of Anderson and Belnap (see Anderson and Belnap 1975). Now we turn to the problem of characterizing “semi-relevance.” Naturally, this should be a weaker notion, for which the notion of relevance is still relevant. Our idea is to look for general conditions, not depending on the properties of any particular connective, which seem relevant. One such condition that seems absolutely necessary was already given in Avron (2014):

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<sup>4</sup>This was observed already in Parks (1972). See also (Anderson and Belnap 1975, p. 148).

**Definition 3.1** A logic  $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$  satisfies the *basic relevance criterion* if for every two theories  $\mathcal{T}_1, \mathcal{T}_2$  and formula  $\psi$ , we have that  $\mathcal{T}_1 \vdash_{\mathbf{L}} \psi$  whenever  $\mathcal{T}_1 \cup \mathcal{T}_2 \vdash_{\mathbf{L}} \psi$  and  $\mathcal{T}_2$  has no atomic formulas in common with  $\mathcal{T}_1 \cup \{\psi\}$ .

*Note 3.1* As explained in Avron (2014), the idea behind the basic relevance criterion is that if a theory  $\mathcal{T}_2$  shares no content with  $\mathcal{T}_1 \cup \{\psi\}$  then it should not be *relevant* to the question whether  $\mathcal{T}_1 \vdash \psi$  or not. These idea and criterion were already implicit in the claim denoted by RM87, on p. 418 of Anderson and Belnap (1975), and almost explicit in the discussion that follows it. It is argued there that this criterion is in fact stronger than the usual relevance criterion (i.e., the variable-sharing property). RM87 (actually, the discussion that follows it) claims that **RM** and **R** satisfy it. Though these claims are correct, their proofs in Anderson and Belnap (1975) are not: they use a false deduction theorem for those logics.<sup>5</sup> Below we provide a correct proof in the case of **RM** (see Proposition 6.5).

The following proposition is an immediate consequence of Definition 3.1.

**Proposition 3.2** Suppose  $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$  is a logic that satisfies the basic relevance criterion. Then:

1. if  $\mathcal{T} \vdash_{\mathbf{L}} \psi$ , then either  $\vdash_{\mathbf{L}} \psi$ , or  $\mathcal{T}$  and  $\psi$  share an atomic formula;
2.  $\mathcal{T} \not\vdash_{\mathbf{L}} q$  whenever  $q$  is an atom that does not occur in any formula of  $\mathcal{T}$ ;
3.  $\mathbf{L}$  is paraconsistent with respect to any (primitive or defined) unary connective  $\neg$  of  $\mathcal{L}$ , i.e.,  $\neg p, p \not\vdash_{\mathbf{L}} q$  in case  $p$  and  $q$  are distinct atoms.

*Example 3.2* 1. Since  $q$  follows from  $\{p, \neg p\}$  in classical logic and in intuitionistic logic, these logics do not satisfy the basic relevance criterion. However, their *positive fragments* are easily seen to satisfy it.

2. Let  $\mathcal{M} = (\{t, \top, f\}, \{\top, \vdash\}, \mathcal{O})$  be a three-valued logic. Assume that all operations of  $\mathcal{O}$  are  $\{\top\}$ -closed (i.e., that  $\hat{\diamond}(\top, \top, \dots, \top) = \top$  for every connective  $\diamond$  of the language). Then  $\mathbf{L}_{\mathcal{M}}$  satisfies the basic relevance criterion. That is, if  $\text{Atoms}(\mathcal{T}_2) \cap \text{Atoms}(\mathcal{T}_1 \cup \{\psi\}) = \emptyset$ , then by assigning  $\top$  to any  $p$  in  $\text{Atoms}(\mathcal{T}_2)$  we can turn any countermodel of  $\mathcal{T}_1 \vdash_{\mathcal{M}} \psi$  into a countermodel of  $\mathcal{T}_1, \mathcal{T}_2 \vdash_{\mathcal{M}} \psi$ .

**Proposition 3.3** Suppose  $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$  is a finitary logic that satisfies the basic relevance criterion. Then  $\mathbf{L}$  has a characteristic matrix.

*Proof* A logic which satisfies the basic relevance criterion is by definition *uniform* (Urquhart 2001), while according to Łoś–Suszko’s Theorem (see Łoś and Suszko 1958; Urquhart 2001), a uniform finitary propositional logic has a single characteristic matrix.  $\square$

<sup>5</sup>Thus if  $\mathcal{T}$  is  $\{p\}$ , and  $\psi$  is  $p \wedge (q \rightarrow q)$ , then  $\psi$  follows from  $\mathcal{T}$  in **RM**, but there is no ‘appropriate form of the deduction theorem’ for either **R** or **RM** that would justify the argument outlined in those proofs.

Example 3.2 shows that we cannot be satisfied with the basic relevance criterion. Thus both of the positive logics mentioned in its first item have  $q \supset (p \supset q)$  as a valid formula, while the rejection of this “paradox of material implication” has been one of the main motivations for developing relevant logics. Proposition 3.3 suggests a natural direction for going beyond the basic relevance criterion: to impose appropriate constraints on the characteristic matrices of the logics which satisfy it (whose existence is guaranteed by that proposition). Next is an analysis which leads to a reasonable constraint of this sort.

Let  $\mathbf{L}$  be one of the three-valued logics mentioned in Example 3.2. Then any two paradoxical formulas are necessarily indistinguishable in  $\mathbf{L}$ . (Formally, this is reflected by the fact that  $p, \neg p, q, \neg q, \psi[p/r] \vdash_{\mathbf{L}} \psi[q/r]$  for every  $p, q, r$  and  $\psi$ .) Intuitively, this state of affairs is in a direct conflict with principles of relevance. More generally, if a logic is induced by a finite matrix with  $n$  elements, then in any state of affairs any set of  $n + 1$  formulas necessarily includes two different formulas which are absolutely indistinguishable in that state of affairs. (Formally, if  $v$  is a valuation, and  $\psi_1, \dots, \psi_{n+1}$  are formulas, then there are  $1 \leq i < j \leq n + 1$  such that for any formula  $\varphi$  and any atom  $p$ ,  $v(\varphi[\psi_i/p]) = v(\varphi[\psi_j/p])$ .) This again is in conflict with the idea of relevance. It seems counterintuitive that there is an a priori, logically dictated, fixed finite bound on the number of distinct propositions (or even just distinct paradoxical propositions). According to this intuition, any characteristic matrix for a relevant logic should necessarily be infinite. Actually, it seems reasonable to make a little bit stronger demand.

**Definition 3.4** (*Minimal semantic relevance criterion*) A logic  $\mathbf{L}$  satisfies the *minimal semantic relevance criterion* if it does not have a finite weakly characteristic matrix.

*Note 3.3* The main reason that the minimal semantic relevance criterion forbids a relevant logic  $\mathbf{L}$  to have even a finite *weakly* characteristic matrix is that the existence of finite weakly characteristic matrix is often reflected in the validity of counterintuitive (from a relevance point of view) formulas. Thus, the existence of a 3-valued weakly characteristic matrix is frequently reflected by a formula of the form

$$(p_1 \leftrightarrow p_2) \vee (p_1 \leftrightarrow p_3) \vee (p_1 \leftrightarrow p_4) \vee (p_2 \leftrightarrow p_3) \vee (p_2 \leftrightarrow p_4) \vee (p_3 \leftrightarrow p_4),$$

where  $\leftrightarrow$  and  $\vee$  are appropriate equivalence and disjunction connectives, respectively, available in the logic.

*Note 3.4* To the best of our knowledge, our minimal semantic relevance criterion has never been suggested before as a criterion for relevance (not even in Avron (2014)). Nevertheless, all the main systems that have been designed to be relevant logics do satisfy it (see Anderson and Belnap 1975).

The two criteria suggested above do not seem sufficient for characterizing relevant logics. However, we believe that they do suffice for characterizing *semi-relevance*.

**Definition 3.5** A logic  $\mathbf{L}$  which satisfies both the basic relevance criterion and the minimal semantic relevance criterion is called *semi-relevant*.

## 4 Introducing $\mathbf{RM}$ and $\mathbf{RM}_{\rightarrow}$

Now we turn at last to the subject of this paper, the logic  $\mathbf{RM}$ .

### Definition 4.1

1.  $\mathbf{HRM}$  is the Hilbert-type system which is obtained from  $\mathbf{HR}$  by replacing the identity axiom  $\varphi \rightarrow \varphi$  by the mingle axiom [Mi] (Definition 2.19).
2.  $\mathbf{RM}$  is the logic in  $\mathcal{L}_R$  which is induced by  $\mathbf{HRM}$ .
3.  $\mathbf{RM}_{\rightarrow}$  is the  $\{\neg, \rightarrow\}$ -fragment of  $\mathbf{RM}$ .

The next proposition lists some of the most characteristic properties of  $\mathbf{RM}$ .

### Proposition 4.2

1.  $\vdash_{\mathbf{RM}} \varphi + \varphi \leftrightarrow \varphi$
2.  $\vdash_{\mathbf{RM}} \varphi \wedge \psi \rightarrow \varphi + \psi$
3. If  $\vdash_{\mathbf{RM}} \varphi$ , and  $\vdash_{\mathbf{RM}} \psi$ , then  $\vdash_{\mathbf{RM}} \varphi + \psi$ .
4. Each of the three equivalent formulas in the last item of Lemma 2.16 is provable in  $\mathbf{RM}$ .

*Proof* 1. Immediate from item 1 of Lemma 2.20.

2. First substitute in item 4 of Lemma 2.16  $\varphi \wedge \psi$  for  $\varphi_1$  and  $\varphi_2$ ,  $\varphi$  for  $\psi_1$ , and  $\psi$  for  $\psi_2$ . Then by using the conjunction axioms of  $\mathbf{HRM}$ , we get that  $\vdash_{\mathbf{RM}} (\varphi \wedge \psi + \varphi \wedge \psi) \rightarrow \varphi + \psi$ . Hence the claim follows from the first part.
3. Immediate from item 2 and the adjunction rule [Ad].
4. From item 3 it follows that  $\vdash_{\mathbf{RM}} (\varphi \rightarrow \varphi) + (\psi \rightarrow \psi)$ . Now apply item 11 of Lemma 2.16.  $\square$

It was observed by Parks (1972) (see also Anderson and Belnap 1975, p. 148) that  $\mathbf{RM}_{\rightarrow}$  is not identical with  $\mathbf{RMI}_{\rightarrow}$ . Indeed, item 3 (or 4) of Proposition 4.2 implies that unlike  $\mathbf{RMI}_{\rightarrow}$ ,  $\mathbf{RM}_{\rightarrow}$  does not have the variable-sharing property for  $\rightarrow$ . Accordingly, our first task is to provide an axiomatization of  $\mathbf{RM}_{\rightarrow}$ . This is what we do next.

**Definition 4.3** 1.  $\mathbf{HRM}_{\rightarrow}$  is the Hilbert-type system that is obtained from  $\mathbf{HR}_{\rightarrow}$  by replacing the identity axiom  $\varphi \rightarrow \varphi$  by axiom [++] below.

$$[++] \quad (\varphi \rightarrow \varphi) + (\psi \rightarrow \psi)$$

2.  $\mathbf{L}_{\mathbf{HRM}_{\rightarrow}}$  is the logic induced by  $\mathbf{HRM}_{\rightarrow}$ .

**Proposition 4.4**  $\mathbf{RMI}_{\rightarrow} \subseteq \mathbf{L}_{\mathbf{HRM}_{\rightarrow}}$ .

*Proof* By substituting  $\varphi$  for  $\psi$  in [++] and in the last item of Lemma 2.16, we get that  $\vdash_{\mathbf{HRM}_{\rightarrow}} \varphi \rightarrow (\varphi \rightarrow \varphi + \varphi)$ . Using contraction, it follows that  $\vdash_{\mathbf{HRM}_{\rightarrow}} \varphi \rightarrow \varphi + \varphi$ . By item 10 of Lemma 2.16 (using item 7 of that lemma), this implies that the mingle axiom [Mi] is provable in  $\mathbf{HRM}_{\rightarrow}$ .  $\square$

**Proposition 4.5** *If  $\mathcal{T} \cup \{\varphi\}$  is in the language  $\{\neg, \rightarrow\}$ , and  $\mathcal{T} \vdash_{HRM_{\rightarrow}} \varphi$ , then  $\mathcal{T} \vdash_{RM_{\rightarrow}} \varphi$ .*

*Proof* Immediate from the last item of Proposition 4.2.  $\square$

That the converse of Proposition 4.5 also holds (and so  $RM_{\rightarrow} = L_{HRM_{\rightarrow}}$ ) will be shown in Theorem 5.11.

## 5 Semantics of RM

In this section, we introduce a semantics for **RM** (and  $RM_{\rightarrow}$ ) for which it is (strongly) complete.

**Definition 5.1** (*Sugihara chains*) A *Sugihara chain* is a triple  $\langle \mathcal{V}, \leq, - \rangle$  such that  $\mathcal{V}$  has at least two elements,  $\leq$  is a linear order on  $\mathcal{V}$ , and  $-$  is an involution for  $\leq$  on  $\mathcal{V}$  (i.e., for every  $a, b \in \mathcal{V}$ ,  $--a = a$ , and  $-b \leq -a$  whenever  $a \leq b$ ).

*Example 5.1* There are plenty of examples of Sugihara chains in all areas of mathematics. The most important for our needs are the following.

- $S_{\mathbb{R}} = \langle \mathbb{R}, \leq, - \rangle$ ,  $S_{\mathbb{Z}} = \langle \mathbb{Z}, \leq, - \rangle$ ,  $S_{\mathbb{Z}^*} = \langle \mathbb{Z} - \{0\}, \leq, - \rangle$ ,  $S_{\mathbb{Q}} = \langle \mathbb{Q}, \leq, - \rangle$ , and  $S_{\mathbb{Q}^*} = \langle \mathbb{Q} - \{0\}, \leq, - \rangle$ , where  $\mathbb{R}$  is the set of real numbers,  $\mathbb{Z}$  is the set of integers,  $\mathbb{Q}$  is the set of rationals,  $\leq$  is the usual order relation on  $\mathbb{R}$ , and  $-a$  is the usual additive inverse of  $a$ .
- The *finite* substructures  $S_{\mathbb{Z}_n} = \langle \mathbb{Z}_n, \leq, - \rangle$  and  $S_{\mathbb{Z}_n^*} = \langle \mathbb{Z}_n^*, \leq, - \rangle$  of  $S_{\mathbb{Z}}$ , where for  $n > 0$   $\mathbb{Z}_n = \{z \in \mathbb{Z} : -n \leq z \leq n\}$ , and  $\mathbb{Z}_n^* = \mathbb{Z}_n - \{0\}$ .
- $S_{[0,1]} = \langle [0, 1], \leq, \lambda x. 1 - x \rangle$ , where  $\leq$  is again the usual order relation. Note that here the underlying ordered set is *bounded and complete*.

The next two lemmas about ordered sets will be useful in the sequel.

**Lemma 5.2** *Let  $n > 0$  be a natural number. Every finite Sugihara chain which has  $2n + 1$  elements is isomorphic to  $S_{\mathbb{Z}_n^*}$ , and every finite Sugihara chain which has  $2n$  elements is isomorphic to  $S_{\mathbb{Z}_n^*}$ .*

*Proof* By an easy induction on  $n$ .  $\square$

**Lemma 5.3** *Every countable Sugihara chain can be embedded in  $S_{[0,1]}$ .*

*Proof* It is well known that every countable linearly ordered set can be embedded in any closed interval  $[a, b]$  of  $\mathbb{R}$ , so that  $a$  is assigned to the minimal element of the set (if such exists), and  $b$  is assigned to the maximal element of the set (if such exists). Now let  $\langle \mathcal{V}, \leq, - \rangle$  be a countable Sugihara chain, and let  $D = \{a \in \mathcal{V} : -a \leq a\}$ . First, suppose that there is  $a \in \mathcal{V}$  such that  $-a = a$ . It is easy to prove that in such a case  $a$  is unique, and it is the minimal element of  $D$ . Let  $f$  be an embedding of

$D$  into  $[1/2, 1]$  such that  $f(a) = 1/2$ , and extend  $f$  to the whole of  $\mathcal{V}$  by letting  $f(x) = -f(-x)$  in case  $x \notin D$ . (Note that if  $x \notin D$  then  $-x \in D$ , because  $\leq$  is linear, and  $- -x = x$ .) If there is no  $a \in \mathcal{V}$  such that  $-a = a$ , then we let  $f$  be any embedding of  $D$  into  $[2/3, 1]$  (say), and we again extend  $f$  to the whole of  $\mathcal{V}$  by letting  $f(x) = -f(-x)$  in case  $x \notin D$ . In both cases,  $f$  is easily seen to be an embedding of  $\langle \mathcal{V}, \leq, - \rangle$  into  $[0, 1]$ .  $\square$

**Definition 5.4** Let  $S = \langle \mathcal{V}, \leq, - \rangle$  be a Sugihara chain, and let  $a, b \in \mathcal{V}$ .

- $a < b$  if  $a \leq b$  and  $a \neq b$ .
- $|a| = \max(-a, a)$ .
- $a \leq_+ b$  iff either  $|a| < |b|$ , or  $|a| = |b|$  and  $a < b$ .

The following lemma is easily verified.

**Lemma 5.5** If  $\langle \mathcal{V}, \leq, - \rangle$  is a Sugihara chain, then  $\leq_+$  linearly orders  $\mathcal{V}$ .

**Definition 5.6** (Sugihara matrix) Let  $S = \langle \mathcal{V}, \leq, - \rangle$  be a Sugihara chain.

- The *multiplicative Sugihara matrix* based on  $S$  is the matrix  $\mathcal{M}_m(S) = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$  for  $\{ \neg, \rightarrow \}$  in which  $\mathcal{D} = \{ a \in \mathcal{V} : -a \leq a \}$  (equivalently,  $\mathcal{D} = \{ a \in \mathcal{V} : |a| = a \}$ ),  $\neg a = -a$ , and  $a \rightarrow b = \max_{\leq_+}(-a, b)$ .
- The *Sugihara matrix*  $\mathcal{M}(S)$  based on  $S$  is the extension of  $\mathcal{M}_m(S)$  to  $\mathcal{L}_R$  in which  $a \wedge b = \min(a, b)$  and  $a \vee b = \max(a, b)$ .
- A matrix  $\mathcal{M}$  for  $\mathcal{L}_R$  (for  $\{ \neg, \rightarrow \}$ ) is a (multiplicative) *Sugihara matrix* if for some Sugihara chain  $S$ ,  $\mathcal{M}$  is the (multiplicative) Sugihara matrix which is based on  $S$ .

*Note 5.2* Obviously, we have that in a (multiplicative) Sugihara matrix  $a \dot{+} b = \max_{\leq_+}(a, b)$ . It is also easy to see that the above definition of  $\rightarrow$  in Sugihara matrices is equivalent to the following original definition from Sugihara (1955):

$$a \rightarrow b = \begin{cases} \max(-a, b) & \text{if } a \leq b, \\ \min(-a, b) & \text{if } a > b. \end{cases}$$

It easily follows that  $a \rightarrow b \in \mathcal{D}$  iff  $a \leq b$ .

*Note 5.3* It is easy to see that the set  $\mathcal{D}$  is upward closed in a Sugihara matrix  $\mathcal{M}$ . That is, if  $a \in \mathcal{D}$  and  $a \leq b$  (where  $\leq$  is the order relation of the Sugihara chain which underlies  $\mathcal{M}$ ), then  $b \in \mathcal{D}$ .

The following observation will be useful in the sequel.

**Proposition 5.7** Let  $S = \langle \mathcal{V}, \leq, - \rangle$  be a Sugihara chain, and suppose that  $\mathcal{V}'$  is a subset of  $\mathcal{V}$  which is closed under  $-$ , and has at least two elements. Then  $S' = \langle \mathcal{V}', \leq, - \rangle$  is also a Sugihara chain, and  $\mathcal{M}(S')$  ( $\mathcal{M}_m(S')$ ) is a submatrix of  $\mathcal{M}(S)$  ( $\mathcal{M}_m(S)$ ).

*Proof* The definitions of the operations immediately imply that if  $\mathcal{V}$  is closed under  $-$ , then it is closed under  $\rightarrow$ ,  $\wedge$ , and  $\vee$  as well. The proposition easily follows from this fact and the definition of the set  $\mathcal{D}$  of designated elements in Sugihara matrices.  $\square$

**Notation** For  $A \in \{\mathbb{R}, \mathbb{Z}, \mathbb{Z}^*, \mathbb{Q}, \mathbb{Q}^*, [0, 1], \mathbb{Z}_n, \mathbb{Z}_n^*\}$  we shall henceforth write just  $\mathcal{M}(A)$  instead of  $\mathcal{M}(S_A)$ , and  $\mathcal{M}_m(A)$  instead of  $\mathcal{M}_m(S_A)$ .

Next we prove a *strong* soundness and completeness theorem for  $HRM_{\rightarrow}$ .

**Theorem 5.8** (Strong soundness and completeness of  $HRM_{\rightarrow}$ )

1.  $HRM_{\rightarrow}$  is strongly sound and complete for the class of multiplicative Sugihara matrices.
2.  $HRM_{\rightarrow}$  is strongly sound and complete for  $\mathcal{M}_m([0, 1])$ .

*Proof* 1. For the soundness part we need to prove that the axioms and rule of  $HRM_{\rightarrow}$  are all valid in any Sugihara matrix. We leave the straightforward but tedious details of this to the reader.

For completeness, assume  $\mathcal{T} \not\vdash_{HRM_{\rightarrow}} \varphi$ . Extend  $\mathcal{T}$  to a maximal theory  $\mathcal{T}^*$  such that  $\mathcal{T}^* \not\vdash_{HRM_{\rightarrow}} \varphi$ . Then the relevant deduction theorem of  $HRM_{\rightarrow}$  (Proposition 2.14) implies that for every sentence  $\psi$ ,  $\psi \notin \mathcal{T}^*$  iff  $\psi \rightarrow \varphi \in \mathcal{T}^*$ . By item 5 of Lemma 2.16, this in turn implies:

- (1) If  $\psi + \sigma \in \mathcal{T}^*$ , then either  $\psi \in \mathcal{T}^*$  or  $\sigma \in \mathcal{T}^*$ .

(1) together with  $[++]$  and items 11 and 6 of Lemma 2.16 imply:

- (2) For every  $\psi, \sigma$ , either  $\psi \rightarrow \sigma \in \mathcal{T}^*$  or  $\sigma \rightarrow \psi \in \mathcal{T}^*$ .
- (3) For every sentence  $\psi$ , either  $\psi \in \mathcal{T}^*$  or  $\neg\psi \in \mathcal{T}^*$ .

- Now construct the Lindenbaum Algebra  $\mathcal{M}_{\mathcal{T}^*}$  of  $\mathcal{T}^*$  in the usual way. We define that  $\psi \equiv \sigma$  iff  $\psi \leftrightarrow \sigma \in \mathcal{T}^*$  (and so both  $\psi \rightarrow \sigma \in \mathcal{T}^*$  and  $\sigma \rightarrow \psi \in \mathcal{T}^*$ , by item 1 of Lemma 2.16). By Proposition 2.18, this is obviously a congruence relation. Let  $\mathcal{V}$  be the set of equivalence classes, and let  $\mathcal{D} = \{[\psi] : \psi \in \mathcal{T}^*\}$ . Define the operations  $\neg$  and  $\rightarrow$  on  $\mathcal{V}$  as  $[\psi] \rightarrow [\sigma] = [\psi \rightarrow \sigma]$  and  $\neg[\psi] = [\neg\psi]$ . To show that the resulting matrix is a multiplicative Sugihara matrix, we let  $[\psi] \leq [\sigma]$  iff  $\psi \rightarrow \sigma \in \mathcal{T}^*$ . These are all legitimate definitions because  $\equiv$  is a congruence relation. It is a standard matter to show that  $\leq$  is a partial order on  $\mathcal{V}$  and that the negation axioms of  $\mathbf{R}_{\rightarrow}$  ensure that  $\neg$  is an involution on  $\langle \mathcal{V}, \leq \rangle$ . (2) above implies that  $\leq$  is also linear. It follows that  $S = \langle \mathcal{V}, \leq, \neg \rangle$  is a Sugihara chain. Next we show that  $\mathcal{M}_{\mathcal{T}^*} = \mathcal{M}_m(S)$ . That  $[\psi] \in \mathcal{D}$  iff  $\neg[\psi] \leq [\psi]$  easily follows from the definitions of  $\mathcal{D}$  and  $\leq$ , and the fact that both  $\psi \rightarrow (\neg\psi \rightarrow \psi)$  and  $(\neg\psi \rightarrow \psi) \rightarrow \psi$  are theorems of  $\mathbf{RMI}_{\rightarrow}$  (Lemma 2.20, 1). It remains to show that the operation  $\rightarrow$  of  $\mathcal{M}_{\mathcal{T}^*}$  is identical to that of  $\mathcal{M}_m(S)$ . We use for that the characterization of  $\mathcal{M}_m(S)$  given in Note 5.2.

- Suppose  $[\psi] \leq [\sigma]$ . Then  $\psi \rightarrow \sigma \in \mathcal{T}^*$ . By items 3 and 2 of Lemma 2.20, it follows that both  $\neg\psi \rightarrow (\psi \rightarrow \sigma)$  and  $\sigma \rightarrow (\psi \rightarrow \sigma)$  are in  $\mathcal{T}^*$ . Hence  $[\psi] \rightarrow [\sigma] \geq \max(\neg[\psi], [\sigma])$ . To prove the converse, note that since  $\leq$  is linear,  $\max(\neg[\psi], [\sigma])$  is either  $[\sigma]$  or  $\neg[\psi]$ . In the first case  $\neg[\psi] \leq [\sigma]$ , and so  $\neg\psi \rightarrow \sigma \in \mathcal{T}^*$ . By item 8 of Lemma 2.16, we get that in this case  $[\psi \rightarrow \sigma] \leq [\sigma]$ . In the second case,  $[\sigma] \leq \neg[\psi]$ , and so  $[\psi] \leq \neg[\sigma]$ , implying that  $\psi \rightarrow \neg\sigma \in \mathcal{T}^*$ . By item 9 of Lemma 2.16, we get that in this case  $[\psi \rightarrow \sigma] \leq \neg[\psi]$ . In both cases, we have that  $[\psi] \rightarrow [\sigma] = [\psi \rightarrow \sigma] \leq \max(\neg[\psi], [\sigma])$ .
- Suppose  $[\psi] \not\leq [\sigma]$ . Then  $\psi \rightarrow \sigma \notin \mathcal{T}^*$ . Hence (3) implies that  $\neg(\psi \rightarrow \sigma) \in \mathcal{T}^*$ . By items 5 and 6 of Lemma 2.20, it follows that both  $(\psi \rightarrow \sigma) \rightarrow \sigma$  and  $(\psi \rightarrow \sigma) \rightarrow \neg\psi$  are in  $\mathcal{T}^*$ . Hence  $[\psi] \rightarrow [\sigma] \leq \min(\neg[\psi], [\sigma])$ . To prove the converse, note that since  $\leq$  is linear,  $\min(\neg[\psi], [\sigma])$  is either  $[\sigma]$  or  $\neg[\psi]$ . In the first case,  $\neg[\psi] \leq [\sigma]$ , and so  $\neg\psi \rightarrow \sigma \in \mathcal{T}^*$ . By item 7 of Lemma 2.20, we get that  $\neg[\psi] \leq [\psi \rightarrow \sigma]$  in this case. In the second case,  $[\sigma] \leq \neg[\psi]$ , and so  $\sigma \rightarrow \neg\psi \in \mathcal{T}^*$ . By item 8 of Lemma 2.20, we get that  $[\sigma] \leq [\psi \rightarrow \sigma]$  in this case. In both cases, we have that  $[\psi] \rightarrow [\sigma] = [\psi \rightarrow \sigma] \geq \min(\neg[\psi], [\sigma])$ .

The end of the proof is now standard. Let  $v(\psi) = [\psi]$ . This is easily seen to be a legitimate valuation (the canonical one) in  $\mathcal{M}_{\mathcal{T}^*}$ . Obviously,  $v$  is a model of  $\psi$  iff  $\psi \in \mathcal{T}^*$ . Hence  $v$  is a model of  $\mathcal{T}$  in the Sugihara matrix  $\mathcal{M}_{\mathcal{T}^*}$  which is not a model of  $\varphi$ .

2. The multiplicative Sugihara matrix constructed in the proof of the first part is countable. Hence the second part follows from the first (and its proof) by Lemma 5.3 and Proposition 5.7.  $\square$

**Proposition 5.9**  $\mathcal{M}_m(\mathbb{Z}_1)$  is weakly characteristic for  $\mathbf{L}_{HRM_{\rightarrow}}$ , but it is not strongly characteristic for it.

*Proof* From the first part of Theorem 5.8 it follows that if  $\vdash_{HRM_{\rightarrow}} \varphi$ , then  $\vdash_{\mathcal{M}_m(\mathbb{Z}_1)} \varphi$ . For the converse, assume that  $\not\vdash_{HRM_{\rightarrow}} \varphi$ . By the second part of Theorem 5.8, it follows that there is an assignment  $v$  in  $\mathcal{M}_m([0, 1])$  such that  $v(\varphi) < 1/2$ . Let  $v(\varphi) = a$ . Without loss of generality, we may assume that  $v(p) \in [a, 1 - a]$  for every  $p \notin \text{Atoms}(\varphi)$ , while the definitions of the operations in  $\mathcal{M}_m([0, 1])$  imply that necessarily  $v(p) \in [a, 1 - a]$  also for every  $p \in \text{Atoms}(\varphi)$ . Hence  $v(\psi) \in [a, 1 - a]$  for every  $\psi$ . Define  $f : [a, 1 - a] \rightarrow \{-1, 0, 1\}$  as

$$f(x) = \begin{cases} 1 & \text{if } x = 1 - a, \\ 0 & \text{if } a < x < 1 - a, \\ -1 & \text{if } x = a. \end{cases}$$

It is easy to verify that  $v^* = f \circ v$  is an assignment in  $\mathcal{M}_m(\mathbb{Z}_1)$  such that  $v^*(\psi) = f(v(\psi))$  for every formula  $\psi$ . In particular,  $v^*(\varphi) = -1$ , and so  $\not\vdash_{\mathcal{M}_m(\mathbb{Z}_1)} \varphi$ .

To see that  $\mathcal{M}_m(\mathbb{Z}_1)$  is not strongly characteristic for  $\mathbf{L}_{HRM_{\rightarrow}}$ , it suffices to note that  $\varphi \otimes \psi \vdash_{\mathcal{M}_m(\mathbb{Z}_1)} \varphi$ , but  $\varphi \otimes \psi \not\vdash_{\mathcal{M}_m([0, 1])} \varphi$ .  $\square$

*Note 5.4* That  $\mathcal{M}_m(\mathbb{Z}_1)$  is weakly characteristic for  $\mathbf{L}_{HRM_{\rightarrow}}$  and  $\mathbf{RM}_{\rightarrow}$  was (essentially) shown in Parks (1972). The fact that it is not strongly characteristic for them was (to my best knowledge) first shown in Avron (1997).

Our next goal is to prove a counterpart of Theorem 5.8 for the whole of **RM**. The main obstacle in doing that is that the relevant deduction theorem, which was used in the proof of Theorem 5.8 for showing the crucial property that was denoted by (1) in that proof, fails for **RM**. Therefore, we shall use instead for that purpose the fact that  $\vee$  is a disjunction for **RM**.

**Theorem 5.10** (Strong soundness and completeness of **RM**)

1. **RM** is strongly sound and complete for the class of Sugihara matrices.
2. **RM** is strongly sound and complete for  $\mathcal{M}([0, 1])$ .

*Proof*

1. Given the strong soundness of  $HRM_{\rightarrow}$  for multiplicative Sugihara matrices (Theorem 5.8), the proof of the strong soundness of **RM** for Sugihara matrices is straightforward, and is left to the reader.

For completeness, assume  $\mathcal{T} \not\vdash_{\mathbf{RM}} \varphi$ . Extend  $\mathcal{T}$  to a maximal theory  $\mathcal{T}^*$  such that  $\mathcal{T}^* \not\vdash_{\mathbf{RM}} \varphi$ . Then  $\psi \notin \mathcal{T}^*$  iff  $\mathcal{T}^*, \psi \vdash_{\mathbf{RM}} \varphi$ . Hence Theorem 2.11 implies that  $\mathcal{T}^*$  is *prime*, i.e., if  $\psi \vee \sigma \in \mathcal{T}^*$ , then either  $\psi \in \mathcal{T}^*$  or  $\sigma \in \mathcal{T}^*$ . Therefore, it follows from Lemma 2.17 that (1) from the proof of Theorem 5.8 holds for  $\mathcal{T}^*$ . From this point on, the proof is almost identical to the proof of the first part of Theorem 5.8, except that we show that  $\mathcal{M}_{\mathcal{T}^*} = \mathcal{M}(S)$  (where  $S$  is defined like in that proof), rather than that  $\mathcal{M}_{\mathcal{T}^*} = \mathcal{M}_m(S)$ . For this, all we have to add to the proof of Theorem 5.8 is that  $[\psi \wedge \sigma] = \min([\psi], [\sigma])$  and  $[\psi \vee \sigma] = \max([\psi], [\sigma])$ . This is obvious from the axioms concerning  $\wedge$  and  $\vee$  of **RM**, and the linearity of  $\leq$ .

2. The proof is identical to that of the second part of Theorem 5.8. □

*Note 5.5* Theorem 5.10 is essentially due to Dunn (1970). However, Dunn used the countable matrix  $\mathcal{M}(\mathbb{Q})$  for strongly characterizing **RM**, rather than the uncountable  $\mathcal{M}([0, 1])$  used by us here.<sup>6</sup>

Now we can at last prove the following theorem.

**Theorem 5.11**  $\mathbf{RM}_{\rightarrow} = \mathbf{L}_{HRM_{\rightarrow}}$ .

*Proof* Immediate from Proposition 4.5, and the second parts of Theorems 5.8 and 5.10. □

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<sup>6</sup>An advantage of choosing  $\mathcal{M}([0, 1])$  is that its use allows us to view **RM** as a fuzzy logic. (See Sect. 7.) Another advantage is that it can be expanded very naturally to provide semantics for first-order **RM**, as well as for the logic that is obtained from **RM** by adding to its language the propositional constants **T** and **F**, together with the axioms  $\mathbf{F} \rightarrow \varphi$  and  $\varphi \rightarrow \mathbf{T}$ .

Next we show that for *weak* completeness the set of finite Sugihara matrices and each of the countable Sugihara matrices  $\mathcal{M}(\mathbb{Z})$  and  $\mathcal{M}(\mathbb{Z}^*)$  suffice.

**Definition 5.12** (The matrices  $\mathcal{RM}_n$ ) For  $k = 1, 2, \dots$ , we let  $\mathcal{RM}_{2k} = \mathcal{M}(\mathbb{Z}_k^*)$  and  $\mathcal{RM}_{2k+1} = \mathcal{M}(\mathbb{Z}_k)$ .

**Proposition 5.13** Every finite Sugihara matrix which has  $n$  elements is isomorphic to  $\mathcal{RM}_n$ . Hence every such matrix is isomorphic to some finite submatrix of  $\mathcal{M}(\mathbb{Z})$ .

*Proof* This is an easy corollary of Lemma 5.2 and Proposition 5.7.  $\square$

**Theorem 5.14** Suppose that  $\text{Atoms}(\mathcal{T} \cup \{\varphi\})$  is finite.

1. Let  $n$  be the number of atomic variables which occur in  $\mathcal{T} \cup \{\varphi\}$ . Then  $\mathcal{T} \vdash_{\mathbf{RM}} \varphi$  iff  $\mathcal{T} \vdash_{\mathcal{RM}_k} \varphi$  for every  $2 \leq k \leq 2n$ .
2.  $\mathcal{T} \vdash_{\mathbf{RM}} \varphi$  iff  $\mathcal{T} \vdash_{\mathcal{M}(\mathbb{Z})} \varphi$ .

*Proof* From the soundness of **RM** for Sugihara matrices, it follows that if  $\mathcal{T} \vdash_{\mathbf{RM}} \varphi$ , then  $\mathcal{T} \vdash_{\mathcal{M}(\mathbb{Z})} \varphi$ , and  $\mathcal{T} \vdash_{\mathcal{RM}_k} \varphi$  for every  $k \geq 2$ . For the converse, assume  $\mathcal{T} \not\vdash_{\mathbf{RM}} \varphi$ . By Theorem 5.10, there is a Sugihara chain  $S = \langle \mathcal{V}, \leq, - \rangle$  and a valuation  $v$  in  $\mathcal{M}(S)$  which is a model of  $\mathcal{T}$  but not of  $\varphi$ . Suppose  $\text{Atoms}(\mathcal{T} \cup \{\varphi\}) = \{p_1, \dots, p_n\}$ , and let  $\mathcal{V}' = \{v(p_1), -v(p_1), \dots, v(p_n), -v(p_n)\}$ . An easy induction on the complexity of a sentence  $\psi$  shows that  $v(\psi) \in \mathcal{V}'$  for every  $\psi$  such that  $\text{Atoms}(\psi) \subseteq \{p_1, \dots, p_n\}$ . Since  $v$  is not a model of  $\varphi$ , this implies that  $\mathcal{V}'$  has at least two elements (and of course not more than  $2n$ ). Hence Proposition 5.7 implies that  $S' = \langle \mathcal{V}', \leq, - \rangle$  is also a Sugihara chain, and  $\mathcal{M}(S')$  is a submatrix of  $\mathcal{M}(S)$ . Let  $v'$  be any valuation in  $\mathcal{M}(S')$  such that  $v'(p_i) = v(p_i)$  for  $1 \leq i \leq n$ . Then  $v'(\psi) = v(\psi)$  for every  $\psi$  such that  $\text{Atoms}(\psi) \subseteq \{p_1, \dots, p_n\}$ . It follows that  $v'$  is a model of  $\mathcal{T}$  in  $\mathcal{M}(S')$  which is not a model of  $\varphi$ . Hence  $\mathcal{T} \not\vdash_{\mathcal{M}(S')} \varphi$ . By Proposition 5.13, this implies that  $\mathcal{T} \not\vdash_{\mathcal{RM}_k} \varphi$  for some  $2 \leq k \leq 2n$ , and that  $\mathcal{T} \not\vdash_{\mathcal{M}(\mathbb{Z})} \varphi$ .  $\square$

**Corollary 5.15** If  $\mathcal{T}$  is a finite theory, then  $\mathcal{T} \vdash_{\mathbf{RM}} \varphi$  iff  $\mathcal{T} \vdash_{\mathcal{M}(\mathbb{Z})} \varphi$ . In particular,  $\mathcal{M}(\mathbb{Z})$  is weakly characteristic for **RM**.

In contrast we have the following.

**Proposition 5.16**  $\mathcal{M}(\mathbb{Z})$  is not strongly characteristic for **RM**.

*Proof* Let  $\mathcal{T} = \{p_i : i \geq 1\} \cup \{(p_i \rightarrow p_{i+1}) \rightarrow p_0 : i \geq 1\}$ , and let  $S = \langle \mathcal{V}, \leq, - \rangle$  be a Sugihara chain. It is not difficult to check that a valuation  $v$  in  $\mathcal{M}(S)$  can be a model of  $\mathcal{T}$  which is not a model of  $p_0$  iff  $v(p_0) < -v(p_0)$ , while for  $i > 0$ ,  $v(p_i) \geq -v(p_i)$  and  $v(p_i) > v(p_{i+1})$ . Such  $v$  does not exist in  $\mathcal{M}(\mathbb{Z})$ , but it does in  $\mathcal{M}([0, 1])$ . Hence  $\mathcal{T} \vdash_{\mathcal{M}(\mathbb{Z})} p_0$ , while  $\mathcal{T} \not\vdash_{\mathbf{RM}} p_0$ .  $\square$

The characterization of **RM** in terms of finite matrices that is given in Theorem 5.14 can in fact be improved using the next proposition.

**Proposition 5.17** *For every  $n \geq 2$ , if  $\vdash_{\mathcal{RM}_{n+1}} \varphi$ , then also  $\vdash_{\mathcal{RM}_n} \varphi$ .*

*Proof* The claim is obvious in case  $n$  is even, since  $\mathcal{RM}_{2k}$  is a submatrix of  $\mathcal{RM}_{2k+1}$  for every  $k \geq 1$ .

Now suppose that  $n = 2k + 1$  for some  $k \geq 1$ , and that  $\not\vdash_{\mathcal{RM}_{2k+1}} \varphi$ . We show that also  $\not\vdash_{\mathcal{RM}_{2k+2}} \varphi$ . Let  $v$  be a valuation in  $\mathcal{RM}_{2k+1}$  such that  $v(\varphi) < 0$ . Define a valuation  $v^*$  in  $\mathcal{RM}_{2k+2}$  by letting  $v^*(p) = v(p) + 1$  in case  $v(p) \geq 0$ , and  $v^*(p) = v(p) - 1$  in case  $v(p) < 0$ . By induction on the complexity of  $\psi$ , it is not difficult to show that for every sentence  $\psi$  we have the following.

- If  $v(\psi) > 0$ , then  $v^*(\psi) = v(\psi) + 1$ .
- If  $v(\psi) = 0$ , then  $v^*(\psi) \in \{-1, 1\}$ .
- If  $v(\psi) < 0$ , then  $v^*(\psi) = v(\psi) - 1$ .

It follows in particular that  $v^*(\varphi) < 0$ . Hence  $\not\vdash_{\mathcal{RM}_{2k+2}} \varphi$ . □

**Corollary 5.18** *If  $n \geq 2$  and  $\vdash_{\mathcal{RM}_n} \varphi$ , then  $\vdash_{\mathcal{RM}_m} \varphi$  for every  $2 \leq m \leq n$ .*

**Proposition 5.19** *Suppose  $|\text{Atoms}(\varphi)| = n$ . Then  $\vdash_{\mathbf{RM}} \varphi$  iff  $\vdash_{\mathcal{RM}_{2n}} \varphi$ .*

*Proof* Immediate from Part 1 of Theorem 5.14 and Corollary 5.18. □

**Proposition 5.20**  *$\mathcal{M}(\mathbb{Z}^*)$  is weakly characteristic for **RM**.*

*Proof* That if  $\vdash_{\mathbf{RM}} \varphi$  then  $\vdash_{\mathcal{M}(\mathbb{Z}^*)} \varphi$  follows from the soundness of **RM** for Sugihara matrices. For the converse, assume  $\not\vdash_{\mathbf{RM}} \varphi$ . Then by Proposition 5.19, there is  $n$  such that  $\not\vdash_{\mathcal{RM}_{2n}} \varphi$ . Since  $\mathcal{RM}_{2n}$  is a submatrix of  $\mathcal{M}(\mathbb{Z}^*)$ , this implies that  $\not\vdash_{\mathcal{M}(\mathbb{Z}^*)} \varphi$ . □

*Note 5.6* The second part of Corollary 5.15, and Propositions 5.16, 5.19, and 5.20 are due to Meyer (see Anderson and Belnap (1975, Sect. 29.3)). Corollary 5.18 and Proposition 5.19 are due to Dunn (see Anderson and Belnap (1975, Sect. 29.4)).

**Corollary 5.21** *If  $\gamma$  is a finite set of sentences, and  $\gamma \vdash_{\mathcal{M}(\mathbb{Z}^*)} \varphi$ , then the rule  $\gamma / \varphi$  is admissible in **RM**.*

*Proof* Let  $\theta$  be a substitution such that  $\vdash_{\mathbf{RM}} \theta(\psi)$  for every  $\psi \in \gamma$ . By Proposition 5.20,  $\vdash_{\mathcal{M}(\mathbb{Z}^*)} \theta(\psi)$  for every  $\psi \in \gamma$ . Since  $\gamma \vdash_{\mathcal{M}(\mathbb{Z}^*)} \varphi$ , it follows that  $\vdash_{\mathcal{M}(\mathbb{Z}^*)} \theta(\varphi)$  as well. Hence  $\vdash_{\mathbf{RM}} \theta(\varphi)$ , by Proposition 5.20 again. □

*Note 5.7* Since  $\neg p, p \vee q \vdash_{\mathcal{M}(\mathbb{Z}^*)} q$ , Corollary 5.21 entails that the disjunctive syllogism is admissible in **RM**. That is, if  $\vdash_{\mathbf{RM}} \neg\varphi$ , and  $\vdash_{\mathbf{RM}} \varphi \vee \psi$ , then  $\vdash_{\mathbf{RM}} \psi$ .<sup>7</sup> On the other hand, it is easy to see that  $\neg p, p \vee q \not\vdash_{\mathcal{M}(\mathbb{Z})} q$ . By Theorem 5.14, this implies that  $\neg p, p \vee q \not\vdash_{\mathbf{RM}} q$ . It follows that the analogue of Theorem 5.14 does not hold for  $\mathcal{M}(\mathbb{Z}^*)$ .

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<sup>7</sup>This is another famous result of Meyer and Dunn. See Meyer and Dunn (1969) and Sect. 25 of Anderson and Belnap (1975). In the latter, two different proofs of this theorem (for the main relevant and semi-relevant logics) are presented.

## 6 The Nice Properties of RM

**RM** has several nice properties. The first we present in this section is one that according to a famous theorem of Urquhart (1984), the main logics developed by Anderson and Belnap's school lack.<sup>8</sup>

**Theorem 6.1** *RM is decidable.*<sup>9</sup>

*Proof* Immediate from Theorem 5.14. (See also Proposition 5.19 for the special case of theoremhood in **RM**.)  $\square$

Our next goal is to show that **RM** is normal.

**Definition 6.2**  $\varphi \supset \psi =_{Df} (\varphi \rightarrow \psi) \vee \psi$

*Note 6.1* It is easy to see that in any Sugihara matrix we have that

$$a \supset b = \begin{cases} -a & \text{if } a \leq b \leq -a, \\ b & \text{otherwise.} \end{cases}$$

**Proposition 6.3**  $\supset$  is an implication for **RM**.

*Proof* By Theorem 2.11,  $\vee$  is a disjunction for **RM**. Given the definition of  $\supset$ , this easily implies that  $\varphi, \varphi \supset \psi \vdash_{\mathbf{RM}} \psi$ . It follows that if  $\mathcal{T} \vdash_{\mathbf{RM}} \varphi \supset \psi$ , then  $\mathcal{T}, \varphi \vdash_{\mathbf{RM}} \psi$ .

For the converse, assume  $\mathcal{T} \not\vdash_{\mathbf{RM}} \varphi \supset \psi$ . By Theorem 5.10, this implies that there is a valuation  $v$  in  $\mathcal{M}([0, 1])$  such that  $v(\sigma) \geq 1/2$  for every  $\sigma \in \mathcal{T}$ , while  $v(\varphi \supset \psi) < 1/2$ . The latter means that  $v(\psi) < 1/2$  and  $v(\varphi) > v(\psi)$ . If  $v(\varphi) \geq 1/2$ , then  $v$  is a model in  $\mathcal{M}([0, 1])$  of  $\mathcal{T} \cup \{\varphi\}$  which is not a model of  $\psi$ , and so  $\mathcal{T}, \varphi \not\vdash_{\mathbf{RM}} \psi$ . So assume  $1/2 > v(\varphi) > v(\psi)$ . Define a new valuation  $v^*$  in  $\mathcal{M}([0, 1])$  as follows.

$$v^*(\sigma) = \begin{cases} 1/2 & \text{if } v(\varphi) \leq v(\sigma) \leq 1 - v(\varphi), \\ v(\sigma) & \text{otherwise.} \end{cases}$$

It is easy to verify that  $v^*$  is indeed a legitimate valuation. Now  $v^*(\sigma) \in \{v(\sigma), 1/2\}$  for every  $\sigma \in \mathcal{T}$ ,  $v^*(\varphi) = 1/2$ , while  $v^*(\psi) = v(\psi) < 1/2$ . Since  $v(\sigma) \geq 1/2$  for every  $\sigma \in \mathcal{T}$ , this implies that  $v^*$  is a model in  $\mathcal{M}([0, 1])$  of  $\mathcal{T} \cup \{\varphi\}$  which is not a model of  $\psi$ , and so again  $\mathcal{T}, \varphi \not\vdash_{\mathbf{RM}} \psi$ .  $\square$

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<sup>8</sup>Here it should be noted that there are many contraction-free logics which are closely related to Anderson and Belnap's relevant logics, and *are* decidable (like **RW** (Brady 1990) or the multiplicative-additive fragment of Girard's linear logic). However, logics without contraction are not relevant logics according to our understanding of this notion (see Avron 2014).

<sup>9</sup>This result too is due to Meyer. See Anderson and Belnap (1975, Sect. 29.3).

*Note 6.2* It is also possible to prove Proposition 6.3 purely syntactically using the standard inductive method of converting a proof in **RM** of  $\psi$  from  $\mathcal{T} \cup \{\varphi\}$  into a proof in **RM** of  $\varphi \supset \psi$  from  $\mathcal{T}$ . In addition to the validity of [MP] for  $\supset$  in **RM** (which was shown above purely syntactically), one should only provide derivations of the following four formulas in **RM**:  $\varphi \supset \varphi$ ,  $\varphi \supset (\psi \supset \varphi)$ ,  $(\varphi \supset (\psi \rightarrow \sigma)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \sigma))$ , and  $(\varphi \supset \psi) \wedge (\varphi \supset \sigma) \supset (\varphi \supset \psi \wedge \sigma)$ . None of these tasks is difficult.

*Note 6.3* The above formulation of Definition 6.2 and Proposition 6.3 are due to Avron (1986). An equivalent definition and proposition have already been given in Dunn and Meyer (1971). However, they were given there only for **RM**<sup>t</sup>, a conservative extension of **RM** which is obtained from **RM** by adding to its language the propositional constant **t** together with the axioms **t** and  $\mathbf{t} \rightarrow (\varphi \rightarrow \varphi)$ . In **RM**<sup>t</sup>  $\varphi \supset \psi$  is equivalent to  $\varphi \wedge \mathbf{t} \rightarrow \psi$ , and this was the definition used in Dunn and Meyer (1971).

**Proposition 6.4** **RM** is normal.

*Proof* The axioms [ConE1], [ConE2], and the adjunction rule [Ad] ensure that  $\wedge$  is a conjunction for every extension of **R**, including **RM**. Hence the proposition follows from Theorem 2.11 and Proposition 6.3.  $\square$

**Proposition 6.5** **RM** satisfies the basic relevance criterion.<sup>10</sup>

*Proof* Suppose  $\mathcal{T}_1, \mathcal{T}_2 \vdash_{\mathbf{RM}} \psi$  and  $\mathcal{T}_2$  has no atomic formulas in common with  $\mathcal{T}_1 \cup \{\psi\}$ . We show that  $\mathcal{T}_1 \vdash_{\mathbf{RM}} \psi$ . Suppose otherwise. Then, by Theorem 5.10, there is a valuation  $v$  in  $\mathcal{M}([0, 1])$  such that  $v(\varphi) \geq 1/2$  for every  $\varphi \in \mathcal{T}_1$ , while  $v(\psi) < 1/2$ . Since  $\mathcal{T}_2$  has no atomic formulas in common with  $\mathcal{T}_1 \cup \{\psi\}$ , we may assume without loss of generality that  $v(p) = 1/2$  for every atom  $p$  which occurs in  $\mathcal{T}_2$ . But then  $v(\varphi) = 1/2$  for every  $\varphi \in \mathcal{T}_2$ , and so  $v$  is a model in  $\mathcal{M}([0, 1])$  of  $\mathcal{T}_1 \cup \mathcal{T}_2$  that is not model of  $\psi$ . By Theorem 5.10 again, this contradicts our assumption that  $\mathcal{T}_1, \mathcal{T}_2 \vdash_{\mathbf{RM}} \psi$ .  $\square$

Next we show that **RM** is a semi-relevant logic (Definition 3.5). For this we need to show that it does not have a weakly characteristic matrix. Actually, we prove something significantly stronger.

**Proposition 6.6** **RM** has no finite weakly characteristic non-deterministic matrix (Nmatrix).<sup>11</sup> In particular, it satisfies the minimal semantic criterion.

*Proof* Assume for contradiction that **RM** has a weakly characteristic Nmatrix  $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ , where the number of elements in  $\mathcal{V}$  is a natural number  $n > 1$ . Let  $p_1, \dots, p_{n+1}$  be  $n + 1$  distinct atomic formulas. Define

<sup>10</sup>As pointed out in Note 3.1, this was first claimed by Meyer, but with a wrong proof, in Anderson and Belnap (1975).

<sup>11</sup>See Avron and Zamansky (2011) about this generalization of the notion of a matrix for a logic, including a lot of examples of logics which do not have a finite weakly characteristic matrix, but do have a finite weakly characteristic Nmatrix.

$$\varphi_n =_{Df} (p_1 \rightarrow p_2) \vee (p_2 \rightarrow p_3) \vee \cdots \vee (p_n \rightarrow p_{n+1}).$$

By assigning  $-i$  to  $p_i$ , we see that  $\varphi_n$  is not valid in  $\mathcal{M}(\mathbb{Z})$ . Hence  $\not\models_{\mathbf{RM}} \varphi_n$ , and so  $\varphi_n$  is not valid in  $\mathcal{M}$ . It follows that there is a valuation  $v$  in  $\mathcal{M}$  such that  $v(\varphi) \notin \mathcal{D}$ . Now by the pigeonhole principle, there are  $1 \leq i < j \leq n+1$  such that  $v(p_i) = v(p_j)$ . Obtain  $\psi_n$  from  $\varphi_n$  by replacing the first occurrence of  $p_j$  in  $\varphi_n$  by  $p_i$ , and define a valuation  $v'$  in  $\mathcal{M}$  by letting  $v'(\sigma) = v(\sigma')$ , where  $\sigma'$  is the formula obtained from  $\sigma$  by replacing in  $\sigma$  each subformula of the form  $p_{j-1} \rightarrow p_i$  by  $p_{j-1} \rightarrow p_j$ . Since  $v(p_i) = v(p_j)$ ,  $v'$  is easily seen to be a legitimate valuation in  $\mathcal{M}$ . Now  $v'(\psi_n) = v(\varphi_n)$ . Hence  $\psi_n$  is not valid in  $\mathcal{M}$ , and so  $\not\models_{\mathbf{RM}} \psi_n$ . On the other hand,  $\psi_n$  is valid in  $\mathcal{M}(\mathbb{Z})$  (since  $(p_i \rightarrow p_{i+1}) \vee \cdots \vee (p_{j-2} \rightarrow p_{j-1}) \vee (p_{j-1} \rightarrow p_i)$  is easily seen to be valid in  $\mathcal{M}(\mathbb{Z})$ ), and so  $\vdash_{\mathbf{RM}} \psi_n$ . A contradiction.  $\square$

*Note 6.4* That **RM** has no finite weakly characteristic *deterministic* (i.e., ordinary) matrix was first observed by Dunn in (1970).

**Theorem 6.7** *RM is a normal semi-relevant logic.*

*Proof* This follows from Propositions 6.4–6.6.  $\square$

Here is another well-known way in which the logic **RM** is “semi-relevant.”

**Proposition 6.8** 1. *RM does not have the variable-sharing property.*

2. *If  $\vdash_{\mathbf{RM}} \varphi \rightarrow \psi$  then either  $\varphi$  and  $\psi$  share an atomic formula, or both  $\neg\varphi$  and  $\psi$  are theorems of **RM**.*

*Proof* 1.  $\neg(p \rightarrow p) \rightarrow (q \rightarrow q)$  is a theorem of  $HRM_{\neg}$ , and so also of **RM**.

2. Suppose that  $\vdash_{\mathbf{RM}} \varphi \rightarrow \psi$ , but  $\varphi$  and  $\psi$  share no atomic formula. We show that both  $\neg\varphi$  and  $\psi$  are theorems of **RM**. Suppose, for example, that  $\neg\varphi$  is not a theorem of **RM**. (The argument in the case where  $\psi$  is not a theorem of **RM** is similar.) Then, by Theorem 5.14, there is valuation  $v$  in  $\mathcal{M}(\mathbb{Z})$  such that  $v(\neg\varphi) < 0$ , and so  $v(\varphi) > 0$ . Without a loss of generality, we may assume that  $v(q) = 0$ , for every atom  $q \notin \text{Atoms}(\varphi)$ . Since  $\varphi$  and  $\psi$  share no atomic formula, this implies that  $v(q) = 0$  for every atom  $q \in \text{Atoms}(\psi)$ . But then  $v(\psi) = 0$ . Since  $v(\varphi) > 0$  this implies that  $v(\varphi \rightarrow \psi) < 0$ , contradicting the assumption that  $\vdash_{\mathbf{RM}} \varphi \rightarrow \psi$ .  $\square$

*Note 6.5* Relevant logics like **R** have the *variable-sharing property*. This means that if  $\varphi \rightarrow \psi$  is a tautology, then  $\varphi$  and  $\psi$  share an atomic formula. On the other hand, in classical logic there are two other possibilities in such a case: first, that  $\neg\varphi$  is a tautology, and second, that  $\psi$  is a tautology. Proposition 6.8 shows that **RM** is intermediate in this respect between relevant logics and classical logic. Intuitively, this provides an additional justification for seeing **RM** as a “semi-relevant” logic. Another one is provided by the following strong, “semi-relevant” version of the Craig interpolation theorem that was shown in Avron (1986) for **RM**: if  $\vdash_{\mathbf{RM}} \varphi \supset \psi$  (where  $\supset$  is the implication for **RM** given in Definition 6.2), then either  $\vdash_{\mathbf{RM}} \psi$ , or there is an interpolant  $\sigma$  such that  $\text{Atoms}(\sigma) \subseteq \text{Atoms}(\varphi) \cap \text{Atoms}(\psi)$ , and both  $\varphi \supset \sigma$  and  $\sigma \supset \psi$  are theorems of **RM**. (In classical logic there is a third possibility:

that  $\vdash \neg\varphi$ .) In connection to this, it is worth mentioning that Meyer has presented in Anderson and Belnap (1975, Sect. 29.3) an example of a case in which the Craig interpolation theorem fails in **RM** for  $\rightarrow$ .

Our next goal is to study the set of simple extensions of **RM**.

**Notation** Let **L** be a logic.  $Th(\mathbf{L}) =_{df} \{ \varphi : \vdash_{\mathbf{L}} \varphi \}$ .

**Theorem 6.9** *Let **L** be a simple strongly proper extension of **RM**. Then there is a natural number  $n \geq 2$  such that  $Th(\mathbf{L}) = Th(\mathcal{RM}_n)$ , i.e.,  $\mathcal{RM}_n$  is weakly characteristic for **L**.*

*Proof* First we prove that all theorems of **L** are valid in  $\mathcal{RM}_2$ . Suppose for contradiction that there is a theorem  $\varphi$  of **L** which is not valid in  $\mathcal{RM}_2$ . Then there is a valuation  $v_0$  in  $\mathcal{RM}_2$  such that  $v_0(\varphi) = -1$ . By substituting  $p_0 \rightarrow p_0$  for every atom  $p$  such that  $v_0(p) = 1$ , and  $\neg(p_0 \rightarrow p_0)$  for every atom  $p$  such that  $v_0(p) = -1$ , we obtain from  $\varphi$  a theorem  $\psi$  of **L** such that  $\text{Atoms}(\psi) = \{p_0\}$ , and  $v(\psi) = -1$  for any valuation  $v$  in  $\mathcal{RM}_2$ . It follows that  $\neg\psi$  is valid in  $\mathcal{RM}_2$ . Therefore, Proposition 5.19 implies that  $\vdash_{\mathbf{RM}} \neg\psi$ . Hence both  $\psi$  and  $\neg\psi$  are theorems of **L**. But because  $\text{Atoms}(\psi) = \{p_0\}$ , the first part of Theorem 5.14 implies that  $\neg\psi, \psi \vdash_{\mathbf{RM}} p_0$ . It follows that  $\vdash_{\mathbf{L}} p_0$ , contradicting the condition of non-triviality in our definition of a logic.

Now let  $A$  be the set of all natural numbers  $n$  such that all theorems of **L** are valid in  $\mathcal{RM}_n$ . By what we have just proved,  $2 \in A$ , and so  $A$  is not empty. On the other hand, the fact that **L** is a simple strongly proper extension of **RM** means that there is a sentence  $\varphi_0$  of  $\mathcal{L}_R$  such that  $\vdash_{\mathbf{L}} \varphi_0$ , but  $\not\vdash_{\mathbf{RM}} \varphi_0$ . Therefore, Proposition 5.19 implies that there is  $n_0 \geq 2$  such that  $\varphi_0$  is not valid in  $\mathcal{RM}_{n_0}$ , and so  $n_0 \notin A$ . It follows, by Corollary 5.18, that  $A$  has a maximal element  $k \geq 2$ . Then by Corollary 5.18 again, every theorem of **L** is valid in  $\mathcal{RM}_j$  for every  $2 \leq j \leq k$ , and there is a theorem of **L** which is not valid in  $\mathcal{RM}_j$  for  $j > k$ . We end the proof by showing that  $\mathcal{RM}_k$  is weakly characteristic for **L**. Since  $k \in A$ , it suffices to show that if  $\not\vdash_{\mathbf{L}} \varphi$ , then  $\varphi$  is not valid in  $\mathcal{RM}_k$ .

So suppose that  $\not\vdash_{\mathbf{L}} \varphi$ , and let  $\text{Atoms}(\varphi) = \{p_1, \dots, p_n\}$ . Define

$$\mathcal{T} = \{ \sigma : \text{Atoms}(\sigma) \subseteq \{p_1, \dots, p_n\} \text{ and } \vdash_{\mathbf{L}} \sigma \}$$

Since  $\not\vdash_{\mathbf{L}} \varphi$ , also  $\mathcal{T} \not\vdash_{\mathbf{RM}} \varphi$ . Therefore, Theorem 5.14 and its proof imply that there is an  $l$  and a valuation  $v_0$  in  $\mathcal{RM}_l$  such that  $v_0$  is a model of  $\mathcal{T}$  in  $\mathcal{RM}_l$  which is not a model of  $\varphi$ , and for every element  $a$  of  $\mathcal{RM}_l$  there is  $1 \leq i \leq n$  such that either  $a = v_0(p_i)$  or  $a = \neg v_0(p_i) = v_0(\neg p_i)$ . We show that  $l \in A$ . So let  $\sigma$  be a theorem of **L**, and let  $v$  be a valuation in  $\mathcal{RM}_l$ . Let  $\theta$  be a substitution that assigns to any atomic formula  $q$  an element  $\tau$  of  $\{p_1, \neg p_1, \dots, p_n, \neg p_n\}$  such that  $v(q) = v_0(\tau)$ . Then for any atomic formula  $q$ ,  $v(q) = v_0 \circ \theta(q)$ . This easily implies that  $v = v_0 \circ \theta$ , and so  $v(\sigma) = v_0(\theta(\sigma))$ . But since **L** is a logic,  $\theta(\sigma)$  is also a theorem of **L**, and by definition of  $\theta$ , this implies that  $\theta(\sigma) \in \mathcal{T}$ . Since  $v_0$  is a model of  $\mathcal{T}$ ,  $v_0(\theta(\sigma))$  is designated, and so  $v(\sigma)$  is designated. This was shown for every valuation  $v$  in

$\mathcal{RM}_l$  and any theorem  $\sigma$  of  $\mathbf{L}$ , and so it follows that indeed  $l \in A$ . Hence  $l \leq k$ . Since  $\varphi$  is not valid in  $\mathcal{RM}_l$  (because  $v_0(\varphi)$  is not designated),  $\varphi$  is not valid in  $\mathcal{RM}_k$  either.  $\square$

**Theorem 6.10**  *$\mathbf{RM}$  has the Scroggs' property, that is, it does not have a finite weakly characteristic matrix, but every strongly proper extension of it does.*

*Proof* This follows from Proposition 6.6 and Theorem 6.9.  $\square$

*Note 6.6* Theorems 6.9 and 6.10 are due to Dunn (see Dunn (1970) and Anderson and Belnap (1975, Sect. 29.4)).

By Theorem 6.9, if  $\mathbf{L}$  is a simple extension of  $\mathbf{RM}$ , then  $Th(\mathbf{L})$  belongs to the sequence  $\{Th(\mathcal{RM}_n)\}_{n=2}^\infty$ . Next we axiomatize each of the elements in this sequence, and show that they are all different from each other.

**Definition 6.11**  $HRM_n$  is the simple axiomatic extension of  $\mathbf{RM}$  which is obtained by adding  $\varphi_n$  (from the proof of Proposition 6.6) to  $HRM$  as an extra axiom schema (i.e., by adding to  $HRM$  all instances of  $\varphi_n$  as new axioms).

### Theorem 6.12

1. For every  $n \geq 2$  and  $\varphi \in \mathcal{L}_R$ ,  $\varphi$  is valid in  $\mathcal{RM}_n$  iff  $\vdash_{HRM_n} \varphi$ . (In other words,  $Th(\mathcal{RM}_n) = Th(HRM_n)$  for every  $n \geq 2$ .)
2. The sequence  $\{Th(\mathcal{RM}_n)\}_{n=2}^\infty$  is strictly decreasing, and includes  $Th(\mathbf{L})$  whenever  $\mathbf{L}$  is a simple strongly proper extension of  $\mathbf{RM}$ .

*Proof* Let  $\varphi_n$  be like in the proof of Proposition 6.6. It is straightforward to check that for every  $n \geq 2$ ,  $\varphi_n$  is valid in  $\mathcal{RM}_n$ , but not in  $\mathcal{RM}_{n+1}$ . Hence  $n$  is the maximal number  $k$  such that  $\varphi_n$  is valid in  $\mathcal{RM}_k$ . Hence the first part follows from the proof of Theorem 6.9. That theorem implies also that the sequence  $\{Th(\mathcal{RM}_n)\}_{n=2}^\infty$  includes every set of the form  $Th(\mathbf{L})$  such that  $\mathbf{L}$  is a simple strongly proper extension of  $\mathbf{RM}$ . That this sequence is decreasing follows from Proposition 5.17. That it is strictly decreasing again follows from the fact that  $\varphi_n$  is valid in  $\mathcal{RM}_n$ , but not in  $\mathcal{RM}_{n+1}$ .  $\square$

Now we turn to what is perhaps the most important property of  $\mathbf{RM}$  (and certainly the main new result in this paper).

**Theorem 6.13**  *$\mathbf{RM}$  is a maximal finitary logic which is both normal and semi-relevant. In other words, every proper simple finitary extension of  $\mathbf{RM}$  is either not normal or not semi-relevant.*

*Proof* Let  $\mathbf{L}$  be a simple finitary extension of  $\mathbf{RM}$  which is both normal and semi-relevant. We show that  $\mathbf{L} = \mathbf{RM}$ . Now by Theorem 6.9, no strongly proper extension of  $\mathbf{RM}$  can be semi-relevant. It follows that  $Th(\mathbf{L}) = Th(\mathbf{RM})$ . Let  $\Rightarrow$  be a defined connective of  $\mathcal{L}_R$  which is an implication for  $\mathbf{L}$ . Then  $\mathcal{T}, \varphi \vdash_{\mathbf{L}} \psi$  iff  $\mathcal{T} \vdash_{\mathbf{L}} \varphi \Rightarrow \psi$ , for every  $\mathcal{T}, \varphi$  and  $\psi$ .

In the sequel, we denote by  $\tilde{\neg}$ ,  $\tilde{\vee}$ ,  $\tilde{\wedge}$ ,  $\tilde{\rightarrow}$ , and  $\tilde{\Rightarrow}$  the interpretations in  $\mathcal{M}(\mathbb{Z})$  of  $\neg$ ,  $\vee$ ,  $\wedge$ ,  $\rightarrow$ , and  $\Rightarrow$ , respectively; and we extensively use the following property of these operations:

(\*) If  $f: \mathbb{Z}^n \rightarrow \mathbb{Z}$  is obtained from  $\tilde{\neg}$ ,  $\tilde{\vee}$ ,  $\tilde{\wedge}$ ,  $\tilde{\rightarrow}$ , and  $\tilde{\Rightarrow}$  using compositions, then  $f(a_1, \dots, a_n) \in \{a_1, -a_1, a_2, -a_2, \dots, a_n, -a_n\}$  for every  $a_1, \dots, a_n \in \mathbb{Z}$ .

Next we prove some properties of  $\tilde{\Rightarrow}$ .

1. For every  $n \geq 0$ ,  $n \tilde{\Rightarrow} n = -n \tilde{\Rightarrow} n = -n \tilde{\Rightarrow} -n = n$ , while  $n \tilde{\Rightarrow} -n = -n$ .

*Proof* Since  $p \vdash_{\mathbf{L}} p$ ,  $\vdash_{\mathbf{L}} p \Rightarrow p$ . Hence  $\vdash_{\mathbf{RM}} p \Rightarrow p$ , and so  $a \tilde{\Rightarrow} a \geq 0$  for every  $a \in \mathbb{Z}$ . By (\*) this implies that  $n \tilde{\Rightarrow} n = -n \tilde{\Rightarrow} -n = n$  for every  $n \geq 0$ . Now the fact that  $\vdash_{\mathbf{L}} p \Rightarrow p$  implies that  $p \vdash_{\mathbf{L}} p \Rightarrow p$ , and so  $\vdash_{\mathbf{L}} p \Rightarrow (p \Rightarrow p)$ . It follows that  $\vdash_{\mathbf{RM}} p \Rightarrow (p \Rightarrow p)$ . Hence for every  $n \geq 0$ ,  $-n \tilde{\Rightarrow} (-n \tilde{\Rightarrow} -n) \geq 0$ , and so  $-n \tilde{\Rightarrow} n \geq 0$ . By (\*), this implies that  $-n \tilde{\Rightarrow} n = n$ . Next we note that since  $\mathbf{L}$  is a logic (i.e., non-trivial), the fact that  $\vdash_{\mathbf{L}} p \Rightarrow p$  implies that  $p \not\vdash_{\mathbf{L}} p$ , and so  $\not\vdash_{\mathbf{L}} (p \Rightarrow p) \Rightarrow p$ . It follows that there is some  $a \in \mathbb{Z}$  such that  $(a \tilde{\Rightarrow} a) \tilde{\Rightarrow} a < 0$ . By what we have already shown, if  $a \geq 0$  then  $(a \tilde{\Rightarrow} a) \tilde{\Rightarrow} a = a \geq 0$ . Hence necessarily  $a < 0$ . So let  $a = -k$  for some  $k > 0$ . Then  $(-k \tilde{\Rightarrow} -k) \tilde{\Rightarrow} -k < 0$ , and so  $k \tilde{\Rightarrow} -k < 0$ . By (\*), this implies that  $k \tilde{\Rightarrow} -k = -k$ . Now by Lemma 5.2, for every  $n > 0$  the submatrix of  $\mathcal{M}(\mathbb{Z})$  induced by  $\{-n, n\}$  is isomorphic to the submatrix of  $\mathcal{M}(\mathbb{Z})$  induced by  $\{-k, k\}$ . It follows that  $n \tilde{\Rightarrow} -n = -n$  for every  $n > 0$ , and (\*) implies that  $n \tilde{\Rightarrow} -n = -n$  also when  $n = 0$ .  $\square$

2.  $a \tilde{\Rightarrow} k \in \{|a|, k\}$  for every  $a \in \mathbb{Z}$  and  $k \geq 0$ .

*Proof* Since  $\vdash_{\mathbf{L}} p \Rightarrow p$ , also  $q \vdash_{\mathbf{L}} p \Rightarrow p$ . Hence  $\vdash_{\mathbf{L}} q \Rightarrow (p \Rightarrow p)$ , and so  $\vdash_{\mathbf{RM}} q \Rightarrow (p \Rightarrow p)$ . Hence  $a \tilde{\Rightarrow} (k \tilde{\Rightarrow} k) \geq 0$  for every  $a \in \mathbb{Z}$  and  $k \geq 0$ . By item 1 above, this means that  $a \tilde{\Rightarrow} k \geq 0$  for every  $a \in \mathbb{Z}$  and  $k \geq 0$ . Hence (\*) implies that  $a \tilde{\Rightarrow} k \in \{|a|, k\}$  for every  $a \in \mathbb{Z}$  and  $k \geq 0$ .  $\square$

3. For every  $a \in \mathbb{Z}$  and  $k \geq 0$ , if  $|a| \leq k$ , then  $-k \tilde{\Rightarrow} a \in \{|a|, k\}$ .

*Proof* Using  $\mathcal{RM}_4$  it is easy to see that  $\vdash_{\mathbf{RM}} \neg((p \rightarrow q) \rightarrow (p \rightarrow q)) \rightarrow p$ . This entails that  $\neg((p \rightarrow q) \rightarrow (p \rightarrow q)) \vdash_{\mathbf{L}} p$ . Hence  $\vdash_{\mathbf{L}} \neg((p \rightarrow q) \rightarrow (p \rightarrow q)) \Rightarrow p$ , and so  $\vdash_{\mathbf{RM}} \neg((p \rightarrow q) \rightarrow (p \rightarrow q)) \Rightarrow p$ . It follows that if  $a \in \mathbb{Z}$  and  $k \geq 0$ , then  $\neg((a \rightarrow k) \rightarrow (a \rightarrow k)) \tilde{\Rightarrow} a \geq 0$ . Now if  $|a| \leq k$ , then  $\neg((a \rightarrow k) \rightarrow (a \rightarrow k)) = -k$ , and so we get that  $-k \tilde{\Rightarrow} a \geq 0$  in such a case. By (\*), this is equivalent to  $-k \tilde{\Rightarrow} a \in \{|a|, k\}$ .  $\square$

4. If  $0 \leq k \leq n$ , then  $k \tilde{\Rightarrow} -n = -n$ .

*Proof* Since  $\mathbf{L}$  is semi-relevant,  $\neg(p \rightarrow p)$ ,  $(p \rightarrow p) \not\vdash_{\mathbf{L}} q$ . Hence  $\neg(p \rightarrow p) \not\vdash_{\mathbf{L}} (p \rightarrow p) \Rightarrow q$ , and so  $\neg(p \rightarrow p) \not\vdash_{\mathbf{RM}} (p \rightarrow p) \Rightarrow q$  as well. By Corollary 5.15, this implies that there is a valuation  $v$  in  $\mathcal{M}(\mathbb{Z})$  which is a model of  $\neg(p \rightarrow p)$ , but not of  $(p \rightarrow p) \Rightarrow q$ . The first fact implies that  $v(p) = 0$ , and so the second one implies that  $0 \tilde{\Rightarrow} v(q) < 0$ . By item 2, this is possible only if  $v(q) = -n$  for some  $n > 0$ . But in such a case it easily follows from Proposition 5.13 that  $0 \tilde{\Rightarrow} -n < 0$  for every  $n > 0$ . By (\*) and item 1, it follows that  $0 \tilde{\Rightarrow} -n = -n$  for every  $n$ .

From the fact shown above that  $\neg(p \rightarrow p) \not\vdash_{\mathbf{L}} (p \rightarrow p) \Rightarrow q$ , it follows that  $\not\vdash_{\mathbf{L}} \neg(p \rightarrow p) \Rightarrow ((p \rightarrow p) \Rightarrow q)$ . Hence  $\not\vdash_{\mathbf{RM}} \neg(p \rightarrow p) \Rightarrow ((p \rightarrow p) \Rightarrow q)$ . Therefore, Proposition 5.19 implies that there is a valuation  $v$  in  $\mathcal{RM}_4$  such that  $v(\neg(p \rightarrow p) \Rightarrow ((p \rightarrow p) \Rightarrow q)) < 0$ . By item 3, it cannot be the case that  $v(\neg(p \rightarrow p)) = -2$ . Hence  $|v(p)| = 1$ , and we get that  $-1 \stackrel{\sim}{\Rightarrow} (1 \stackrel{\sim}{\Rightarrow} v(q)) < 0$ . By items 1 and 2, this is impossible if  $v(q) \in \{-1, 1, 2\}$ . It follows that  $v(q) = -2$ , and so  $-1 \stackrel{\sim}{\Rightarrow} (1 \stackrel{\sim}{\Rightarrow} -2) < 0$ . This in turn implies (by items 1 and 2 again) that  $1 \stackrel{\sim}{\Rightarrow} -2 = -2$ . As usual, by Proposition 5.13 this means that  $k \stackrel{\sim}{\Rightarrow} -n = -n$  in case  $0 < k < n$ . By item 1 and what we have shown above about  $0 \stackrel{\sim}{\Rightarrow} -n$ , this equation holds also in the cases where  $k = n$  or  $k = 0$ . Hence  $k \stackrel{\sim}{\Rightarrow} -n = -n$  whenever  $0 \leq k \leq n$ .  $\square$

5. If  $0 < n < k$ , then  $k \stackrel{\sim}{\Rightarrow} -n < 0$ .

*Proof*  $(p \wedge \neg p) \vee (p \wedge \neg p \rightarrow q)$  is not a tautology of **RM** in case  $p \neq q$ . (Take  $v(p) = 1$  and  $v(q) = -2$  in  $\mathcal{M}(\mathbb{Z})$ .) Hence it is not provable in **L** either, and so also  $q \rightarrow q \not\vdash_{\mathbf{L}} (p \wedge \neg p) \vee (p \wedge \neg p \rightarrow q)$ . It follows that  $\not\vdash_{\mathbf{L}} (q \rightarrow q) \Rightarrow (p \wedge \neg p) \vee (p \wedge \neg p \rightarrow q)$ , and so  $\not\vdash_{\mathbf{RM}} (q \rightarrow q) \Rightarrow (p \wedge \neg p) \vee (p \wedge \neg p \rightarrow q)$ . Therefore, Proposition 5.19 implies that there is a valuation  $v$  in  $\mathcal{RM}_4$  such that  $v((q \rightarrow q) \Rightarrow (p \wedge \neg p) \vee (p \wedge \neg p \rightarrow q)) < 0$ . By item 2, this is possible only if  $v((p \wedge \neg p) \vee (p \wedge \neg p \rightarrow q)) < 0$ . An easy check shows that this is the case only if  $v(q) = -2$  and  $|v(p)| = 1$ . Hence the fact that  $v((q \rightarrow q) \Rightarrow (p \wedge \neg p) \vee (p \wedge \neg p \rightarrow q)) < 0$  means that  $2 \stackrel{\sim}{\Rightarrow} -1 < 0$ . By Proposition 5.13 again, it follows that  $k \stackrel{\sim}{\Rightarrow} -n < 0$  whenever  $0 < n < k$ .

Next we show that  $[MP]$  for  $\Rightarrow$  is valid in **RM**, i.e.,  $\varphi, \varphi \Rightarrow \psi \vdash_{\mathbf{RM}} \psi$  for every  $\varphi$  and  $\psi$ . Suppose otherwise. Then from Corollary 5.15 it follows that there is a valuation  $v$  in  $\mathcal{M}(\mathbb{Z})$  such that  $v(\varphi) \geq 0$ ,  $v(\psi) < 0$ , and  $v(\varphi \Rightarrow \psi) \geq 0$ . But this is impossible, by items 4 and 5 of the above list of properties of  $\stackrel{\sim}{\Rightarrow}$ .

Finally, we prove that  $\mathbf{L} = \mathbf{RM}$ . Since **L** is an extension of **RM**, it suffices to show that if  $\mathcal{T} \vdash_{\mathbf{L}} \varphi$ , then  $\mathcal{T} \vdash_{\mathbf{RM}} \varphi$ . So assume that  $\mathcal{T} \vdash_{\mathbf{L}} \varphi$ . Since **L** is finitary, there are  $\psi_1, \dots, \psi_n \in \mathcal{T}$  such that  $\{\psi_1, \dots, \psi_n\} \vdash_{\mathbf{L}} \varphi$ . It follows that  $\vdash_{\mathbf{L}} \psi_1 \Rightarrow (\psi_2 \Rightarrow \dots (\psi_n \Rightarrow \varphi) \dots)$ . This in turn implies that  $\vdash_{\mathbf{RM}} \psi_1 \Rightarrow (\psi_2 \Rightarrow \dots (\psi_n \Rightarrow \varphi) \dots)$ . But we have shown that  $[MP]$  for  $\stackrel{\sim}{\Rightarrow}$  is valid in **RM**. Therefore  $\{\psi_1, \dots, \psi_n\} \vdash_{\mathbf{RM}} \varphi$ , and so  $\mathcal{T} \vdash_{\mathbf{RM}} \varphi$ .  $\square$

*Note 6.7* The fact that **L** is semi-relevant was used in the last proof only for deriving 4. Since  $\neg p, p \vdash_{\mathbf{RM}} \neg(p \rightarrow p)$ , while  $\neg(p \rightarrow p) \vdash_{\mathbf{RM}} p$  and  $\neg(p \rightarrow p) \vdash_{\mathbf{RM}} \neg p$ , an almost identical proof shows that if **L** is a finitary proper simple extension of **RM** which is both normal and paraconsistent, then **L** has a finite weakly characteristic matrix. In other words, **RM** is a maximal normal paraconsistent logic that satisfies the minimal semantic relevance criterion.

*Note 6.8* It is worth noting that in addition to its nice semantic properties and maximality properties as described in this section, **RM** is nice also from a proof-theoretical

point of view, since it has a corresponding cut-free Gentzen-type system  $GRM$  with the subformula property.  $GRM$  employs *hypersequents*, rather than ordinary sequents, and its logical rules are identical to those used in classical logic (with caution about the chosen form of each rule, namely, whether the rule is multiplicative or additive). See Avron (1987) for details.

## 7 RM as a fuzzy logic

Fuzzy logics are logics that are designed to deal with propositions that involve imprecise concepts, like “tall” or “old.” Their semantics is based on the idea of *degrees of truth*, according to which the truth-value assigned to a proposition of this sort might not be one of the two classical values 0 and 1, but any real number between them. Now, in all the standard fuzzy logics investigated in the literature (see Cintula et al. (2011) for an extensive survey), the consequence relation is based on preserving absolute truth, i.e., 1 is taken as the only designated value. This choice implies that none of these logics is paraconsistent. Therefore, the obvious way to develop useful paraconsistent fuzzy logics is to use a more comprehensive set of designated values. This is precisely what is done in the semantics of **RM** as given in the second part of Theorem 5.10 (i.e., the matrix  $\mathcal{M}([0, 1])$ ). Hence **RM** can serve as an excellent candidate for paraconsistent fuzzy logic.<sup>12</sup> However, to view and use **RM** as a fuzzy logic it would be better to take  $\supset$  (rather than  $\rightarrow$ ) as a primitive connective. This is possible, since by the next proposition this choice does not affect the expressive power of the language.

**Proposition 7.1** *The connective  $\rightarrow$  of **RM** is definable in  $\{\neg, \supset, \wedge, \vee\}$ .*

*Proof* By using  $\mathcal{M}([0, 1])$ , it is easy to check that  $\varphi \rightarrow \psi$  is equivalent in **RM** to  $(\varphi \supset \psi) \wedge (\neg\psi \supset \neg\varphi)$ .<sup>13</sup>  $\square$

Next we show that not only is **RM** a fuzzy logic according to the above characterization of this notion, but it (more exactly, its natural conservative extension **RM**<sup>F</sup> defined below) is in fact a conservative extension of one of the three most basic *standard* fuzzy logics (Cintula et al. 2011), namely, of the Gödel–Dummett logic **G**<sub>∞</sub>.

**Definition 7.2** Let  $\mathcal{L}_R^F = \mathcal{L}_R \cup \{F\}$ .  $HRM^F$  is the extension of  $HRM$  by the axiom  $F \rightarrow \varphi$ . **RM**<sup>F</sup> is the logic in  $\mathcal{L}_R^F$  that is induced by  $HRM^F$ .

<sup>12</sup>Slaney’s logic **F** (Slaney 2010) is another recent work on substructural fuzzy logics.

<sup>13</sup>In Avron (1986), it is noted that  $\varphi \rightarrow \psi$  is equivalent in **RM** also to  $\neg(\varphi \supset \psi) \supset \neg(\psi \supset \varphi)$ , so it is definable in terms of just  $\neg$  and  $\supset$ .

### Definition 7.3

- A Sugihara chain  $\langle \mathcal{V}, \leq, - \rangle$  is *bounded* if  $\langle \mathcal{V}, \leq \rangle$  has a minimal element.<sup>14</sup>
- A *bounded Sugihara matrix* for  $\mathcal{L}_R^F$  is a Sugihara matrix which is based on a bounded Sugihara chain  $\langle \mathcal{V}, \leq, - \rangle$ , and in which the interpretation  $\tilde{F}$  of  $F$  is the minimal element of  $\langle \mathcal{V}, \leq \rangle$ .

Here is a particularly important example of a bounded Sugihara matrix.

**Definition 7.4**  $\mathcal{M}^F([0, 1])$  is the extension of  $\mathcal{M}([0, 1])$  to  $\mathcal{L}_R^F$  that is obtained by letting  $\tilde{F}$  (the interpretation of  $F$ ) be 0.

### Theorem 7.5

1.  $\mathbf{RM}^F$  is strongly sound and complete for bounded Sugihara matrices.
2.  $\mathbf{RM}^F$  is strongly sound and complete for  $\mathcal{M}^F([0, 1])$ .

*Proof* A straightforward extension of the proof of Theorem 5.10. □

**Corollary 7.6**  $\mathbf{RM}^F$  is a conservative extension of  $\mathbf{RM}$ .

**Definition 7.7** (Gödel–Dummett logic  $\mathbf{G}_\infty$ ) Let  $\mathcal{IL} = \{\supset, \wedge, \vee, F\}$ , and let  $HIL$  be some standard Hilbert-type system in  $\mathcal{IL}$  for intuitionistic logic.  $HG_\infty$  is the extension of  $HIL$  by the following linearity axiom.

$$[Li] \quad (\varphi \supset \psi) \vee (\psi \supset \varphi)$$

$\mathbf{G}_\infty$  is the logic in  $\mathcal{IL}$  which is induced by  $HG_\infty$ , and  $\mathbf{G}_\infty^+$  is its positive (i.e.,  $F$ -free) fragment.

**Theorem 7.8**  $\mathbf{RM}^F$  is a conservative extension of  $\mathbf{G}_\infty$ , and  $\mathbf{RM}$  is a conservative extension of  $\mathbf{G}_\infty^+$ .

*Proof* We show the first part. The proof of the second part is almost identical.

Using Proposition 6.3 (and the fact that  $\wedge$  and  $\vee$  are, respectively, conjunction and disjunction for  $\mathbf{RM}$ ), it is easy to show that  $HIL$  is included in  $\mathbf{RM}^F$ . It is also easy to verify that the extra axiom  $[Li]$  of  $HG_\infty$  is a theorem of  $\mathbf{RM}^F$  too. Hence  $\mathbf{RM}^F$  is an extension of  $\mathbf{G}_\infty$ .

To show that  $\mathbf{RM}^F$  conservatively extends  $\mathbf{G}_\infty$ , assume that  $\mathcal{T} \not\vdash_{HG_\infty} \psi$ , where both  $\mathcal{T}$  and  $\psi$  are in  $\mathcal{IL}$ . Like in the proof of Theorem 5.10, we get an extension  $\mathcal{T}^*$  of  $\mathcal{T}$  such that

1.  $\mathcal{T}^* \not\vdash_{HG_\infty} \psi$ ;
2. for every  $\varphi$  and  $\tau$ ,  $\mathcal{T}^* \vdash_{HG_\infty} \varphi \wedge \tau$  iff both  $\mathcal{T}^* \vdash_H \varphi$  and  $\mathcal{T}^* \vdash_{HG_\infty} \tau$ ;
3. for every  $\varphi$  and  $\tau$ ,  $\mathcal{T}^* \vdash_{HG_\infty} \varphi \vee \tau$  iff either  $\mathcal{T}^* \vdash_{HG_\infty} \varphi$  or  $\mathcal{T}^* \vdash_{HG_\infty} \tau$ .

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<sup>14</sup>Obviously, if  $a$  is a minimal element then  $-a$  is a maximal one. Hence a Sugihara chain is bounded according to Definition 7.3 iff it is bounded in the usual sense of having both a minimal element and a maximal one.

Now define  $\psi \equiv \sigma$  iff both  $\vdash_{HG_\infty} \psi \supset \sigma$  and  $\vdash_{HG_\infty} \sigma \supset \psi$ . Since  $HG_\infty$  is an (axiomatic simple) extension of  $HIL$ ,  $\equiv$  is an equivalence relation (indeed, a congruence relation). Let  $\mathcal{V}$  be the set of equivalence classes, and define  $\leq$  on  $\mathcal{V}$  by letting  $[\tau] \leq [\sigma]$  iff  $\vdash_{HG_\infty} \tau \supset \sigma$ . The fact that  $HG_\infty$  is an extension of  $HIL$  easily implies this time that  $\leq$  is well defined, and is a partial order on  $\mathcal{V}$ . In addition, the  $\vee$ -primeness of  $T^*$  (item 3 above) and the special axiom [Li] of  $HG_\infty$  entail that  $\leq$  is a *linear* order. Obviously,  $[F]$  is the minimal element of  $\mathcal{V}$  according to this linear order, while axiom  $[\supset 1]$  of  $HIL$  ensures that  $\{\varphi : T^* \vdash_{HG_\infty} \varphi\}$  is its maximal element. Since  $\mathcal{V}$  is countable, these facts imply (see the beginning of the proof of Lemma 5.3) that there is a function  $e : \mathcal{V} \rightarrow [0, 1/2]$  such that  $e$  is order preserving,  $e([F]) = 0$ , and  $e(\{\varphi : T^* \vdash_{HG_\infty} \varphi\}) = 1/2$ . Define a valuation  $v$  in  $\mathcal{M}([0, 1])$  by letting  $v(p) = e([p])$  for every atom  $p$ . We show that the following is true for every formula  $\varphi$  of  $\mathcal{IL}$ :

- (a) If  $T^* \vdash_{HG_\infty} \varphi$ , then  $v(\varphi) \geq 1/2$ .
- (b) If  $T^* \not\vdash_{HG_\infty} \varphi$ , then  $v(\varphi) = e([\varphi])$  (and so  $v(\varphi) < 1/2$ ).

Since  $\mathcal{T} \subseteq T^*$  and  $T^* \not\vdash_{HG_\infty} \psi$ , these two facts imply that  $v$  is a model of  $\mathcal{T}$  in  $\mathcal{M}([0, 1])$  that is not a model of  $\psi$ . Hence Theorem 7.5 entails that  $T^* \not\vdash_{\mathbf{RM}^F} \psi$ , which is what we wanted to prove.

We prove (a) and (b) by induction on the complexity of  $\varphi$ .

- The case where  $\varphi$  is an atomic variable or the constant  $F$  easily follows from the definition of  $v$ , and the properties of  $e$  mentioned above.
- Suppose that  $\varphi = \tau \supset \sigma$ .
  - (a) Suppose  $T^* \vdash_{HG_\infty} \varphi$ . Then  $[\tau] \leq [\sigma]$ , and so  $e([\tau]) \leq e([\sigma])$ . If  $v(\sigma) \geq 1/2$ , then  $v(\varphi) \geq 1/2$  (see Note 6.1). If not, then  $T^* \not\vdash_{HG_\infty} \sigma$  by (a) of the induction hypothesis, and so  $T^* \not\vdash_{HG_\infty} \tau$ . Hence (b) of the induction hypothesis implies that  $v(\tau) = e([\tau])$  and  $v(\sigma) = e([\sigma])$ . Therefore  $v(\tau) \leq v(\sigma)$ , and so  $v(\varphi) \geq 1/2$ .
  - (b) Suppose  $T^* \not\vdash_{HG_\infty} \varphi$ . Because of Axiom  $[\supset 1]$ , this implies that also  $T^* \not\vdash_{HG_\infty} \sigma$ , and so  $v(\sigma) = e([\sigma]) < 1/2$  by (a). The assumption also implies that  $[\tau] \not\leq [\sigma]$ , and so  $e([\sigma]) < e([\tau])$ . Since by (a) and (b)  $v(\tau) \geq 1/2$  or  $v(\tau) = e([\tau])$ , it follows that  $v(\sigma) < v(\tau)$ , and so (see Note 6.1)  $v(\varphi) = v(\sigma) = e([\sigma])$ . It remains to show that  $e([\varphi]) = e([\sigma])$  in this case, i.e., that  $\varphi \equiv \sigma$ . That  $T^* \vdash_{HG_\infty} \sigma \supset \varphi$  is immediate from Axiom  $[\supset 1]$ . For the converse implication, note that since  $\tau \supset (\tau \supset \sigma) \vdash_{HIL} \tau \supset \sigma$  (immediate from the deduction theorem of  $HIL$ ), our assumption implies that  $T^* \not\vdash_{HG_\infty} \tau \supset (\tau \supset \sigma)$ . Hence Axiom [Li] and the  $\vee$ -primeness of  $T^*$  entail that  $T^* \vdash_{HG_\infty} (\tau \supset \sigma) \supset \tau$ . But  $\vdash_{HIL} ((\tau \supset \sigma) \supset \tau) \supset ((\tau \supset \sigma) \supset \sigma)$ . It follows that  $T^* \vdash_{HG_\infty} (\tau \supset \sigma) \supset \sigma$ , i.e.,  $T^* \vdash_{HG_\infty} \varphi \supset \sigma$ .
- Suppose that  $\varphi = \tau \vee \sigma$ .
  - (a) Suppose  $T^* \vdash_{HG_\infty} \varphi$ . Then the  $\vee$ -primeness of  $T^*$  implies that either  $T^* \vdash_{HG_\infty} \tau$  or  $T^* \vdash_{HG_\infty} \sigma$ . It follows by (a) of the induction hypothesis that either  $v(\tau) \geq 1/2$  or  $v(\sigma) \geq 1/2$ . In both cases, also  $v(\varphi) \geq 1/2$ .
  - (b) Suppose that  $T^* \not\vdash_{HG_\infty} \varphi$ . Then property 3 of  $T^*$  implies that  $T^* \not\vdash_{HG_\infty} \tau$  and  $T^* \not\vdash_{HG_\infty} \sigma$ . It follows by (b) of the induction hypothesis that  $v(\tau) = e([\tau])$

and  $v(\sigma) = e([\sigma])$ . Assume, without loss of generality, that  $[\sigma] \leq [\tau]$ . Then  $T^* \vdash_{HG_\infty} \sigma \supset \tau$ , and  $e([\sigma]) \leq e([\tau])$ . The former fact implies (with the help of the Axioms  $[\supset \vee]$  and  $[\vee \supset]$ ) that  $\varphi \equiv \tau$ , and so  $e([\varphi]) = e([\tau])$ . The latter fact implies that  $v(\varphi) = e([\tau])$ , hence  $v(\varphi) = e([\varphi])$ .

- We leave the case where  $\varphi = \tau \wedge \sigma$  to the reader.

This ends the proof of (a) and (b), and so of the theorem.  $\square$

*Note 7.1* The connection between **RM** and **G<sub>∞</sub>** was first observed by Dunn and Meyer in (1971), where it was proved that **RM<sup>t</sup>** (see Note 6.3) is a *weakly* conservative extension of the positive fragment of **G<sub>∞</sub>**.

*Note 7.2* The standard semantics of Gödel–Dummett logic **G<sub>∞</sub>**, as described in the literature on fuzzy logics, is provided by the matrix  $\langle [0, 1], 1, \mathcal{O} \rangle$ , where the interpretations in  $\mathcal{O}$  of  $\vee$ ,  $\wedge$ , and **F** are like in  $\mathcal{M}([0, 1])$  (the strongly characteristic matrix for **RM<sup>F</sup>**), while  $a \widetilde{\supset} b$  is 1 if  $a \leq b$ , and  $b$  otherwise. However, the last theorem shows that when we use **G<sub>∞</sub>**, it is not essential at all to take 1 as the only designated value. It is also interesting to note that the interpretation of  $\neg$  in  $\mathcal{M}([0, 1])$  is identical to that used in the most famous fuzzy logic (except perhaps **G<sub>∞</sub>**): Łukasiewicz’s logic. (In **G<sub>∞</sub>** itself  $\neg\varphi$  is usually taken as an abbreviation for  $\varphi \supset F$ .)

*Note 7.3* A Hilbert-type system  $HRM^\supset$  in  $\{\neg, \vee, \wedge, \supset\}$  which is strongly sound and complete for **RM** has been given in Avron (1986).  $HRM^\supset$  is obtained from  $HG_\infty$ , by adding to it axioms connected with  $\neg$ . By adding **F**  $\supset \varphi$  and  $\varphi \supset \neg F$  as axioms to  $HRM^\supset$ , we get a Hilbert-type system in  $\mathcal{IL} \cup \{\neg\}$  that is strongly sound and complete for **RM<sup>F</sup>**.

## References

- Anderson, A. R., & Belnap, N. D. (1975). *Entailment: The logic of relevance and necessity* (Vol. I). Princeton: Princeton University Press.
- Avron, A. (1984). Relevant entailment—semantics and formal systems. *Journal of Symbolic Logic*, 49, 334–342.
- Avron, A. (1986). On an implication connective of RM. *Notre Dame Journal of Formal Logic*, 27, 201–209.
- Avron, A. (1987). A constructive analysis of RM. *Journal of Symbolic Logic*, 52, 939–951.
- Avron, A. (1997). Multiplicative conjunction as an extensional conjunction. *Logic Journal of the IGPL*, 5, 181–208.
- Avron, A. (2014). What is relevance logic? *Annals of Pure and Applied Logic*, 165, 26–48.
- Avron, A. & Zamansky, A. (2011). Non-deterministic semantics for logical systems —a survey, In D. Gabbay & F. Guenther (Eds.), *Handbook of philosophical logic* (2nd ed., Vol. 16, pp. 227–304). Springer.
- Brady, R. T. (1990). The gentzenization and decidability of RW. *Journal of Philosophical Logic*, 19, 35–73.
- Cintula, P., Hájek, P. & Noguera, C. (2011). *Handbook of mathematical fuzzy logic*, Volume 37–38 of Studies in Logic, College Publications.

- Dunn, J. M. (1970). Algebraic completeness results for R-Mingle and its extensions. *Journal of Symbolic Logic*, 35, 1–13.
- Dunn, J. M., & Meyer, R. K. (1971). Algebraic completeness results for Dummett's LC and its extensions. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 17, 225–230.
- Dunn, J. M., & Restall, G. (2002). Relevance logic. In D. Gabbay & F. Guenther (Eds.), *Handbook of philosophical logic* (2nd ed., Vol. 6, pp. 1–128). Amsterdam: Kluwer.
- Łoś, J., & Suszko, R. (1958). Remarks on sentential logics. *Indagationes Mathematicae*, 20, 177–183.
- Meyer, R. K., & Dunn, J. M. (1969). E, R and  $\gamma$ . *Journal of Symbolic Logic*, 34, 460–474.
- Parks, R. Z. (1972). A note on R-Mingle and Sobociński's three-valued logic. *Notre Dame Journal of Formal Logic*, 13, 227–228.
- Slaney, J. K. (2010). A logic for vagueness. *Australian Journal of Logic*, 8, 100–134.
- Sugihara, T. (1955). Strict implication free from implicational paradoxes. *Memoirs of the Faculty of Liberal Arts, Fukui University, Series*, 1(4), 55–59.
- Urquhart, A. (1984). The undecidability of entailment and relevant implication. *Journal of Symbolic Logic*, 49, 1059–1073.
- Urquhart, A. (2001). Many-valued logic, In D. Gabbay & F. Guenther (Eds.), *Handbook of philosophical logic* (Vol. 11, pp. 249–295). Kluwer

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