Preface

The Stochastic Equation. The authors of this text (called DB, ED and TM) started their collaboration with the paper Buraczewski et al. [76] in 2011. We studied large deviations and ruin probabilities for the solution (X_t) to Kesten's stochastic recurrence equation

$$X_t = A_t X_{t-1} + B_t, \qquad t \in \mathbb{Z}. \tag{1}$$

Despite our cultural differences (DB and ED come from the area of harmonic analysis, TM from applied probability theory) we managed to overcome problems caused by distinct mathematical terminology and found a lot of common scientific ground. What might seem an obstacle—the diversity of our background—proved very fruitful due to the variety of ideas and techniques at hand. Soon we decided to write this text about the solution to the equation in law

$$X \stackrel{d}{=} AX + B \tag{2}$$

and the very closely related solution to the stochastic recurrence equation (1). The two equations (1) and (2) have attracted a lot of attention. On the one hand, they are widely used to draw fractal images. On the other hand, when the solution X to the equation (2) is unbounded, Kesten [175] in 1973 derived asymptotics for the tails of the solution X in the univariate and multivariate cases. The most surprising aspect of his results is the fact that, under general conditions, the tails of X_t are of power-law-type: light-tailed input variables (A_t, B_t) in (1) may cause rather heavy-tailed output X_t.

The highly praised Kesten paper has motivated several generations of researchers to work on closely related topics. One branch of research moved in the direction of improving and simplifying the conditions of Kesten's results. Goldie's

1991 paper [128] is a benchmark. He gave an alternative proof of Kesten's main result in the univariate case. Based on some ideas of Grincevičius and using implicit renewal theory, he proved the asymptotic tail relations

$$\mathbb{P}(\pm X > x) \sim c_{\pm} x^{-a}, \qquad x \to \infty, \tag{3}$$

for some $\alpha > 0$ and determined the constants $c_{\pm}$.[1] Over the years, Alsmeyer, Babillot, Bougerol, Brofferio, Collamore, Diaconis, Elie, Freedman, Grey, Grincevičius, Grübel, Guivarc'h, Hitchenko, Iksanov, Klüppelberg, Le Page, Letac, Mentemeier, Picard, Vervaat, Wesołowski, Zienkiewicz, DB, ED, TM, and many others have been contributing to a better understanding of the tail behavior of (X_t).

As it happens, none of these papers is easy to read, and each involves complicated methods and techniques from Markov chain and renewal theory as well as results for products of random matrices. One of the goals of this text is to present (sketches of) proofs of the Kesten and Goldie results in a reader-friendly way. As regards Goldie's result we were quite successful. Even 40 years after the publication of Kesten's paper there exists no "easy" proof of his multivariate results. Therefore we focused on the main arguments of the proof, illustrating the method. Recently, Guivarc'h and Le Page [142] published a very impressive paper (consisting of 110 pages) which contains a complete proof of the tail asymptotics for quite general matrices.

ARCH and GARCH Processes. In 1996, in the process of writing Section 8.4 of the monograph Embrechts et al. [112], TM became aware of the Kesten–Goldie results. In [112] the relationship between power-law-type tails of X and the extremes of the solution (X_t) was investigated. Early on, in 1989, de Haan et al. [146] had proved that the normalized maxima of this sequence converge in distribution to a Fréchet distribution. The authors of [146] also applied their results to ARCH(1) processes, which were not well known among mathematicians at that time.

Processes of ARCH-type were introduced by Engle [113] in 1982 and extended to GARCH processes by Bollerslev [46] in 1986. ARCH and GARCH processes and their numerous modifications have been major successes as models for the log-returns of speculative prices. In 2003, Robert Engle was awarded the Bank of Sweden Prize in Memory of Alfred Nobel for his contributions to financial econometrics, in particular for the ARCH-GARCH benchmark model. Since its discovery in 1982 the ARCH-GARCH model has triggered a steadily increasing number of papers on the topic which after 2003 turned into an avalanche of scientific articles. The fact that a stochastic recurrence equation of type (1) lies at the heart of this process was discovered early on; see the papers by Bougerol and Picard [51, 52], which dealt with the stationarity problem for such processes, and

[1]Here and in what follows, $f(x) \sim g(x)$ as $x \to \infty$ for positive functions f and g means that $f(x)/g(x) \to 1$.

the papers by de Haan et al. [146] and Goldie [128] who mentioned the ARCH(1) case as a special case of (1).

The power-law tail behavior of the marginal distribution of stationary GARCH processes is part of the folklore among experts in extreme value theory and time series analysis; see Davis and Mikosch [96], Mikosch and Stărică [210], Basrak et al. [27]. Although the publication of the book [112] in 1997 contributed to spreading the message about the power-law tails of ARCH processes, this property is still not well known in the econometrics community. The theoretical properties of ARCH-GARCH processes play a major role in this text. We illustrate in great detail the consequences of the underlying stochastic recurrence equation structure for the GARCH processes and we hope that these parts of the book will be particularly useful for time series analysts, statisticians, and econometricians.

Boundaries of Harmonic Functions. In the 1970s, a group of French mathematicians, including Guivarc'h, Raugi, Elie, Babillot, Bougerol, started studying so-called μ-harmonic functions on Lie groups. Boundaries and Poisson representation were problems of interest. At first glance, these problems have nothing in common with equations (1) and (2) but this is not true. The impressive paper by Raugi [235] gave a complete description of the Poisson boundary-reproducing μ-harmonic functions. The Poisson boundary for a measure μ on a Lie group G is a topological G-space M equipped with a probability measure ν (the Poisson kernel) such that all bounded μ-harmonic functions are Poisson integrals of functions in $L^\infty(M,\nu)$. Raugi described such spaces for measures μ that are spread out.[2] In some cases, the Poisson kernel is the stationary solution to equation (2). Indeed, suppose that $B \in \mathbb{R}^d$ and $A \in GL(d,\mathbb{R})$ the set of $d \times d$ real invertible matrices. The smallest closed subgroup generated by the support of the law μ of (A,B) is a Lie subgroup of the semi-direct product of $GL(d,\mathbb{R})$ and $\mathbb{R}^d$. For some classes of A, the Poisson boundary-reproducing bounded μ-harmonic functions coincide with $\mathbb{R}^d$, the Poisson kernel is the stationary solution to (2), and the techniques applied in both cases are close.

Along this path, DB and ED became interested in the stochastic equation (1). In the 1980s and 1990s, ED together with Andrzej Hulanicki studied the boundaries of harmonic functions (with respect to some sub-elliptic operators L) on solvable NA groups[3] and tried to understand the tail behavior of the corresponding Poisson kernel. Such functions become μ-harmonic via the heat semigroup μ_t generated by L (more precisely, μ_t-harmonic for every t). Probabilistic techniques are very natural in this context. But there is one more advantage: μ_t has a smooth density and so does the law of the solution X to (2). Therefore, instead of the tail asymptotics (3) pointwise estimates of this density were derived; definite results can be found in Buraczewski et al. [74]. The Wrocław group used to meet Yves

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[2]We refer to Example 2.2.11 for more details on μ-harmonic functions and a concrete example of a group and its Poisson boundary.

[3]One may think of (1) with upper triangular matrices A; see Damek [91] for the passage from Raugi's theory to differential operators.

Guivarc'h at conferences and, at some point in 2002, everybody understood that there was common ground for collaboration related to (2) with similarities or non-homogeneous dilations A (see Buraczewski et al. [72]), and not necessarily related to differential operators. In this way DB and ED got interested in the equation (2).

Iterated-function Systems and Regularly Varying Sequences. The model (1) is a particular random iterated-function system: it is based on iterated affine mappings. Over the last years, there has been an increasing interest in random iterated-function systems whose properties are often investigated by starting from the affine case treated in this book. Another class of stochastic processes with a wider range than those provided by the stochastic recurrence equation (1) consists of the regularly varying stationary processes (X_t). It has been developed over the last 30–40 years. These processes have power-law tails of the marginal and finite-dimensional distributions and they constitute a major class of heavy-tailed processes which have found applications in telecommunication models, financial econometrics, and queuing theory. In this text, the reader will learn many interesting facts about the calculus of regularly varying random structures and iterated-function systems.

What this Book is About. We mentioned that one of the objectives of writing this text was to overcome language problems between different groups of mathematicians who use rather distinct mathematical terminology. Often groups work in parallel, not being aware of similar results of other groups. We have been using the language of an applied probabilist in this text to make the results better known to a wider audience. It is not a priori clear that we will be successful with our approach; the topics treated in this book are not simple and proofs are long and technical. Every result requires knowledge of rather distinct areas of probability theory. Graduate students and researchers may find this text useful either for teaching a graduate course or as a collection of classical and more recent results on stochastic recurrence equations. These results are spread over the literature and we provide some guidance to access the wealth of material. We require that, to some extent, the reader is familiar with Markov chain theory, renewal theory, and regular variation calculus. We provide plenty of references to sources where one can read about these topics.

When reading the original papers about the stochastic recurrence equation (1) or the identity in law (2), one gets confronted with a large manifold of technical conditions on the distribution of (A_t, B_t). In this text, we have tried to give minimal conditions and to explain their meaning.

The process $(X_t)_{t \geq 0}$ from (1) is a particular Markov chain. It has a "simple" structure. This fact enables one to verify various properties by calculation. These include, for example, conditions for the stationarity of (X_t), properties of the support of (X_t), tails, moments, and mixing properties. The reader will learn about many properties of a particular Markov chain but, in the process of reading, he/she will also get familiar with many techniques and tools which can be applied in much broader contexts.

Although we study general properties of the solution (X_t) to (1) such as stationarity, moments, support, the main focus of this text is on the tails of the marginal and finite-dimensional distributions of (X_t). In particular, we are interested in power-law tail behavior as provided by the aforementioned results of Kesten and Goldie and in the consequences of such tail behavior in various applications. Topics include extreme value theory for (X_t) and convergence of the point process of exceedances (see de Haan et al. [146]), infinite-variance stable central limit theory (Bartkiewicz et al. [26], Buraczewski et al. [71]), large deviation and ruin probabilities (Konstantinides and Mikosch [185], Buraczewski et al. [76]).

The authors of this book did not intend to write an encyclopedia about the topic of stochastic recurrence equations. There is no doubt that we could have included various closely related topics such as parameter estimation for recurrence equations —Straumann's lecture notes [254] give a good introduction—or exponential functionals of Lévy processes which can be considered as continuous-time analogs of the time series model (1); see the survey paper by Bertoin and Yor [39] and the recent papers by Behme et al. [34] and Behme and Lindner [33].

This monograph has been written over a period of 3 years. We frequently visited each other at the Departments of Mathematics of the Universities of Copenhagen and Wrocław. We are grateful for generous support from both universities. We also take pleasure to thank the Polish National Science Centre (UMO-2011/01/M/ST1/04604) and the Danish Research Council (DFF) (DFF-4002-00435) for financial support. We thank Sara Brofferio, Piotr Dyszewski,[4] Konrad Kolesko, Sebastian Mentemeier, Muneya Matsui, Olivier Wintenberger, and Johannes Heiny for valuable discussions and corrections on earlier versions of this text.

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[4]Piotr kindly contributed Section 5.5. There he reports about his recent research which is part of a Ph.D. project in Wrocław.

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