Preface

Henri Poincaré is considered to be one of the great minds of mathematics, physics, and astronomy. Apart from his rigorous mathematical and analytical style, he was also renowned for his deep insights into science and the philosophy of science. He developed and contributed to many important scientific achievements, and his works on the foundations of science, scientific hypothesis, and scientific method were written with elegance and style. Even more significantly, perhaps, he came to bear upon recent scientific achievements when he put forward the Poincaré conjecture, thereby introducing geometry and topology into the analysis of shape and form. The Poincaré conjecture and his work on the three-body problem are considered to constitute the foundations of the modern chaos theory.

This book *The Foundations of Chaos Revisited: From Poincaré to Recent Advancements* was motivated by the CHAOS 2015 International Conference at the Henri Poincaré Institute in Paris. This was undoubtedly the best place to gain insight into chaos theory as inspired by the Poincaré tradition in a place that must be considered as the home of Poincaré or, better, the home of mathematics in Paris.

In order to explore the foundations of chaos theory in greater depth, the aim was to approach the main theme with the style and elegance of Henri Poincaré, as exemplified in his mathematical-analytical formulation. Chaos theory provides a link between science and the humanities. It is one of the few scientific topics that tends to unify the different areas of science and to connect them with society as a whole and with a language, CHAOS, that is generally accepted as providing a common substrate, even if this substrate can be seen as mathematics, geometry, graphs, or linguistic material, depending on your viewing point. However, all would accept that chaos theory brings together a very broad range of fields.

Following a proposal by Christian Caron from Springer, we have asked the plenary and keynote speakers of the conference to contribute to a book with an extended version of their presentations, the aim being to connect Poincaré's contributions with today's achievements. We are happy that we have already received contributions of high caliber that will take the reader on a fascinating tour of chaos theory. Important applications integrating traditional and modern chaos theory are included in the final chapters of this book.

Ferdinand Verhulst has already published several contributions on the Henri Poincaré legacy. With his elegant style and deep understanding of the state of science, especially in mathematics and physics, both during and prior to the days when Poincaré was active, he presents a brilliant paper entitled "Henri Poincaré's Inventions in Dynamical Systems and Topology." He explains how Poincaré's broad knowledge of the existing literature led to such outstanding contributions to dynamical systems and topology. The latter achievement was also built upon the foundations in geometry and geometric representations of mathematical problems prevalent in the French school. The Poincaré map exemplifies Poincaré's deep insight into the way geometric visualization can lead to progress in mathematical modeling and especially chaotic modeling.

Jean-Mark Ginoux, a biographical expert on Poincaré who has made good use of the "Archives Henri Poincaré," has contributed a paper entitled "From Nonlinear Oscillations to Chaos Theory." Following on from the first chapter by Ferdinand Verhulst, he proceeds to explain how Poincaré's mathematical concept of limit cycle and the existence of sustained oscillations representing a stable regime of sustained waves contributed to the advancement of theory and practice in radio communications. The author provides documentation and an excellent presentation of the three main devices, the series-dynamo machine, the singing arc, and the triode, over a period ranging from the end of the nineteenth century till the end of the Second World War. He shows how Van der Pol's study of the oscillations of two coupled triodes and the forced oscillations of a triode led, at the end of the Second World War, to Mary Cartwright and John Littlewood's characterization of the related oscillating behavior as "bizarre." This behavior would later be identified as "chaotic." However, the basis of this achievement was set forty years earlier by Poincaré in his work *La Théorie de Maxwell et les oscillations Hertziennes: la télegraphie sans fil* (Gauthier-Villars, 3e ed. (Paris), 1907).

The early 1940s were a milestone for the characterization of nonlinear and "bizarre" oscillations, or better "fine structure solutions," to use the more elegant terminology for chaotic solutions in wave modeling in telecommunications. Then, in 1941 the Russian researcher A.N. Kolmogorov began modeling the chaotic phenomenon in fluid flow known as turbulence. It was an important step to pass from oscillations to waves in flows and turbulence. However, the limit cycles introduced by Poincaré in the solution of differential equations were a key achievement underpinning progress that would be made some decades later. And even more important was his paper on rotating fluids: "Sur la stabilité de l'équilibre des figures piriformes affectées par une masse fluide en rotation," Poincaré, H. (1901) *Philosophical Transactions A* **198**, 333–373. David Ruelle contributes to this important topic with an extended paper from the honorary presentation for his eightieth birthday at the CHAOS 2015 International Conference at the Henri Poincaré Institute in Paris. This paper follows up with further comments by Giovanni Gallavotti and Pedro Garrido, who also discuss related computer applications. From the early 1970s, with their seminal paper "On the Nature of Turbulence," Ruelle and Takens helped to bring forward Kolmogorov's ideas, while over the last few years (2012, 2014), David Ruelle has extended his contributions to the nonequilibrium statistical mechanics of turbulence. Note that the related work of Kolmogorov was mainly on an ideal form of homogeneous and isotropic turbulence, whereas Ruelle is working on the problem of real nonhomogeneous turbulence, where the lack of homogeneity is called *intermittency*. According to David Ruelle, his paper integrates ideas of turbulence and heat flow:

> Translating a nonequilibrium problem (turbulence) into another nonequilibrium problem (heat flow) is in principle an interesting idea, but there are two obvious difficulties:
>
> - Expressing the fluid Hamiltonian as the Hamiltonian of a coupled system of nodes is likely to give complicated results.
> - The rigorous study of heat flow is known to be extremely hard.
>
> What we shall do is to use crude (but physically motivated) approximations, with the hope that the results obtained are in reasonable agreement with experiments. This is indeed the conclusion of our study, indicating that turbulence lies naturally within accepted ideas of nonequilibrium statistical mechanics.

Giovanni Gallavotti and Pedro Garrido follow Ruelle's paper "Non-equilibrium Statistical Mechanics of Turbulence" with "Comments on Ruelle's Intermittency Theory." Giovanni Gallavotti has made significant contributions to chaos theory and applications in the late 1970s and has published a book entitled *Foundations of Fluid Dynamics*. Here, in this joint paper with Garrido, they present an intermittency correction term to the classical Kolmogorov law. Many calculations are presented for various cases of turbulence and for different Reynold's numbers, thus strengthening the related theory.

Following the previous papers, Roger Lewandowski and Benoît Pinier contribute with a paper "The Kolmogorov Law of Turbulence: What Can Rigorously Be Proved?" They consider how homogeneity and isotropy are introduced into turbulence and give a mathematical proof of the famous -5/3 Kolmogorov law. Their aim is to:

1. Carefully express the appropriate similarity assumption that a homogeneous and isotropic turbulent flow must satisfy in order to derive the -5/3 law
2. Derive the -5/3 law theoretically from the similarity assumption
3. Discuss the numerical validity of such a law from a numerical simulation in a test case, using the software BENFLOW 1.0, developed at the Institute of Mathematical Research in Rennes

They use the Navier-Stokes equations and refer to work by Boussinesq: "Essai sur la théorie des eaux courantes." *Mémoires présentés par divers savants à l'Académie des Sciences* (Paris, 23.1.1877, 1–660). Another approach is given in "Sur la stabilité de l'équilibre des figures piriformes affectées par une masse fluide en rotation," Poincaré, H. (1901), *Philosophical Transactions A* **198**, 333–373.

Pierre Coullet and Yves Pomeau present a very important topic under the title "History of Chaos from a French Perspective." This is an exceptional paper, deserving much attention. Every point is presented with clarity and a deep insight into the subject. They start with Poincaré and the French tradition in dynamical systems. As they explain:

> The history of chaos begins with Poincaré. His PhD thesis can be seen as the very beginning of dynamics as we know it. He invented powerful geometrical methods to understand "qualitatively" the behavior of solutions of ordinary differential equations. His message remains alive, because of the power of his methods. As a side remark it is curious to see his basic concepts rediscovered again and again. The saddle-node bifurcation (noeud-col in Poincaré thesis) has grown popular in this respect and lately has acquired various fancy new names. Poincaré not only pioneered qualitative methods for the analysis of differential equations, but he also began to study dissipative dynamical systems that differed from the (far more complex) methods of Lagrangian dynamics (a topic where he also brought fundamental ideas).

In the same style they continue with a fascinating presentation, discussing authors and researchers, theoreticians and experimentalists, and the interaction between them, as well as scientific progress in the field of chaos. They conclude:

> Clearly, chaos theory and experiment has not suffered from lack of attractiveness. Nowadays it has morphed into the wider field of nonlinear science, drawing in many bright young colleagues. We hope this tree will continue to blossom.

Orbits and periodic orbits in a topological environment, maps, and related presentations all started with Poincaré, to be expanded later in a well-known paper by V. Arnold entitled: "Small Denominators. I. Mapping of the Circumference onto Itself" (*Amer. Math. Soc. Transl.* (2), 46:213–284, 1965). Quasiperiodicity is explored in the paper by Suddhasattwa Das, Yoshitaka Saiki, Evelyn Sander, and James A. Yorke. They provided a one-dimensional quasiperiodic map as an example and showed that their weighted averages converged far faster than the usual rate of O(1 N), provided f was sufficiently differentiable. They used this method for efficient numerical computation of rotation numbers, invariant densities, and conjugacies of quasiperiodic systems and also to provide evidence that the changes of variables were (real) analytic. James Yorke was an invited plenary speaker at the CHAOS 2015 International Conference. He is one of the main contributors to chaos theory with many papers to his name. Two of the best are "Period Three Implies Chaos," T.Y. Li, and J.A. Yorke, *American Mathematical Monthly* 82, 985 (1975), and "Controlling Chaos," E. Ott, C. Grebogi, and J.A. Yorke, *Phys. Rev. Lett.* 64, 1196–1199 (1990).

Alexander Ramm has explored the problem of heat transfer in a complex medium. He has already investigated the scattering of acoustic and electromagnetic waves by small bodies of arbitrary shapes and discussed applications to the creation of new engineered materials. These are very important contributions to a subject that has many practical applications in the production of modern materials with special characteristics.

Theory and practice suggests that time delays are connected with chaotic behavior, and this is explained in the paper by V.J. Law, W.G. Graham, and D.P. Dowling entitled "Plasma Hysteresis and Instability: A Memory Perspective". They start with a historical review of the significance of Duddell's "singing arc" and its application to deleterious effects in the control of both hysteresis and spatiotemporal stability as the two-electrode valve evolved into the three-electrode or triode vacuum tube. They illustrate the use of oscillograph Lissajous figures in the I-V plane, the Q-V plane, and the harmonic plane to investigate these deleterious effects in modern low-pressure parallel-plate systems and atmospheric pressure plasma systems and compare the hysteresis and stability within the "singing arc." They discuss developments from the original oscillograph measurement to today's analog, digital, and software methods. They also ask whether the "singing arc" and other plasma systems fall in the category of a memory element. The authors explain Poincaré's achievements in this area:

> A recent reevaluation of the work of Henri Poincaré has revealed that he too played a significant role in the mathematical understanding of the arc's stable regime using limit cycles and their deviation from that regime. Even though Poincaré did not study the triode vacuum tube, the review claims that the two-electrode "singing arc" is analogous to the three-electrode or triode vacuum tube. Given the extended triode development time line, it would seem unlikely that, at Poincaré's wireless telegraphy conference in 1908 or at the time close to his death in 1912, he was able to deduce or describe the behavior of early triode vacuum tubes that operated under soft or hard vacuum conditions. Nevertheless, Poincaré's closed limit cycles do predate the work of Van de Pol and J. Van de Mark along with Andronov self-oscillations.

The Indian scientist Sir Chandrasekhara Venkata Raman earned the 1930 Nobel Prize in physics for his work in the field of light scattering and the development of the so-called Raman amplifiers. Following this discovery, several theoretical and applied studies led to the construction of new scientific fields, including the fiber Raman amplifiers presented in a paper by Vladimir L. Kalashnikov and Sergey V. Sergeyev entitled "Stochastic Anti-resonance in Polarization Phenomena." To treat this problem, the authors based their work on the classical Poincaré sphere, an analytic tool first developed in Poincaré's publication: "Les methodes nouvelles de la mecanique celeste" (Tome I, Paris, 1892, Gauthier-Villars). The authors put forward a more general analytic framework, useful in many topics, as discussed in their paper:

> Here we shall demonstrate a cooperation between analytical multi-scale techniques and direct numerical simulations of SDEs that reveals a quite nontrivial phenomenon, stochastic antiresonance (SAR). This can be characterized by different signatures, including the Hurst parameter, the Kramers length, the standard deviation, etc. This phenomenon can be treated as a noise-driven escape from a metastable state which is intrinsic to diffusion in crystals, protein-folding, activated chemical reactions, and many other contexts. As a test bed, we consider a fiber Raman amplifier with random birefringence, a device with a direct practical impact on the development of high-transmission-rate optical networks.

Many applications of chaos are based on differential equations and systems of differential equations. Right from the beginning, when methods were first introduced to solve differential equations, it was evident that exact solutions would not generally exist in the majority of applications. Still other scientific advancements relating to second-order differentials had to wait until Ito and Stratonovich came on the scene in the twentieth century, establishing the stochastic theory already introduced in another form by Paul Langevin (1908). Poincaré's great achievement is illustrated by the fact that, very early in his career, in fact, in his PhD dissertation, he had suggested a qualitative approach to solving differential equations, including limit cycles and singular or stationary points, while he had introduced the term "bifurcation" in his first paper on mathematics (1885). It is interesting to see how important these tools have become today. The paper by Irene M. Moroz, Roger Cropp, and John Norbury entitled "A Simple Plankton Model with Complex Behaviour" includes all the recipes provided by Poincaré to deal with a coupled system of four nonlinear differential equations, including phase portraits, critical or equilibrium points, bifurcation diagrams, and chaotic oscillations. This paper is a typical example of the importance of Poincaré's findings across a broad range of theoretical and applied fields in science.

An interesting application, entitled "Fractal Radar: Towards 1980–2015," is included in the paper by Alexander A. Potapov, along with an interesting approach to the theory of fractional measure and nonintegral dimension. According to the author:

> The main feature of fractals is the nonintegral value of its dimension. The development of dimension theory began with the work of Poincaré, Lebesgue, Brauer, Urysohn, and Menger. Sets which are negligibly small and indistinguishable in one way or another in the sense of Lebesgue measure arise in different fields of mathematics. To distinguish such sets with a pathologically complicated structure, one should use unconventional characteristics of smallness, for example, Hausdorff's capacity, potential, measures, dimension, and so on. The application of the fractional Hausdorff dimension associated with entropy, fractals, and strange attractors has turned out to be most fruitful in dynamical systems theory.

Irina N. Pankratova and Pavel A. Inchin explore a "Simulation of Multidimensional Nonlinear Dynamics by One-Dimensional Maps with Many Parameters." They propose a class of discrete dynamical systems as nonlinear matrix models to describe multidimensional multiparameter nonlinear dynamics. In their article, they simulate the system's asymptotic behavior by introducing a two-step algorithm to compute ω-limit sets of dynamical systems. They propose a qualitative theory allocating invariant subspaces of the system matrix that contain cycles of rays on which the ω-limit sets of the dynamical systems are situated, and they introduce dynamical parameters to describe the system behavior. The ω-limit set of the system trajectory is computed using the analytical form of the one-dimensional nonlinear Poincaré map determined by the dynamical parameters.

The paper "Sudden Cardiac Death and Turbulence" authored by Guillaume Attuel, Oriol Pont, Binbin Xu, and Hussein Yahia is another important application of the theories presented in the first part of this book, including Poincaré's methods and Y. Pomeau's conjecture regarding hydrodynamic intermittency. This is a clear and concise discussion of one of the main causes of death in our societies. Many of the theoretical tools of chaos theory are used, including abnormal oscillations, fluctuations, and limit cycles. A system of four coupled differential equations is introduced, and Poincaré section plots are presented, along with an analysis of the onset of turbulence.

The paper by Philippe Beltrame entitled "Absolute Negative Mobility in a Ratchet Flow" relates to the papers of David Ruelle, Jean-Mark Ginoux, and Alexander Ramm. The problem is modeled by a simple system and by a system of four coupled nonlinear differential equations. Bifurcation diagrams, perioddoubling cascades, critical values, strange attractors, and Poincaré sections are presented along with a discussion of the chaotic transition.

Given the selection of papers in the book, the aim here is to reach a broad scientific and general audience. Indeed, it is directed not only at researchers and scientists in almost every field but also at a wider audience interested in discovering and exploring the way modern chaos theory was brought into being some 120 years ago by a brilliant scientist who had the intellectual ability and the scientific knowledge to reach both the heart and the boundaries of this theory.

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