

Chapter 2

Chains, Antichains, and Fences

Chains and antichains are arguably the most common kinds of ordered sets in mathematics. The elementary number systems \mathbb{N} , \mathbb{Z} , \mathbb{Q} , and \mathbb{R} (but not \mathbb{C}) are chains. Chains are also at the heart of set theory. The Axiom of Choice is equivalent to Zorn's Lemma, which we will adopt as an axiom, and the Well-Ordering Theorem. The latter two results are both about chains.

Antichains on the other hand are common when people are not talking about order, as an antichain is essentially an “unordered” set. Dilworth's Chain Decomposition Theorem (see Theorem 2.26) shows how these two concepts are linked together. Fences are not as widely known as either chains or antichains. Yet they do play a fundamental role, because they are the analogue of paths in graph theory.

2.1 Chains and the Rank of an Element

In a chain, any two elements are comparable. This means that the hierarchy is total. When people talk about ranking objects, they typically are talking about a chain, which may be why chains seem to be the most natural orders.

Definition 2.1. An ordered set C is called a **chain** (or a **totally ordered set** or a **linearly ordered set**) iff, for all $p, q \in C$, we have $p \sim q$.

Example 2.2. The following are all examples of chains.

1. The natural numbers, integers, rational and real numbers with their natural orders.
2. Every set of ordinal numbers.
3. The set $\mathbb{R} \times \mathbb{R}$ with the order $(a, b) \sqsubseteq (c, d)$ iff $a < c$ or $(a = c \text{ and } b \leq d)$.
4. If $f : P \rightarrow P$ is order-preserving and $p \in P$ is such that $p \leq f(p)$, then $\{f^n(p) : n \in \mathbb{N}\}$ is a chain. \square

Proof. We will only prove 4. First note that, by induction on n , for all $n \in \mathbb{N}$, we have $p \leq f^n(p)$: The base step $n = 0$ is trivial, because $p = f^0(p)$. For the induction step $n \rightarrow (n + 1)$, assume $p \leq f^n(p)$. Then $f(p) \leq f(f^n(p)) = f^{n+1}(p)$ and, by assumption, $p \leq f(p)$. Hence $p \leq f^{n+1}(p)$.

Now let $n, m \in \mathbb{N}$ with $m \leq n$. Then we have $p \leq f^{n-m}(p)$ and hence we conclude $f^m(p) \leq f^m(f^{n-m}(p)) = f^n(p)$. ■

Remark 2.3. In relation to problems 1.19, 1.24, and 1.34 we can record:

1. Not every chain has the fixed point property. Simply consider \mathbb{N} with the map $f(x) = x + 1$, which is order-preserving and fixed point free. It is possible to classify exactly those chains that have the fixed point property, and we will do so in Theorem 8.10.
2. Finite chains have exactly one automorphism, the identity. Infinite chains can have fixed point free automorphisms: Consider \mathbb{Z} with the map $f(x) = x + 1$.
3. Finite chains with at least four points are reconstructible by Proposition 1.37.

Chains and their lengths are also closely intertwined with the rank, which indicates the “height” at which we can find an element in a finite ordered set.

Definition 2.4. Let P be an ordered set. An element $m \in P$ is called **minimal** iff, for all $p \in P$ with $p \sim m$, we have $p \geq m$. We denote the set of minimal elements of P by $\text{Min}(P)$.

Note that we were already talking about minimal elements in the proof of Proposition 1.37.

Definition 2.5. Let P be a finite ordered set. For $p \in P$, we define the **rank** $\text{rank}(p)$ of p recursively as follows. If p is minimal, let $\text{rank}(p) := 0$. If the elements of rank $< n$ have been determined and p is minimal in $P \setminus \{q \in P : \text{rank}(q) < n\}$, then we set $\text{rank}(p) := n$.

The rank can be used to somewhat standardize the drawing of a finite ordered set. As was stated in the drawing procedure on page 6, $p < q$ forces that p is drawn with a smaller y-coordinate than q in the diagram. Using the rank function, a drawing of a diagram can be standardized by drawing all points of the same rank at the same height. In fact, this convention has been used in all drawings of diagrams of finite ordered sets so far. The ordered set in Figure 2.1 part a) shows once more that this convention can lead to good drawings of an ordered set. In part b) of Figure 2.1, we show that this convention should not be mechanically applied at all times. The down-up bias of ordering the points by rank can mask symmetries that are not compatible with the rank.

Example 2.6. It is possible for two points in an ordered set to be adjacent and yet have their ranks differ by more than 1. Just consider d and h in the ordered set in part b) of Figure 2.1. □

Returning to chains, we record the following.

Definition 2.7. Let C be a finite chain. The **length** of C is $l(C) := |C| - 1$.

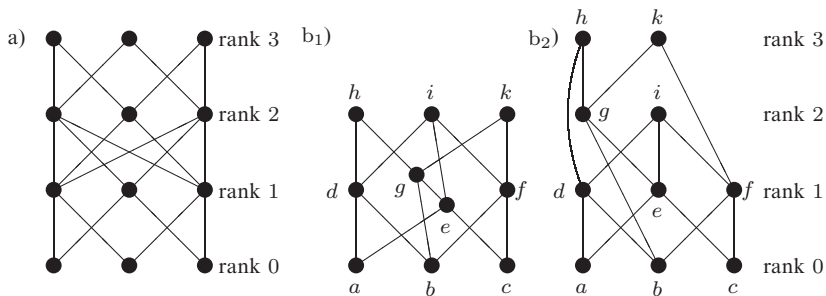


Fig. 2.1 Two ordered sets for which drawing points with the same rank at the same height has different effects. The set in a) has a very orderly drawing revealing several symmetries. The second set is drawn twice. Once (in b₁) to reveal its vertical symmetry (it is isomorphic to its own dual) and once (in b₂) by drawing points of the same rank at the same height

Proposition 2.8. *Let P be a finite ordered set and let $p \in P$. The rank of p is the length of the longest chain in P that has p as its largest element.*

Proof. The proof is an induction on the rank of p . For the base step, $\text{rank}(p) = 0$, there is nothing to prove.

For the induction step, let $\text{rank}(p) = k > 0$ and assume the statement is true for all elements of P of rank $< k$. Then p has a lower cover l of rank $k - 1$. By induction hypothesis, there is a chain C of length $k - 1$ that has l as its largest element. Thus $\{p\} \cup C$ is a chain of length k with p as its largest element.

Now suppose for a contradiction that there is a chain $K \subseteq P$ of length $> k$ such that p is the largest element of K . Let c be the unique lower cover of p in K . Then c is of rank $< k$ and yet $K \setminus \{p\}$ is a chain of length $\geq k$ with largest element c , a contradiction. Thus the largest possible length of a chain in P that has p as its largest element is k , and we are done. ■

Proposition 2.9. *Let P, Q be finite ordered sets and let $\Phi : P \rightarrow Q$ be an isomorphism. Then, for all $p \in P$, we have $\text{rank}_P(p) = \text{rank}_Q(\Phi(p))$.*

Proof. First note that the image of a chain under an order-preserving map is again a chain (details are left to Exercise 2-2). Thus, because Φ is injective, the image of any k -element chain under Φ is again a k -element chain. By Proposition 2.8, this means that $\text{rank}_P(p) \leq \text{rank}_Q(\Phi(p))$.

Now Φ^{-1} is an isomorphism, too, so the result of the previous paragraph also applies to Φ^{-1} . This means that

$$\text{rank}_P(p) \leq \text{rank}_Q(\Phi(p)) \leq \text{rank}_P(\Phi^{-1}(\Phi(p))) = \text{rank}_P(p),$$

which implies $\text{rank}_P(p) = \text{rank}_Q(\Phi(p))$. ■

As we have seen, the elements of the largest rank in the ordered set determine how high the drawing of the order is. This motivates the following definition, which also makes sense for infinite ordered sets.

Definition 2.10. *Let P be an ordered set. The **height** of P is the length of the longest chain in P . If there are chains of arbitrary length in P , we will say that P is of **infinite height**.*

*The height of an ordered set is sometimes also called its **length**.*

Proposition 2.11. *The height of a finite ordered set with at least four elements is reconstructible from the deck of the ordered set.*

Proof. Let P be a finite ordered set with at least four elements. If P is a chain, then P is reconstructible by Proposition 1.37 and we are done. If not, the height of P is the maximum of the heights of the cards $P \setminus \{x\}$. ■

We can now use the ideas of chains and the rank function to establish that every ordered set has an exponential number of order-preserving self-maps.

Theorem 2.12 (Compare with [75]). *Let P be a finite ordered set with $n > 1$ elements and height h . Then P has at least $2^{\frac{h}{h+1}n}$ order-preserving self-maps that are not automorphisms.*

Proof. The case that P is a chain is left to Exercise 2-4. (Also see Exercise 2-6 for a recursive formula for the number of order-preserving maps between chains.) If $h = 0$, there is nothing to prove. Thus we can assume in the following that P is not a chain, and $h > 0$. Let $C = \{c_0 < c_1 < \dots < c_h\}$ be a chain with $h + 1$ elements.

Let r_0, r_1, \dots, r_h be the number of elements of rank h in P . Let $j \in \{0, \dots, h\}$ be so that $r_j \leq r_i$ for all i . Let R_0, \dots, R_h be the sets of elements of rank h in P . Then, for any choices of numbers $u_i \in \{0, \dots, r_i\}$ for $i < j$ and numbers $d_i \in \{0, \dots, r_i\}$ for $i > j$, we have that the function that maps

- For each $i < j$, u_i elements of R_i to c_{i+1} and the remaining $r_i - u_i$ elements of R_i to c_i ,
- For each $i > j$, d_i elements of R_i to c_{i-1} and the remaining $r_i - d_i$ elements of R_i to c_i ,

is an order-preserving map that is not an automorphism. The number of such functions is $\prod_{i=0}^{j-1} 2^{r_i} \cdot \prod_{i=j+1}^h 2^{r_i} = 2^{n-r_j} \geq 2^{n-\frac{n}{h+1}} = 2^{\frac{h}{h+1}n}$. ■

Because, for height 0, every non-bijective function is order-preserving, the above shows that every ordered set with more than one element has at least $2^{\frac{n}{2}}$ order-preserving maps that are not automorphisms. In [75] the better lower bound $2^{\frac{2}{3}n}$ is proved. You can do this later, in Exercise 4-14.

Because every automorphism is an order-preserving map, too, Theorem 2.12 shows that there will always be more order-preserving maps than automorphisms. The exponential lower bound on the difference seems substantial. Moreover, existence of a nontrivial automorphism, that is, symmetry, appears to be a rare property, a fact that we will substantiate later in Corollary 13.25. Hence, it is natural to ask if the ratio goes to zero.

Definition 2.13. Let P be an ordered set. Then $\text{Aut}(P)$ denotes the set of all automorphisms of P . Moreover, $\text{End}(P)$ denotes the set of all order-preserving self-maps, or (order) endomorphisms of P .

Open Question 2.14. The automorphism problem. (See [253].) Is it true that

$$\lim_{|P| \rightarrow \infty} \frac{|\text{Aut}(P)|}{|\text{End}(P)|} := \lim_{n \rightarrow \infty} \max_{|P|=n} \frac{|\text{Aut}(P)|}{|\text{End}(P)|} = 0 ?$$

The automorphism conjecture states that the above limit is indeed zero.

Natural stronger versions of the automorphism problem would include precise upper bounds on the quotient, even for restricted classes of ordered sets. The automorphism conjecture appears reasonable. Experiences with another problem, namely, how many maps of an ordered set are fixed point free, should caution us against jumping to conclusions, however.

Indeed, it seems that most endomorphisms of an ordered set should have a fixed point. It was even conjectured (very briefly) that the ratio of fixed point free endomorphisms to all endomorphisms should converge to zero. In [253], it is observed that, for an ordered set of height 0, that is, an ordered set in which no two elements are comparable, the limit of the quotient of the number of fixed point free order-preserving maps and the number of all order-preserving maps is $\frac{1}{e}$. It is not known if this is the largest possible quotient or not. Incidentally, the consideration of fixed point free endomorphisms versus all endomorphisms was the context in which the automorphism conjecture first arose. Please consider Open Problem 1 at the end of this chapter for the statement and references on this problem.

Exercises

- 2-1. Prove Proposition 2.11 using the Kelly Lemma.
- 2-2. Chains and order-preserving maps. Let P, Q be ordered sets, $f : P \rightarrow Q$ be order-preserving and let $C \subseteq P$ be a chain. Prove that $f[C]$ is a chain.
- 2-3. Prove that, for every finite ordered set P , there is an order-preserving map from P onto a $|P|$ -element chain.
- 2-4. Let C be a chain of length ℓ . Prove that C has at least $2^\ell = 2^{\lfloor \frac{\ell}{2} \rfloor} |C|$ order-preserving self-maps that are not automorphisms.
- 2-5. Prove that, for any finite ordered set P and any two distinct elements $x, r \in P$, we have that $\text{rank}_{P \setminus \{x\}}(r) \in \{\text{rank}_P(r), \text{rank}_P(r) - 1\}$. Give an example of an ordered set P and two points x and r such that the rank of r in $P \setminus \{x\}$ is less than the rank of r in P .
- 2-6. Let $MC_{n,m}$ be the number of order-preserving maps from an n -chain to an m -chain. Prove that $MC_{n,m} = MC_{n-1,m} + MC_{n,m-1}$.
- 2-7. Determine the largest class of ordered sets in which a sensible rank function can be defined.
- 2-8. Prove Proposition 1.10 for ordered sets that have no infinite chains.

2.2 A Remark on Duality

Definitions 2.4 and 2.5 clearly have a “down-up bias”: We build the rank function by first defining rank zero, then rank one, etc. There is nothing wrong with this approach, as, for example, a house is built in the same “down-up” fashion. In order theory, however, going up and going down are closely related. Indeed, if we were to reverse all comparabilities, we would obtain equally sensible definitions. For example, if we reverse the comparabilities in Definition 2.4, we obtain the definition of a maximal element.

Definition 2.15. *Let P be an ordered set. An element $m \in P$ is called **maximal** iff, for all $p \in P$ with $p \sim m$, we have $p \leq m$. We denote the set of maximal elements of P by $\text{Max}(P)$.*

Note that the only difference between Definitions 2.4 and 2.15 is that the one inequality in the definition is reversed. This is why minimal and maximal elements are “at opposite ends” of an ordered set. For example, in the ordered set in Figure 2.1 part a), the elements of rank 0 are (naturally) the minimal elements, while the elements of rank 3 happen to be the maximal elements. Rank and maximality do not have a simple relationship, though. The maximal elements of the ordered set in Figure 2.1 part b) are h, i, k , which do not all have the same rank.

Taking an order-theoretical statement and reversing all inequalities as we did to obtain the definition of maximal elements from that of minimal elements is called **dualizing** the statement. Properties like minimality and maximality, that are obtained from each other by reversing all comparabilities are called **dual properties**. In part 8 of Example 1.2, we mentioned that, for each ordered set P , there is another ordered set P^d , called its dual ordered set, that is obtained by reversing all comparabilities. This means that the minimal elements of P are the maximal elements of P^d and vice versa.

Another example of dual properties are lower bounds and upper bounds.

Definition 2.16. *Let P be an ordered set. If $A \subseteq P$, then $l \in P$ is called a **lower bound** of A iff, for all $a \in A$, we have $l \leq a$*

Definition 2.17. *Let P be an ordered set. If $A \subseteq P$, then $u \in P$ is called an **upper bound** of A iff, for all $a \in A$, we have $u \geq a$.*

If l is a lower bound of A , we will also write $l \leq A$, and if L is a set of lower bounds of A we will write $L \leq A$. Similarly, or, better, dually, we can define $u \geq A$ and $U \geq A$.

Duality is a powerful tool when results are proved that are biased in one direction. Instead of re-stating and re-proving everything, one can simply invoke duality. For example, for a finite ordered set, we can also define a dual rank function that has a “top-down bias.”

There are notions that are their own duals, such as the notion of being order-preserving. If $f : P \rightarrow Q$ is order-preserving, then so is $f : P^d \rightarrow Q^d$. Thus, for example, being an isomorphism is a self-dual notion and any result proved about the relation between isomorphisms and, say, upper bounds is also a result

on isomorphisms and lower bounds. You are invited to state (and briefly prove) the duals of the results we have proved so far and in the future. In this fashion, when the need for dualization arises, it will not hold any surprises. We conclude this section with one simple example on the use of duality, complete with formal proof. Future uses of duality will not be elaborated as much.

Proposition 2.18. *Let P, Q be finite ordered sets and let $\Phi : P \rightarrow Q$ be an isomorphism. Then, for each $p \in P$, the image $\Phi(p)$ has as many upper bounds in Q as p has upper bounds in P . The same statement holds for lower bounds.*

Proof. Let $p \in P$ and let $x \in P$ be such that $x \geq p$. Then, because Φ is order-preserving, $\Phi(x) \geq \Phi(p)$. Moreover, because Φ is injective, no two upper bounds of p are mapped to the same point. Thus $\Phi(p)$ has at least as many upper bounds as p . Now suppose that $\Phi(p)$ has an upper bound q that is not the image of an upper bound of p . Then $\Phi^{-1}(q) \not\geq p$ even though $q \geq \Phi(p)$, a contradiction. This proves that the numbers of upper bounds of p and of $\Phi(p)$ are equal.

To prove the same statement for lower bounds, we could simply follow the above argument with reversed comparabilities. A formal proof using duality could work as follows. Note that Φ is an isomorphism between P^d and Q^d , too. Thus $\Phi(p)$ has as many Q^d -upper bounds in Q^d as p has P^d -upper bounds in P^d . For any ordered set R , an R^d -upper bound in R^d is an R -lower bound in R . Thus $\Phi(p)$ has as many lower bounds in Q as p has lower bounds in P . In the future, such arguments will be replaced with saying “By duality, the statement for the lower bounds holds.” ■

Exercises

- 2-9. Let P be a finite ordered set of height h . Let $C \subseteq P$ be a chain of length h . Prove each of the following.
- The largest element of C is maximal in P .
 - The smallest element of C is minimal in P .
 - If x is a lower cover of y in C , then x is a lower cover of y in P .
- 2-10. Find the maximal elements of each of the following ordered sets.
- The ordered set in Exercise 1-1a.
 - The ordered sets in Figure 1.1.
 - The ordered set in Figure 1.3.
- 2-11. Prove that, in a finite ordered set, every element is below at least one maximal element. Use this to prove quickly that, in a finite ordered set, every element is above at least one minimal element.
- 2-12. For the ordered set in Figure 2.1 part b), find
- The upper bounds of the set $\{e\}$,
 - The upper bounds of the set $\{a, b, c\}$,
 - The upper bounds of the set $\{f, g\}$,
 - The upper bounds of the set $\{d, f, g\}$,
 - The lower bounds of the set $\{g, h, k\}$,

2.3 Antichains and Dilworth's Chain Decomposition Theorem

Logically, antichains are the simplest possible ordered sets, because we impose no comparabilities at all on the points. If chains are totally ordered, one could say that antichains are totally unordered.

Definition 2.19. An ordered set P is called an **antichain** iff, for all $p, q \in P$ with $p \neq q$, we have $p \not\prec q$.

Proposition 2.20. Let P be a finite ordered set and let $A \subseteq P$ be an antichain. Then there is an antichain $B \subseteq P$ that is maximal with respect to set inclusion and contains A . Antichains such as B are also called **maximal antichains**.

Proof. Left to the reader as Exercise 2-13. Note that, for infinite ordered sets, this result is surprisingly challenging, requiring the use of Zorn's Lemma (see Exercise 2-39). ■

Antichains can be used to pictorially characterize chains via a “forbidden subset” (also compare with Theorem 11.5).

Proposition 2.21. An ordered set C is a chain iff C does not contain any two-element antichains. ■

Just as chains lead to a natural notion of height in ordered sets, antichains lead to a notion of width.

Definition 2.22. Let P be an ordered set. We define the **width** $w(P)$ of P to be the size of the largest antichain in P , if such an antichain exists, and to be ∞ otherwise.

Note that, although chains allow us to define a natural vertical ranking of ordered sets, there is no possibility to define a horizontal ranking using antichains. We can, however, reconstruct the width just as easily as the height.

Proposition 2.23. The width of a finite ordered set with at least four elements is reconstructible from the deck.

Proof. If P is not an antichain, then the width of P is the largest width of any card of P . If P is an antichain, then all its cards are antichains. Because only antichains can have a deck consisting of antichains, this would mean that P is reconstructible as the unique (up to isomorphism) antichain with $|P|$ elements. ■

A nice connection between chains and antichains is provided by Dilworth's Chain Decomposition Theorem, which says that any ordered set of width k can be written as the union of k chains. This fact seems quite obvious. Call our ordered set P and consider the following.

Definition 2.24. Let P be an ordered set. A chain C in P will be called a **maximal chain** iff, for all chains $K \subseteq P$ with $C \subseteq K$, we have $C = K$.

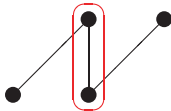


Fig. 2.2 An ordered set for which there is a maximal chain (marked) whose removal does not decrease the width

Every chain in an ordered set is contained in a maximal chain (see Exercise 2-15 for the case of finite ordered sets and Proposition 2.55 for the surprisingly subtle infinite case). Let C be a maximal chain. Then $P \setminus C$ should have width $w(P) - 1$ and should thus be (if we argue inductively) the union of $w(P) - 1$ chains. Throw in C , and P is the union of $w(P)$ chains. Unfortunately what we just gave was a “poof,” not a proof. Figure 2.2 shows an example of a maximal chain whose removal does not decrease the width. Of course, in this example, we simply did not remove the “right” maximal chain. A formalization of what the “right” maximal chain may be appears to be quite hard. For our proof (which is very similar to the proofs that can be found in [27], Chapter III.4, Theorem 11, [311], Theorem 3.3, or [314]; for another proof consider [220]), it will be useful to have the following notation at hand.

Definition 2.25. Let P be an ordered set. For $p \in P$, we define the **filter** or **up-set** of p to be $\uparrow p := \{q \in P : q \geq p\}$ and the **ideal** or **down-set** of p to be $\downarrow p := \{q \in P : q \leq p\}$. Finally, we define $\downarrow p \cup \uparrow p$ and call it the **neighborhood** of p .

Theorem 2.26 (Dilworth's Chain Decomposition Theorem, see [63] or Exercise 2-43 for the proof for infinite ordered sets). Let P be a finite ordered set of width k . Then P is the union of k chains. That is, there are chains $C_1, \dots, C_k \subseteq P$ such that $P = \bigcup_{i=1}^k C_i$.

Proof. The proof is an induction on $n := |P|$. For $n = 1$, the result is obvious.

For the induction step, assume Dilworth's Chain Decomposition Theorem holds for ordered sets with $\leq n$ elements and let P be an ordered set of width k with $(n+1)$ elements. Let us first assume there is an antichain $A = \{a_1, \dots, a_k\}$ with k elements in P such that at least one a_j is not maximal and at least one a_j is not minimal. Then the sets $L := \bigcup_{i=1}^k \downarrow a_i$ and $U := \bigcup_{i=1}^k \uparrow a_i$ both have width k and $\leq n$ elements. Thus U is the union of k chains U_1, \dots, U_k and L is the union of k chains L_1, \dots, L_k . Without loss of generality we can assume that a_i is the largest element of L_i and the smallest element of U_i . However then each $L_i \cup U_i$ is a chain and $P = \bigcup_{i=1}^k (L_i \cup U_i)$.

If the only antichain(s) with k elements in P are the antichain of maximal elements or the antichain of minimal elements, let $C \subseteq P$ be a maximal chain. Then C contains a maximal element and a minimal element. Thus we have $|P \setminus C| \leq n$ and $w(P \setminus C) = k - 1$. By the induction hypothesis, there are chains C_1, \dots, C_{k-1} such that $P \setminus C = \bigcup_{i=1}^{k-1} C_i$. Now we set $C_k := C$ and we are done. ■

Dilworth's Chain Decomposition Theorem can be used to provide a surprisingly easy solution to the order-theoretical analogue of a hard graph-theoretical problem. The graph-theoretical analogue of a chain is a **complete graph**, that is, a graph in which any two vertices are connected by an edge. Complete graphs with s vertices are also denoted K_s . The natural analogue of antichains are graphs in which no two vertices are connected by an edge. These are called **discrete graphs**.

Definition 2.27 (For an introduction on Ramsey numbers see [27], Chapter VI; for an introduction to more sophisticated Ramsey Theory for ordered sets see [311], Chapter 10, Section 5). *Let $s, t \in \mathbb{N}$, $s, t \geq 2$. Then the **Ramsey number** $R(s, t)$ is the smallest natural number n such that any graph G with $\geq n$ vertices contains a complete subgraph K_s or a discrete subgraph with t elements.*

One of Ramsey's theorems (see [241]) states that the Ramsey numbers actually are finite. Very few Ramsey numbers are known. In fact, the precise value for $R(5, 5)$ is still unknown. Dilworth's Chain Decomposition Theorem allows us to easily solve the analogous problem for ordered sets.

Proposition 2.28. *Let $s, t \in \mathbb{N}$, $s, t \geq 2$ and let the **ordered set Ramsey number** $R_{\text{ord}}(s, t)$ be the smallest natural number n such that every ordered set with $\geq n$ elements contains a chain with s elements or an antichain with t elements. Then $R_{\text{ord}}(s, t) = (s - 1)(t - 1) + 1$.*

Proof. To see that $R_{\text{ord}}(s, t) > (s - 1)(t - 1)$ consider the ordered set that consists of $t - 1$ pairwise disjoint chains with $s - 1$ elements each and no further comparabilities involved. It has $(s - 1)(t - 1)$ elements and no s -element chain and no t -element antichain.

Now suppose P has $(s - 1)(t - 1) + 1$ elements. We must show that P contains an s -element chain or a t -element antichain. If P contains an antichain with t elements, there is nothing to prove. If every antichain in P has at most $t - 1$ elements, then P has width at most $t - 1$. By Dilworth's Chain Decomposition Theorem, P is the union of at most $t - 1$ chains. However, then one of these chains must have more than $s - 1$ elements. ■

We can immediately conclude that infinite size means we have an infinite chain or an infinite antichain.

Corollary 2.29. *Every infinite ordered set contains an infinite chain or an infinite antichain.*

Proof. Let P be an infinite ordered set. From Proposition 2.28, we conclude that P must contain chains of arbitrary length or antichains of arbitrary length. Indeed, if the longest chain in P was of length c and the longest antichain in P was of length a , then P would have at most ca elements.

If P contains an infinite chain, then there is nothing to prove. If P does not contain any infinite chains, we want to define a rank function. To make sure the rank function is well defined, we must first prove that there cannot be any elements that are top elements of chains of arbitrary length unless there are infinite antichains: Suppose

there is an x so that, for every n , there is a chain X_n with at least n elements and top element x so that X_n is a maximal chain in $\downarrow x$. Then the chain $X_{\sum_{i=1}^n |X_n|+1}$ contains an element that is not in any of the chains X_1, \dots, X_n . This fact can be used to inductively construct a nested sequence of antichains whose union is an infinite antichain. Hence, for every $p \in P$, the maximum length of a chain with top element p is finite. For each element $p \in P$, we define $\text{rank}_P(p)$ as the length of a longest chain that has p as its largest element. If, for any k , P has infinitely many elements of rank k , we are done.

Finally, in case P has no infinite chains and, for each k , only finitely many elements of rank k , there must be an element of rank k for each $k \in \mathbb{N}$. Then there must be an infinite sequence k_1, k_2, \dots of natural numbers, such that, for each i , there is a maximal element m_{k_i} of rank k_i . This sequence is constructed as follows. Start with a minimal element b_1 . Because b_1 is not the bottom element of an infinite chain, there is a maximal element above b_1 . The rank of this maximal element is k_1 ; we call the element m_{k_1} . Once k_i is found, let b_i be a non-maximal element of rank k_i . There is a maximal element above b_i . The rank of said maximal element is k_{i+1} ; we call the element $m_{k_{i+1}}$.

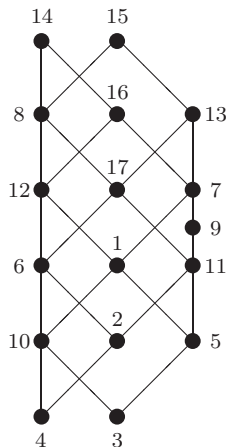
Because no two maximal elements are comparable, the set $\{m_{k_i} : i \in \mathbb{N}\}$ is an infinite antichain. ■

Exercises

- 2-13. Prove Proposition 2.20.
- 2-14. Prove Proposition 2.23 using the Kelly Lemma.
- 2-15. Let P be a finite ordered set and let $C \subseteq P$ be a chain. Prove that C is contained in a maximal chain.
- 2-16. Write the following ordered sets as a union of as few chains as possible.
 - a. The ordered set in Figure 1.3,
 - b. The ordered sets in Figure 1.4,
 - c. The ordered sets in Figure 2.1.
- 2-17. A subset S of an ordered set P is called an **N** iff $S = \{a, b, c, d\}$ and $a < b > c < d$ with no further comparabilities. An ordered set that does not contain an **N** is called **N-free**.
 - a. Prove that, if a finite ordered set is **N-free**, then the removal of any maximal chain decreases the width.
 - b. Give an example that shows that the converse is not true.
- 2-18. In the following, we present an *incorrect proof of Dilworth's Chain Decomposition Theorem* (see Theorem 2.26). Find the mistake.

Induction on $n := |P|$, $n = 1$ is obvious. For the induction step $n \rightarrow (n + 1)$, let us assume that Dilworth's Chain Decomposition Theorem holds for ordered sets with n elements and let P be an ordered set with $(n + 1)$ elements. Let $m \in P$ be a maximal element. Then $|P \setminus \{m\}| = n$. By the induction hypothesis, there are chains $C_1, \dots, C_{w(P \setminus \{m\})}$ such that $P \setminus \{m\} = \bigcup_{i=1}^{w(P \setminus \{m\})} C_i$. If $w(P \setminus \{m\}) = k - 1$, we set $C_k := \{m\}$ and we are done. Otherwise, m has strict lower bounds and thus, for some $i_0 \in \{1, \dots, w(P \setminus \{m\})\}$, we have that m is an upper bound of C_{i_0} . Then $C_{i_0} \cup \{m\}$ is a chain and $C_1, \dots, C_{i_0-1}, C_{i_0} \cup \{m\}, C_{i_0+1}, \dots, C_{w(P \setminus \{m\})}$ are the desired chains. ■??

Fig. 2.3 An ordered set
(from [76]) without fibre of
size $\leq \frac{|P|}{2}$



- 2-19. Consider the ordered set P_0 in Figure 2.3 (also see Remark 11 at the end of this chapter).
- Prove that $\{1, 9, 17\}$ is a maximal antichain in P_0 .
 - Find all two-element maximal antichains in P_0 .
 - Prove that every set in P_0 that intersects all maximal antichains and that contains an odd-numbered element must have at least 9 elements.
 - Prove that every set in P_0 that intersects all maximal antichains must have at least nine elements.
- 2-20. Prove that, for every finite ordered set P of height h , there are antichains $A_0, \dots, A_h \subseteq P$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and such that $P = \bigcup_{i=0}^h A_i$.
- 2-21. Prove that, for an ordered set of width 3, we have that $|\text{Aut}(P)| \leq 6^{\lceil \frac{n}{3} \rceil}$. Use the result for the number of endomorphisms for sets of height h in Theorem 2.12 to show that, for ordered sets of width 3, we have $\frac{|\text{Aut}(P)|}{|\text{End}(P)|} \rightarrow 0$ as $|P| \rightarrow \infty$.

2.4 Dedekind Numbers

Antichains also give rise to a simple-looking, but still baffling, counting problem.

Open Question 2.30 (See [232]). *Dedekind's problem.* What is the number of antichains in the power set \mathcal{P}_n of an n -element set (ordered of course by inclusion)?

Although a formula for the number of antichains in \mathcal{P}_n is given in [160], because the computational effort for this formula is so vast, the problem can still be considered unsolved. (To date, the numbers are only known up to $n = 8$, see [322] and Remark 4 at the end of this chapter.) In this section, we record two possible ways to start on this problem. One is to break the problem into smaller subproblems, the other is to translate the problem into another venue. These very common ideas can at least give us a flavor of the problem and of two common techniques. We start with a possible breakup of the problem into smaller problems.

Definition 2.31. Let $n \in \mathbb{N}$. We define

1. D_n to be the number of antichains in $\mathcal{P}(\{1, \dots, n\})$; D_n is also called the n^{th} **Dedekind number**.
2. T_n to be the number of antichains in $\mathcal{P}(\{1, \dots, n\})$ such that the union of these antichains is $\{1, \dots, n\}$.

Clearly $D_0 = 1$, $T_0 = 1$, because, vacuously, the empty set is an antichain, $D_1 = 2$ and $T_1 = 1$. The connection between the D_n and the T_n is the following.

Proposition 2.32. $D_n = \sum_{k=0}^n \binom{n}{k} T_k$.

Proof. Let \mathcal{A}_k be the set of all antichains in $\mathcal{P}(\{1, \dots, n\})$ whose union is a k -element set. Clearly $D_n = \sum_{k=0}^n |\mathcal{A}_k|$. For $B \subseteq \{1, \dots, n\}$, let \mathcal{A}_B be the set of all antichains in $\mathcal{P}(\{1, \dots, n\})$ whose union is B . Then $|\mathcal{A}_k| = \sum_{|B|=k} |\mathcal{A}_B|$. Now if $B = \{b_1, \dots, b_j\}$, then $\mathcal{P}(B)$ is order-isomorphic to $\mathcal{P}(\{1, \dots, j\})$ via $\varphi : \mathcal{P}(\{1, \dots, j\}) \rightarrow \mathcal{P}(B)$, $\{z_1, \dots, z_i\} \mapsto \{b_{z_1}, \dots, b_{z_i}\}$. Hence $|\mathcal{A}_B| = T_{|B|}$. Because an n -element set has $\binom{n}{k}$ k -element subsets, we infer $|\mathcal{A}_k| = \binom{n}{k} T_k$, which directly implies the result. ■

Thus the task of computing the D_n has been reduced to the task of computing the T_n , which is equally formidable. We can record the following.

Definition 2.33. For $n \geq 1$ let T_n^j be the number of j -element antichains of $\mathcal{P}(\{1, \dots, n\})$ whose union is $\{1, \dots, n\}$.

Proposition 2.34. For $n \geq 1$, we have $T_n = \sum_{j=1}^n T_n^j$. ■

As is often done in a counting task, the above has reduced our task to a number of more specific counting tasks. Unfortunately, such reductions do not always lead to success, and Dedekind's problem is one example in which this is the case. The only further (very modest) advance I can report here is the following.

Proposition 2.35. For $n \geq 1$, we have $T_n^2 = \frac{1}{2} (3^n - 2^{n+1} + 1)$.

Proof. We will count the number of all pairs

$$(F, S) \in \mathcal{P}(\{1, \dots, n\}) \times \mathcal{P}(\{1, \dots, n\})$$

such that $\{F, S\}$ is an antichain and $F \cup S = \{1, \dots, n\}$. Clearly this number is $2T_n^2$. We first count the ways in which F can be chosen and then we multiply, for each possible choice of F , with the number of ways S can be chosen.

If the first set F has k elements, then $k \neq 0$ and $k \neq n$. Hence we must sum from $k = 1$ to $n - 1$ and, for each k , there are $\binom{n}{k}$ possible choices for F . Once F

is chosen, the second set S must contain $\{1, \dots, n\} \setminus F$. For $F \cap S$ there are $2^k - 1$ possibilities (only $F \cap S = F$ cannot happen). Hence

$$\begin{aligned} 2T_n^2 &= \sum_{k=1}^{n-1} \binom{n}{k} (2^k - 1) = \sum_{k=1}^{n-1} \binom{n}{k} 2^k 1^{n-k} - \sum_{k=1}^{n-1} \binom{n}{k} 1^k 1^{n-k} \\ &= (3^n - 2^n - 1) - (2^n - 2) = 3^n - 2^{n+1} + 1. \end{aligned}$$

■

It is now easy to see that a similar process will give us the values for T_n^3 , T_n^4 , and so on. Unfortunately, the formulas become so unwieldy that they are not very useful any more. For example, I was unable to express the T_n^j recursively in terms of T_k^i with $k < n$ and $i < j$. For a computational approach to the problem and a list of the known Dedekind numbers so far (also given in Remark 4 at the end of this chapter), see [322].

Another time honored approach to difficult problems is to translate them into a different venue, in this case, order-preserving maps.

Proposition 2.36. *Let P be a finite ordered set. Then the number of antichains in P is equal to the number of order-preserving maps from P into the two-element chain.*

Proof. We will show that there is a bijective map B from the set $\mathcal{A}(P)$ of all antichains in P , to the set $\text{Hom}(P, \{0, 1\})$ of order-preserving maps from P into the two-element chain $\{0, 1\}$. Let $A = \{a_1, \dots, a_m\} \in \mathcal{A}(P)$. Define $L(A) := \bigcup_{i=1}^m \downarrow a_i$ (note that, for $A = \emptyset$, we have that $L(A) = \emptyset$) and define

$$f_A(p) := \begin{cases} 1; & \text{if } p \notin L(A), \\ 0; & \text{if } p \in L(A). \end{cases}$$

Then $f_A : P \rightarrow \{0, 1\}$ is order-preserving. Indeed, if $p \leq q$ and $f_A(q) = 1$, there is nothing to prove, while if $p \leq q$ and $f_A(q) = 0$, then $q \in L(A)$, which implies $p \in L(A)$ and $f_A(p) = 0$.

Define $B(A) := f_A$. Then B is an injective function, because, for $A_1 \neq A_2$, we have that $(L(A_1) \setminus L(A_2)) \cup (L(A_2) \setminus L(A_1)) \neq \emptyset$, which means that there will be a point $p \in (L(A_1) \setminus L(A_2)) \cup (L(A_2) \setminus L(A_1))$ such that $f_{A_1}(p) \neq f_{A_2}(p)$. To prove that B is surjective, let $f : P \rightarrow \{0, 1\}$ be given. If f is identical to 1, then $f = B(\emptyset)$. Otherwise, let A be the set of maximal elements of $f^{-1}[0]$. Then $f^{-1}[0] = L(A)$ and $f = f_A = B(A)$. Thus B is the desired bijection. ■

We will consider the problem of counting maps in more detail in Chapter 5. The translation in Proposition 2.36 reaches further than Dedekind's problem, because it applies to arbitrary finite ordered sets. Thus, a natural generalization of Dedekind's problem is to ask for the number of antichains in an arbitrary ordered set. Beyond what is presented in this text, I am unaware of any classes of ordered sets for which this question was asked or answered. However, it was proved in [230] that the computational enumeration must be considered "hard."

Exercises

2-22. Prove that the n^{th} Dedekind number D_n is at least $2^{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$.

2-23. a. Compute the number of all antichains in $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$, and \mathcal{P}_4 .

Hint. The numbers are given in Remark 4. (You need to justify why they are right.)

b. Attempt to find the number of antichains in \mathcal{P}_n for $n \geq 5$ via computer (see [322]).
[Publish the result if you get past 8.]

2.5 Fences and Crowns

The next-simplest ordered structures after chains and antichains are fences and crowns. Fences can be considered analogous to paths connecting distinct points in topology or graph theory. Crowns are very much analogous to closed, non-self-intersecting paths. We start with fences.

Definition 2.37. Let P be an ordered set. An $(n + 1)$ -**fence** (see Figure 2.4) is an ordered set $F = \{f_0, \dots, f_n\}$ such that $f_0 > f_1, f_1 < f_2, f_2 > f_3, \dots, f_{n-1} < f_n$ or $f_0 < f_1, f_1 > f_2, f_2 < f_3, \dots, f_{n-1} > f_n$ if n is even, respectively, $f_0 < f_1, f_1 > f_2, f_2 < f_3, \dots, f_{n-1} < f_n$ or $f_0 > f_1, f_1 < f_2, f_2 > f_3, \dots, f_{n-1} > f_n$ if n is odd, and such that these are all comparabilities between the points. The **length** of the fence is n . The points f_0 and f_n are called the **endpoints** of the fence.

Remark 2.38. Note that the “spider” in Figure 1.1 f) is made up of countably many fences with their left endpoints “glued together.”

Proposition 2.39. Every fence has the fixed point property.

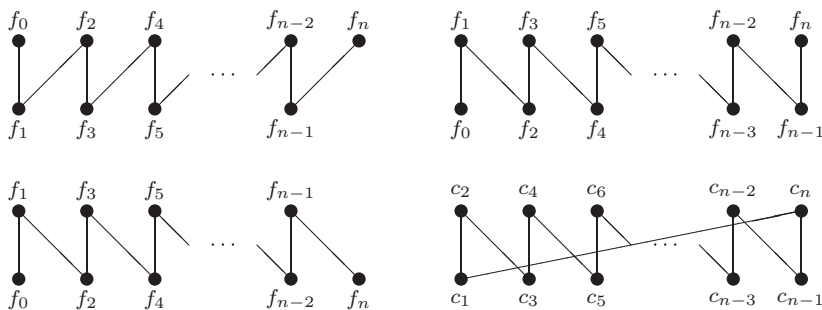


Fig. 2.4 Fences and crowns. On the left are the two nonisomorphic fences with an odd number of elements, top right is the unique (up to isomorphism) fence with an even number of elements. The bottom right shows the unique (up to isomorphism) crown with n elements (n even)

Proof. We will soon have more sophisticated tools to prove this result. However, the following direct proof nicely demonstrates the analogy between fences and intervals on the real line. It is also practice in working with order-preserving maps.

Let $P = \{p_0, p_1, p_2, \dots, p_n\}$ be a fence and assume, without loss of generality, that $p_0 < p_1$. Suppose $f : P \rightarrow P$ was a fixed point free order-preserving map. For each $k \in \{0, \dots, n\}$, let $g(k)$ be the number l such that $f(p_k) = p_l$. Then, for $|k_1 - k_2| \leq 1$, we have $|g(k_1) - g(k_2)| \leq 1$, because points in P are comparable iff their indices are adjacent. Moreover if $|k - g(k)| \leq 1$, then $p_k \sim f(p_k)$ and $f(p_k)$ is a fixed point of f . Thus $|k - g(k)| \geq 2$ for all k .

Let m be the smallest number such that $g(m) \leq m$. Then $g(m) \leq m - 2$ and $g(m - 1) \geq m + 1$, a contradiction to $|g(m) - g(m - 1)| \leq 1$. Thus P cannot have any fixed point free order-preserving self-maps f . ■

Just as we can turn a path from one point to another into a closed path by joining the endpoints, we can turn a fence with an even number of points into a crown by making the endpoints comparable in the appropriate manner.

Definition 2.40. Let $n \in \mathbb{N}$ be even and ≥ 4 . An n -crown (see Figure 2.4) is an ordered set C_n with point set $\{c_1, \dots, c_n\}$ such that $c_1 < c_2$, $c_2 > c_3$, $c_3 < c_4$, \dots , $c_{n-2} > c_{n-1}$, $c_{n-1} < c_n$, $c_n > c_1$ are the only strict comparabilities.

Just as moving from injective paths with distinct endpoints to closed paths destroys the fixed point property in topology, moving from fences to crowns makes us lose the fixed point property, too.

Proposition 2.41. Let $n \in \mathbb{N}$ be even and ≥ 4 . Then C_n does not have the fixed point property.

Proof. The map that maps $c_k \mapsto c_{k+2}$ for $k = 1, \dots, n - 2$, $c_{n-1} \mapsto c_1$, and $c_n \mapsto c_2$ is order-preserving and fixed point free. ■

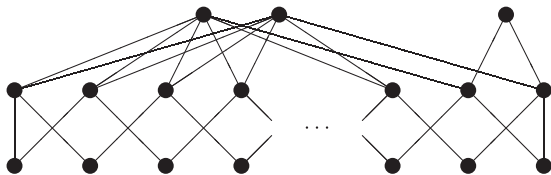
Pictorially, it is easy to see that fences are parts of crowns and that crowns cannot be parts of fences. This observation can be formalized as follows.

Proposition 2.42. Any fence of length k can be embedded into an l -crown with $l > k + 1$. However, no n -crown can be embedded into a fence of any length.

Proof. Let $F = \{f_0, \dots, f_k\}$ be a fence of length k and let $C = \{c_1, \dots, c_l\}$ be an l -crown with $l > k + 1$. Assume without loss of generality that $f_0 < f_1$. Then the map $f_i \mapsto c_{i+1}$ is an embedding.

Now suppose for a contradiction that $C = \{c_1, \dots, c_n\}$ is an n -crown and that the function $f : C \rightarrow F$ is an embedding. Let $f_k := f(c_1)$ and assume without loss of generality that $f(c_2) = f_{k+1}$. Then, for all $i \in \{1, \dots, n\}$, we must have $f_{k+i-1} = f(c_i)$. In particular $f(c_n) = f_{k+n-1} \not\sim f_k = f(c_1)$ and f could not have been an embedding. ■

Fig. 2.5 A class of ordered sets that generalizes the ordered set in Figure 1.3



Exercises

- 2-24. Prove that fences are reconstructible.
- 2-25. The number of antichains in crowns and fences. For a finite ordered set P , we define $\#A(P)$ to be the number of antichains in P .
- (See [19].) Let P be a finite ordered set. Prove that $\#A(P) = \#A(P \setminus \{x\}) + \#A(P \setminus \downarrow x)$.
 - Prove that $\#A(F_n) = \#A(F_{n-1}) + \#A(F_{n-2})$.
 - Prove that $\#A(F_n)$ is the $(n+2)$ nd Fibonacci number.
 - Prove that $\#A(C_n) = \#A(F_{n-1}) + \#A(F_{n-3})$.
- 2-26. Prove that the number of order-preserving self-maps of fences with n elements is bounded by $n3^{n-1}$.
- 2-27. Prove that crowns are reconstructible.
- 2-28. Prove that all fixed point free order-preserving self-maps of a crown are automorphisms.
- 2-29. Consider the ordered sets in Figure 2.5.
- Prove that each of these sets has the fixed point property.
 - Find at least one more class of ordered sets that generalizes the ordered sets in Figure 2.5 and in which each ordered set has the fixed point property.
- 2-30. For $j \in \{1, \dots, m\}$ let $C_j := \{c_1^j, \dots, c_{2n}^j\}$ be crowns with $2n$ elements such that, for all $i \in \{1, \dots, n\}$ and $j \in \{2, \dots, m\}$, we have $c_{2i}^{j-1} = c_{2i-1}^j$. Let \prec_j be the lower cover relation of C_j . Equip $CT_{2n}^m := \bigcup_{j=1}^m C_j$ with the order induced by the lower cover relation which is the union $\bigcup_{j=1}^m \prec_j$ of the individual lower cover relations. The ordered set CT_{2n}^m thus obtained is called a **$2n$ -crown tower** with m crowns, or a **$(2n)$ -crown tower** when m or $2n$ are not needed. Prove that any $2n$ -crown tower has a fixed point free automorphism.
- 2-31. Prove that a 6-crown tower with 3 crowns T_6^3 has more than two fixed point free order-preserving maps.
- 2-32. Prove that $\#A(CT_4^m) = 3m + 4$.

2.6 Connectivity

The notion of connectivity for ordered sets (as well as the analogous notion for graphs) is inspired by what is called pathwise connectivity in topology.

Definition 2.43. Let P be an ordered set. P is called **connected** iff, for all $a, b \in P$, there is a fence $F \subseteq P$ with endpoints a and b . An ordered set that is not connected is called **disconnected**.

Similar to topology, the fixed point property implies connectivity.

Proposition 2.44. *Let P be an ordered set with the fixed point property. Then P is connected.*

Proof. Suppose for a contradiction that P is not connected. Let $a, b \in P$ be two points such that there is no fence with endpoints a and b . Define

$$f(p) := \begin{cases} a; & \text{if there is a fence with endpoints } b \text{ and } p, \\ b; & \text{otherwise.} \end{cases}$$

Then f is order-preserving and has no fixed points, a contradiction. ■

A related problem is the question what properties the set of fixed points of an order-preserving mapping has when it is nonempty. Although there are nice positive results (see, for example, Theorem 8.10 and Section 8.2.2), we present a negative result here.

Example 2.45. Even a finite ordered set P with the fixed point property can have a self-map $f : P \rightarrow P$ such that $\text{Fix}(f)$ is not connected. (See Exercise 2-36.) □

Analogous to topology, we can define the connected components of an ordered set. Components and the metric notions defined thereafter will be useful for our proof that disconnected ordered sets are reconstructible.

Proposition 2.46. *Let P be an ordered set. If $S \subseteq P$ is a connected subset of P , then there is a maximal (with respect to inclusion) connected subset C of P such that $S \subseteq C$.*

Proof. Let \mathcal{C} be the set of all connected subsets $C_x \subseteq P$ that contain S . Then $C := \bigcup \mathcal{C}$ is a connected subset of P : Indeed, for $a, b \in C$, let s be an arbitrary point in S . Let $C_a \in \mathcal{C}$ be such that $a \in C_a$ and let $C_b \in \mathcal{C}$ be such that $b \in C_b$. Then there is a fence F_a in C_a from a to s and there is a fence F_b in C_b from s to b . Thus, after possibly removing some elements from $F_a \cup F_b$, there is a fence in C from a to b and C is connected.

The definition of C shows that C is the largest connected subset of P that contains S . ■

Definition 2.47. *The maximal (with respect to inclusion) connected subsets of an ordered set are called the (**connected**) **components** of the ordered set.*

Metric notions such as distance and diameter arise in ordered sets through the lengths of fences.

Definition 2.48. *The **distance** $\text{dist}(a, b)$ between two points $a, b \in P$ is the length of the shortest fence from a to b . If a and b are in different components of P , we will say the distance is infinite. The **diameter** $\text{diam}(P)$ of an ordered set P is the largest distance between any two points in P . If P contains points that are arbitrarily far apart or if P is disconnected, we say the diameter of P is infinite.*

- Example 2.49.* 1. The “one-way infinite fence” $F_\omega := \{f_0 < f_1 > f_2 < f_3 > \cdots\}$ is connected and has infinite diameter and infinite width.
2. The “spider” in Figure 1.1 is an ordered set with infinite diameter that contains no infinite fences.
3. Let A be an antichain. Obtain the set V by adding an element to A that is an upper cover of all elements of A . Then V has width $|A|$, but its diameter is 2. \square

Diameter and width appear to be related notions, as they both measure how far away an ordered set is from being a chain. However, both notions are distinct, and there are no globally tight inequalities that relate either to each other. Indeed, Example 2.49, part 3 shows that there is no possibility to bound the width of an ordered set by using the diameter. Proposition 2.50 gives an inequality in the opposite direction. Although the inequality cannot be improved in general, Example 2.49, part 3 shows that it need not be very good in special cases.

Proposition 2.50. *Let P be an ordered set of finite diameter. Then the diameter of P is bounded by twice the width of P minus 1 and this inequality cannot be improved.*

Proof. A fence of length n contains an antichain of length $\lceil \frac{n+1}{2} \rceil$. Therefore $\lceil \frac{\text{diam}(P)+1}{2} \rceil \leq w(P)$, which implies $\text{diam}(P) \leq 2w(P) - 1$.

The fences with $2k$ elements all have length $2k - 1$ and width k . Thus the above inequality cannot be improved. \blacksquare

All the above notions merge nicely in the proof that disconnected ordered sets are reconstructible.

Proposition 2.51. *Let P be a disconnected finite ordered set with $|P| \geq 4$. Then P is reconstructible from its deck.*

Proof. First, we prove that disconnected ordered sets are recognizable. If P is disconnected, then all cards of P are disconnected unless P has a component with one point. In this case, all but one card of P are disconnected.

On the other hand, if Q is a connected ordered set, we can show that Q has at least two connected cards: Let $a, b \in P$ be such that $\text{dist}(a, b) = \text{diam}(P)$. Then $P \setminus \{a\}$ must be connected. Indeed, otherwise there is a $c \in P$ such that every fence from c to b goes through a , which means $\text{dist}(c, b) > \text{dist}(a, b)$, contradicting the choice of a and b . Similarly we prove that $P \setminus \{b\}$ is connected.

Thus disconnected ordered sets are recognizable: They are the ordered sets whose decks contain at most one connected card. For reconstruction, note that, if the deck of a disconnected ordered set D contains a connected card, then the components of D are the connected card and a singleton. This leaves the case in which all cards of D are disconnected. Find a card C of D such that the sum of the squares of the component sizes is maximal. This card must have been obtained by removing an element from a minimum-sized component. Thus C has a unique smallest component, which is the unique component of C that is not a component of D . Let s be the size of the unique smallest component of C . Then the last component of D is the unique connected ordered set S with $s + 1 < |D|$ elements such that D

contains more isomorphic copies of S than C . Because the number of isomorphic copies of S in D can be determined via the Kelly Lemma (see Proposition 1.40), we have reconstructed D . ■

Exercises

- 2-33. Prove that a one-way infinite fence does not have the fixed point property.
- 2-34. Prove that, if P is connected and $f : P \rightarrow Q$ is order-preserving, then $f[P]$ is connected.
- 2-35. Show that there are disconnected ordered sets P such that every automorphism of P has a fixed point. (Also see Exercise 12-25.)
- 2-36. For each of the following ordered sets, find an order-preserving self-map with a disconnected set of fixed points.
- The ordered set in Figure 1.1 b).
 - The ordered set in Figure 1.3.
 - The ordered sets in Figure 2.5.
- 2-37. The reconstruction problem has a negative answer for infinite ordered sets: Let P be a connected ordered set of height 1 such that P has no crowns, every maximal element has countably many lower covers and every minimal element has countably many upper covers. Let Q be an ordered set with two connected components that are isomorphic to P . Show that P and Q have the same (infinite) deck. (Of course the deck would have to be a function that assigns every class $[C]$ of ordered sets the cardinality of $\{p \in P : P \setminus \{p\} \in [C]\}$.)
- 2-38. Some notes on reconstruction of the diameter.
- Let P be an ordered set and let $A \subseteq P$ be a subset of P . Prove that $\text{diam}(A) \geq \text{diam}(P)$.
 - Give an example of an ordered set P and a point $x \in P$ so that $\text{diam}(P \setminus \{x\}) > \text{diam}(P)$.
 - Give an example of an ordered set P so that, for all $x \in P$, we have the strict inequality $\text{diam}(P \setminus \{x\}) > \text{diam}(P)$.

Note. I do not know of any proof that the diameter is reconstructible. Exercise 2-38a gives an obvious starting point. Exercise 2-38c shows that a proof would not be totally trivial.

2.7 Maximal Elements in Infinite Sets: Zorn's Lemma

In a finite ordered set, every element is below at least one maximal element (see Exercise 2-11). For infinite ordered sets, as long as we don't have chains without upper bounds (like in the natural numbers \mathbb{N}), the same thing *should* hold. We can prevent the existence of chains without upper bounds by demanding that our set is inductively ordered.

Definition 2.52. Let P be an ordered set. Then P is called **inductively ordered** iff every nonempty chain $C \subseteq P$ has an upper bound.

Surprisingly, even when we exclude chains without upper bounds, existence of maximal elements cannot be derived from simpler axioms of set theory. Hence, the existence of maximal elements in inductively ordered sets actually has the status of an axiom.

Axiom 2.53. Zorn's Lemma. *Let P be an inductively ordered set. Then P contains a maximal element M .*

Zorn's Lemma is a standard tool for mathematicians who freely use the Axiom of Choice (to which Zorn's Lemma is equivalent, see Exercise 2-44). It is used, for example, in algebra to establish the existence of maximal ideals in rings with unity, or in functional analysis in the proof of the Hahn–Banach Theorem (see Exercise 2-41). On the other hand, the Axiom of Choice can be used to establish the existence of non-measurable sets and such counterintuitive things as the Banach–Tarski Paradox (see [317]). This is why some mathematicians decide to avoid its use. The philosophical issues that arise from using the infinitary generalization of a statement that is obvious in finite structures (see Exercise 2-11) are beyond the scope of this text. We will freely use Zorn's Lemma and anything equivalent here.

The structure of a proof involving Zorn's Lemma is fairly standard: The key is to design an appropriate nonempty set of objects that is inductively ordered and such that the desired element (if it exists) is maximal in it. The appropriate order often is set inclusion. We give examples in the following.

Our first example is the Axiom of Choice. The proof will feature all the characteristics of a proof using Zorn's Lemma without any additional order theory involved. Subsequent examples will (naturally) be examples that apply to order theory. These examples will often work with several orders at once and we will need to distinguish these orders carefully. Thus the Axiom of Choice is a good “warm-up” to the slightly more complex arguments that follow. Note that we do not give an *absolute* proof of the Axiom of Choice. All that is shown in the following is that the Axiom of Choice is true *if Zorn's Lemma is true*.

Theorem 2.54 (Axiom of Choice.). *If $\{P_\alpha\}_{\alpha \in I}$ is an indexed family of nonempty sets, then there is a function $f : I \rightarrow \bigcup_{\alpha \in I} P_\alpha$, also called a **choice function**, such that, for all $\alpha \in I$, we have $f(\alpha) \in P_\alpha$.*

Proof (using Zorn's Lemma). Let \mathcal{P} be the set of all functions $f : D \rightarrow \bigcup_{\alpha \in I} P_\alpha$ such that $D \subseteq I$ and, for all $\alpha \in D$, we have $f(\alpha) \in P_\alpha$. Clearly this set is not empty, because, for any finite set $D \subseteq I$, such functions exist.

Because every function f is just the set $\{(\alpha, f(\alpha)) : \alpha \in \text{domain}(f)\}$, we have that \mathcal{P} is ordered by set inclusion. If you prefer to think about restrictions to subsets we can say that, if $f_i : D_i \rightarrow \bigcup_{\alpha \in I} P_\alpha$, $i = 1, 2$ are functions in \mathcal{P} , then $f_1 \subseteq f_2$ iff $D_1 \subseteq D_2$ and $f_2|_{D_1} = f_1$. All following arguments that use unions can be re-written in language similar to the previous sentence. We choose not to do so, as we will see that the formation of unions is a much more compact way of arguing.

To show that \mathcal{P} is inductively ordered, let $\mathcal{C} \subseteq \mathcal{P}$ be a nonempty chain. Define $g := \bigcup \mathcal{C}$. Then g is well-defined: Indeed, let $\alpha \in \text{domain}(g)$. Then, for all functions $f \in \mathcal{C}$ for which $f(\alpha)$ is defined, we have $(\alpha, f(\alpha)) \in g$, and these are all the pairs in g with first component α . If $f_1, f_2 \in \mathcal{C}$ satisfy $\alpha \in \text{domain}(f_1) \cap \text{domain}(f_2)$, then we can assume without loss of generality that $f_1 \subseteq f_2$. But that means $f_1(\alpha) = f_2(\alpha)$ and so g is well-defined.

Moreover, g is a (partial) choice function. For each $\alpha \in \text{domain}(g)$, there is an $f \in \mathcal{C}$ with $\alpha \in \text{domain}(f)$. This means that $g(\alpha) = f(\alpha) \in P_\alpha$.

By Zorn's Lemma, we can now conclude that \mathcal{P} has a maximal element F . To show that F is a choice function on I and not just on some subset $D \subset I$, suppose for a contradiction that $\text{domain}(F) \neq I$. Then there is an $\alpha \in I \setminus \text{domain}(F)$. Let $p_\alpha \in P_\alpha$ and define $G := F \cup \{(\alpha, p_\alpha)\}$. Then $G \in \mathcal{P}$ and $G \supsetneq F$ is a strict upper bound of F , contradicting the maximality of F . Thus $\text{domain}(F) = I$ and F is the desired choice function. ■

Having warmed up to Zorn's Lemma we now focus on maximal chains once more.

Proposition 2.55. *Let (P, \leq) be a nonempty ordered set and let $C_0 \subseteq P$ be a chain. Then there is a chain $M \subseteq P$ which is maximal with respect to set inclusion and such that $C_0 \subseteq M$. That is, there is a maximal chain that contains C_0 .*

Proof. Let \mathcal{C} be the (nonempty) set of all chains $C \subseteq P$ with $C_0 \subseteq C$. This set is ordered by set inclusion \subseteq .

To show that \mathcal{C} is inductively ordered, let $\mathcal{K} \subseteq \mathcal{C}$ be a chain with respect to inclusion. Consider the set $K := \bigcup \mathcal{K}$ (with the induced order from P). To see that $K \in \mathcal{C}$, we must show that K is a chain. Let $x, y \in K$. Then there are $C_x, C_y \in \mathcal{K}$ such that $x \in C_x$ and $y \in C_y$. Because \mathcal{K} is a chain, we can assume without loss of generality that $C_x \subseteq C_y$. Thus $x, y \in C_y$ and, because C_y is a chain, we have $x \sim_P y$. Thus K is a chain and hence K is in \mathcal{C} . Because, for all $C \in \mathcal{K}$, we trivially have $K \supseteq C$, we conclude that K is an upper bound of \mathcal{K} in \mathcal{C} . Thus \mathcal{C} is inductively ordered.

Therefore, by Zorn's Lemma, \mathcal{C} has a maximal element M . By choice of \mathcal{C} , M is a chain that contains C_0 and is maximal with respect to inclusion. ■

In the next section, we will see another application of Zorn's Lemma when we prove that every set can be well-ordered.

Exercises

- 2-39. Let P be an infinite ordered set and let $A \subseteq P$ be an antichain. Prove that there is an antichain $B \subseteq P$ that is maximal with respect to set inclusion and contains A .
- 2-40. Let V be a vector space. A **basis** of V is a subset $B \subseteq V$ such that any finite subset of B is linearly independent and such that any $v \in V$ has a (necessarily unique) representation as a finite linear combination of elements of B . Prove that every vector space has a basis.
- 2-41. Hahn–Banach Theorem. Let V be a normed vector space over $F = \mathbb{R}$ or $F = \mathbb{C}$. A **(continuous) linear functional** is a linear function $\Phi : V \rightarrow F$ such that there is a $c > 0$ such that, for all $v \in V$, we have $|\Phi(v)| \leq c\|v\|$.
Let $W \subset V$ be a linear subspace of V and let $v \in V \setminus W$. Prove that there is a continuous linear functional $\Phi : V \rightarrow F$ such that $\Phi|_W = 0$ and $\Phi(v) = 1$.
- 2-42. State the definition of a dual inductive order and state and prove the dual of Zorn's Lemma.

2-43. Let us now finally prove the general version of Dilworth's Chain Decomposition Theorem. It states that *any* ordered set of width k can be written as the union of k chains. We follow Dilworth's original idea (see [63], p. 163).

The idea is of course an induction on k with $k = 1$ still being trivial, so assume that the result holds for sets of width $k - 1$. Define a chain $C \subseteq P$ to be *strongly dependent* iff, for every finite subset $S \subseteq P$, there is a representation of $S = K_1 \cup \dots \cup K_l$ as a union of chains K_i such that $S \cap C \subseteq K_i$ for some $i \in \{1, \dots, l\}$.

- a. Prove that there is a maximal strongly dependent chain C_1 in P .
- b. Prove that $P \setminus C_1$ has width $k - 1$. To do so, assume that $\{a_1, \dots, a_k\}$ is an antichain in $P \setminus C_1$.
 - For each i find a finite set $S_i \subseteq P$ such that no chain decomposition of S_i contains $S_i \cap (C_1 \cup \{a_i\})$ in one chain.
 - Apply the property of strong dependence of C_1 to $S = \bigcup_{i=1}^k S_i$.
 - Use the insight gained to find a j such that S_j has a chain decomposition such that $S_j \cap (C_1 \cup \{a_j\})$ is contained in exactly one chain.

2-44. Prove that the Axiom of Choice is equivalent to Zorn's Lemma.

Note. This is quite challenging, see, for example, [283].

2.8 Well-Ordered Sets

Well-ordered sets are a particularly nice type of chain. Ordinal numbers in set theory are examples of well-ordered sets. In fact, they are, up to isomorphism, *all* well-ordered sets.

Definition 2.56. Let P be an ordered set and let $S \subseteq P$. Then $s \in S$ is called the **smallest element** of S iff $s \leq x$ for all $x \in S$.

Definition 2.57. An ordered set W is called **well-ordered** iff each nonempty subset $A \subseteq W$ has a smallest element.

It is easy to see that well-ordered sets are chains and that not all chains are well-ordered. The next proposition shows that well-ordered sets are very closely related to the notion of counting. For every non-maximal element there will be a “next” element.

Proposition 2.58. Let W be a well-ordered set. Then every non-maximal element $w \in W$ has an **immediate successor**. That is, for every non-maximal $w \in W$, there is an element w^+ such that $w < w^+$ and, for all $p > w$, we have $p \geq w^+$.

Proof. Let $w \in W$ be not maximal. Then $\{p \in W : p > w\}$ is not empty and it therefore has a smallest element. This smallest element is w^+ . ■

Example 2.59. Every finite chain is well-ordered. The natural numbers \mathbb{N} are the smallest infinite well-ordered set. It would be tempting to try to prove that, just as in \mathbb{N} , every element of a well-ordered set has an immediate predecessor, too. An immediate predecessor of w would of course be an element w^- such that $w > w^-$ and, for all $p < w$, we have $p \leq w^-$. The well-ordered set $\mathbb{N} \oplus \{\infty\}$, consisting of \mathbb{N}

with a largest element ∞ attached, shows that not every element of a well-ordered set has an immediate predecessor: In this example, ∞ does not have an immediate predecessor. \square

Informally speaking, well-ordered sets can be built as we just described. Pick a well-ordered set and attach a new largest element to get a new well-ordered set. This generates chains of well-ordered sets, which then can be united to form even bigger well-ordered sets. How far we can push this process will depend on how strong a version of the Axiom of Choice we are willing to accept. The standard class of examples of well-ordered sets (and, see Exercise 2-49, the only class of examples) is the class of ordinal numbers.

Example 2.60 (See [117], p. 75.). An **ordinal number** is a well-ordered set α such that, for each $\xi \in \alpha$, we have that $\xi^+ = \{\eta \in \alpha : \eta \leq \xi\}$. That is, the immediate successor of each ordinal number ξ is the set of all ordinal numbers up to and including ξ . Because we can form sets of existing objects, this is a well-defined operation that allows formation of ordinal numbers. The set-theoretical problems start once we encounter infinite ordinals.

The **first infinite ordinal number** is (isomorphic to) the set of natural numbers. In ordinal arithmetic, it is denoted by ω . By simply defining the successor via the above equation, we obtain ω^+ , $(\omega^+)^+$, and so on. In ordinal arithmetic, the n^{th} successor of an ordinal α is called $\alpha + n$. Note that ω does not have an immediate predecessor. Indeed, any ordinal number that is less than ω is finite and thus its successor is finite, too, and not equal to ω . Ordinal numbers that do not have an immediate predecessor are also called **limit ordinals**.

The indicated counting process continues through all $\omega + n$ and the next limit ordinal is 2ω . Continuing in the above fashion we reach the limit ordinals $3\omega, 4\omega, \dots$, until we reach ω^2 . What follows are $\omega^2 + 1, \omega^2 + 2, \dots, \omega^2 + \omega, \dots, \omega^2 + 2\omega, \dots, \omega^3, \dots, \omega^4, \dots, \omega^\omega$. As in [325], p. 10 we will denote the **first uncountable ordinal number** by ω_1 . \square

Proposition 2.58 and Examples 2.59 and 2.60 show that well-ordered sets are a natural generalization of the natural numbers. In a well-ordered set, there is a natural notion of counting (the immediate successor of each element is the “next” element as we count) and we can count “past infinity” if the well-ordered set is big enough. On the other hand, well-ordered sets can arise anywhere, if we believe Zorn’s Lemma: With Zorn’s Lemma, we can show that any set can be well-ordered.

Definition 2.61. Let W be a well-ordered set. A well-ordered subset $V \subseteq W$ is called an **initial segment** of W iff $V \subseteq W$ and, for all $w \in W \setminus V$, we have that w is an upper bound of V .

Theorem 2.62. Well-Ordering Theorem. For every set S , there is an order relation $\leq \subseteq S \times S$ such that (S, \leq) is well-ordered.

Proof (using Zorn’s Lemma). Let \mathcal{W} be the set of all pairs (\leq, M) such that $M \subseteq S$ and $\leq \subseteq M \times M$ is a well-ordering. Then \mathcal{W} is not empty, because, for each $s \in S$, the pair $(\{(s, s)\}, \{s\})$ is in \mathcal{W} . Order \mathcal{W} by $(\leq_1, M_1) \sqsubseteq (\leq_2, M_2)$ iff

1. $M_1 \subseteq M_2$,
2. $\leq_1 \subseteq \leq_2$ (as relations),
3. M_1 is an initial segment of M_2 ordered by \leq_2 .

To prove that \mathcal{W} is inductively ordered, let \mathcal{K} be a \sqsubseteq -chain in \mathcal{W} . Let

$$M_t := \bigcup \{M : (\leq, M) \in \mathcal{K}\}$$

and equip it with the relation

$$\leq_t := \bigcup \{\leq : (\leq, M) \in \mathcal{K}\}.$$

We must show that M_t is well-ordered by \leq_t and that, for all $(\leq, M) \in \mathcal{K}$, we have that M is an initial segment of M_t . We shall first show that M_t is totally ordered. Let $x, y, z \in M_t$. Then there are M_x, M_y , and M_z so that $x \in M_x$, $y \in M_y$, and $z \in M_z$. Without loss of generality, assume that $(\leq_x, M_x) \sqsubseteq (\leq_y, M_y) \sqsubseteq (\leq_z, M_z)$. Thus $x \leq_x x$, which implies $x \leq_t x$ for all $x \in M_t$. If $x \geq_t y$ and $x \leq_t y$, then $x \geq_y y$ and $x \leq_y y$ (there is a small argument here, see Exercise 2-47), which implies $x = y$. If $x \leq_t y$ and $y \leq_t z$, then $x \leq_z z$ and $y \leq_z z$, which implies $x \leq_z z$ and then $x \leq_t z$. Finally, for all $x, y \in M$, we have either $x \geq_y y$ or $x \leq_y y$ which means $x \geq_t y$ or $x \leq_t y$ and \leq is a total order on M .

Now we will show that, for every $(\leq, M) \in \mathcal{K}$, and every $b \in M_t \setminus M$, we have that b is an \leq_t -upper bound of M . Let $(\leq, M) \in \mathcal{K}$ and let $b \in M_t \setminus M$. Then there is a $(\leq', M') \in \mathcal{K}$ so that $b \in M'$. Because $b \in M' \setminus M$, we must have $(\leq, M) \sqsubseteq (\leq', M')$. By condition 3, we infer that b is an \leq' -upper bound of M and hence it is an \leq_t -upper bound of M .

To show that \leq_t is a well-ordering, let $A \subseteq M_t$ be nonempty. Then there is a $(\leq, M) \in \mathcal{K}$ such that $A \cap M \neq \emptyset$. Let $a \in A$ be the \leq -smallest element of A . Then, for all $b \in A \cap M$, we have $b \geq a$, hence $b \geq_t a$. For $b \in A \setminus M$, the preceding paragraph shows $b \geq_t a$. Thus a is the \leq_t -smallest element of A . Hence \leq_t is a well-ordering and, via the preceding paragraph, (\leq_t, M_t) is a \sqsubseteq -upper bound of \mathcal{K} . This means $(\mathcal{W}, \sqsubseteq)$ is inductively ordered.

Let $(\leq, M) \in \mathcal{W}$ be a \sqsubseteq -maximal element as guaranteed by Zorn's Lemma. If $M = S$, we are done. Assume that there is an $s \in S$ that is not in M . Define

$$\leq' := \leq \cup \{(x, s) : x \in M \text{ or } x = s\} \subseteq (M \cup \{s\}) \times (M \cup \{s\}).$$

Then \leq' is easily verified to be a well-ordering. But then $(\leq', M \cup \{s\}) \in \mathcal{W}$ is a \sqsubseteq -upper bound of (\leq, M) that is not equal to (\leq, M) , which is a contradiction. Thus \leq must be a well-ordering of $S = M$. ■

Note that, in the proof of the Well-Ordering Theorem, it is not possible to avoid the complicated definition of \sqsubseteq . Indeed, if we just used containment of the orders as \sqsubseteq , then the candidates for upper bounds of chains as defined in the proof need not be well-ordered. That is, they might not be in \mathcal{W} . For illustration, consider the set of chains $\mathcal{B} := \{\{\pm \frac{1}{k} : k = 1, \dots, n\} : n \in \mathbb{N}\}$, with each individual chain ordered

with the order inherited from \mathbb{Q} . Set containment induces a total order on this set of chains and any two chains in this set are well-ordered (after all, all chains in \mathcal{B} are finite). Yet the union of these chains is the set $\bigcup \mathcal{B} = \{\pm \frac{1}{n} : n \in \mathbb{N}\}$, which is not well-ordered. So we need condition 3 in the definition of \sqsubseteq to prevent chains that we unify from “filling in holes” instead of “building upwards in a well-ordered fashion.”

A final observation about well-ordered sets is that they have many of the properties that increasing sequences have. In the near future (see Theorem 4.17), we will encounter situations in which we are only interested in “where the tops of certain chains go.” In such situations it will be helpful to replace the chains we have with chains that have the same “growth” or “convergence behavior” and are otherwise well-behaved. The result that allows us to do so is the fact that any chain has a cofinal (see Definition 2.63) well-ordered subchain (see Proposition 2.64).

Definition 2.63. *Let P be an ordered set and let $A \subseteq B \subseteq P$. Then A is called **cofinal (coinital)** in B iff, for every $b \in B$, there is an $a \in A$ such that $a \geq b$ ($a \leq b$).*

Proposition 2.64. *Let P be an ordered set. Then, for every chain $C \subseteq P$, there is a well-ordered cofinal subchain $W \subseteq C$.*

Proof. Left as Exercise 2-45. Hint: Order the well-ordered subchains of C with an order like in the proof of Theorem 2.62. ■

Exercises

- 2-45. Prove Proposition 2.64.
- 2-46. Prove that a well-ordered set has the fixed point property iff it has a largest element.
- 2-47. In the proof of the Well-Ordering Theorem, we claim that, if $x \in M_x$ (ordered by \leq_x) and $y \in M_y$ (ordered by \leq_y) with $(\leq_x, M_x) \sqsubseteq (\leq_y, M_y)$ and $x \leq_l y$, then $x \leq_y y$. Formally we only know that $x \leq y$ for some (\leq, M) where $x, y \in M$. Prove that $x \leq_y y$.
- 2-48. Prove that the Well-Ordering Theorem implies Zorn’s Lemma.
Hint. Use the result from Exercise 2-44.
- 2-49. Prove that every well-ordered set is isomorphic to one of the ordinal numbers in Example 2.60.
- 2-50. Let (Ω, Σ, μ) be a measure space and let $p \in [1, \infty)$. We only consider real valued functions. Prove that if $W \subseteq L^p(\Omega, \Sigma, \mu)$ is well-ordered without a largest element, then W is countable and there is a cofinal subchain $N \subseteq W$ that is isomorphic to \mathbb{N} .

Remarks and Open Problems

This is the first time in this text that the remarks section also includes open problems. The open problems presented in this section are special cases of the main open questions or modifications of them.

1. The original questions that motivated the automorphism problem: Let $\text{Fpf}(P)$ be the set of fixed point free maps of the ordered set P .
 - a. Find an overall bound for $\frac{|\text{Fpf}(P)|}{|\text{End}(P)|}$.
 - b. Find $\limsup_{|P| \rightarrow \infty} \frac{|\text{Fpf}(P)|}{|\text{End}(P)|}$.
 - c. Find the above quantities when P is restricted to a special class of ordered sets.

I am not aware of any progress in this direction beyond [253]. Natural candidates for first partial results might be ordered sets of small width, as the analogous automorphism problem also seems to be solvable there (see Exercises 2-21 and 9-7).

2. The ordered set in Figure 2.2 also appears in [64], where R. P. Dilworth recounts the history of Theorem 2.26. For more on the work of R. P. Dilworth, consider [25]. For connections between graph theory and Dilworth's Theorem, consider Section 2.1 in [319].
3. Dilworth's Theorem cannot be extended to ordered sets of infinite width. In [220], it is proved that, for every infinite cardinal c , there is an ordered set of cardinality c without infinite antichains, which cannot be decomposed into less than c chains. This construction is presented in Exercise 12-10.
4. The first eight Dedekind numbers have been found via computation. The list goes as follows (see [322]) 2; 3; 6; 20; 168; 7,581; 7,828,354; 2,414,682,040,998, and 56,130,437,228,687,557,907,788.

Proofs of asymptotic formulas for Dedekind numbers can be found in [164, 171].

5. A conjecture on reconstruction of infinite ordered sets states that the example in Exercise 2-37 is characteristic for infinite ordered sets. The conjecture says that, if P, Q are infinite ordered sets that are not isomorphic and have the same decks, then there is a $p \in P$ such that $P \setminus \{p\}$ contains an isomorphic copy of Q or there is a $q \in Q$ such that $Q \setminus \{q\}$ contains an isomorphic copy of P .
6. Find formulas for T_n^j for $j \geq 3$ that allow (much) faster computation than [160].
7. Find easy-to-compute formulas for the number of antichains in classes of ordered sets other than fences or crowns. This might give ideas for the solution of Dedekind's problem. See Theorem 11.12 for a formula for interval ordered sets.
8. Characterize the ordered sets of height 2 or those of width 3 that have the fixed point property. We will consider width 2 in Theorem 4.34 and height 1 in Theorem 4.37. I conjecture that there is a polynomial algorithm to determine the fixed point property for ordered sets of width 3. In the light of the proof of Theorem 7.32, which says that it is NP-complete to decide if an ordered set of height 5 has a fixed point free order-preserving self-map, and in light of Exercise 7-27, I have no intuition what might happen for height 2.

9. Prove that ordered sets of width 3 or of height 1 are reconstructible. We will consider width 2 in Exercise 3-8. I believe that a proof of reconstructibility of ordered sets of width 3 should be possible with the reconstruction tools available today (also see [278] for a start). Reconstruction of ordered sets of height 1 on the other hand appears almost as hard as the reconstruction problem in general.
10. For Ramsey-type order-theoretical results beyond Proposition 2.28, see [97, 213, 214, 225]. One of Ramsey's theorems (the one which guarantees the existence of finite Ramsey numbers, see [241]) essentially says that certain structures (complete graphs and discrete graphs) are so plentiful, that a representative of a certain size can be found in any graph. Embed the graph $G = (V, E)$ into a complete graph with $|V|$ vertices and color the edges of G red and the remaining edges blue. Then Ramsey's theorem says that any such coloring will always allow for a monochromatic complete subgraph of a certain size.

The mentioned papers investigate and prove the following. Fix $r, s \in \mathbb{N}$. For every ordered set P it is possible to find an ordered set P' such that for any r -coloring χ of the s -chains of P' , there is an embedding e of P into P' such that the s -chains of $e[P]$ are monochromatic under χ . So there is an ordered set that contains so many copies of P that even a partition through coloring will still allow us to find a copy in one of the elements of the partition.

11. Define a **fiber** of an ordered set to be a subset $F \subseteq P$ such that F intersects every maximal antichain with at least two elements.¹ Pictorially, the maximal antichains of an ordered set represent "horizontal separators" in the order. That is, every element of the ordered set will be either above or below some element of the maximal antichain. The notion of a fiber is then a subset that "stabs through all the separators."

It was conjectured in [193] that every finite ordered set has a fiber whose complement also is a fiber. Consequently, every ordered set would have a fiber of size at most $\frac{|P|}{2}$. This is not true, because, by Exercise 2-19 the smallest size for a fiber of the ordered set P_0 in Figure 2.3 is $\frac{9}{17}|P_0|$.

The natural question that now arises is the following. What is the smallest λ such that any ordered set P is guaranteed to have a fiber of size at most $\lambda|P|$?

In [199] an iterative construction using the set in Figure 2.3 is used to show that $\lambda \geq \frac{8}{15}$. In [69], Theorem 1, it is shown that the elements of an ordered set can be 3-colored so that all nontrivial maximal antichains receive at least two colors. This means $\lambda \leq \frac{2}{3}$. The exact value of λ remains unknown.

12. The automorphism conjecture can be settled trivially for classes of ordered sets for which $|\text{End}(P)|$ grows faster than $n!$. Unfortunately the hope that this is true in general is false. Fences and crowns have $< O(3^n) \ll n!$ (for large n) endomorphisms.

¹Sometimes, sets that intersect *every* maximal antichain are called fibers. The overall upper bound on the size of such a fiber is $|P|$, as can be seen considering chains. Upper and lower bounds on the fiber size for individual ordered sets in given classes of ordered sets can be interesting.

13. What general results are there to compute the number of homomorphisms from one ordered set to another? I am not aware of any results beyond [86, 280] and Exercise 2-6. A good list of earlier references is in [86]. Related to this, for each n , what is the ordered set P of size n with the fewest endomorphisms?
14. The **endomorphism spectrum** (see [111]) of a finite ordered set P is the set $S := \{s \in \{1, \dots, |P|\} : (\exists f \in \text{End}(P)) |f[P]| = s\}$. That is, it is the set of possible range sizes for order-preserving self-maps. Aside from trivial choices such as $1, 2, |P|$, what numbers are guaranteed to always be in the spectrum? What are examples of ordered sets with “small” spectra? In [70], Theorem 1.1, it is shown that every ordered set has an endomorphism with $|P|^{\frac{1}{2}}$ elements in its image. It is also shown in [70], Theorem 1.2, that there is a $c > 0$ such that, for each n , there is an ordered set of size n such that every endomorphism that is not the identity has at most $c(n \log(n))^{\frac{1}{3}}$ elements in its image.
15. For every k in the spectrum of P , let e_k^P be the number of endomorphisms with image of size k . The spectrum analyzes the zeroes of this sequence. What more can be said about the properties of the sequence e_k^P ?

Ordered Sets

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