Chapter 2
Stability of Interconnected Systems

Consider the interconnection in Fig. 2.1 where each subsystem G_i, $i = 1, \ldots, N$, is described by

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i(t) = f_i(x_i(t), u_i(t)) \tag{2.1}$$

$$y_i(t) = h_i(x_i(t), u_i(t)) \tag{2.2}$$

with $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, $y_i(t) \in \mathbb{R}^{p_i}$, $f_i(0, 0) = 0$, $h_i(0, 0) = 0$.

The static matrix M defines the coupling of these subsystems: the input u_i to G_i depends on the outputs y_j of other subsystems by

$$u = My \tag{2.3}$$

where $u = [u_1^T \cdots u_N^T]^T$ and $y = [y_1^T \cdots y_N^T]^T$. We assume that the interconnection is well-posed; that is, upon the substitution $y_i = h_i(x_i, u_i)$ the Eq. (2.3) admits a unique solution for u as a function x.

2.1 Compositional Stability Certification

Our goal is to derive a bottom-up stability test using dissipativity properties and the interconnection structure of the subsystems. Dissipativity serves as an abstraction of the subsystem models (Fig. 1.1) and allows us to study interconnections whose combined dynamical equations are too large to analyze directly. The use of input/output properties and interconnection matrices for network stability tests dates back to the early Refs. [1, 2].

M. Arcak et al., *Networks of Dissipative Systems*,
SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-29928-0_2

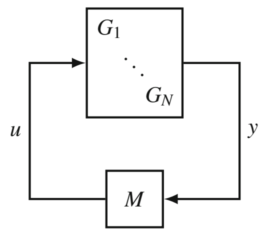

Fig. 2.1 An interconnection of subsystems $G_1, \ldots, G_N$. The inputs depend on the outputs of other subsystems by $u = My$ where M is a static matrix

We assume each subsystem is dissipative with a positive definite, continuously differentiable storage function $V_i(\cdot)$ and a quadratic supply rate:

$$s_i(u_i, y_i) = \begin{bmatrix} u_i \\ y_i \end{bmatrix}^T X_i \begin{bmatrix} u_i \\ y_i \end{bmatrix} = \begin{bmatrix} u_i \\ y_i \end{bmatrix}^T \begin{bmatrix} X_i^{11} & X_i^{12} \\ X_i^{21} & X_i^{22} \end{bmatrix} \begin{bmatrix} u_i \\ y_i \end{bmatrix} \tag{2.4}$$

where X_i^{jk}, $j, k \in \{1, 2\}$, are conformal block partitions of X_i. We then search for a weighted sum of storage functions

$$V(x) = p_1 V_1(x_1) + \cdots + p_N V_N(x_N) \quad p_i > 0, \ i = 1, \ldots, N \tag{2.5}$$

that serves as a Lyapunov function for the interconnection. To this end we ask that the right-hand side of the inequality

$$\sum_{i=1}^{N} p_i \nabla V_i(x_i)^T f_i(x_i, u_i) \le \sum_{i=1}^{N} p_i \begin{bmatrix} u_i \\ y_i \end{bmatrix}^T X_i \begin{bmatrix} u_i \\ y_i \end{bmatrix} \tag{2.6}$$

be negative semidefinite in y when u is eliminated with the substitution $u = My$. Rewriting the right-hand side of (2.6) as

$$\begin{bmatrix} u_1 \\ \vdots \\ u_N \\ y_1 \\ \vdots \\ y_N \end{bmatrix}^T \underbrace{\begin{bmatrix} p_1 X_1^{11} & & & p_1 X_1^{12} & & \\ & \ddots & & & \ddots & \\ & & p_N X_N^{11} & & & p_N X_N^{12} \\ p_1 X_1^{21} & & & p_1 X_1^{22} & & \\ & \ddots & & & \ddots & \\ & & p_N X_N^{21} & & & p_N X_N^{22} \end{bmatrix}}_{\triangleq \mathbf{X}(p_1 X_1, \ldots, p_N X_N)} \begin{bmatrix} u_1 \\ \vdots \\ u_N \\ y_1 \\ \vdots \\ y_N \end{bmatrix} \tag{2.7}$$

$$= y^T \begin{bmatrix} M \\ I \end{bmatrix}^T \mathbf{X}(p_1 X_1, \ldots, p_N X_N) \begin{bmatrix} M \\ I \end{bmatrix} y$$

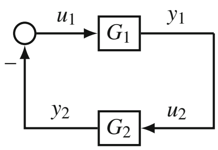

Fig. 2.2 When M is as in (2.9), $u = My$ describes a negative feedback interconnection of two subsystems where $u_1 = -y_2$ and $u_2 = y_1$

we obtain the following stability criterion:

Proposition 2.1 *If there exist $p_i > 0$, $i = 1, \ldots, N$, such that*

$$\begin{bmatrix} M \\ I \end{bmatrix}^T \mathbf{X}(p_1X_1, \ldots, p_NX_N) \begin{bmatrix} M \\ I \end{bmatrix} \leq 0 \tag{2.8}$$

where $\mathbf{X}(p_1X_1, \ldots, p_NX_N)$ is as defined in (2.7), then $x = 0$ is stable for the interconnected system (2.1)–(2.3) and (2.5) is a Lyapunov function.

For memoryless subsystems of the form $y_i(t) = h_i(u_i(t))$ we take the corresponding storage function in (2.5) to be zero.

Asymptotic stability requires additional assumptions, such as strict inequality in (2.8) accompanied with an argument that $x(t) = 0$ is the only solution satisfying $h_i(x_i(t), 0) = 0$, $i = 1, \ldots, N$, for all t.

Note that (2.8) is a linear matrix inequality (LMI) and the search for $p_i > 0$ satisfying this inequality can be performed with convex optimization packages [3, 4].

Below we assume each subsystem is single-input single-output and specialize the LMI (2.8) to particular types of dissipativity. This allows us to derive analytical feasibility conditions for special interconnection matrices M. Of particular interest is

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.9}$$

which describes the negative feedback loop of two subsystems (Fig. 2.2), commonly studied in control theory.

2.2 Small Gain Criterion

Suppose each subsystem possesses a finite L_2 gain; that is, the supply rate in (2.4) is

$$X_i = \begin{bmatrix} \gamma_i^2 & 0 \\ 0 & -1 \end{bmatrix}.$$

Defining $P \triangleq \mathrm{diag}(p_1, \ldots, p_N)$ and $\Gamma \triangleq \mathrm{diag}(\gamma_1, \ldots, \gamma_N)$ we get

$$\mathbf{X}(p_1 X_1, \ldots, p_N X_N) = \begin{bmatrix} \Gamma P \Gamma & 0 \\ 0 & -P \end{bmatrix}$$

and (2.8) becomes

$$(\Gamma M)^T P (\Gamma M) - P \leq 0. \tag{2.10}$$

Thus a diagonal matrix $P > 0$ satisfying this LMI certifies the stability of the interconnection.

When M is as in (2.9), the LMI (2.10) becomes

$$\begin{bmatrix} p_2\gamma_2^2 & 0 \\ 0 & p_1\gamma_1^2 \end{bmatrix} - \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \leq 0$$

which consists of two simultaneous inequalities, $p_2\gamma_2^2 \leq p_1$ and $p_1\gamma_1^2 \leq p_2$. We rewrite them as

$$\gamma_2^2 \leq \frac{p_1}{p_2} \leq \frac{1}{\gamma_1^2}$$

and note that such $p_1 > 0$ and $p_2 > 0$ exist if and only if $\gamma_2^2 \leq \frac{1}{\gamma_1^2}$, that is

$$\gamma_1\gamma_2 \leq 1. \tag{2.11}$$

This condition restricts the loop gain in Fig. 2.2 and is known as a "small gain" criterion.

Note that the derivation above yields the same condition, (2.11), when adapted to the *positive* feedback interconnection where

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This means that the small gain criterion is oblivious to the feedback sign.

2.3 Passivity Theorem

We now specialize Proposition 2.1 to passivity where

$$X_i = \begin{bmatrix} 0 & 1/2 \\ 1/2 & -\varepsilon_i \end{bmatrix} \quad \varepsilon_i \geq 0.$$

With $P \triangleq \mathrm{diag}(p_1, \ldots, p_N)$ and $E \triangleq \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_N)$ we get

$$\mathbf{X}(p_1 X_1, \ldots, p_N X_N) = \frac{1}{2} \begin{bmatrix} 0 & P \\ P & -2PE \end{bmatrix}$$

which means that (2.8) is equivalent to

$$P(M-E) + (M-E)^T P \leq 0 \tag{2.12}$$

and a diagonal matrix $P > 0$ satisfying this LMI certifies the stability of the interconnected system.

From matrix Hurwitz stability theory, (2.12) with $P > 0$ implies that all eigenvalues of $M - E$ are within the closed left half-plane. Thus, if $M - E$ has an eigenvalue with a strictly positive real part, there is no $P > 0$ satisfying (2.12). However, we cannot confirm the feasibility of (2.12) with a *diagonal* $P > 0$ from the eigenvalues alone.

Below we exhibit practically important classes of interconnection structures for which (2.12) admits a diagonal solution $P > 0$.

2.3.1 Skew Symmetric Interconnections

The stability criterion (2.12) holds trivially with $P = I$ when M is skew symmetric:

$$M + M^T = 0.$$

There is no restriction on the number or the gains of subsystems, which makes passivity ideally suited to large-scale systems with a skew symmetric coupling structure.

In Chap. 4 we show that this structure arises naturally in distributed control of vehicle platoons and in Internet congestion control. A simpler example of a skew symmetric interconnection is the negative feedback interconnection of two subsystems (Fig. 2.2) where M is as in (2.9). The stability of this interconnection with passive subsystems is a classical result known as the passivity theorem.

2.3.2 Negative Feedback Cyclic Interconnection

To derive another special case of the stability criterion (2.12), we consider a negative feedback loop of N subsystems where the interconnection matrix is

$$M = \begin{bmatrix} 0 & \cdots & 0 & \delta_1 \\ \delta_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \delta_N & 0 \end{bmatrix} \quad \text{with} \quad \prod_{i=1}^{N} \delta_i = -1. \tag{2.13}$$

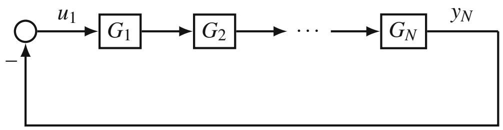

Fig. 2.3 A negative feedback cyclic interconnection of N subsystems. In this example M is as in (2.13) with $\delta_1 = -1$, $\delta_2 = \cdots = \delta_N = 1$

One such interconnection is shown in Fig. 2.3 where $\delta_1 = -1$, $\delta_2 = \cdots = \delta_N = 1$.

We prove in Sect. 7.2 that (2.12) admits a diagonal solution $P > 0$ for the class of matrices (2.13) if and only if

$$\prod_{i=1}^{N} \varepsilon_i \geq \cos^N(\pi/N). \tag{2.14}$$

In addition, it was shown in [5] that (2.12) holds with strict inequality if and only if (2.14) is strict.

For $N = 2$ the condition (2.14) recovers the classical passivity theorem: $\cos(\pi/2) = 0$ and passivity ($\varepsilon_i \geq 0$) guarantees stability. For $N \geq 3$, $\cos(\pi/N) > 0$ and (2.14) demands output strict passivity ($\varepsilon_i > 0$).

To compare (2.14) to the small gain criterion, we recall from Sect. 1.1 that output strict passivity implies an L_2 gain of $\gamma_i = 1/\varepsilon_i$ and rewrite (2.14) as

$$\prod_{i=1}^{N} \gamma_i \leq \sec^N(\pi/N) \tag{2.15}$$

where $\sec(\cdot) = 1/\cos(\cdot)$. Unlike the small gain criterion which restricts the feedback loop gain by one, the "secant condition" (2.15) offers the relaxed bound $\sec^N(\pi/N)$ which is equal to 8 when $N = 3$, and decreases asymptotically to one as $N \to \infty$. This sharper bound is due to the output strict passivity assumption which restricts the subsystems further than an L_2 gain property.

Example 2.1 Consider the following model for a *ring oscillator* circuit (Fig. 2.4) that consists of a feedback loop of three inverters:

$$\begin{aligned}
\tau_1 \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= -x_1(t) - h_3(x_3(t)) \\
\tau_2 \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= -x_2(t) - h_1(x_1(t)) \\
\tau_3 \frac{\mathrm{d}x_3(t)}{\mathrm{d}t} &= -x_3(t) - h_2(x_2(t))
\end{aligned} \tag{2.16}$$

where $\tau_i = R_i C_i > 0$, $i = 1, 2, 3$, and x_i represent voltages. The functions $h_i(\cdot)$ depend on the inverter characteristics and satisfy

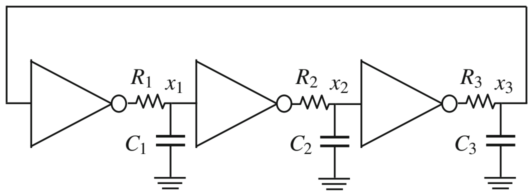

Fig. 2.4 Schematic of a three-stage ring oscillator circuit

$$h_i(0) = 0, \quad xh_i(x) > 0 \quad \forall x \neq 0, \tag{2.17}$$

as in the commonly used model

$$h_i(x) = \alpha_i \tanh(\beta_i x) \quad \alpha_i > 0, \beta_i > 0. \tag{2.18}$$

We decompose (2.16) into the subsystems

$$G_i\colon \quad \tau_i \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -x_i(t) + u_i(t) \quad y_i(t) = h_i(x_i(t))$$

interconnected according to $u = My$ where $M \in \mathbb{R}^{3\times 3}$ is as in (2.13) with $\delta_1 = \delta_2 = \delta_3 = -1$.

Next, we note from (1.14) with $f_0(x) = -x$ that the subsystems are output strictly passive if

$$\varepsilon_i x h_i(x) \leq x^2.$$

This inequality, combined with (2.17), restricts the graph of $h_i(\cdot)$ to the sector in Fig. 1.2 (middle) with slope $\gamma_i = 1/\varepsilon_i$. An example of such a function is (2.18) where $\gamma_i = \alpha_i \beta_i$.

Then, an application of (2.15) with $N = 3$ shows that the equilibrium of the interconnection $x = 0$ is stable when

$$\gamma_1 \gamma_2 \gamma_3 \leq 8 \tag{2.19}$$

and a weighted sum of storage functions, each constructed as in (1.13), serves as a Lyapunov function:

$$V(x) = \sum_{i=1}^{3} p_i \int_0^{x_i} h_i(z)\mathrm{d}z.$$

The weights $p_i > 0$ are obtained from the LMI (2.12) which is guaranteed to have a diagonal solution $P > 0$ by (2.19). When the inequality (2.19) is strict we conclude asymptotic stability because (2.12) is negative definite, which means that (2.7) is a negative definite function of y and, further, $y_i = h_i(x_i) = 0 \Rightarrow x_i = 0$ by (2.17).

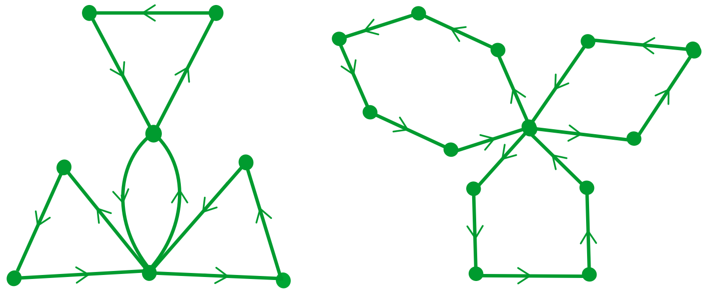

Fig. 2.5 Examples of cactus graphs

When $\tau_1 = \tau_2 = \tau_3$, the secant condition (2.19) is also necessary for stability [5]. Once the loop gain exceeds 8, the equilibrium loses its stability and a limit cycle emerges, hence, the term "ring oscillator."

2.3.3 Extension to Cactus Graphs

To describe a broader interconnection structure that encompasses the cyclic interconnection above, we define an incidence graph for M by directing an edge from vertex j to i if and only if $m_{ij} \neq 0$. This graph is said to be a *cactus graph* if any pair of distinct simple cycles[1] have at most one common vertex, as in the examples of Fig. 2.5.

For matrices M with this structure and $E \triangleq \text{diag}(\varepsilon_1, \ldots, \varepsilon_N) > 0$, a procedure was developed in [6] to determine the range of the entries of M and E for which a diagonal $P > 0$ satisfies (2.12) with strict inequality. This procedure assigns the weight m_{ij}/ε_i to the edge connecting vertex j to i and calculates the gain Γ_c for each cycle $c = 1, \ldots, C$ by multiplying the weights along the cycle. It then restricts the cycle gains according to the specific topology of the graph.

When applied to the subclass of cactus graphs where *all* cycles intersect at one common vertex as in Fig. 2.5 (right), this procedure yields the condition

$$\sum_{c=1}^{C} \alpha_c \Gamma_c < 1 \quad \text{where} \quad \alpha_c = \begin{cases} 1 & \text{if } \Gamma_c > 0 \\ -\cos^{n_c}(\pi/n_c) & \text{if } \Gamma_c < 0 \end{cases} \tag{2.20}$$

and n_c is the number of edges on cycle c. For a single cycle ($C = 1$) with negative gain $\Gamma < 0$ and N edges, (2.20) becomes

$$\alpha\Gamma = |\Gamma| \cos^N(\pi/N) < 1,$$

thus recovering the strict form of the secant condition.

[1] Simple cycles are cycles with no repeated vertices other than the starting and ending vertexes.

Although the feasibility of (2.12) with diagonal $P > 0$ can be checked numerically, algebraic conditions like (2.20) that explicitly display the range of feasibility are beneficial when the parameters exhibit wide uncertainty, as in typical biological models. Such conditions further give insight into the interplay between network structure and stability properties.

References

1. Moylan, P., Hill, D.: Stability criteria for large-scale systems. IEEE Trans. Autom. Control **23**(2), 143–149 (1978)
2. Vidyasagar, M.: Input-Output Analysis of Large Scale Interconnected Systems. Springer, Berlin (1981)
3. Grant, M., Boyd, S.: CVX: Matlab software for disciplined convex programming. http://cvxr.com/cvx (2014)
4. Löfberg, J.: Yalmip: a toolbox for modeling and optimization in MATLAB. In: Proceedings of the CACSD Conference, Taipei, Taiwan (2004)
5. Arcak, M., Sontag, E.: Diagonal stability of a class of cyclic systems and its connection with the secant criterion. Automatica **42**(9), 1531–1537 (2006)
6. Arcak, M.: Diagonal stability on cactus graphs and application tonetwork stability analysis. IEEE Trans. Autom. Control **56**(12), 2766–2777 (2011). doi:10.1109/TAC.2011.2125130

http://www.springer.com/978-3-319-29927-3

Networks of Dissipative Systems
Compositional Certification of Stability, Performance, and Safety
Arcak, M.; Meissen, C.; Packard, A.
2016, X, 102 p. 19 illus., 6 illus. in color. With online files/update., Softcover
ISBN: 978-3-319-29927-3