Chapter 2
Notational Conventions

For convenience of the reader we now summarize most of our notational conventions used throughout this manuscript.

We find it convenient to employ the abbreviations, $\mathbb{N}_{\geqslant k} := \mathbb{N} \cap [k, \infty)$, $k \in \mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{C}_{\mathrm{Re}>a} := \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > a\}$, $a \in \mathbb{R}$.

The identity matrix in $\mathbb{C}^r$ will be denoted by I_r, $r \in \mathbb{N}$.

Let $\mathscr{H}$ be a separable complex Hilbert space, $(\cdot\,,\cdot)_{\mathscr{H}}$ the scalar product in $\mathscr{H}$ (linear in the second factor), and $I_{\mathscr{H}}$ the identity operator in $\mathscr{H}$.

Next, let T be a linear operator mapping (a subspace of) a Banach space into another, with $\mathrm{dom}(T)$, $\ker(T)$, and $\mathrm{ran}(T)$ denoting the domain, kernel (i.e., null space), and range of T. The spectrum and resolvent set of a closed linear operator in $\mathscr{H}$ will be denoted by $\sigma(\cdot)$ and $\varrho(\cdot)$. For resolvents of closed operators T acting on $\mathrm{dom}(T) \subseteq \mathscr{H}$, we will frequently write $(T - z)^{-1}$ rather than the precise $(T - zI_{\mathscr{H}})^{-1}$, $z \in \varrho(T)$.

The Banach spaces of bounded and compact linear operators in $\mathscr{H}$ are denoted by $\mathscr{B}(\mathscr{H})$ and $\mathscr{B}_\infty(\mathscr{H})$, respectively. The Schatten–von Neumann ideals of compact linear operators on $\mathscr{H}$ corresponding to ℓ^p-summable singular values will be denoted by $\mathscr{B}_p(\mathscr{H})$ or, if the Hilbert space under consideration is clear from the context (and, especially, for brevity in connection with proofs) just by $\mathscr{B}_p$, $p \in [1, \infty)$. The norms on the respective spaces will be noted by $\|T\|_{\mathscr{B}_p(\mathscr{H})}$ for $T \in \mathscr{B}_p(\mathscr{H})$, $p \in [1, \infty)$, and for ease of notation we will occasionally identify $\|T\|_{\mathscr{B}(\mathscr{H})}$ with $\|T\|_{\mathscr{B}_\infty(\mathscr{H})}$ for $T \in \mathscr{B}(\mathscr{H})$, but caution the reader that it is the set of compact operators on $\mathscr{H}$ that is denoted by $\mathscr{B}_\infty(\mathscr{H})$. Similarly, $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ and $\mathscr{B}_\infty(\mathscr{H}_1, \mathscr{H}_2)$ will be used for bounded and compact operators between two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$. Moreover, $\mathscr{X}_1 \hookrightarrow \mathscr{X}_2$ denotes the continuous embedding of the Banach space $\mathscr{X}_1$ into the Banach space $\mathscr{X}_2$. Throughout this manuscript, if $\mathscr{X}$ denotes a Banach space, $\mathscr{X}^*$ denotes the *adjoint space* of continuous conjugate linear functionals on $\mathscr{X}$, that is, the *conjugate dual space* of $\mathscr{X}$ (rather than the usual dual space of continuous linear functionals on $\mathscr{X}$). This avoids the well-known awkward distinction between adjoint operators in Banach and Hilbert

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spaces (cf., e.g., the pertinent discussion in [39, p. 3–4]). In connection with bounded linear functionals on $\mathscr{X}$ we will employ the usual bracket notation $\langle \cdot\,,\cdot \rangle_{\mathscr{X}^*,\mathscr{X}}$ for pairings between elements of $\mathscr{X}^*$ and $\mathscr{X}$.

Whenever estimating the operator norm or a particular trace ideal norm of a finite product of operators, $A_1 A_2 \cdots A_N$, with $A_j \in \mathscr{B}(\mathscr{H})$, $j \in \{1,\dots,N\}$, $N \in \mathbb{N}$, we will simplify notation and write

$$\prod_{j=1}^{N} A_j, \tag{2.1}$$

disregarding any noncommutativity issues of the operators A_j, $j \in \{1,\dots,N\}$. This is of course permitted due to standard ideal properties and the associated (noncommutative) Hölder-type inequalities (see, e.g., [55, Sect. III.7], [92, Ch. 2]). The same convention will be applied if operators mapping between several Hilbert spaces are involved.

We use the *commutator* symbol

$$[A,B] := AB - BA \tag{2.2}$$

for suitable operators A, B. For unbounded A and B the natural domain of $[A,B]$ is the intersection of the respective domains of AB and BA. In particular, $[A,B]$ is not closed in general. However, in the situations we are confronted with, we shall always be in the situation that $[A,B]$ is densely defined and bounded, in particular, it is closable with bounded closure. As this is always the case, we shall—in order to reduce a clumsy notation as much as possible—typically omit the closure bar (i.e., we use $[A,B]$ rather than $\overline{[A,B]}$). In fact, most of the operators under consideration can be extended to suitable distribution spaces, such that seemingly formal computations can be justified in the appropriate distribution space.

w-lim and s-lim denote weak and strong limits in $\mathscr{H}$ as well as limits in the weak and strong operator topology for operators in $\mathscr{B}(\mathscr{H})$, n-lim denotes the norm limit of bounded operators in $\mathscr{H}$ (i.e., in the topology of $\mathscr{B}(\mathscr{H})$).

$C_0^\infty(\mathbb{R}^n)$ denotes the space of infinitely often differentiable functions with compact support in $\mathbb{R}^n$. We typically suppress the Lebesgue measure in L^p-spaces, $L^p(\mathbb{R}^n) := L^p(\mathbb{R}^n; d^n x)$, $\|\cdot\|_{L^p(\mathbb{R}^n; d^n x)} := \|\cdot\|_p$, and similarly, $L^p(\Omega) := L^p(\Omega; d^n x)$, $\Omega \subseteq \mathbb{R}^n$, $p \in [1,\infty) \cup \{\infty\}$. To avoid too lengthy expressions, we will frequently just write I rather than the precise $I_{L^2(\mathbb{R}^n)}$, etc.

Sometimes we use the symbol $\langle \cdot\,,\cdot \rangle_{L^2(\mathbb{R}^n)}$ (or, for brevity, especially in proofs, simply $\langle \cdot\,,\cdot \rangle$), to indicate the fact that the scalar product $(\cdot\,,\cdot)_{L^2(\mathbb{R}^n)}$ in $L^2(\mathbb{R}^n)$ has been continuously extended to the pairing on the entire Sobolev scale, that is, we abbreviate $\langle \cdot\,,\cdot \rangle_{L^2(\mathbb{R}^n)} := \langle \cdot\,,\cdot \rangle_{H^{-s}(\mathbb{R}^n), H^s(\mathbb{R}^n)}$, $s \geqslant 0$.

The unit sphere in $\mathbb{R}^n$ is denoted by $S^{n-1} := \{x \in \mathbb{R}^n \,|\, |x| = 1\}$, with $d^{n-1}\sigma(\cdot)$ representing the surface measure on S^{n-1}, $n \in \mathbb{N}_{\geqslant 2}$. The open ball in $\mathbb{R}^n$ centered at $x_0 \in \mathbb{R}^n$ of radius $r_0 > 0$ is denoted by $B(x_0, r_0)$.

Since various matrix structures and tensor products are naturally associated with the Dirac-type operators studied in this manuscript, we had to simplify the notation in several respects to avoid entirely unmanageably long expressions. For example, given $d, \hat{n} \in \mathbb{N}$, spaces such as $L^2(\mathbb{R}^n) \otimes \mathbb{C}^d$, $L^2(\mathbb{R}^n) \otimes \mathbb{C}^{2^{\hat{n}}}$, and $L^2(\mathbb{R}^n) \otimes \mathbb{C}^{2^{\hat{n}}} \otimes \mathbb{C}^d$ (and analogously for Sobolev spaces) will simply be denoted by $L^2(\mathbb{R}^n)^d$, $L^2(\mathbb{R}^n)^{2^{\hat{n}}}$, and $L^2(\mathbb{R}^n)^{2^{\hat{n}}d}$, respectively.

In addition, given a $d \times d$ matrix $\Phi : \mathbb{R}^n \to \mathbb{C}^{d \times d}$ with entries given by bounded measurable functions, and given an element $\psi \otimes \phi \in L^2(\mathbb{R}^n)^{2^{\hat{n}}} \otimes \mathbb{C}^d$, we will frequently adhere to a slight abuse of notation and employ the symbol Φ also in the context of the operation

$$\Phi : \psi \otimes \phi \mapsto (x \mapsto \psi(x) \otimes \Phi(x)\phi)\,, \tag{2.3}$$

and accordingly then shorten this even further to

$$\Phi : \psi\, \phi \mapsto (x \mapsto \psi(x)\Phi(x)\phi)\,, \tag{2.4}$$

Moreover, in connection with constant, invertible $m \times m$ matrices $\alpha \in \mathbb{C}^{m \times m}$ and scalar differential expressions such as ∂_j, Δ, etc., we will use the notation

$$\alpha\partial_j = \partial_j\alpha, \quad \alpha\Delta = \Delta\alpha \tag{2.5}$$

(with equality of domains) when applying these differential expressions to sufficiently regular functions of the type $\eta(\cdot) \otimes c$, $c \in \mathbb{C}^m$, abbreviated again by $\eta(\cdot)\, c$.

In the context of matrix-valued operators we also agree to use the following notational conventions: Given a scalar function f on $\mathbb{R}^n$, or a scalar linear operator R in $L^2(\mathbb{R}^n)$, we will frequently identify f or R with the diagonal matrices $f\, I_m$ or $R\, I_m$ in $L^2(\mathbb{R}^n)^{m \times m}$ for appropriate $m \in \mathbb{N}$.

Remark 2.1 We will identify a function Φ with its corresponding multiplication operator of multiplying by this function in a suitable function space. In doing so, for a differential operator $\mathcal{Q}$, we will distinguish between the expression $\mathcal{Q}\Phi$ and $(\mathcal{Q}\Phi)$ and, similarly, for other differential operators. Namely, $\mathcal{Q}\Phi$ denotes the *composition* of the two operators $\mathcal{Q}$ and Φ, whereas $(\mathcal{Q}\Phi)$ denotes the *multiplication operator* of multiplying by the function $x \mapsto (\mathcal{Q}\Phi)(x)$. $\diamond$

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