# Continuum Mechanics

**Marcelo A. Savi**

**Abstract** This chapter presents an introduction of the fundamentals of continuum mechanics. It starts with a revision of tensor analysis that discusses the definition of tensor and coordinate transformations. In the sequence, continuum motion is treated discussing the kinematics or the geometry of motion. Definitions of strain tensors are of concern. Material derivative and Reynolds transport theorem is also treated. Afterward, a discussion about stress is presented presenting the Cauchy principle. The definition of stress tensors is established presenting Cauchy and Piola-Kirchhoff tensors. Conservation principles are then analyzed: linear and angular momentum; mass; and energy. The principle of entropy is also treated. After these definitions, it is presented a summary of fundamental equations of mechanics, discussing the importance of constitutive equations. The generalized standard material approach is discussed as a framework to elaborate constitutive equations that respect the thermodynamical principles. As examples, it is discussed the elasticity, elastoplasticity, and also smart materials phenomena as piezoelectricity, pseudoelasticity, and shape memory effect.

**Keywords** Continuum mechanics • Tensor analysis • Indicial notation • Thermodynamics • Conservation principle • Constitutive models • Elasticity

## 1 Introduction

Mechanics is the science that treats motions and forces, establishing the relations between them. In brief, it is possible to imagine that a body is subjected to external effects that can arise from different sources as forces, movements, interactions with other bodies, gravitational forces, chemical interactions, electromagnetic effects, thermal changes, among other possibilities.

M.A. Savi, Ph.D. (✉)
COPPE—Department of Mechanical Engineering, Center for Nonlinear Mechanics, Universidade Federal do Rio de Janeiro, P.O. Box 68.503, Rio de Janeiro, RJ 21.941.972, Brazil
e-mail: savi@mecanica.ufrj.br



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Molecules and atoms compose the matter structure and the description of the interactions among them can define the mechanical description of motions and forces. Although appropriate, this point of view has the inconvenience of the huge number of equations to be treated. An alternative approach is to discard the atomistic structure of the matter, representing the physical phenomena by using a macroscopic point of view. This is the main idea of continuum mechanics that is limited to situations where the smallest characteristic length is much larger than the size of an atom.

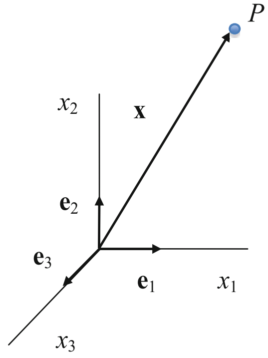
The study of continuum mechanics implies the use of tensor quantities and, because of that, it is important to have a background in tensor analysis. Continuum mechanics can be presented by introducing motion, treating the geometry of the movement, and forces that causes this motion. The conservation principles are the essential part of the mechanics defining the laws of nature. The mechanical problem is a well-posed system if constitutive equations are stated. They are built upon the main features of material behavior, establishing a connection among mechanical quantities based on experimental macroscopic observations.

This chapter presents a general overview of the fundamentals of continuum mechanics. The following references are employed: Borisneko and Tarapov (1968), Crandall et al. (1978), Currie (1974), Ertuk and Inman (2011), Eringen (1967), Fung (1965, 1969), Germain (1962), Gurtin (1981), Malvern (1969), Mase (1970), Reddy (2013), Shames (1992), Sokolnikoff (1956, 1964), Timoshenko and Goodier (1970), Valanis (1972), Ziegler (1977). Initially, it presents a brief overview of tensor analysis, presenting the index notation. The basic notion of motion is then presented, introducing the idea of deformation and strain tensors, material derivative and Reynolds transport theorem. Afterward, the influence of external forces is discussed introducing the concept of stress, presenting different representations. The conservation principles of mechanics are then discussed: linear and angular momenta, mass and energy conservations. The entropy principle is also discussed presenting the second law of thermodynamics. The necessity of the use of constitutive equations is presented and an approach to obtain these equations is shown. Some examples of constitutive models are treated: elasticity, elastoplasticity, piezoelectricity, pseudoelasticity and shape memory effect.

## 2 Tensor Analysis

Physical entities have different aspects and their mathematical representation needs to reflect their main characteristics. In this regard, an observation of some common mechanical systems allows one to identify scalar and vector quantities. Mass and temperature are typical scalar quantities while force and velocity are typical vector quantities. Observing carefully, it is possible to find other quantities that need a more complex representation. A generalization of physical quantities representation involves the definition of tensors. This generalization defines scalars as zero-order tensors, an entity that needs $1 = 3^0$ components to be represented; vectors are

**Fig. 1** Cartesian frame




first-order tensors, an entity that needs $3 = 3^1$ components to be represented; and so on. Hence, an $N$-order tensor is an entity that needs $3^N$ components to be represented. Tensor may be understood as a mathematical entity that represents all kinds of physical quantities. In mathematical point of view, its definition is related to the algebra that represents a generalization of scalar and vector algebra.

The nature description usually needs the definition of a coordinate system in a chosen frame. Besides, nature laws should be independent of the choice of a coordinate system. Cartesian coordinate frame is probably the most common reference frame, being close related to our physical intuition. The main idea is to represent a point in an $N$-dimensional space by a set of $N$ numbers. In the usual 3D (3-dimensional) space, the representation corresponds to $(x_1, x_2, x_3)$. This representation is similar to consider a position vector, $\mathbf{x}$, that can be described by using a Cartesian basis (Fig. 1):

$$\mathbf{x} = (x_1, x_2, x_3) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \tag{1}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the basis vectors.

This equation suggests different ways to represent a tensor quantity. Symbolic representation is related to symbols that describe the tensor and all its operations. In this example, $\mathbf{x}$ is a symbolic representation of the vector. On the other hand, it is possible to represent the vector by its components, $x_i$. In this case, it is implicit that index $i$ varies from 2.1 to 2.3.

Index notation establishes a compact way to deal with tensor calculus. Summation convention is a usual way to facilitate the representation of all tensor operations. Essentially, this convention establishes that the repetition of an index denotes a summation with respect to that index over its range (1, 2, 3 in 3D space). An index that is summed is called *dummy* index. The one that is not summed is called *free* index. Under this assumption, the vector representation is the following:

$$\mathbf{x} \equiv x_i\mathbf{e}_i \tag{2}$$

Note that this is equivalent to: $x_i\mathbf{e}_i = \sum_{i=1}^{3} x_i\mathbf{e}_i$.

Since a dummy index just indicates summation, it does not matter what symbol is used. Therefore,

$$\mathbf{x} \equiv x_i \mathbf{e}_i = x_k \mathbf{e}_k = x_j \mathbf{e}_j \tag{3}$$

An important point should be highlighted in terms of summation convention. It is not possible to use more than two dummy indexes since it implies an inconsistent representation.

An $N$-order tensor can be represented as $\mathbf{A}$ in symbolic notation or as follows, using $N$ indexes:

$$\mathbf{A} \equiv A_{ijkl\ldots} \tag{4}$$

It should be pointed out that a tensor is an abstract object whose properties are independent of reference frame used to describe the object. A tensor is represented by its components and therefore, there is a transformation law that connects different frames.

All tensors operations can be represented by the use of index notation. Nevertheless, it is important to define some tensors that help this representation.

## *2.1 Kronecker Delta Tensor*

Kronecker delta is a second-order tensor equivalent to the identity matrix, being defined as follows:

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{5}$$

An important characteristic of the Kronecker delta is to represent the scalar product of a Cartesian basis vectors:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{6}$$

In this regard, observe that the scalar product between two vectors is given by:

$$\mathbf{u} \cdot \mathbf{v} = u_i \mathbf{e}_i \cdot v_j \mathbf{e}_j = u_i v_j \left( \mathbf{e}_i \cdot \mathbf{e}_j \right) = u_i v_j \delta_{ij} = u_i v_i \tag{7}$$

Here it is important to highlight a special characteristic of the Kronecker delta tensor—the index change. Since when $i \neq j$ its value vanishes, it is possible to neglect all possibilities different from $i = j$ in the summation. This is equivalent to change the index.

## 2.2 Permutation Tensor

Permutation tensor is a third-order tensor defined as follows:

$$\xi_{ijk} = \begin{cases} 0, & \text{if there are any equal indexes } (112,\ 121,\ 233,\ \ldots) \\ +1, & \text{for even permutation } (123,\ 312,\ 231,\ \ldots) \\ -1, & \text{for odd permutation } (132,\ 321,\ 213,\ \ldots) \end{cases} \quad (8)$$

An important characteristic of the permutation tensor is to represent the vector product of Cartesian basis vectors:

$$\mathbf{e}_i \times \mathbf{e}_j = \xi_{ijk}\mathbf{e}_k \quad (9)$$

In this regard, observe that the vector product between two vectors is given by:

$$\mathbf{u} \times \mathbf{v} = u_i\mathbf{e}_i \times v_j\mathbf{e}_j = u_i v_j\left(\mathbf{e}_i \times \mathbf{e}_j\right) = \xi_{ijk}u_i v_j\mathbf{e}_k \quad (10)$$

The $\xi$–$\delta$ identity establishes a relationship between the permutation symbol and the Kronecker delta:

$$\xi_{miq}\xi_{jkq} = \delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij} \quad (11)$$

## 2.3 Coordinate Transformations

Since a physical quantity, as a velocity, is an intrinsic property of the body, it needs to present an invariance related to reference frame. Nevertheless, its representation is dependent of this frame. Therefore, it is important to map the variation between them, defining a proper relationship. In this regard, consider a vector quantity represented by $\mathbf{v} = v_i$. Two different frames are employed to describe this vector: original, $X_i$, and new, $x_i$. Figure 2 shows this situation presenting a vector $\mathbf{v}$ and two reference frames.

The representation of the vector can be done as follows:

$$\mathbf{v} \equiv V_i\mathbf{E}_i \ \ (\text{original frame}) \quad (12)$$

$$\mathbf{v} \equiv \mathrm{v}_i\mathbf{e}_i \ \ (\text{new frame}) \quad (13)$$

Since the vector is the same, it is possible to write

$$\mathbf{v} \equiv V_i\mathbf{E}_i = \mathrm{v}_i\mathbf{e}_i \quad (14)$$

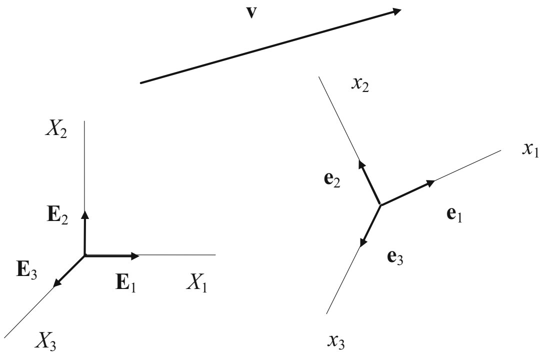



**Fig. 2** Vector representation in different frames

Performing a scalar product with $\mathbf{E}_j$:

$$\mathrm{V}_i\mathbf{E}_i \cdot \mathbf{E}_j = \mathrm{v}_i\mathbf{e}_i \cdot \mathbf{E}_j \tag{15}$$

Since the following expressions are valid,

$$\mathbf{E}_i \cdot \mathbf{E}_j = \delta_{ij} \tag{16}$$

$$\mathbf{e}_i \cdot \mathbf{E}_j = Q_{ij} = \cos\left(\mathbf{e}_i, \mathbf{E}_j\right) \tag{17}$$

the transformation from the new to the original frame is given by

$$V_j = Q_{ij}v_i \tag{18}$$

The inverse transformation, from the original to the new frame, can be obtained in an analogous way by performing the scalar product of $\mathbf{e}_j$:

$$\mathrm{V}_i\mathbf{E}_i \cdot \mathbf{e}_j = \mathrm{v}_i\mathbf{e}_i \cdot \mathbf{e}_j \tag{19}$$

since,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{20}$$

$$\mathbf{E}_i \cdot \mathbf{e}_j = \cos\left(\mathbf{E}_i, \mathbf{e}_j\right) = Q_{ji} \tag{21}$$

the transformation is given by

$$v_j = Q_{ji}V_i \tag{22}$$

Note that, transformation matrices define both operations, being formed by the angles between both frames. Since orthogonal systems are adopted, the inverse is related to the transpose of the transformation matrices. Figure 3 illustrates the transformation between two reference frames.

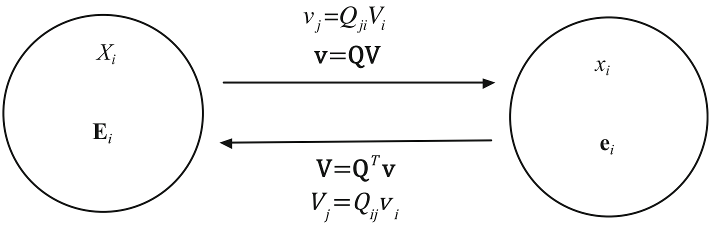



**Fig. 3** Transformation between two reference frames

Similar transformations can be defined for higher order tensors. By considering that a second-order tensor is built from vectors, it is possible to write:

$$C_{ij} = A_iB_j = a_ib_j \tag{23}$$

Since,

$$\begin{aligned} A_i &= Q_{ki}a_i \\ B_j &= Q_{mj}b_m \end{aligned} \tag{24}$$

the transformation of the second-order tensor is then given by:

$$C_{ij} = (Q_{ki}a_k)\big(Q_{mj}b_m\big) = Q_{ki}Q_{mj}(a_kb_m) = Q_{ki}Q_{mj}c_{km} \tag{25}$$

The generalization for a $N$-order tensor is automatic:

$$C_{ijk\ldots} = Q_{mj}Q_{nj}Q_{ok}\ldots c_{mno\ldots} \tag{26}$$

## 3 Motion

The kinematics analysis is related to the geometry of the motion being an essential part of the mechanical modeling. In this regard, consider a continuum body that evolves from an original, initial, or reference configuration to new positions due to the action of some external stimulus. This stimulus does not matter in the geometrical analysis. In order to map the continuum evolution, it is necessary to establish the relationship between its initial and subsequent states. Two frames are considered for this aim (Fig. 4): original or initial and deformed configurations.

In general, the motion can be split into rigid body (translation and rotation) and local strain that represent the relative motion. In order to map the body evolution, consider the position of two points in the original configuration at $t_0$, $A$ and $B$, that evolves to the deformed configuration at instant $t_1$, being represented by $A'$ and $B'$. It should be pointed out that reference frame has an important aspect in the

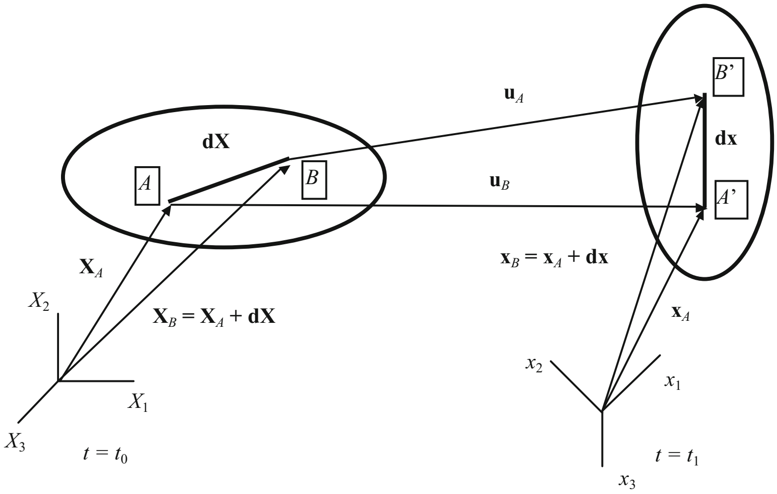



**Fig. 4** Continuum motion

description. It is possible to use either original ($X_i$) or deformed ($x_i$) frames to describe each quantity involved. Besides, two descriptions are possible: *material* or *Lagrangian* description; and *spatial* or *Eulerian* description.

*Material* or *Lagrangian* description is essentially based on material points and therefore assumes that initial state is known. The idea is to map a general position of a specific material point from its initial position:

$$x_i = x_i(X_i, t) \tag{27}$$

On the other hand, *spatial* or *Eulerian* description is essentially based on a specific location. Hence, a position is known, and one needs to map the initial configuration of this spatial point. Note that the spatial description is, in general, related to different material points at different times.

$$X_i = X_i(x_i, t) \tag{28}$$

Motion analysis maps the deformed configuration from the original one (or vice-versa) and the description of the segments $dX_i$ and $dx_i$ allows one to evaluate how the motion evolves. The evaluation of this evolution implies the definition of the *material deformation gradient*, $F_{ij}$:

$$dx_i = \frac{\partial x_i}{\partial X_j} dX_j = F_{ij} dX_j \tag{29}$$

Similar definition is established for the *spatial deformation gradient*, $H_{ij}$:

$$dX_i = \frac{\partial X_i}{\partial x_j} dx_j = H_{ij} dx_j \tag{30}$$

Note that these tensors define the mapping between the two configurations, and therefore one is the inverse of the other:

$$F_{ik} H_{kj} = \frac{\partial x_i}{\partial X_k} \frac{\partial X_k}{\partial x_j} = \delta_{ij} \tag{31}$$

The Jacobian of the transformation is defined as the determinant of the *material deformation gradient*, $F_{ij}$:

$$J \equiv \det(F_{ij}) = \xi_{ijk} \frac{\partial x_1}{\partial X_i} \frac{\partial x_2}{\partial X_j} \frac{\partial x_3}{\partial X_k} \tag{32}$$

The Jacobian establishes a relationship between original and deformed volumes.

$$dv = J\, dV \tag{33}$$

Based on these definitions, it is possible to establish a proper motion description and the definition of deformation and strain are essential. A metric should be used for this aim. Here, two possible situations are treated and each one of them can be described using either the original or the deformed frame. Another important aspect related to the motion description is the definition of the displacement vector:

$$u_i = x_i - X_i \tag{34}$$

The definitions of the displacement gradients are given by the following expressions, being respectively presented with respect to original and deformed configurations:

$$\nabla_x \mathbf{u} = \frac{\partial u_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j} - \frac{\partial X_i}{\partial x_j} = F_{ij} - \delta_{ij} \tag{35}$$

$$\nabla_x \mathbf{u} \equiv \frac{\partial u_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \frac{\partial X_i}{\partial X_j} = \delta_{ij} - H_{ij} \tag{36}$$

## *3.1 Deformation Tensors*

Deformation tensors can be defined from a specific metric. In essence, consider the vector $dX_i$ and $dx_i$ that has, respectively, the magnitudes $dS$ and $ds$. It is convenient to establish the following definitions:

$$dS^2 = dX_k dX_k = H_{ki} H_{kj} dx_i dx_j = c_{ij} dx_i dx_j \tag{37}$$

$$ds^2 = dx_k dx_k = F_{ki} F_{kj} dX_i dX_j = C_{ij} dX_i dX_j \tag{38}$$

Based on these metrics, two deformation tensors are defined:

$$C_{ij} = F_{ki} F_{kj} \quad \text{Green's deformation tensor} \tag{39}$$

$$c_{ij} = H_{ki} H_{kj} \quad \text{Cauchy's deformation tensor} \tag{40}$$

The displacement vector can be employed to rewrite these tensors as follows:

$$C_{ij} = F_{ki} F_{kj} = \left(\frac{\partial u_k}{\partial X_i} + \delta_{ki}\right)\left(\frac{\partial u_k}{\partial X_j} + \delta_{kj}\right) = \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j} + \delta_{ij} \tag{41}$$

$$c_{ij} = H_{ki} H_{kj} = \left(\delta_{ki} - \frac{\partial u_k}{\partial x_i}\right)\left(\delta_{kj} - \frac{\partial u_k}{\partial x_j}\right) = \delta_{ij} - \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} \tag{42}$$

## 3.2 Strain Tensors

In an analogous way, strain tensors can be defined by using a different metric.

$$ds^2 - dS^2 = dx_i dx_i - dX_i dX_i = \left(C_{ij} - \delta_{ij}\right) dX_i dX_j = 2E_{ij} dX_i dX_j \tag{43}$$

$$ds^2 - dS^2 = dx_i dx_i - dX_i dX_i = \left(\delta_{ij} - c_{ij}\right) dx_i dx_j = 2e_{ij} dx_i dx_j \tag{44}$$

Based on these metrics, two strain tensors are defined:

$$2E_{ij} = C_{ij} - \delta_{ij} \quad \text{Lagrange's strain tensor} \tag{45}$$

$$2e_{ij} = \delta_{ij} - c_{ij} \quad \text{Euler's deformation tensor} \tag{46}$$

Once again, the displacement vector can be employed to rewrite the strain tensors as follows:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \tag{47}$$

$$c_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \tag{48}$$

### 3.3 Infinitesimal Strain Tensors

A usual approximation in terms of strain description is the infinitesimal strains. Basically, two essential simplifications are adopted for this aim. The first one is related to the fact that both configurations (original and deformed) are the same. Hence, $X_i$ and $x_i$ are the same, being represented by $x_i$. Besides, the nonlinear terms of the strain definitions are neglected. Hence, the following definitions can be presented for the infinitesimal Lagrange-Euler tensor:

$$\hat{E}_{ij} = \hat{e}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{49}$$

While the infinitesimal Green and Cauchy's tensors are given by:

$$\hat{C}_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \delta_{ij} \tag{50}$$

$$\hat{c}_{ij} = \delta_{ij} - \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \tag{51}$$

An intuitive form of understanding the consequences of the infinitesimal strain simplification is observing that the displacement gradient, a second-order tensor, can be written as a combination of a symmetric and an anti-symmetric tensor:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \hat{E}_{ij} + \hat{\omega}_{ij} \tag{52}$$

where $\hat{\omega}_{ij}$ is an anti-symmetric tensor related to rotation. Since,

$$\frac{\partial u_j}{\partial x_i} = \hat{E}_{ij} - \hat{\omega}_{ij} \tag{53}$$

the Lagrange's strain tensor can be written in terms of the infinitesimal strains as follows:

$$E_{ij} = \frac{1}{2}\left[\left(\hat{E}_{ij} + \hat{\omega}_{ij}\right) + \left(\hat{E}_{ij} - \hat{\omega}_{ij}\right) + \left(\hat{E}_{ki} + \hat{\omega}_{ki}\right)\left(\hat{E}_{kj} + \hat{\omega}_{kj}\right)\right] \tag{54}$$

which results to,

$$E_{ij} = \hat{E}_{ij} + \frac{1}{2}\left[\hat{E}_{ki}\hat{E}_{kj} + \hat{E}_{ki}\hat{\omega}_{ki} + \hat{E}_{kj}\hat{\omega}_{ki} + \hat{\omega}_{ki}\hat{\omega}_{kj}\right] \tag{55}$$

Based on that, $E_{ij} = \hat{E}_{ij}$ if there are infinitesimal strains and rotations.

### 3.4 Principal Strains

All tensors obey the coordinate transformation defined in the previous section. Since strain tensors are second-order tensors, their transformation are represented by:

$$\varepsilon_{ij}^{'} = Q_{ik} Q_{jm} \varepsilon_{km} \tag{56}$$

where the symbol $\varepsilon_{ij}^{'}$ represents any strain or deformation tensor at a general configuration while $\varepsilon_{km}$ is the same tensor in the initial configuration. There is a special transformation that has as characteristic that the strain vector is aligned with the normal vector. This situation is investigated from the eigenvalue problem. Figure 5 shows a geometrical interpretation of an eigenvalue problem that governs this situation. Note that in 2D space, there are two possible situations for the alignment state.

$$\left(\varepsilon_{ij} - \lambda \delta_{ij}\right) n_j = 0 \tag{57}$$

The eigenvalue problem is a search for non-trivial situations, established by:

$$\det\left(\varepsilon_{ij} - \lambda \delta_{ij}\right) = 0$$

This situation establishes the characteristic polynomial:

$$\lambda^3 - I_\varepsilon \lambda^2 + II_\varepsilon \lambda - III_\varepsilon = 0 \tag{58}$$

where $I_\varepsilon$, $II_\varepsilon$, $III_\varepsilon$ are the tensor invariants that are unchanged under coordinate transformation, defined as follows:

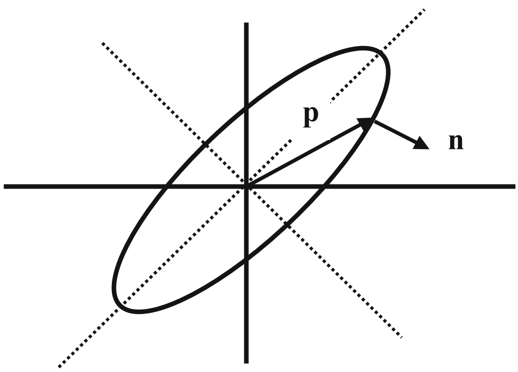



**Fig. 5** Geometrical interpretation of the eigenvalue problem

$$
\begin{aligned}
I_\varepsilon &= \varepsilon_{ii} \\
II_\varepsilon &= \frac{1}{2}\left(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ji}\right) \\
III_\sigma &= \xi_{ijk}\varepsilon_{1i}\varepsilon_{2j}\varepsilon_{3k} \equiv \det(\varepsilon)
\end{aligned}
\tag{59}
$$

## 3.5 Material Derivative and Reynolds Transport Theorem

The physical definition of time derivative is essentially related to the material or Lagrangian description since the time limit of a certain quantity is evaluated at a specific $X$, being related to the same material point. This is different to evaluate the limit at a spatial point $x$, since distinct material points are at this point at different times. In this regard, it is essential to establish a proper definition of time derivative that is called *material derivative*. Let $\varphi = \varphi(t)$ be a function of time. Its time derivative is, by definition:

$$
\frac{D\varphi}{Dt} = \left.\frac{\partial \varphi}{\partial t}\right|_x \tag{60}
$$

which is essentially related to a material description. Nevertheless, by considering spatial description, the time derivative consists of two parts: the local change and the change due to the particle motion. This can be evaluated by considering the chain rule as follows:

$$
\frac{D\varphi}{Dt} = \left.\frac{\partial \varphi}{\partial t}\right|_x + \frac{\partial \varphi}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial \varphi}{\partial t} + v_i \frac{\partial \varphi}{\partial x_i} \tag{61}
$$

Note that the first term is a *local* or *spatial derivative* while the second term is the *convective derivative*, since $v_i = \partial x_i / \partial t$ is the velocity. This term allows one to follow the particle, establishing a proper definition of the time derivative.

Consider a quantity $\Phi$ that is represented by its specific value $\varphi$ in such a way that:

$$
\Phi = \int_V \varphi \, dv \tag{62}
$$

The material derivative of this quantity is given by:

$$
\frac{D\Phi}{Dt} = \frac{D}{Dt}\left(\int_V \varphi dv\right) = \frac{D}{Dt}\left(\int_V \varphi J \, dV\right) = \int_V \left(\frac{D\varphi}{Dt} J + \frac{DJ}{Dt}\varphi\right) dV \tag{63}
$$

since $\frac{DJ}{Dt} = J\frac{\partial v_k}{\partial x_k}$ the following expression is obtained, being known as the *Reynolds transport theorem*:

$$\frac{D}{Dt}\left(\int_V \varphi dv\right) = \int_V \left(\frac{D\varphi}{Dt} + \varphi \frac{\partial v_i}{\partial x_i}\right) dv \tag{64}$$

By using the definition of material derivative, it is possible to rewrite this equation as follows:

$$\frac{D}{Dt}\left(\int_V \varphi dv\right) = \int_V \left(\frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial x_i}(\varphi v_i)\right) dv \tag{65}$$

# 4 Stress

The geometry of the continuum motion was mainly discussed until this moment. Now, a different perspective is investigated incorporating the forces that are causing this motion. Basically, contact and body forces can be imagined. Consider a continuum subjected to external forces, $\mathbf{f}^{(k)}$ (Fig. 6). It is important to consider a portion of the continuum body, defined by an arbitrary volume $V$, surrounded by an area $A$. As a consequence, forces are transmitted from one portion to another establishing an interaction between internal and external portions.

Hence, consider an arbitrary area that defines two portions. The interaction between them occurs at area $A$, defined by the unit normal vector, $n_i$. Then, a generic point $P$ of an area element $\Delta A$ of $A$ is subjected to a resultant force $\Delta f_i$. The average of the force per unit of area is given by $\Delta f_i/\Delta A$. The *Cauchy's stress principle* establishes that this average tends to a value at $P$ when the area $\Delta A$ tends to zero. Based on that, the stress vector $t_i$ at $P$ is defined as follows:

$$t_i = \lim_{\Delta A \to 0} \frac{\Delta f_i}{\Delta A} \tag{66}$$

At point $P$, there is a vector $t_i$ associated with a normal vector $n_i$, $t_i = t_i(n_i)$. Since there are an infinite number of possibilities of the normal vector, there are an infinite

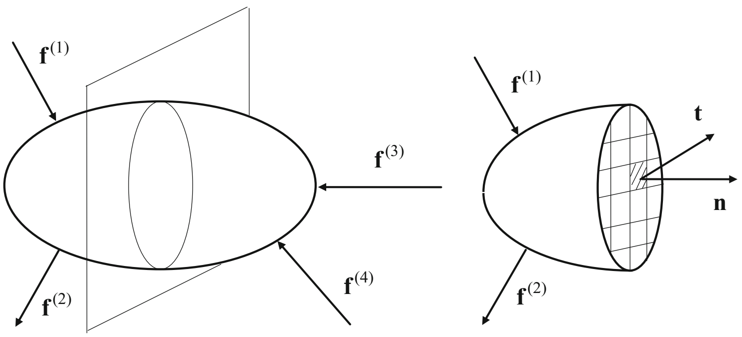



**Fig. 6** Continuum media subjected to external forces

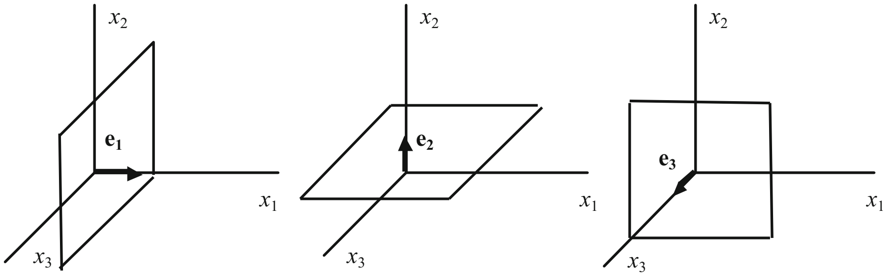



**Fig. 7** Three linear independent cuts

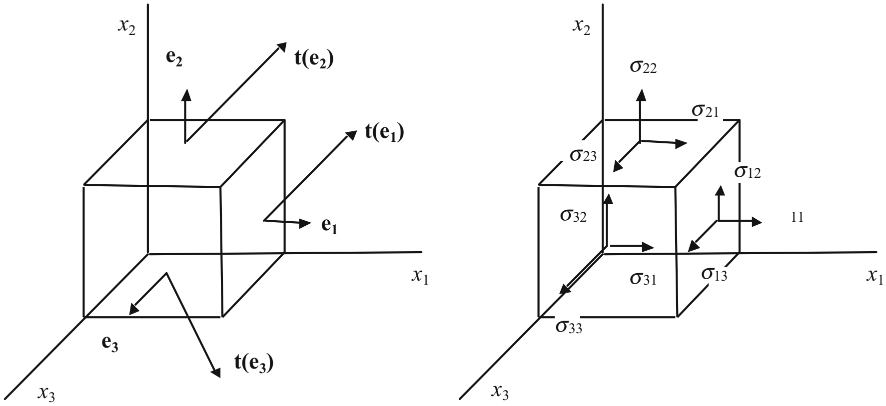



**Fig. 8** State of stress

number of stress vector at this point. The totality of possibilities defines a *stress state* that can be completely defined by three normal vectors, evaluated in linear independent directions (Fig. 7). This is equivalent to enclose the point $P$ inside a cubic element (Fig. 8). The projections of each stress vector define the *Cauchy's stress tensor*.

$$\sigma_{ij} = t_j(\mathbf{e}_i)\,\mathbf{e}_j \tag{67}$$

## *4.1 Coordinate Transformations*

Once the stress state is defined from three different stress vectors, any vector is known by considering coordinate transformations. Hence, consider an arbitrary area, expressed by a normal vector, which defines a tetrahedron that encloses the specific point where the stress state is considered (Fig. 9). By performing the summation of each direction yields to:

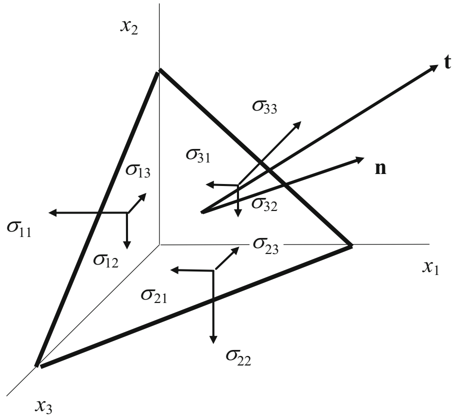



**Fig. 9** Arbitrary stress vector

$$t_i dA = \sigma_{ji} dA_j \tag{68}$$

since $dA_j = n_j dA$ it is possible to establish a relationship between the stress vector at an arbitrary area and the stress tensor, known as *Cauchy's stress formula*:

$$t_i = \sigma_{ji} n_j \tag{69}$$

## 4.2 Principal Stress

Since the stress tensor is a second-order tensor, its coordinate change is similar to the one presented for strain tensor. Therefore, similar analysis can be done in terms of principal stress that are defined by the eigenvalue problem,

$$\left(\sigma_{ij} - \lambda \delta_{ij}\right) n_j = 0 \tag{70}$$

which establishes the characteristic polynomial,

$$\lambda^3 - I_\sigma \lambda^2 + II_\sigma \lambda - III_\sigma = 0 \tag{71}$$

where $I_\sigma, II_\sigma, III_\sigma$ are the stress invariants.

## 4.3 Piola-Kirchhoff Tensors

The Cauchy's stress tensor treated until now considers that both the normal vector and the area are evaluated in the deformed configuration. This is a particular situation that can be conveniently changed when necessary. Hence, consider a

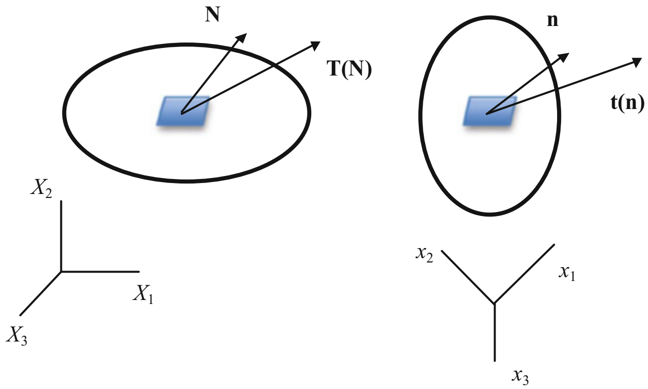



**Fig. 10** Different representations of the stress vector

force $f_i$ that can be expressed in terms of the stress vector either of the deformed configuration or of the undeformed configuration. Figure 10 shows this idea considering two different configurations and the definition of stress vectors at both configurations.

Based on that, it is possible to write:

$$f_i = t_i da = T_i dA \tag{72}$$

Each of the stress vectors can be related to stress tensors as follows:

$$t_i = \sigma_{ji} n_j \tag{73}$$

$$T_i = \hat{S}_{ji} N_j \tag{74}$$

where a new stress tensor $\hat{S_{ji}}$, known as the *first Piola-Kirchhoff tensor*, is defined from the normal vector at the undeformed configuration. On this basis, it is possible to write:

$$\sigma_{ji} n_j da = \hat{S}_{ij} N_j dA \tag{75}$$

Since the relation between areas is given by: $n_j da = J H_{kj} N_k dA$
It follows that:

$$\left(\sigma_{ji} J H_{kj} - \hat{S}_{ij} \delta_{kj}\right) N_k dA = 0 \tag{76}$$

and the *first Piola-Kirchhoff stress tensor* is given by

$$\hat{S}_{ij} = J \sigma_{ik} H_{jk} \tag{77}$$

Hence, this tensor is defined at the undeformed configuration from a force at the deformed configuration, having mixture characteristics. In order to use a force at the undeformed configuration, an extra coordinate transformation can be done, defining the *second Piola-Kirchhoff stress tensor*,

$$S_{mk} = H_{mi}\hat{S}_{ki} = JH_{mi}\sigma_{ik}H_{jk} \tag{78}$$

# 5 Conservation Principles

The conservation principles of mechanics involve laws that govern the general interaction between forces and motions, representing the nature laws. In essence, these principles are:

1. Conservation of linear momentum
2. Conservation of angular momentum
3. Conservation of mass
4. Conservation of energy

The conservation of linear momentum is the Newton's second law while the conservation of energy is the first law of thermodynamics. To these principles, it is important to add the principle of entropy, associated with the second law of thermodynamics, in order to obtain a proper description of mechanical processes. The following sections present these conservation laws.

## 5.1 Conservation of Linear Momentum

The conservation of linear momentum establishes the balance between linear momentum and external forces acting in a body represented by surface, $t_i$, and body, $b_i$, forces. The *Newton's second law* can be written as follows:

$$\frac{D}{Dt}(G_i) = T_i + B_i \tag{79}$$

where

$$G_i = \int_V \rho v_i dv \quad \text{Linear momentum}$$

$$T_i = \int_A t_i da \quad \text{Surface force}$$

$$B_i = \int_V b_i dv \quad \text{Body forces}$$

Using the integral form, the Newton's second law is written as follows:

$$\frac{D}{Dt}\int_V \rho v_i dv = \int_A t_i da + \int_V b_i dv \tag{80}$$

The divergence theorem can be evoked in order to transform the area integral into volume integral,

$$\int_A t_i da = \int_A \sigma_{ji} n_j da = \int_V \frac{\partial \sigma_{ji}}{\partial x_j} dv \tag{81}$$

Based on that, the conservation principle is given by:

$$\int_V \left[\frac{\partial \sigma_{ji}}{\partial x_j} + b_i - \rho \ddot{u}_i\right] dv = 0 \tag{82}$$

The local form of the linear momentum conservation establishes that the principle is valid for arbitrarily small neighborhood, being written as follows:

$$\frac{\partial \sigma_{ji}}{\partial x_j} + b_i - \rho \ddot{u}_i = 0 \tag{83}$$

## *5.2 Conservation of Angular Momentum*

The conservation of angular momentum establishes the balance between angular momentum and external moments acting in a body. From the second Newton's law, it is possible to write the following equation where **p** is the position vector with respect to a specific point $O$.

$$\mathbf{p} \times \frac{D}{Dt}(\mathbf{G}) = \mathbf{p} \times \mathbf{T} + \mathbf{p} \times \mathbf{B} \tag{84}$$

In this regard, it is important to define the angular momentum as follows:

$$\overline{G}_i^0 = \int_V \rho \xi_{ikl} x_k v_l \, dv \equiv \int_V \mathbf{p} \times \rho \boldsymbol{v} \, dv \quad \text{Angular momentum}$$

The balance of angular momentum and external forces are given by:

$$\int_V \xi_{ijk} x_j \rho \ddot{u}_k dv = \int_A \xi_{ijk} x_j t_k da + \int_V \xi_{ijk} x_j b_k dv \tag{85}$$

The divergence theorem is evoked in order to transform the area integral into volume integral:

$$\begin{aligned} \int_A \xi_{ijk} x_j t_k da &= \int_A \xi_{ijk} x_j \sigma_{mk} n_m da = \int_V \frac{\partial}{\partial x_m} \left( \xi_{ijk} x_j \sigma_{mk} \right) dv \\ &= \int_V \xi_{ijk} \left( \frac{\partial x_j}{\partial x_m} \sigma_{mk} + \frac{\partial \sigma_{mk}}{\partial x_m} x_j \right) dv = \int_V \xi_{ijk} \left( \sigma_{jk} + \frac{\partial \sigma_{mk}}{\partial x_m} x_j \right) dv \end{aligned} \tag{86}$$

Based on that, the conservation principle is given by:

$$\int_V \left[ \xi_{ijk} \sigma_{jk} + \xi_{ijk} x_j \left( \frac{\partial \sigma_{mk}}{\partial x_m} + b_k - \rho \ddot{u}_k \right) \right] dv = 0 \tag{87}$$

This equation contains the conservation of linear momentum. Therefore, the conservation of angular momentum establishes the balance of moments:

$$\int_V \xi_{ijk} \sigma_{jk} \, dv = 0 \tag{88}$$

The local form of the conservation of angular momentum establishes that:

$$\xi_{ijk} \sigma_{jk} = 0 \tag{89}$$

which means that the Cauchy's tensor is symmetric:

$$\sigma_{jk} = \sigma_{kj} \tag{90}$$

Note that other stress tensors cannot be considered symmetric by definition. By observing the Piola-Kirchhoff tensors, for instance, it is possible to observe that the second tensor is symmetric but the first is not.

## 5.3 Conservation of Mass

The conservation of mass establishes that the mass of a body, $m$, is unchanged during the motion. This principle may be expressed by the material derivative as follows:

$$\frac{D}{Dt}(m) = 0 \tag{91}$$

where the mass is defined from the material density:

$$m = \int_V \rho \, dv$$

Using an integral equation,

$$\frac{D}{Dt}\int_V \rho \, dv = 0 \tag{92}$$

This implies a direct application of the Reynolds transport theorem,

$$\frac{D}{Dt}\left(\int_V \rho \, dv\right) = \int_V \left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i)\right) dv = 0 \tag{93}$$

The local form of the mass conservation establishes that:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0 \tag{94}$$

## 5.4 Conservation of Energy

The energy conservation is related to the first law of thermodynamics that establishes that the rate of change of body energy needs to balance with the rate of external work and all other energies that enter or leave the body. In general, assuming that $K$ is the kinetic energy, $U$ is the internal energy, $W$ is the rate of work (power) of external forces, $Q$ is the heat flux, and $R$ is the rate of heat generation, the following balance should be established:

$$\frac{D}{Dt}(K + U) = W + Q + R \tag{95}$$

By defining the energy quantities as integral expressions,

$$K = \int_V \frac{\rho}{2} v_i v_i dv \quad \text{Kinetic energy} \tag{96}$$

$$U = \int_V \rho \vartheta \, dv \quad \text{Internal energy} \tag{97}$$

$$W = \int_A t_i v_i da + \int_V b_i v_i dv \quad \text{Power of external forces} \tag{98}$$

$$Q = -\int_A q_i n_i da \quad \text{Heat flux} \tag{99}$$

$$R = \int_V \rho r\, dv \quad \text{Heat generation} \tag{100}$$

Using these expressions on the first law of thermodynamics,

$$\frac{D}{Dt}\int_V \rho\left(\frac{v_i v_i}{2} + \vartheta\right) dv = -\int_A q_i n_i da + \int_A t_i v_i da \int_V b_i v_i dv + \int_V \rho r\, dv \tag{101}$$

The left hand of the equation implies the use of the Reynolds transport theorem:

$$\frac{D}{Dt}\int_V \rho\left(\frac{v_i v_i}{2} + \vartheta\right) dv = \int_V \left[\rho \frac{D}{Dt}\left(\frac{v_i v_i}{2} + \vartheta\right) + \left(\frac{v_i v_i}{2} + \vartheta\right)\left(\frac{D\rho}{Dt} + \rho \frac{\partial v_i}{\partial x_i}\right)\right] dv \tag{102}$$

Since the last term is related to the conservation of mass (needs to vanish), this equation is reduced to:

$$\frac{D}{Dt}\int_V \rho\left(\frac{v_i v_i}{2} + \vartheta\right) dv = \int_V \rho \frac{D}{Dt}\left(\frac{v_i v_i}{2} + \vartheta\right) dv \tag{103}$$

The area integrals need to be transformed into volume integrals with the aid of the divergence theorem. Hence,

$$-\int_A q_i n_i\, da = -\int_V \frac{\partial q_i}{\partial x_i} dv \tag{104}$$

$$\int_A t_i v_i da = \int_A \sigma_{ji} n_j v_i da = \int_V \frac{\partial\left(\sigma_{ji} v_i\right)}{\partial x_j} dv \tag{105}$$

The energy conservation is then rewritten as:

$$\int_V \left[\rho \frac{D}{Dt}\left(\frac{v_i v_i}{2} + \vartheta\right) + \frac{\partial q_i}{\partial x_i} - b_i v_i - \frac{\partial\left(\sigma_{ji} v_j\right)}{\partial x_i} - \rho r\right] dv = 0 \tag{106}$$

The local form of the conservation of energy is then given by:

$$\rho \frac{D}{Dt}\left(\frac{v_i v_i}{2} + \vartheta\right) + \frac{\partial q_i}{\partial x_i} - b_i v_i - \frac{\partial\left(\sigma_{ji} v_j\right)}{\partial x_i} - \rho r = 0 \tag{107}$$

Since,

$$\rho \frac{D}{Dt}\left(\frac{v_i v_i}{2} + \vartheta\right) = \rho \frac{D\vartheta}{Dt} + \rho\left(v_i \frac{Dv_i}{Dt}\right) \tag{108}$$

$$\frac{\partial\left(\sigma_{ji} v_j\right)}{\partial x_i} = \frac{\partial \sigma_{ji}}{\partial x_i} v_j + \sigma_{ji} \frac{\partial v_j}{\partial x_i} \tag{109}$$

the equation can be rewritten and, using the conservation of linear momentum, the new version of the local form of the energy equation is obtained:

$$\rho \frac{D\vartheta}{Dt} = -\frac{\partial q_i}{\partial x_i} + \sigma_{ij} \frac{\partial v_i}{\partial x_i} + \rho r \tag{110}$$

The second-order tensor related to the velocity gradient may be written as a combination of a symmetric and an anti-symmetric tensors:

$$\frac{\partial v_i}{\partial x_j} = D_{ij} + \Omega_{ij} = 0 \tag{111}$$

where

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{112}$$

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \tag{113}$$

Under this assumption,

$$\sigma_{ji} = \frac{\partial v_j}{\partial x_i} = \sigma_{ji} D_{ji} + \sigma_{ji} \Omega_{ji} = \sigma_{ji} D_{ji} \tag{114}$$

since $\sigma_{ji}\Omega_{ji} = 0$ due to the fact that it represents a product between a symmetric and an anti-symmetric tensors.

Hence, the conservation of energy or the first law of thermodynamics has the following form:

$$\rho \frac{D\vartheta}{Dt} = -\frac{\partial q_i}{\partial x_i} + \sigma_{ij} D_{ij} + \rho r \tag{115}$$

## 5.5 Principle of Entropy

The principle of entropy introduces the idea of the irreversibility of thermodynamical processes. This is established by the *second law of thermodynamics* with the objective to properly describe the natural processes that obey the first law, but do not occur in nature. In essence, the second law of thermodynamics states that entropy is always greater than or equal to zero. This is expressed by an expression that computes the variation of entropy and the interactions with the neighborhood. The sum of all terms should be always greater than or equal to zero:

$$\frac{D}{Dt}(S) + \Xi \geq 0 \tag{116}$$

where $S$ is the *entropy* and $\Xi$ is the *entropy input rate* that defines the interactions with the neighborhood. The description of this law implies the following definitions:

$$S = \int_V \rho s \, dv \quad \text{Entropy}$$

$$\Xi = \int_V \frac{\rho r}{T} dv - \int_A \frac{q_i}{T} n_i da \quad \text{Entropy input rate}$$

Using these expressions, an integral expression is obtained:

$$\frac{D}{Dt}\int_V \rho s \, dv - \int_A \frac{q_i}{T} n_i da + \int_V \frac{\rho r}{T} dv \geq 0 \tag{117}$$

Using the Reynolds transport theorem in the first integral together with the mass conservation:

$$\frac{D}{Dt}\int_V \rho s \, dv = \int_V \left[\frac{D(\rho s)}{Dt} + \rho s \frac{\partial v_i}{\partial x_i}\right] dv = \int_V \rho \frac{Ds}{Dt} dv \tag{118}$$

The divergence theorem is applied in the second integral, transforming area into volume integral:

$$\int_A \frac{q_i}{T} n_i da = \int_V \frac{\partial}{\partial X_i}\left(\frac{q_i}{T}\right) dv \tag{119}$$

Under these considerations, an expression for the second law of thermodynamics is obtained:

$$\int_V \left[ \rho \frac{Ds}{Dt} + \frac{\partial}{\partial x_i}\left(\frac{q_i}{T}\right) - \frac{\rho r}{T} \right] dv \geq 0 \tag{120}$$

The local form of the second law of thermodynamics is known as the *Clausius-Duhem inequality*:

$$\rho \frac{Ds}{Dt} + \frac{\partial}{\partial x_i}\left(\frac{q_i}{T}\right) - \frac{\rho r}{T} \geq 0 \tag{121}$$

By considering that:

$$\frac{\partial}{\partial x_i}\left(\frac{q_i}{T}\right) = \frac{1}{T}\frac{\partial q_i}{\partial x_i} - \frac{1}{T^2} q_i \frac{\partial T}{\partial x_i} = \frac{1}{T}\left(\frac{\partial q_i}{\partial x_i} - q_i g_i\right) \tag{122}$$

where $g_i = -\frac{1}{T}\frac{\partial T}{\partial x_i}$.

It follows that:

$$\rho T \frac{Ds}{Dt} + \frac{\partial q_i}{\partial x_i} - q_i g_i - \rho r \geq 0 \tag{123}$$

Since the first law of thermodynamics is given by:

$$\frac{\partial q_i}{\partial x_i} - \rho r = -\rho \frac{D\vartheta}{Dt} + \sigma_{ij} D_{ij} \tag{124}$$

The second law is rewritten as follows:

$$\sigma_{ij} D_{ij} + \rho\left(T\dot{s} - \dot{\vartheta}\right) - q_i g_i \geq 0 \tag{125}$$

By defining the Helmholtz free energy density, $\Psi = \vartheta - Ts$, the second law is rewritten as

$$\sigma_{ji} D_{ji} + \rho\left(\dot{\Psi} + \dot{T} s\right) - q_i g_i \geq 0 \tag{126}$$

Similar consideration can be done by the definition of the Gibbs free energy density, $\Gamma = \vartheta - \frac{1}{\rho}\sigma_{ij}\varepsilon_{ij} - \rho T$, resulting in the following form of the inequality:

$$\dot{\sigma}_{ij}\varepsilon_{ij} - \rho\left(\dot{\Gamma} + s\dot{T}\right) - q_i g_i \geq 0 \tag{127}$$

### 5.6 Summary of the Fundamental Equations

Based on the presentation of the conservation principles, it is possible to present the following summary of the essential laws of mechanics.

Conservation of linear momentum: $\frac{\partial \sigma_{ji}}{\partial x_j} + b_i - \rho \ddot{u}_i = 0$

Conservation of angular momentum: $\sigma_{jk} = \sigma_{kj}$

Conservation of mass: $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial X_i}(\rho v_i) = 0$

Conservation of energy: $\rho \frac{D\vartheta}{Dt} = -\frac{\partial q_i}{\partial x_i} + \sigma_{ij} D_{ij} + \rho r$

Second law of thermodynamics: $\sigma_{ij} D_{ij} + \rho\left(T\dot{s} - \dot{\vartheta}\right) - q_i g_i \geq 0$

The external forces that cause the motion of a continuum body are related to several sources. They can be mechanical, electrical, magnetic, among other possibilities. Therefore, it is important to understand that the conservation principles have multiphysic characteristics. In this regard, some couplings between usually independent fields are necessary for a general description. Additional conservation laws should also be necessary in these cases. The conservation of electrical charge is an illustrative example. This is of special interest in terms of smart materials that have as an essential property the coupling between different fields.

Note that the fundamental principles furnish a set of 11 equations and 1 inequality that are related to 21 unknown variables. Therefore, there is a need of six extra equations in order to have a well-posed system. This is furnished by constitutive equations that establish a connection between unknown variables of the mechanical problem.

## 6 Constitutive Equations

Constitutive equations are mathematical models that describe the main features of the material behavior, establishing a connection among mechanical quantities. In general, they are idealized models based on experimental macroscopic observations. The formulation of constitutive equations should follow some cares in order to avoid inconsistent description. *Admissibility* and *objectivity* are some special aspects that need to be observed. Admissibility establishes that constitutive equations must be consistent with fundamental principles. Objectivity defines conditions where the equations must be invariant through rigid motion of the reference frame.

The elaboration of constitutive equations should follow a proper formalism avoiding inconsistent equations that, for instances, disrespect the fundamental principles of mechanics. An interesting procedure is the framework of continuum mechanics employing the generalized standard material approach (Lemaitre and

Chaboche 1990). On this basis, the thermomechanical behavior of a continuum may be modeled from a free energy density (Helmholtz free energy, $\varPsi$, or Gibbs free energy, $\varGamma$) and the pseudo-potential of dissipation, $\varPhi$, in order to satisfy the second law of thermodynamics. A brief discussion about this procedure is now presented. Consider the Clausius-Duhem inequality, assuming that $D_{ij} = \dot{\varepsilon}_{ij}$:

$$\sigma_{ij}\dot{\varepsilon}_{ij} - \rho\left(\dot{\varPsi} + s\dot{T}\right) - q_i g_i \geq 0 \tag{128}$$

Stress and strain tensors should be energetically conjugated, meaning that their product defines energy. Hence, it is convenient to use description in the same frame.

As a first hypothesis concerning the constitutive modeling, it is assumed that the Helmholtz free energy density is a function of a finite set of variables:

$$\varPsi = \varPsi\left(\varepsilon_{ij}, T, \beta\right) \tag{129}$$

where $\beta$ represents a set of internal variables. Since $\dot{\varPsi} = \frac{\partial \varPsi}{\partial \varepsilon_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial \varPsi}{\partial T}\dot{T} + \frac{\partial \varPsi}{\partial \beta}\dot{\beta}$, the Clausius-Duhem inequality is rewritten as follows:

$$\left(\sigma_{ij} - \rho\frac{\partial \varPsi}{\partial \varepsilon_{ij}}\right)\dot{\varepsilon}_{ij} - \rho\left(s + \frac{\partial \varPsi}{\partial T}\right)\dot{T} - \rho\frac{\partial \varPsi}{\partial \beta}\dot{\beta} - q_i g_i \geq 0 \tag{130}$$

This form motivates the following definitions of the thermodynamical forces:

$$\sigma_{ij}^R = \rho\frac{\partial \varPsi}{\partial \varepsilon_{ij}};\ \ B = -\rho\frac{\partial \varPsi}{\partial \beta};\ \ s^R = -\frac{\partial \varPsi}{\partial T} \tag{131}$$

In order to describe irreversible processes, complementary laws are defined from a pseudo-potential of dissipation that is a function of internal variables:

$$\varPhi = \varPhi\left(\dot{\varepsilon}_{ij}, \dot{\beta}, \dot{T}, q_i\right) \tag{132}$$

The thermodynamical formalism establishes thermodynamics fluxes as follows (Lemaitre and Chaboche 1990):

$$\sigma_{ij}^I = \frac{\partial \varPhi}{\partial \dot{\varepsilon}_{ij}};\quad B = \frac{\partial \varPhi}{\partial \dot{\beta}};\quad s^I = -\frac{\partial \varPhi}{\partial \dot{T}};\quad g_i = -\frac{\partial \varPhi}{\partial q_i} \tag{133}$$

Alternatively, these thermodynamic fluxes may be obtained from the dual of the potential of dissipation $\varPhi^*(\sigma_{ij}^I, B, g_i)$ allowing the definitions:

$$\dot{\varepsilon}^I_{ij} = \frac{\partial \Phi^*}{\partial \sigma^I_{ij}}; \quad \dot{\beta} = \frac{\partial \Phi^*}{\partial B}; \quad q_i = -\frac{\partial \Phi^*}{\partial g_i} \tag{134}$$

where $\varepsilon^I_{ij}$ is the inelastic strain.

On this basis, a complete set of constitutive equations is defined:

$$\sigma_{ij} = \rho \frac{\partial \Psi}{\partial \varepsilon_{ij}} + \frac{\partial \Phi}{\partial \dot{\varepsilon}_{ij}} \tag{135}$$

$$B = -\rho \frac{\partial \Psi}{\partial \beta} = \frac{\partial \Phi}{\partial \dot{\beta}} \tag{136}$$

$$s = -\frac{\partial \Psi}{\partial T} - \frac{\partial \Phi}{\partial \dot{T}} \tag{137}$$

$$g_i = -\frac{\partial \Phi}{\partial q_i} \tag{138}$$

In general, if the pseudo-potential $\Phi$ is a positive convex function that vanishes at the origin, the Clausius-Duhem inequality is automatically satisfied.

The description of thermomechanical couplings must consider the energy conservation equation given by the first law of thermodynamics:

$$\rho \dot{\Psi} = \sigma_{ij} \dot{\varepsilon}_{ij} - \frac{\partial q_i}{\partial x_i} - \rho T \dot{s} - \rho \dot{T} s \tag{139}$$

By considering a single point description, spatial variations are neglected. Besides, a convection boundary condition is assumed. Therefore, the first law of thermodynamics has the following form:

$$\rho c_p \dot{T} = -h(T - T_\infty) + \sigma_{ij} \dot{\varepsilon}_{ij} + B\dot{\beta} + T\left[\frac{\partial \sigma_{ij}}{\partial T}\left(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^I_{ij}\right) - \frac{\partial B}{\partial T}\dot{\beta}\right] \tag{140}$$

where $c_p$ is the specific heat at constant pressure, $h$ is the convection coefficient, and $T_\infty$ is the environmental temperature. The first term on the equation right side is the convection term whereas the others are associated with the thermomechanical couplings.

The following sections present basic examples of constitutive equations: elasticity and elastoplasticity. Afterward, piezoelectricity and pseudoelasticity are treated showing examples of smart materials constitutive relations.

## *6.1 Elasticity*

Elastic materials are characterized by reversibility where all effects finish when the stimulus is over. In general, elasticity may have linear or nonlinear behaviors. The general linear constitutive equation for the three-dimensional media establishes that stress components are built from a linear combination of strain components. This is equivalent to consider a quadratic energy function, $\Psi = \frac{1}{2}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$, and the pseudo-potential of dissipation $\Phi$ vanishes. Therefore,

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} \tag{141}$$

or in the inverse form,

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \tag{142}$$

where $E_{ijkl}$ is the elastic tensor while $S_{ijkl}$ is the compliance tensor. They are fourth-order tensors that have 81 components. Due to symmetry reasons, it is possible to conclude that only 36 components are independent. Therefore, it is possible to rewrite the equation as follows:

$$\sigma_I = E_{IJ}\varepsilon_J \tag{143}$$

or

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yx} \\ \tau_{xx} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ E_{21} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & E_{36} \\ E_{41} & E_{42} & E_{43} & E_{44} & E_{45} & E_{46} \\ E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & E_{56} \\ E_{61} & E_{62} & E_{63} & E_{64} & E_{65} & E_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \varepsilon_{yx} \\ \varepsilon_{xx} \\ \varepsilon_{xy} \end{Bmatrix} \tag{144}$$

where the indexes are replaced as presented in Table 1.

Since this elastic matrix is symmetric, there are actually 21 independent components. This general behavior establishes that normal stress causes normal and

**Table 1** Index conversion

| *ij* | *I* |
|---|---|
| 11 | 1 |
| 22 | 2 |
| 33 | 3 |
| 23 | 4 |
| 13 | 5 |
| 12 | 6 |

shear strains. This is a typical anisotropic behavior where the material presents different properties for different directions.

By assuming that all material behaviors are the same for all directions, this general anisotropic behavior is reduced to simpler situations. The simplest case is the isotropic media where the stress–strain relation is given by:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk} \tag{145}$$

where $\mu$ and $\lambda$ are the Lamé coefficients. Note that there are only two independent coefficients. The inverse equation is given by

$$\varepsilon_{ij} = \frac{(1+\nu)}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} \tag{146}$$

where $E$ and $\nu$ are the engineering constants, together with $G$, defined as follows:

$$G = \frac{E}{2(1+\nu)} \tag{147}$$

The relation between the Lamé and engineering coefficients are given by:

$$\begin{aligned} \lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)} \\ \mu = G &= \frac{E}{2(1+\nu)} \end{aligned} \tag{148}$$

Note that, the use of elastic constitutive equations together with the fundamental principles allows one to completely describe an elastic material system.

## 6.2 *Elastoplasticity*

Elastoplastic behavior is an inelastic irreversible process promoted by the discordances movements. This kind of behavior occurs for stress levels over critical values that define the yield surface. There are several idealizations to establish elastoplastic models. Ideal plasticity is the simplest model where yield stress is the maximum limit. A more sophisticate model considers hardening effect, meaning that plastic strains influence the yield surface. The three-dimensional description usually considers an equivalent stress employed to compare the three-dimensional state with an equivalent one-dimensional case, obtained from experimental tests. A one-dimensional version is discussed here.

The elastoplastic model with kinematic and isotropic hardening can be represented by the model presented in Fig. 11. Kinematic hardening is related to

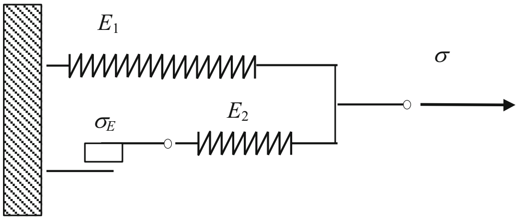



**Fig. 11** Elastoplastic model with hardening

the translation of yield surface while isotropic hardening defines the expansion of this surface due to plastic strains.

A simple one-dimensional constitutive model to describe this behavior is written considering the following variables: total strain, $\varepsilon$, and plastic strain, $\varepsilon^{\mathrm{p}}$, isotropic hardening, $\alpha$, and kinematic hardening, $\beta$. Hence, the stress–strain relationship is given by:

$$\sigma = E(\varepsilon - \varepsilon^{\mathrm{p}}) \tag{149}$$

The evolution equations are described by the following equations:

$$\dot{\varepsilon}^{\mathrm{p}} = \dot{\gamma}\,\mathrm{sign}(\sigma - \beta) \tag{150}$$

$$\dot{\alpha} = |\dot{\varepsilon}^{\mathrm{p}}| \tag{151}$$

$$\dot{\beta} = H\dot{\varepsilon}^{\mathrm{p}} \tag{152}$$

where $H$ is a parameter and $\dot{\gamma}$ represents plastic strain rate.

The yield surface is defined by the function,

$$h(\sigma, \alpha, \beta) = |\sigma - \beta| - (\sigma_E + K\alpha) \tag{153}$$

where $K$ is the plastic parameter. The irreversibility of the plastic flux is represented by the constraints,

$$\begin{aligned} &\gamma \geq 0 \\ &\gamma h(\sigma, \alpha, \beta) = 0 \\ &\gamma \dot{h}(\sigma, \alpha, \beta) = 0 \ \text{ if } \ h(\sigma, \alpha, \beta) = 0 \end{aligned} \tag{154}$$

## 6.3 *Piezoelectricity*

Piezoelectric materials have a reciprocal electro-mechanical coupling. Hence, once an electrical field is applied, the material exhibits a mechanical deformation; on the other hand, when the material undergoes a mechanical load, an electrical potential is generated.

The description of the three-dimensional behavior of piezoelectric materials involves the connection of both electrical and mechanical quantities. Hence, besides the strain, $\varepsilon_I$, and the stress $\sigma_I$, it is necessary to consider the electric displacement, $D_I$, and the applied field, $V_I$. Therefore, the 3D linear constitutive equation to describe the direct effect, where mechanical loads generates electrical field, is given by:

$$D_M = d_{MI}\sigma_I + \epsilon_{MK}V_K \tag{155}$$

The inverse effect converts electrical field into mechanical energy being described as follows:

$$\varepsilon_I = S_{IJ}\sigma_J + d_{MI}V_M \tag{156}$$

where $d_{IJ}$ is the piezoelectric coupling tensor and $\epsilon_{IJ}$ is the permittivity tensor. It is essential for a proper description to identify the poling direction, perpendicular to directions 1 and 2. On the other hand, the shear planes are indicated by the subscripts 4, 5, and 6.

### *6.4 Pseudoelasticity and Shape Memory Effect*

Shape memory alloys (SMAs) present a mechanical–temperature coupling motivated by solid phase transformations. These materials have the ability to recover a shape previously defined, when subjected to an appropriate thermomechanical loading process. Besides, they present other phenomena as pseudoelasticity.

The constitutive modeling of SMAs is very complex due to several thermomechanical phenomena involved. Among many alternatives, there is a class known as models with assumed phase transformation kinetics that are popular in the literature (Lagoudas 2008; Paiva and Savi 2006). The main idea related to these models is to consider pre-established mathematical functions to describe the phase transformation kinetics. Here, a one-dimensional version is presented. In this regard, besides strain, $\varepsilon$, and temperature, $T$, an internal variable, $\beta$, is used to represent the martensitic volume fraction. The constitutive relation between stress and state variables is considered in the rate form as follows:

$$\dot{\sigma} = E\dot{\varepsilon} - \alpha\dot{\beta} - \Omega\dot{T} \tag{157}$$

where $E$ represents the elastic modulus, $\alpha$ corresponds to the phase transformation parameter, and $\Omega$ is associated with the thermoelastic expansion. Due to martensitic transformation non-diffusive nature, the martensitic volume fraction can be expressed as function of current values of stress and temperature $\beta = \beta(\sigma, T)$. Brinson (1993) proposed a split of this volume fraction into two distinct martensitic fractions: temperature induced, $\beta_T$, and stress induced, $\beta_S$, in such a way that

$\beta = \beta_T + \beta_S$. Moreover, different elastic moduli for austenite, $E_A$, and martensite, $E_M$, are considered being given by a linear combination such that: $E(\beta) = E_A + \beta(E_M - E_A)$.

The kinetics of the Brinson's model considers that the martensitic transformation evolution is expressed by:

$$\beta_S = \frac{1-\beta_{S_0}}{2} \cos\left\{\frac{\pi}{\sigma_s^{CRIT} - \sigma_f^{CRIT}}\left[\sigma - \sigma_f^{CRIT} - C_M(T - M_s)\right]\right\} + \frac{1+\beta_{S_0}}{2}$$
$$\beta_T = \beta_{T_0} - \frac{\beta_{T_0}}{1-\beta_{S_0}}\left(\beta_S - \beta_{S_0}\right) \tag{158}$$

Both equations hold for: $\sigma_s^{CRIT} + C_M(T - M_s) < \sigma < \sigma_f^{CRIT} + C_M(T - M_s)$ and $T > M_s$.

For $T < M_s$ and $\sigma_s^{CRIT} < \sigma < \sigma_f^{CRIT}$, the martensitic transformation is given by

$$\begin{aligned} \beta_S &= \frac{1-\beta_{S_0}}{2} \cos\left[\frac{\pi}{\sigma_s^{CRIT} - \sigma_f^{CRIT}}\left(\sigma - \sigma_f^{CRIT}\right)\right] + \frac{1+\beta_{S_0}}{2} \\ \beta_T &= \beta_{T_0} - \frac{\beta_{T_0}}{1-\beta_{S_0}}\left(\beta_S - \beta_{S_0}\right) + \Delta_T \end{aligned} \tag{159}$$

where $\Delta_T = \begin{cases} \frac{1-\beta_{T_0}}{2}\{\cos\left[a_M(T - M_f)\right] + 1\} & \text{if } M_f < T < M_s \text{ and } T < T_0 \\ \text{else}: & 0 \end{cases}$.

The reverse transformation holds for $C_A(T - A_f) < \sigma < C_A(T - A_s)$ and $T > A_s$ being defined as:

$$\begin{aligned} \beta_S &= \frac{\beta_{S_0}}{2}\left\{\cos\left[a_A\left(T - A_s - \frac{\sigma}{C_A}\right)\right] + 1\right\} \\ \beta_T &= \frac{\beta_{T_0}}{2}\left\{\cos\left[a_A\left(T - A_s - \frac{\sigma}{C_A}\right)\right] + 1\right\} \end{aligned} \tag{160}$$

where $a_M$ and $a_A$ are material coefficients. $\beta_{S_0}$ and $\beta_{T_0}$ represent, respectively, the stress induced and the temperature induced martensitic volume fractions immediately before transformations begin.

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