Tangles and Connectivity in Graphs

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Abstract. This paper is a short introduction to the theory of tangles, both in graphs and general connectivity systems. An emphasis is put on the correspondence between tangles of order k and k-connected components. In particular, we prove that there is a one-to-one correspondence between the triconnected components of a graph and its tangles of order 3.

1 Introduction

Tangles, introduced by Robertson and Seymour in the tenth paper [21] of their graph minors series [20], have come to play an important part in structural graph theory. For example, Robertson and Seymour's structure theorem for graphs with excluded minors is phrased in terms of tangles in its general form [22]. Tangles have also played a role in algorithmic structural graph theory (for example in [3,7,8,11,14]).

Tangles describe highly connected regions in a graph. In a precise mathematical sense, they are "dual" to decompositions (see Theorem 23). Intuitively, a graph has a highly connected region described by a tangle if and only if it does not admit a decomposition along separators of low order. By decomposition I always mean a decomposition in a treelike fashion; formally, this is captured by the notions of tree decomposition or branch decomposition.

However, tangles describe regions of a graph in an indirect and elusive way. This is why we use the unusual term "region" instead of "subgraph" or "component". The idea is that a tangle describes a region by pointing to it. A bit more formally, a *tangle of order k* assigns a "big side" to every separation of order less than k. The big side is where the (imaginary) region described by the tangle is supposed to be. Of course this assignment of "big sides" to the separations is subject to certain consistency and nontriviality conditions, the "tangle axioms".

To understand why this way of describing a "region" is a good idea, let us review decompositions of graphs into their k-connected components. It is well known that every graph can be decomposed into its connected components and into its biconnected components. The former are the (inclusionwise) maximal connected subgraphs, and the latter the maximal 2-connected subgraphs. It is also well-known that a graph can be decomposed into its triconnected components, but the situation is more complicated here. Different from what one might guess, the triconnected components are not maximal 3-connected subgraphs; in fact they are not even subgraphs, but just topological subgraphs (see Sect. 2 for a definition of topological subgraphs). Then what about 4-connected components?

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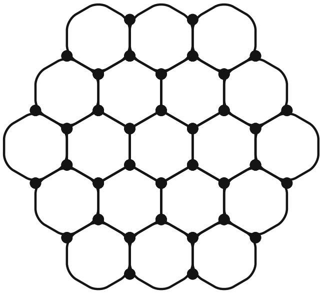

Fig. 1. A hexagonal grid

It turns out that in general a graph does not have a reasonable decomposition into 4-connected components (neither into k-connected components for any $k \geq 5$), at least if these components are supposed to be 4-connected and some kind of subgraph. To understand the difficulty, consider the hexagonal grid in Fig. 1. It is 3-connected, but not 4-connected. In fact, for any two nonadjacent vertices there is a separator of order 3 separating these two vertices. Thus it is not clear what the 4-connected components of a grid could possibly be (except, of course, just the single vertices, but this would not lead to a meaningful decomposition). But maybe we need to adjust our view on connectivity: a hexagonal grid is fairly highly connected in a "global sense". All its low-order separations are very unbalanced. In particular, all separations of order 3 have just a single vertex on one side and all other vertices on the other side. This type of global connectivity is what tangles are related to. For example, there is a unique tangle of order 4 in the hexagonal grid: the big side of a separation of order 3 is obviously the side that contains all but one vertex. The "region" this tangle describes is just the grid itself. This does not sound particularly interesting, but the grid could be a subgraph of a larger graph, and then the tangle would identify it as a highly connected region within that graph. A key theorem about tangles is that every graph admits a canonical tree decomposition into its tangles of order k [1,21]. This can be seen as a generalisation of the decomposition of a graph into its 3-connected components. A different, but related generalisation has been given in [2].

The theory of tangles and decompositions generalises from graphs to an abstract setting of *connectivity systems*. This includes nonstandard notions of connectivity on graphs, such as the "cut-rank" function, which leads to the notion of "rank width" [16,17], and connectivity functions on other structures, for example matroids. Tangles give us an abstract notion of "k-connected components" for these connectivity systems. The canonical decomposition theorem can be generalised from graphs to this abstract setting [5,13].

This paper is a short introduction to the basic theory of tangles, both for graphs and for general connectivity systems. We put a particular emphasis on the correspondence between tangles of order k and k-connected components of a graph for $k \leq 3$, which gives some evidence to the claim that for all k, tangles of order k may be viewed as a formalisation of the intuitive notion of "k-connected component".

The paper provides background material for my talk at LATA. The talk itself will be concerned with more recent results [6] and, in particular, computational aspects and applications of tangles [9–11].

2 Preliminaries

We use a standard terminology and notation (see [4] for background); let me just review a few important notions. All graphs considered in this paper are finite and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The *order* of G is $|G| := |V(G)|$. For a set $W \subseteq V(G)$, we denote the *induced subgraph* of G with vertex set W by $G[W]$ and the induced subgraph with vertex set $V(G) \setminus W$ by $G \setminus W$. The *(open) neighbourhood* of a vertex v in G is denoted by $N^G(v)$, or just $N(v)$ if G is clear from the context. For a set $W \subseteq V(G)$ we let $N(W) := \Big(\bigcup_{v \in W} N(v)\Big) \setminus W$, and for a subgraph $H \subseteq G$ we let $N(H) := N(V(H))$. The *union* of two graphs A, B is the graph $A \cup B$ with vertex set $V(A) \cup V(B)$ and edge set $E(A) \cup E(B)$, and the *intersection* $A \cap B$ is defined similarly.

A *separation* of G is a pair (A, B) of subgraphs of G such that $A \cup B = G$ and $E(A) \cap E(B) = \emptyset$. The *order* of the separation (A, B) is $\mathrm{ord}(A, B) := |V(A) \cap V(B)|$. A separation (A, B) is *proper* if $V(A) \setminus V(B)$ and $V(B) \setminus V(A)$ are both nonempty. A graph G is *k-connected* if $|G| > k$ and G has no proper $(k-1)$-separation.

A *subdivision* of G is a graph obtained from G by subdividing some (or all) of the edges, that is, replacing them by paths of length at least 2. A graph H is a *topological subgraph* of G if a subdivision of H is a subgraph of G.

3 Tangles in a Graph

In this section we introduce tangles of graphs, give a few examples, and review a few basic facts about tangles, all well-known and at least implicitly from Robertson and Seymour's fundamental paper on tangles [21] (except Theorem 7, which is due to Reed [19]).

Let G be a graph. A *G-tangle* of order k is a family $\mathcal{T}$ of separations of G satisfying the following conditions.

(GT.0) The order of all separations $(A, B) \in \mathcal{T}$ is less than k.
(GT.1) For all separations (A, B) of G of order less than k, either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$.
(GT.2) If $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ then $A_1 \cup A_2 \cup A_3 \neq G$.
(GT.3) $V(A) \neq V(G)$ for all $(A, B) \in \mathcal{T}$.

Observe that (GT.1) and (GT.2) imply that for all separations (A, B) of G of order less than k, exactly one of the separations $(A, B), (B, A)$ is in $\mathcal{T}$.

We denote the order of a tangle $\mathcal{T}$ by $\mathrm{ord}(\mathcal{T})$.

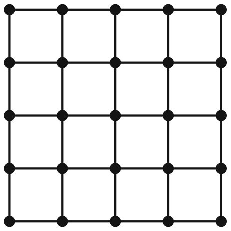

Fig. 2. A (5×5)-grid

Example 1. Let G be a graph and $C \subseteq G$ a cycle. Let $\mathcal{T}$ be the set of all separations (A, B) of G of order 1 such that $C \subseteq B$. Then $\mathcal{T}$ is a G-tangle of order 2.

To see this, note that $\mathcal{T}$ trivially satisfies (GT.0). It satisfies (GT.1), because for every separation (A, B) of G of order 1, either $C \subseteq A$ or $C \subseteq B$. To see that $\mathcal{T}$ satisfies (GT.3), let $(A_i, B_i) \in \mathcal{T}$ for $i = 1, 2, 3$. Note that it may happen that $V(A_1) \cup V(A_2) \cup V(A_3) = V(G)$ (if $|C| = 3$). However, no edge of C can be in $E(A_i)$ for any i, because $C \subseteq B_i$ and $|A_i \cap B_i| \leq 1$. Hence $E(A_1) \cup A(A_2) \cup E(A_3) \neq E(G)$, which implies (GT.2). Finally, $\mathcal{T}$ satisfies (GT.3), because $V(C) \setminus V(A) \neq \emptyset$ for all $(A, B) \in \mathcal{T}$. ⌟

Example 2. Let G be a graph and $X \subseteq V(G)$ a clique in G. Note that for all separations (A, B) of G, either $X \subseteq V(A)$ or $X \subseteq V(B)$. For every $k \geq 1$, let $\mathcal{T}_k$ be the set of all separations (A, B) of G of order less than k such that $X \subseteq V(B)$.

Then if $k < \frac{2}{3}|X| + 1$, the set $\mathcal{T}_k$ is a G-tangle of order k. We omit the proof, which is similar to the proof in the previous example.

Instead, we prove that $\mathcal{T}_k$ is not necessarily a G-tangle if $k = \frac{2}{3}|X| + 1$. To see this, let G be a complete graph of order $3n$, $k := 2n + 1$, and $X := V(G)$. Suppose for contradiction that $\mathcal{T}_k$ is a G-tangle of order k. Partition X into three sets X_1, X_2, X_3 of size n. For $i \neq j$, let $A_{ij} := G[X_i \cup X_j]$ and $B_{ij} := G$. Then (A_{ij}, B_{ij}) is a separation of G of order $2n < k$. By (GT.1) and (GT.3), we have $(A_{ij}, B_{ij}) \in \mathcal{T}_k$. However, $A_{12} \cup A_{13} \cup A_{23} = G$, and this contradicts (GT.2). ⌟

Example 3. Let G be a graph and $H \subseteq G$ a $(k \times k)$-grid (see Fig. 2). Let $\mathcal{T}$ be the set of all separations (A, B) of G of order at most $k - 1$ such that B contains some row of the grid. Then $\mathcal{T}$ is a G-tangle of order k. (See [21] for a proof.) ⌟

The reader may wonder why in (GT.2) we take three separations, instead of two or four or seventeen. The following lemma gives (some kind of) an explanation: we want our tangles to be closed under intersection, in the weak form stated as assertion (3) of the lemma; this is why taking just two separations in (GT.2) would not be good enough. Three is just enough, and as we do not want to be unnecessarily restrictive, we do not take more than three separations.

Lemma 4. *Let $\mathcal{T}$ be a G-tangle of order k.*

(1) If (A, B) *is a separation of* G *with* $|V(A)| < k$ *then* $(A, B) \in \mathcal{T}$*.*
(2) If $(A, B) \in \mathcal{T}$ *and* (A', B') *is a separation of* G *of order* $< k$ *such that* $B' \supseteq B$*, then* $(A', B') \in \mathcal{T}$*.*
(3) If $(A, B), (A', B') \in \mathcal{T}$ *and* $\mathrm{ord}(A \cup A', B \cap B') < k$ *then* $(A \cup A', B \cap B') \in \mathcal{T}$*.*

Proof. We leave the proofs of (1) and (2) to the reader. To prove (3), let $(A, B), (A', B') \in \mathcal{T}$ and $\mathrm{ord}(A \cup A', B \cap B') < k$. By (GT.1), either $(A \cup A', B \cap B') \in \mathcal{T}$ or $(B \cup B', A \cap A') \in \mathcal{T}$. As $A \cup A' \cup (B \cup B') = G$, by (GT.2) we cannot have $(B \cup B', A \cap A') \in \mathcal{T}$. □

Corollary 5. *Let* $\mathcal{T}$ *be a* G*-tangle of order* k*. Let* $(A, B), (A', B') \in \mathcal{T}$*. Then* $|B \cap B'| \geq k$*.*

The following lemma will allow us, among other things, to give an alternative characterisation of tangles in terms of so-called brambles.

Lemma 6. *Let* $\mathcal{T}$ *be a* G*-tangle of order* k*. Then for every set* $S \subseteq V(G)$ *of cardinality* $|S| < k$ *there is a unique connected component* $C(\mathcal{T}, S)$ *of* $G \setminus S$ *such that for all separations* (A, B) *of* G *with* $V(A) \cap V(B) \subseteq S$ *we have* $(A, B) \in \mathcal{T} \iff C(\mathcal{T}, S) \subseteq B$*.*

Proof. Let $C_1, \ldots, C_m$ be the set of all connected components of $G \setminus S$. For every $I \subseteq [m]$, let $C_I := \bigcup_{i \in I} C_i$. We define a separation (A_I, B_I) of G as follows. B_I is the graph with vertex set $S \cup V(C_I)$ and all edges that have at least one endvertex in $V(C_I)$, and A_I is the graph with vertex set $S \cup V(C_{[m] \setminus I})$ and edge set $E(G) \setminus E(B_I)$. Note that $V(A_I) \cap V(B_I) = S$ and thus $\mathrm{ord}(A_I, B_I) < k$. Thus for all I, either $(A_I, B_I) \in \mathcal{T}$ or $(B_I, A_I) \in \mathcal{T}$. It follows from Lemma 4(1) and (GT.2) that $(B_I, A_I) \in \mathcal{T}$ implies $(A_{[m] \setminus I}, B_{[m] \setminus I}) \in \mathcal{T}$, because $(G[S], G) \in \mathcal{T}$ and $B_I \cup B_{[m] \setminus I} \cup G[S] = G$. Furthermore, it follows from Lemma 4(3) that $(A_I, B_I), (A_J, B_J) \in \mathcal{T}$ implies $(A_{I \cap J}, B_{I \cap J}) \in \mathcal{T}$. By (GT.3) we have $(A_{[m]}, B_{[m]}) \in \mathcal{T}$ and $(A_\emptyset, B_\emptyset) \notin \mathcal{T}$.

Let $I \subseteq [m]$ be of minimum cardinality such that $(A_I, B_I) \in \mathcal{T}$. Since (A_I, B_I), $(A_J, B_J) \in \mathcal{T}$ implies $(A_{I \cap J}, B_{I \cap J}) \in \mathcal{T}$, the minimum set I is unique. If $|I| = 1$, then we let $C(\mathcal{T}, S) := C_i$ for the unique element $i \in I$. Suppose for contradiction that $|I| > 1$, and let $i \in I$. By the minimality of $|I|$ we have $(A_{\{i\}}, B_{\{i\}}) \notin \mathcal{T}$ and thus $(A_{[m] \setminus \{i\}}, B_{[m] \setminus \{i\}}) \in \mathcal{T}$. This implies $(A_{I \setminus \{i\}}, B_{I \setminus \{i\}}) \in \mathcal{T}$, contradicting the minimality of $|I|$. □

Let G be a graph. We say that subgraphs $C_1, \ldots, C_m \subseteq G$ *touch* if there is a vertex $v \in \bigcap_{i=1}^{m} V(C_i)$ or an edge $e \in E(G)$ such that each C_i contains at least one endvertex of e. A family $\mathcal{C}$ of subgraphs of G *touches pairwise* if all $C_1, C_2 \in \mathcal{C}$ touch, and it *touches triplewise* if all $C_1, C_2, C_3 \in \mathcal{C}$ touch. A *vertex cover* (or *hitting set*) for $\mathcal{C}$ is a set $S \subseteq V(G)$ such that $S \cap V(C) \neq \emptyset$ for all $C \in \mathcal{C}$.

Theorem 7 (Reed [19]). *A graph* G *has a* G*-tangle of order* k *if and only if there is a family* $\mathcal{C}$ *of connected subgraphs of* G *that touches triplewise and has no vertex cover of cardinality less than* k*.*

In fact, Reed [19] defines a tangle of a graph G to be a family $\mathcal{C}$ of connected subgraphs of G that touches triplewise and its order to be the cardinality of a minimum vertex cover. A *bramble* is a family $\mathcal{C}$ of connected subgraphs of G that touches pairwise. In this sense, a tangle is a special bramble.

Proof (of Theorem 7). For the forward direction, let $\mathcal{T}$ be a G-tangle of order k. We let

$$\mathcal{C} := \{C(\mathcal{T}, S) \mid S \subseteq V(G) \text{ with } |S| < k\}.$$

$\mathcal{C}$ has no vertex cover of cardinality less than k, because if $S \subseteq V(G)$ with $|S| < k$ then $S \cap V(C(\mathcal{T}, S)) = \emptyset$. It remains to prove that $\mathcal{C}$ touches triplewise. For $i = 1, 2, 3$, let $C_i \in \mathcal{C}$ and $S_i \subseteq V(G)$ with $|S_i| < k$ such that $C_i = C(\mathcal{T}, S_i)$. Let B_i be the graph with vertex set $V(C_i) \cup S$ and all edges of G that have at least one vertex in $V(C_i)$, and let A_i be the graph with vertex set $V(G) \setminus V(C_i)$ and the remaining edges of G. Since $C(\mathcal{T}, S_i) = C_i \subseteq B_i$, we have $(A_i, B_i) \in \mathcal{T}$. Hence $A_1 \cup A_2 \cup A_3 \neq G$ by (GT.2), and this implies that C_1, C_2, C_3 touch.

For the backward direction, let $\mathcal{C}$ be a family of connected subgraphs of G that touches triplewise and has no vertex cover of cardinality less than k. We let $\mathcal{T}$ be the set of all separations (A, B) of G of order less than k such that $C \subseteq B \setminus V(A)$ for some $C \in \mathcal{C}$. It is easy to verify that $\mathcal{T}$ is a G-tangle of order k. $\square$

Let $\mathcal{T}, \mathcal{T}'$ be κ-tangles. If $\mathcal{T}' \subseteq \mathcal{T}$, we say that $\mathcal{T}$ is an *extension* of $\mathcal{T}'$. The *truncation* of $\mathcal{T}$ to order $k \leq \mathrm{ord}(\mathcal{T})$ is the set $\{(A, B) \in \mathcal{T} \mid \mathrm{ord}(A, B) < k\}$, which is obviously a tangle of order k. Observe that if $\mathcal{T}$ is an extension of $\mathcal{T}'$, then $\mathrm{ord}(\mathcal{T}') \leq \mathrm{ord}(\mathcal{T})$, and $\mathcal{T}'$ is the truncation of $\mathcal{T}$ to order $\mathrm{ord}(\mathcal{T}')$.

4 Tangles and Components

In this section, we will show that there is a one-to-one correspondence between the tangles of order at most 3 and the connected, biconnected, and triconnected components of a graph. Robertson and Seymour [21] established a one-to-one correspondence between tangles of order 2 and biconnected component. Here, we extend the picture tangles of order 3.[1]

4.1 Biconnected and Triconnected Components

Let G be a graph. Following [2], we call a set $X \subseteq V(G)$ *k-inseparable* in G if $|X| > k$ and there is no separation (A, B) of G of order at most k such that $X \setminus V(B) \neq \emptyset$ and $X \setminus V(A) \neq \emptyset$. A *$k$-block* of G is an inclusionwise maximal k-inseparable subset of $V(G)$. We call a k-inseparable set of cardinality greater than $k + 1$ a *proper* k-inseparable set and, if it is a k-block, a *proper* k-block. (Recall that a $(k+1)$-connected graph has order greater than $k+1$ by definition.)

[1] My guess is that the result for tangles of order 3 is known to other researchers in the field, but I am not aware of it being published anywhere.

We observe that every vertex x in a proper k-inseparable set X has degree at least $(k+1)$, because it has $(k+1)$ internally disjoint paths to $X \setminus \{x\}$.

A *biconnected component* of G is a subgraph induced by a 1-block, which is usually just called a *block*.[2] It is easy to see that a biconnected component B either consists of a single edge that is a bridge of G, or it is 2-connected. In the latter case, we call B a *proper biconnected component*.

The definition of triconnected components is more complicated, because the subgraph induced by a 2-block is not necessarily 3-connected (even if it is a proper 2-block).

Example 8. Let G be a graph obtained from the complete graph K_4 by subdividing each edge once. Then the vertices of the original K_4, which are precisely the vertices of degree 3 in G, form a proper 2-block, but the subgraph they induce has no edges and thus is certainly not 3-connected.

It can be shown, however, that every proper 2-block of G is the vertex set of a 3-connected topological subgraph. For a subset $X \subseteq V(G)$, we define the *torso* of X in G to be the graph $G[\![X]\!]$ obtained from the induced subgraph $G[X]$ by adding an edge vw for all distinct $v, w \in X$ such that there is a connected component C of $G \setminus X$ with $v, w \in N(C)$. We call the edges in $E(G[\![X]\!]) \setminus E(G)$ the *virtual edges* of $G[\![X]\!]$. It is not hard to show that if X is a 2-block of G then for every connected component C of $G \setminus X$ it holds that $N(C) \leq 2$; otherwise X would not be an *inclusionwise maximal* 2-inseparable set. This implies that $G[\![X]\!]$ is a topological subgraph of G: if, for some connected component C of $G \setminus X$, $N(C) = \{v, w\}$ and hence vw is a virtual edge of the torso, then there is a path from v to w in C, which may be viewed as a subdivision of the edge vw of $G[\![X]\!]$. We call the torsos $G[\![X]\!]$ for the 2-blocks X the *triconnected components* of G. We call a triconnected component *proper* if its order is at least 4.

It is a well known fact, going back to MacLane [15] and Tutte [25], that all graphs admit tree decompositions into their biconnected and triconnected components. Hopcroft and Tarjan [12,24] proved that the decompositions can be computed in linear time.

4.2 From Components to Tangles

Lemma 9. *Let G be a graph and $X \subseteq V(G)$ a $(k-1)$-inseparable set of order $|X| > \frac{3}{2} \cdot (k-1)$. Then*

$$\mathcal{T}^{(k)}(X) := \{(A, B) \mid (A, B) \textit{separation of } G \textit{ of order } < k \textit{ with } X \subseteq V(B)\}$$

is a G-tangle of order k.

Proof. $\mathcal{T}^{(k)}(X)$ trivially satisfies (GT.0). It satisfies (GT.1), because the $(k-1)$-inseparability of X implies that for every separation (A, B) of G of order $< k$ either $X \subseteq V(A)$ or $X \subseteq V(B)$.

[2] There is a slight discrepancy to standard terminology here: a set consisting of a single isolated vertex is usually also called a block, but it is not a 1-block, because its size is not greater than 1.

To see that $\mathcal{T}^{(k)}(X)$ satisfies (GT.2), let $(A_i, B_i) \in \mathcal{T}^k(X)$ for $i = 1, 2, 3$. Then $|V(A_i) \cap X| \leq k-1$, because $V(A_i) \cap X \subseteq V(A_i) \cap V(B_i)$. As $|X| > \frac{3}{2} \cdot (k-1)$, there is a vertex $x \in X$ such that x is contained in at most one of the sets $V(A_i)$. Say, $x \notin V(A_2) \cup V(A_3)$. If $x \notin V(A_1)$, then $V(A_1) \cup V(A_2) \cup V(A_3) \neq V(G)$. So let us assume that $x \in V(A_1)$.

Let $y_1, \ldots, y_{k-1} \in X \setminus \{x\}$. As X is $(k-1)$-inseparable, for all i there is a path P_i from x to y_i such that $V(P_i) \cap V(P_j) = \{x\}$ for $i \neq j$. Let w_i be the last vertex of P_i (in the direction from x to y_i) that is in $V(A_1)$. We claim that $w_i \in V(B_1)$. This is the case if $w_i = y_i \in X \subseteq V(B_1)$. If $w_i \neq y_i$, let z_i be the successor of w_i on P_i. Then $z_i \in V(B_1) \setminus V(A_1)$, and as $w_i z_i \in E(G)$, it follows that $w_i \in V(B_1)$ as well.

Thus $\{x, w_1, \ldots, w_{k-1}\} \subseteq V(A_1) \cap V(B_1)$, and as $|V(A_1) \cap V(B_1)| \leq k-1$, it follows that $w_i = x$ for some i. Consider the edge $e = xz_i$. We have $e \notin E(A_1)$ because $z_i \notin V(A_1)$ and $e \notin E(A_2) \cup E(A_3)$ because $x \notin V(A_2) \cup V(A_3)$. Hence $E(A_1) \cup E(A_2) \cup E(A_3) \neq E(G)$, and this completes the proof of (GT.2).

Finally, $\mathcal{T}^{(k)}(X)$ satisfies (GT.3), because for every $(A, B) \in \mathcal{T}$ we have $|V(A) \cap X| \leq k - 1 < |X|$. □

Corollary 10. *Let G be a graph and $X \subseteq V(G)$.*

(1) If X is the vertex set of a connected component of G (that is, a 0-block), then $\mathcal{T}^1(X)$ is a G-tangle of order 1.
(2) If X is the vertex set of a biconnected component of G (that is, a 1-block), then $\mathcal{T}^2(X)$ is a G-tangle of order 2.
(3) If X is the vertex set of a proper triconnected component of G (that is, a 2-block of cardinality at least 4), then $\mathcal{T}^3(X)$ is a G-tangle of order 3.

Let us close this section by observing that the restriction to *proper* triconnected components in assertion (3) of the corollary is necessary.

Lemma 11. *Let G be a graph and $X \subseteq V(G)$ be a 2-block of cardinality 3. Then $\mathcal{T}^3(X)$ is not a tangle.*

Proof. Let $\mathcal{T} := \mathcal{T}^3(X)$. Suppose that $X = \{x_1, x_2, x_3\}$. For $i \neq j$, let $S_{ij} := \{x_i, x_j\}$, and let Y_{ij} be the union of the vertex sets of all connected components C of $G \setminus X$ with $N(C) \subseteq S_{ij}$, and let $Z_{ij} := V(G) \setminus (Y_{ij} \cup S_{ij})$. Let $A_{ij} := G[Y_{ij} \cup S_{ij}]$, and let B_{ij} be the graph with vertex set $S_{ij} \cup Z_{ij}$ and edge set $E(G) \setminus E(A_{ij})$. Then $(A_{ij}, B_{ij}) \in \mathcal{T}$, because $X \subseteq V(B_{ij})$. As X is a 2-block, for every connected component C of $G \setminus X$ it holds that $|N(C)| \leq 2$, and hence $C \subseteq A_{ij}$ for some i, j. It is not hard to see that this implies $A_{12} \cup A_{13} \cup A_{23} = G$. Thus $\mathcal{T}$ violates (GT.2). □

4.3 From Tangles to Components

For a G-tangle $\mathcal{T}$, we let

$$X_{\mathcal{T}} := \bigcap_{(A,B) \in \mathcal{T}} V(B).$$

In general, $X_{\mathcal{T}}$ may be empty; an example is the tangle of order k associated with a $(k \times k)$-grid for $k \geq 5$ (see Example 3). However, it turns out that for tangles of order $k \leq 3$, the set $X_{\mathcal{T}}$ is a $(k-1)$-block. This will be the main result of this section.

Lemma 12. *Let $\mathcal{T}$ be a G-tangle of order k. If $|X_{\mathcal{T}}| \geq k$, then $X_{\mathcal{T}}$ is a $(k-1)$-block of G and $\mathcal{T} = \mathcal{T}^k(X_{\mathcal{T}})$.*

Proof. Suppose that $|X_{\mathcal{T}}| \geq k$. If (A, B) is a separation of G of order less than k then either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$, which implies $X_{\mathcal{T}} \subseteq V(B)$ or $X_{\mathcal{T}} \subseteq V(A)$. Thus $X_{\mathcal{T}}$ is $(k-1)$-inseparable. If $X \supset X_{\mathcal{T}}$, say, with $x \in X \setminus X_{\mathcal{T}}$, then there is some separation $(A, B) \in \mathcal{T}$ with $x \in V(A) \setminus V(B)$ and $X_{\mathcal{T}} \subseteq V(B)$, and this implies that X is not $(k-1)$-inseparable. Hence $X_{\mathcal{T}}$ is a k-block.

We have $\mathcal{T} = \mathcal{T}^k(X_{\mathcal{T}})$, because $X_{\mathcal{T}} \subseteq V(B)$ for all $(A, B) \in \mathcal{T}$, and for a separation (A, B) of order at most $k-1$ we cannot have $X_{\mathcal{T}} \subseteq V(A) \cap V(B)$. □

Let $\mathcal{T}$ be a G-tangle. A separation $(A, B) \in \mathcal{T}$ is *minimal* in $\mathcal{T}$ if there is no $(A', B') \in \mathcal{T}$ such that $B' \subset B$. Clearly, $X_{\mathcal{T}}$ is the intersection of all sets $V(B)$ for minimal $(A, B) \in \mathcal{T}$. Hence if we want to understand $X_{\mathcal{T}}$, we can restrict our attention to the minimal separations in $\mathcal{T}$. Let $(A, B) \in \mathcal{T}$ be minimal and $S := V(A) \cap V(B)$. It follows from Lemma 6 that $B \setminus S = C := C(\mathcal{T}, S)$, and it follows from the minimality that $S = N\big(C\big)$ and that $E(B)$ consists of all edges with one endvertex in $V(C)$. Hence B is connected.

Theorem 13 (Robertson and Seymour [21]). *Let G be a graph.*

(1) For every G-tangle $\mathcal{T}$ of order 1, the set $X_{\mathcal{T}}$ is a vertex set of a connected component of G, and we have $\mathcal{T} = \mathcal{T}^1(X_{\mathcal{T}})$.
(2) For every G-tangle $\mathcal{T}$ of order 2, the set $X_{\mathcal{T}}$ is the vertex set of a biconnected component of G, and we have $\mathcal{T} = \mathcal{T}^2(X_{\mathcal{T}})$.

Proof. To prove (1), let $\mathcal{T}$ be a G-tangle of order 1. Let $C = C(\mathcal{T}, \emptyset)$. Then $(G \setminus V(C), C)$ is the unique minimal separation in $\mathcal{T}$, and thus we have $X_{\mathcal{T}} = V(C)$.

To prove (2), let $\mathcal{T}$ be a G-tangle of order 2. By Lemma 12, it suffices to prove that $|X_{\mathcal{T}}| \geq 2$. Let $\mathcal{T}'$ be the truncation of $\mathcal{T}$ to order 1. Then $W := X_{\mathcal{T}'}$ is the vertex set of a connected component C of G, and we have $X_{\mathcal{T}} \subseteq W$. Moreover, for every minimal $(A, B) \in \mathcal{T}$ we have $B \subseteq C$, because B is connected and $B \cap C \neq \emptyset$ by (GT.2).

Claim 1. Let $(A_1, B_1), (A_2, B_2) \in \mathcal{T}$ be distinct and minimal in $\mathcal{T}$. Then $A_1 \cap C \subseteq B_2$ and $A_2 \cap C \subseteq B_1$.

Proof. We have $\operatorname{ord}(A_1 \cup A_2, B_1 \cap B_2) \geq 2$, because otherwise $(A_1 \cup A_2, B_1 \cap B_2) \in \mathcal{T}$ by Lemma 4(3), which contradicts the minimality of the separations (A_i, B_i). Suppose that $V(A_i) \cap V(B_i) = \{s_i\}$. As

$$V(A_1 \cup A_2) \cap (V(B_1 \cap B_2) \subseteq V(A_1 \cap B_1) \cup V(A_2 \cap B_2) = \{s_1, s_2\},$$

we must have $s_1 \neq s_2$ and $V(A_1 \cup A_2) \cap V(B_1 \cap B_2) = \{s_1, s_2\}$ (see Fig. 3). This implies $V(A_1 \cap A_2) \cap V(B_1 \cup B_2) = \emptyset$. Then $(A_1 \cap A_2, B_1 \cup B_2)$ is a separation of G

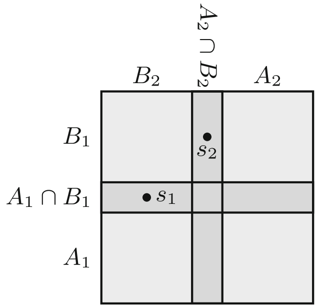

Fig. 3. Proof of Theorem 13

of order 0, and as C is connected and $(B_1 \cup B_2) \cap C \neq \emptyset$, we have $A_1 \cap A_2 \cap C = \emptyset$. The assertion of the claim follows. ⌟

Let $(A_1, B_1), \ldots, (A_m, B_m)$ be an enumeration of all minimal separations in $\mathcal{T}$ of order 1. Even if C is 1-inseparable, there is such a separation: $\big(G \setminus (V(C) \setminus \{v\}), C\big)$ for an arbitrary $v \in V(C)$. Thus $m \geq 1$. If $m = 1$, then $X_{\mathcal{T}} = V(B_1)$ and thus $|X_{\mathcal{T}}| \geq 2$ by Lemma 4(1).

If $m \geq 2$, let $A_i \cap B_i = \{s_i\}$. We can assume the s_i to be mutually distinct, because if $s_i = s_j$ then $B_i = B_j$. It follows from Claim 1 that $s_1, \ldots, s_m \in \bigcap_i V(B_i) = X_{\mathcal{T}}$. This implies $|X_{\mathcal{T}}| \geq 2$. □

To extend Theorem 13 to tangles of order 3, we first prove a lemma, which essentially says that we can restrict our attention to 2-connected graphs. Let G be graph and $X \subseteq V(G)$. For every $A \subseteq G$, let $A \cap X := A\big[V(A) \cap X\big]$. Note that if (A, B) is a separation of G, then $(A \cap X, B \cap X)$ is a separation of $G[X]$ with $\mathrm{ord}(A \cap X, B \cap X) \leq \mathrm{ord}(A, B)$.

Lemma 14. *Let $\mathcal{T}$ be a G-tangle of order 3. Let $\mathcal{T}'$ be the truncation of $\mathcal{T}$ to order 2, and let $W := X_{\mathcal{T}'}$. Let $\mathcal{T}[W]$ be the set of all separations $(A \cap W, B \cap W)$ of $G[W]$ where $(A, B) \in \mathcal{T}$. Then $\mathcal{T}[W]$ is a $G[W]$-tangle of order 3. Furthermore, $X_{\mathcal{T}} = X_{\mathcal{T}[W]}$.*

Proof. By Theorem 13, $G[W]$ is a biconnected component of G. This implies that $|W| \geq 2$ and $|N(C)| \leq 1$ for every connected component C of $G \setminus W$. For every $w \in W$, we let Y_w be union of the vertex sets of all connected components C of $G \setminus W$ with $N(C) \subseteq \{w\}$. Then $V(G) = W \cup \bigcup_{w \in W} Y_w$. Let $Z_w := V(G) \setminus (Y_w \cup \{w\})$. Let $A_w := G[Y_w \cup \{w\}]$ and $B_w := G[Z_w \cup \{w\}]$. Then $W \subseteq V(B_w)$ and thus $(A_w, B_w) \in \mathcal{T}^2(W) = \mathcal{T}' \subseteq \mathcal{T}$.

Claim 1. Let $(A, B) \in \mathcal{T}$. Then $W \setminus V(A) \neq \emptyset$.

Proof. Suppose for contradiction that $W \subseteq V(A)$. Let $S := V(A) \cap V(B)$ and suppose that $S = \{s_1, s_2\}$. Let $w_i \in W$ such that $s_i \in Y_{w_i} \cup \{w_i\}$. Then $A \cup A_{w_1} \cup A_{w_2} = G$, which contradicts (GT.2). This proves that $W \setminus V(A) \neq \emptyset$. ⌟

It is now straightforward to prove that $\mathcal{T}[W]$ satisfies the tangle axioms (GT.0), (GT.1), and (GT.3). To prove (GT.2), let $(A_i, B_i) \in \mathcal{T}$ for $i = 1, 2, 3$. We need to prove that $(A_1 \cap W) \cup (A_2 \cap W) \cup (A_3 \cap W) \neq G[W]$. Without loss of generality we may assume that (A_i, B_i) is minimal in $\mathcal{T}$. Then $C_i := B_i \setminus V(A_i)$ is connected. By Claim 1, $V(C_i) \cap W \neq \emptyset$. This implies that if $V(C_i) \cap Y_w \neq \emptyset$ for some $w \in W$, then $w \in V(C_i)$.

As $\mathcal{T}$ satisfies (GT.2), $A_1 \cup A_2 \cup A_3 \neq G$, and thus there either is a vertex in $V(C_1) \cap V(C_2) \cap V(C_3)$ or an edge with an endvertex in every $V(C_i)$. Suppose first that $v \in V(C_1) \cap V(C_2) \cap V(C_3)$. If $v \in W$ then

$$V\big((A_1 \cap W) \cup (A_2 \cap W) \cup (A_3 \cap W)\big) \neq W = V\big(G[W]\big).$$

Otherwise, $v \in Y_w$ for some $w \in W$, and we have $w \in V(C_1) \cap V(C_2) \cap V(C_3)$. Similarly, if $e = vv'$ has an endvertex in every $V(C_i)$, then we distinguish between the case that $v, v' \in W$, which implies $E\big((A_1 \cap W) \cup (A_2 \cap W) \cup (A_3 \cap W)\big) \neq E\big(G[W]\big)$, and the case that $e \in E(A_w)$ for some $w \in W$, which implies $w \in V(C_1) \cap V(C_2) \cap V(C_3)$ and thus $V\big((A_1 \cap W) \cup (A_2 \cap W) \cup (A_3 \cap W)\big) \neq W = V\big(G[W]\big)$. This proves (GT.2) and hence that $\mathcal{T}[W]$ is a tangle.

The second assertion $X_{\mathcal{T}} = X_{\mathcal{T}[W]}$ follows from the fact that $X_{\mathcal{T}} \subseteq X_{\mathcal{T}'} = W$. □

Theorem 15. *Let G be a graph. For every G-tangle $\mathcal{T}$ of order 3, the set $X_{\mathcal{T}}$ is a vertex set of a proper triconnected component of G.*

Proof. Let $\mathcal{T}$ be a G-tangle of order 3. It suffices to prove that $|X_{\mathcal{T}}| \geq 3$. Then by Lemma 12, $X_{\mathcal{T}}$ is a 3-block and $\mathcal{T} = \mathcal{T}^3(X_{\mathcal{T}})$, and by Lemma 11, $X_{\mathcal{T}}$ is proper 3-block, that is, the vertex set of a proper triconnected component.

By the previous lemma, we may assume without loss of generality that G is 2-connected. The rest of the proof follows the lines of the proof of Theorem 13. The core of the proof is again an "uncrossing argument" (this time a more complicated one) in Claim 1.

Claim 1. Let $(A_1, B_1), (A_2, B_2) \in \mathcal{T}$ be distinct and minimal in $\mathcal{T}$. Then $V(A_1) \subseteq V(B_2)$ and $V(A_2) \subseteq V(B_1)$.

Proof. Let $S_i := V(A_i) \cap V(B_i)$ and $Y_i := V(A_i) \setminus S_i$ and $Z_i := V(B_i) \setminus S_i$ (see Fig. 4(a)). By the minimality of (A_i, B_i), we have $Z_i = V\big(C(\mathcal{T}, S_i)\big)$ and $S_i = N(Z_i)$. Thus $S_1 \neq S_2$ and $Z_1 \neq Z_2$, because the two separations are distinct.

It follows that $(A_1 \cup A_2, B_1 \cap B_2)$ is a separation with $B_1 \cap B_2 \subset B_i$, and by the minimality of (A_i, B_i) this separation is not in $\mathcal{T}$. By Lemma 4(3), this means that its order is at least 3. Thus

$$|S_1 \cap Z_2| + |S_1 \cap S_2| + |Z_1 \cap S_2| = |V(A_1 \cup A_2) \cap V(B_1 \cap B_2)| \geq 3. \quad (\star)$$

As $|S_i| \leq 2$ and $S_1 \neq S_2$, it follows that

$$|S_1 \cap Y_2| + |S_1 \cap S_2| + |Y_1 \cap S_2| = |V(A_1 \cap A_2) \cap V(B_1 \cup B_2)| \leq 1.$$

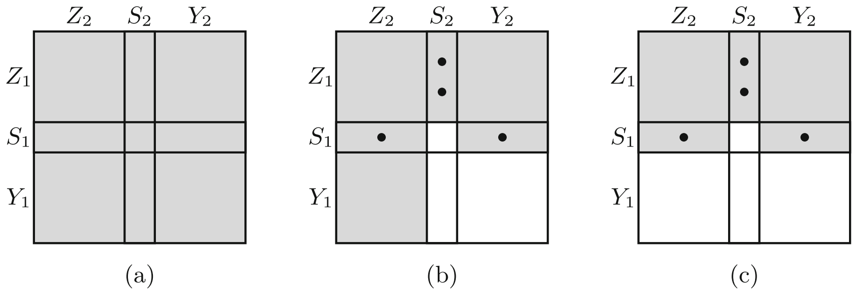

Fig. 4. Uncrossing minimal separations of order 2

Hence $(A_1 \cap A_2, B_1 \cup B_2)$ is a separation of order at most 1. As G is 2-connected, the separation is not proper, which means that either $V(A_1 \cap A_2) = V(G)$ or $V(B_1 \cup B_2) = V(G)$. By Lemma 4(2), we have $(A_1 \cap A_2, B_1 \cup B_2) \in \mathcal{T}$ and thus $V(A_1 \cap A_2) \neq V(G)$. Thus $V(B_1 \cup B_2) = V(G)$, and this implies $Y_1 \cap Y_2 = \emptyset$.

To prove that $V(A_i) = S_i \cup Y_i \subseteq V(B_{3-i}) = S_{3-i} \cup Z_{3-i}$, we still need to prove that $S_i \cap Y_{3-i} = \emptyset$. Suppose for contradiction that $S_1 \cap Y_2 \neq \emptyset$. Then $(\star)$ implies $|S_1 \cap Y_2| = 1$ and $|S_1 \cap Z_2| = 1$ and $|S_2 \cap Z_1| = 2$ and $S_1 \cap S_2 = Y_1 \cap S_2 = \emptyset$ (see Fig. 4(b)). Note that $(Y_1 \cup S_1) \cap Z_2 = V(A_1) \setminus V(A_2)$. It follows that $(A_1 \setminus V(A_2), B_1)$ is a separation of G of order 1, and we have $(A_1 \setminus V(A_2), B_1) \in \mathcal{T}$. Thus $Y_1 \cap Z_2 = \emptyset$, which implies $V(B_2) = Z_2 \cup S_2 \subset Z_1 \cup S_1 = V(B_1)$ (see Fig. 4(c)). This contradicts the minimality of (A_1, B_1). Hence $S_1 \cap Y_2 = \emptyset$, and similarly $Y_1 \cap S_2 = \emptyset$. ⌟

Let $(A_1, B_1), \ldots, (A_m, B_m)$ be an enumeration of all minimal separations in $\mathcal{T}$ of order 2. Note that there is at least one minimal separation of order 2 even if G has no proper separations of order 2. Thus $m \geq 1$.

Let $S_i := V(A_i) \cap V(B_i)$. Then the sets S_i are all distinct, because two minimal separations in $\mathcal{T}$ with the same separators are equal. It follows from Claim 1 that $S_i \subseteq V(B_j)$ for all $j \in [m]$ and thus

$$S_1 \cup \ldots \cup S_m \subseteq X_{\mathcal{T}}.$$

If $m \geq 2$ this implies $|X_{\mathcal{T}}| \geq 3$. If $m = 1$, then $X_{\mathcal{T}} = V(B_1)$ and thus $|X_{\mathcal{T}}| \geq 3$ by Lemma 4. □

The results of this section clearly do not extend beyond tangles of order 3. For example, the hexagonal grid H in Fig. 1 has a (unique) tangle $\mathcal{T}$ of order 4. But the set $X_{\mathcal{T}}$ is empty, and the graph H has no 3-inseparable set of cardinality greater than 1.

Nevertheless, it is shown in [6] that there is an extension of the theorem to tangles of order 4 if we replace 4-connectivity by the slightly weaker "quasi-4-connectivity": a graph G is *quasi-4-connected* if it is 3-connected and for all separations (A, B) of order 3, either $|V(A) \setminus V(B)| \leq 1$ or $|V(B) \setminus V(A)| \leq 1$.

For example, the hexagonal grid H in Fig. 1 is quasi-4-connected. It turns out that there is a one-to-one correspondence between the tangles of order 4 and (suitably defined) quasi-4-connected components of a graph.

5 A Broader Perspective: Tangles and Connectivity Systems

Many aspects of "connectivity" are not specific to connectivity in graphs, but can be seen in an abstract and much more general context. We describe "connectivity" on some structure as a function that assigns an "order" (a nonnegative integer) to every "separation" of the structure. We study symmetric connectivity functions, where the separations (A, B) and (B, A) have the same order. The key property such connectivity functions need to satisfy is submodularity.

Separations can usually be described as partitions of a suitable set, the "universe". For example, the separations of graphs we considered in the previous sections are essentially partitions of the edge set. Technically, it will be convenient to identify a partition $(\overline{X}, X)$ with the set X, implicitly assuming that $\overline{X}$ is the complement of X. This leads to the following definition.

A *connectivity function* on a finite set U is a symmetric and submodular function $\kappa\colon 2^U \to \mathbb{N}$ with $\kappa(\emptyset) = 0$. *Symmetric* means that $\kappa(X) = \kappa(\overline{X})$ for all $X \subseteq U$; here and whenever the ground set U is clear from the context we write $\overline{X}$ to denote $U \setminus X$. *Submodular* means that $\kappa(X)+\kappa(Y) \geq \kappa(X\cap Y)+\kappa(X\cup Y)$ for all $X, Y \subseteq U$. The pair (U, κ) is sometimes called a *connectivity system*.

The following two examples capture what is known as *edge connectivity* and *vertex connectivity* in a graph.

Example 16 (Edge connectivity). Let G be a graph. We define the function $\nu_G : 2^{V(G)} \to \mathbb{N}$ by letting $\nu_G(X)$ be the number of edges between X and $\overline{X}$. Then ν_G is a connectivity function on $V(G)$. ⌟

Example 17 (Vertex connectivity). Let G be a graph. We define the function $\kappa_G : 2^{E(G)} \to \mathbb{N}$ by letting $\kappa_G(X)$ be the number of vertices that are incident with an edge in X and an edge in $\overline{X}$. Then κ_G is a connectivity function on $E(G)$.

Note that for all separations (A, B) of G we have $\kappa_G(E(A)) = \kappa_G(E(B)) \leq \mathrm{ord}(A, B)$, with equality if $V(A) \cap V(B)$ contains no isolated vertices of A or B. For $X \subseteq E(G)$, let us denote the set of endvertices of the edges in X by $V(X)$. Then for all $X \subseteq E(G)$ we have $\kappa_G(X) = \mathrm{ord}(A_X, B_X)$, where $B_X = (V(X), X)$ and $A_X = (V(\overline{X}), \overline{X})$. The theory of tangles and decompositions of the connectivity function of κ_G is essentially the same as the theory of tangles and decompositions of G (partially developed in the previous sections). ⌟

Example 18. Let G be a graph. For all subsets $X, Y \subseteq V(G)$, we let $M = M_G(X, Y)$ be the $X \times Y$-matrix over the 2-element field $\mathbb{F}_2$ with entries $M_{xy} = 1 \iff xy \in E(G)$. Now we define a connectivity function ρ_G on $V(G)$ by letting $\rho_G(X)$, known as the *cut rank* of X, be the row rank of the matrix $M_G(X, \overline{X})$. This connectivity function was introduced by Oum and Seymour [17] to define the *rank width* of graphs, which approximates the *clique width*, but has better algorithmic properties. ⌟

Let us also give an example of a connectivity function not related to graphs.

Example 19. Let M be a matroid with ground set E and rank function r. (The rank of a set $X \subseteq E$ is defined to be the maximum size of an independent set contained in X.) The connectivity function of M is the set function $\kappa_M : E \to \mathbb{N}$ defined by $\kappa_M(X) = r(X) + r(\overline{X}) - r(E)$ (see, for example, [18]). ⌟

5.1 Tangles

Let κ be a connectivity function on a set U. A *κ-tangle* of order $k \geq 0$ is a set $\mathcal{T} \subseteq 2^U$ satisfying the following conditions.

(T.0) $\kappa(X) < k$ for all $X \in \mathcal{T}$,
(T.1) For all $X \subseteq U$ with $\kappa(X) < k$, either $X \in \mathcal{T}$ or $\overline{X} \in \mathcal{T}$.
(T.2) $X_1 \cap X_2 \cap X_3 \neq \emptyset$ for all $X_1, X_2, X_3 \in \mathcal{T}$.
(T.3) $\mathcal{T}$ does not contain any singletons, that is, $\{a\} \notin \mathcal{T}$ for all $a \in U$.

We denote the order of a κ-tangle $\mathcal{T}$ by $\operatorname{ord}(\mathcal{T})$.

We mentioned in Example 17 that the theory of κ_G-tangles is essentially the same as the theory of tangles in a graph. Indeed, κ_G-tangles and G-tangles are "almost" the same. The following proposition makes this precise.

We call an edge of a graph *isolated* if both of its endvertices have degree 1. We call an edge *pendant* if it is not isolated and has one endvertex of degree 1.

Proposition 20. *Let G be a graph and $k \geq 0$.*

(1) If $\mathcal{T}$ is a κ_G-tangle of order k, then

$$\mathcal{S} := \big\{(A,B) \bigm| (A,B) \textit{separation of } G \textit{ of order } < k \textit{ with } E(B) \in \mathcal{T}\big\}$$

is a G-tangle of order k.

(2) If $\mathcal{S}$ is a G-tangle of order k, then

$$\mathcal{T} := \big\{E(B) \bigm| (A,B) \in \mathcal{S}\big\}$$

is a κ_G-tangle of order k, unless
(i) either $k = 1$ and there is an isolated vertex $v \in V(G)$ such that $\mathcal{S}$ is the set of all separations (A,B) of order 0 with $v \in V(B) \setminus V(A)$,
(ii) or $k = 1$ and there is an isolated edge $e \in E(G)$ such that $\mathcal{S}$ is the set of all separations (A,B) of order 0 with $e \in E(B)$,
(iii) or $k = 2$ and there is an isolated or pendant edge $e = vw \in E(G)$ and $\mathcal{S}$ is the set of all separations (A,B) of order at most 1 with $e \in E(B)$.

We omit the straightforward (albeit tedious) proof.

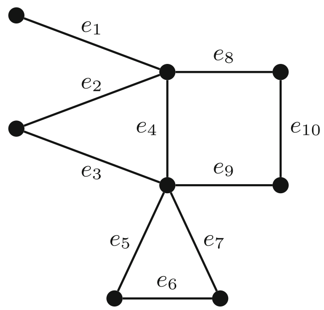

Fig. 5. A graph G with three G tangles of order 2 and two κ_G-tangles of order 2

Example 21. Let G be the graph shown in Fig. 5. G has one tangle of order 1 (since it is connected) and three tangles of order 2 corresponding to the three biconnected components. The G-tangle corresponding to the "improper" biconnected component consisting of the edge e_1 and its endvertices does not correspond to a κ_G-tangle (by Proposition 20(2-iii)). ⌟

A *star* is a connected graph in which at most 1 vertex has degree greater than 1. Note that we admit degenerate stars consisting of a single vertex or a single edge.

Corollary 22. *Let G be a graph that has a G-tangle of order k. Then G has a κ_G-tangle of order k, unless $k = 1$ and G only has isolated edges or $k = 2$ and all connected components of G are stars.*

6 Decompositions and Duality

A *cubic tree* is a tree where every node that is not a leaf has degree 3. An *oriented edge* of a tree T is a pair (s, t), where $st \in E(T)$. We denote the set of all oriented edges of T by $\overrightarrow{E}(T)$ and the set of leaves of T by $L(T)$. A *branch decomposition* of a connectivity function κ over U is a pair (T, ξ), where T is a cubic tree and ξ a bijective mapping from $L(T)$ to U. For every oriented edge $(s, t) \in \overrightarrow{E}(T)$ we define $\widetilde{\xi}(s, t)$ to be the set of all $\xi(u)$ for leaves $u \in L(T)$ contained in the same connected component of $T - \{st\}$ as t. Note that $\widetilde{\xi}(s, t) = \overline{\widetilde{\xi}(t, s)}$. We define the *width* of the decomposition (T, ξ) be the maximum of the values $\kappa(\widetilde{\xi}(t, u))$ for $(t, u) \in \overrightarrow{E}(T)$. The *branch width* of κ, denoted by $\mathrm{bw}(\kappa)$, is the minimum of the widths of all its branch decompositions.

The following fundamental result relates tangles and branch decompositions; it is one of the reasons why tangles are such interesting objects.

Theorem 23 (Duality Theorem; Robertson and Seymour [21]). *The branch width of a connectivity function κ equals the maximum order of a κ-tangle.*

We omit the proof.

Let G be a graph. A *branch decomposition* of G is defined to be a branch decomposition of κ_G, and the *branch width* of G, denoted by $\mathrm{bw}(G)$, is the branch width of κ_G.

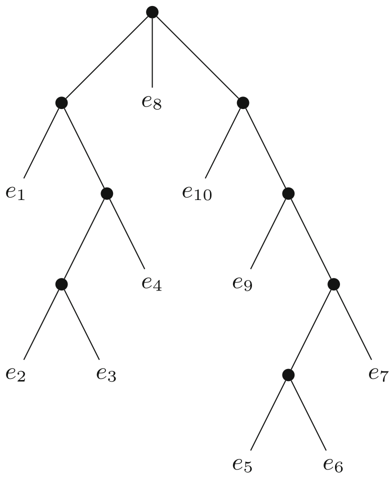

Fig. 6. A branch decomposition of width 2 of the graph shown in Fig. 5

Example 24. Let G be the graph shown in Fig. 5. Figure 6 shows a branch decomposition of G of width 2. Thus $\mathrm{bw}(G) \leq 2$. As G has a tangle of order 2 (see Example 21), by the Duality Theorem we have $\mathrm{bw}(G) = 2$. ⌟

The branch width of a graph is closely related to the better-known *tree width* $\mathrm{tw}(G)$: it is not difficult to prove that

$$\mathrm{bw}(G) \leq \mathrm{tw}(G) + 1 \leq \max\left\{\frac{3}{2}\,\mathrm{bw}(\kappa_G),\ 2\right\}$$

(Robertson and Seymour [21]). Both inequalities are tight. For example, a complete graph K_{3n} has branch width $2n$ and tree width $3n-1$, and a path of length 3 has branch width 2 and tree width 1. There is also a related duality theorem for tree width, due to Seymour and Thomas [23]: $\mathrm{tw}(G)+1$ equals the maximum order of bramble of G. (Recall the characterisation of tangles that we gave in Theorem 7 and the definition of brambles right after the theorem.)

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