Aggregation

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For small or highly volatile portfolios the standard methods of loss reserving tend to produce highly volatile predictors of future losses and hence of reserves. One might be tempted to combine a small or highly volatile portfolio with a large and stable one and to apply the corresponding methods to the resulting total portfolio. A typical example of such a situation is the combination of bodily injury claims with pure property damage claims in *motor third party liability insurance* insurance.

However, aggregation of sub-portfolios to a total portfolio turns out to be problematic since it can lead to a systematic distortion of the predictors. This is, in particular, the case when the sub-portfolios show different development patterns and develop differently also over the accident years. Moreover, if the standard methods of loss reserving are interpreted not just as algorithms but rather as statistical methods based on a stochastic model, then the problem arises that a model which is acceptable for each of the sub-portfolios will not necessarily be appropriate for the total portfolio. We discuss these aspects of aggregation for the chain ladder method and the additive method.

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Consider the run-off square of incremental losses:

Accident year	**Development year** 0	1	…	k	…	$n-i$	…	$n-1$	n
0	$Z_{0,0}$	$Z_{0,1}$	…	$Z_{0,k}$	…	$Z_{0,n-i}$	…	$Z_{0,n-1}$	$Z_{0,n}$
1	$Z_{1,0}$	$Z_{1,1}$	…	$Z_{1,k}$	…	$Z_{1,n-i}$	…	$Z_{1,n-1}$	$Z_{1,n}$
⋮	⋮	⋮		⋮		⋮		⋮	⋮
i	$Z_{i,0}$	$Z_{i,1}$	…	$Z_{i,k}$	…	$Z_{i,n-i}$	…	$Z_{i,n-1}$	$Z_{i,n}$
⋮	⋮	⋮		⋮		⋮		⋮	⋮
$n-k$	$Z_{n-k,0}$	$Z_{n-k,1}$	…	$Z_{n-k,k}$	…	$Z_{n-k,n-i}$	…	$Z_{n-k,n-1}$	$Z_{n-k,n}$
⋮	⋮	⋮		⋮		⋮		⋮	⋮
$n-1$	$Z_{n-1,0}$	$Z_{n-1,1}$	…	$Z_{n-1,k}$	…	$Z_{n-1,n-i}$	…	$Z_{n-1,n-1}$	$Z_{n-1,n}$
n	$Z_{n,0}$	$Z_{n,1}$	…	$Z_{n,k}$	…	$Z_{n,n-i}$	…	$Z_{n,n-1}$	$Z_{n,n}$

We assume that the incremental losses $Z_{i,k}$ are observable for $i + k \leq n$ and that they are non-observable for $i + k \geq n + 1$. For $i, k \in \{0, 1, \dots, n\}$, let

$$S_{i,k} := \sum_{l=0}^{k} Z_{i,l}$$

denote the cumulative loss from accident year i in development year k.

Chain Ladder Method

The chain ladder method is usually described by means of the cumulative losses. It is based on the *chain ladder factors*

$$\varphi_k^{\mathrm{CL}} := \frac{\sum_{j=0}^{n-k} S_{j,k}}{\sum_{j=0}^{n-k} S_{j,k-1}}$$

with $k \in \{1, \dots, n\}$ and it consists primarily in the prediction of the future cumulative losses $S_{i,k}$ with $i + k \geq n + 1$ by the *chain ladder predictors*

$$S_{i,k}^{\mathrm{CL}} := S_{i,n-i} \prod_{l=n-i+1}^{k} \varphi_l^{\mathrm{CL}}$$

For the prediction of the future incremental losses $Z_{i,k}$ with $i + k \geq n + 1$ one uses the *chain ladder predictors*

$$Z_{i,k}^{\mathrm{CL}} := S_{i,n-i}\, (\varphi_k^{\mathrm{CL}} - 1) \prod_{l=n-i+1}^{k-1} \varphi_l^{\mathrm{CL}}$$

(with $Z^{\mathrm{CL}}_{i,n-i+1} = S_{i,n-i}\,(\varphi^{\mathrm{CL}}_{n-i+1} - 1)$) from which the *chain ladder predictors* of the reserves result by summation.

By analogy with the chain ladder factors, we define for $i \in \{1, \dots, n\}$ the *dual chain ladder factors*

$$\psi_i^{\mathrm{CL}} := \frac{\sum_{j=0}^{i} S_{j,n-i}}{\sum_{j=0}^{i-1} S_{j,n-i}}$$

Here the analogy and the notion of duality result from the identities

$$\varphi_k^{\mathrm{CL}} = \frac{\sum_{j=0}^{n-k} \sum_{l=0}^{k} Z_{j,l}}{\sum_{j=0}^{n-k} \sum_{l=0}^{k-1} Z_{j,l}} \qquad \text{and} \qquad \psi_i^{\mathrm{CL}} = \frac{\sum_{l=0}^{n-i} \sum_{j=0}^{i} Z_{j,l}}{\sum_{l=0}^{n-i} \sum_{j=0}^{i-1} Z_{j,l}}$$

The dual chain ladder factors are exactly the chain ladder factor in the *reflected run-off triangle* of incremental losses, in which the roles of accident years and of development years are interchanged. Therefore they describe the development over accident years instead of development years.

We consider now two sub-portfolios with the respective incremental losses $\bar{Z}_{i,k} > 0$ and $\tilde{Z}_{i,k} > 0$ as well as the total portfolio with the incremental losses $Z_{i,k} := \bar{Z}_{i,k} + \tilde{Z}_{i,k}$. We also denote all other quantities of the sub-portfolios in the same way as the incremental losses.

Theorem.

(1) *If $\bar{\varphi}_k^{\mathrm{CL}} > \tilde{\varphi}_k^{\mathrm{CL}}$ and $\bar{\psi}_i^{\mathrm{CL}} > \tilde{\psi}_i^{\mathrm{CL}}$ holds for all $i, k \in \{1, \dots, n\}$, then the inequality*

$$\bar{Z}_{i,k}^{\mathrm{CL}} + \tilde{Z}_{i,k}^{\mathrm{CL}} > Z_{i,k}^{\mathrm{CL}}$$

holds for all $i, k \in \{1, \dots, n\}$ such that $i + k \geq n + 1$.

(2) If $\bar{\varphi}_k^{\mathrm{CL}} = \tilde{\varphi}_k^{\mathrm{CL}}$ *holds for all $i, k \in \{1, \dots, n\}$, then the identity*

$$\bar{Z}_{i,k}^{\mathrm{CL}} + \tilde{Z}_{i,k}^{\mathrm{CL}} = Z_{i,k}^{\mathrm{CL}}$$

holds for all $i, k \in \{1, \dots, n\}$ such that $i + k \geq n + 1$.

(3) If $\bar{\varphi}_k^{\mathrm{CL}} < \tilde{\varphi}_k^{\mathrm{CL}}$ *and $\bar{\psi}_i^{\mathrm{CL}} > \tilde{\psi}_i^{\mathrm{CL}}$ holds for all $i, k \in \{1, \dots, n\}$, then the inequality*

$$\bar{Z}_{i,k}^{\mathrm{CL}} + \tilde{Z}_{i,k}^{\mathrm{CL}} < Z_{i,k}^{\mathrm{CL}}$$

holds for all $i, k \in \{1, \dots, n\}$ such that $i + k \geq n + 1$.

By summation, the results of the theorem for the chain ladder predictors of incremental losses yield corresponding results for the chain ladder predictors of cumulative losses and for the chain ladder reserves.

Example. Sub-portfolio I: Incremental losses and predictors of incremental losses:

Accident year i	**Development year** k 0	1	2	3	$\bar{\psi}_i^{\mathrm{CL}}$
0	230	110	60	20	
1	240	120	80	*22*	*2.10*
2	230	120	*70*	*21*	*1.50*
3	280	*140*	*84*	*25*	*1.40*
$\bar{\varphi}_k^{\mathrm{CL}}$		*1.50*	*1.20*	*1.05*	

Sub-portfolio II: Incremental losses and predictors of incremental losses:

Accident year i	**Development year** k 0	1	2	3	$\tilde{\psi}_i^{\mathrm{CL}}$
0	780	140	80	10	
1	760	120	100	*10*	*1.98*
2	410	130	*54*	*6*	*1.30*
3	390	*78*	*47*	*5*	*1.20*
$\tilde{\varphi}_k^{\mathrm{CL}}$		*1.20*	*1.10*	*1.01*	

Sums of the predictors of the two sub-portfolios:

Accident year i	**Development year** k 0	1	2	3	
0					
1				*32*	
2			*124*	*27*	
3		*218*	*131*	*30*	

Total portfolio: Incremental losses and predictors of incremental losses:

Accident year i	**Development year** k 0	1	2	3	
0	1010	250	140	30	
1	1000	240	180	*30*	
2	640	250	*114*	*21*	
3	670	*187*	*110*	*21*	
φ_k^{CL}		*1.28*	*1.13*	*1.02*	

The results confirm assertion (1) of the theorem.

The chain ladder method is based on the assumption of the existence of a development pattern for factors.

- If the existence of a development pattern for factors is assumed for each of the sub-portfolios, then there exist parameters $\bar{\varphi}_k$ and $\tilde{\varphi}_k$ such that

$$E[\bar{S}_{i,k}] = E[\bar{S}_{i,k-1}]\,\bar{\varphi}_k$$
$$E[\tilde{S}_{i,k}] = E[\tilde{S}_{i,k-1}]\,\tilde{\varphi}_k$$

  holds for all $k \in \{1, \dots, n\}$ and $i \in \{0, 1, \dots, n\}$.

- If the existence of a development pattern for factors is assumed for the total portfolio, then there exist parameters φ_k such that

$$E[S_{i,k}] = E[S_{i,k-1}]\,\varphi_k$$

  holds for all $k \in \{1, \dots, n\}$ and $i \in \{0, 1, \dots, n\}$.

It thus follows that a development pattern for factors exists for each of the sub-portfolios and also for the total portfolio if and only if there exists, for every $k \in \{1, \dots, n\}$, some c_{k-1} such that the identity

$$\frac{E[\bar{S}_{i,k-1}]}{E[\tilde{S}_{i,k-1}]} = c_{k-1}$$

holds for all $i \in \{0, 1, \dots, n\}$. As this proportionality condition is not plausible in general, this raises the problem of a consistent modelling of the sub-portfolios and the total portfolio.

One possibility of a consistent modelling of the sub-portfolios and the total portfolio is provided by the *multivariate chain ladder model*, which provides a justification of the *multivariate chain ladder method*. The multivariate chain ladder model describes not only the individual sub-portfolios, but also the correlations between the sub-portfolios.

In actuarial practice, the application of the multivariate chain ladder method may cause problems, but the method represents a *benchmark* and in many cases the multivariate chain ladder predictors are approximated quite well by the univariate chain ladder predictors for the individual sub-portfolios.

Additive Method

The additive method uses known *volume measures* $v_0, v_1, \dots, v_n$ of the accident years. It is based on the *additive incremental loss ratios*

$$\zeta_k^{\mathrm{AD}} := \frac{\sum_{j=0}^{n-k} Z_{j,k}}{\sum_{j=0}^{n-k} v_j}$$

with $k \in \{0, 1, \dots, n\}$ and it consists primarily in the prediction of the future incremental losses $Z_{i,k}$ with $i + k \geq n + 1$ by the *additive predictors*

$$Z_{i,k}^{\mathrm{AD}} := v_i \zeta_k^{\mathrm{AD}}$$

from which the *additive predictors* of the future cumulative losses and of the reserves result by summation.

We consider now two sub-portfolios with the respective incremental losses $\bar{Z}_{i,k} > 0$ and $\tilde{Z}_{i,k} > 0$ and the respective volume measures $\bar{v}_i > 0$ and $\tilde{v}_i > 0$ as well as the total portfolio with the incremental losses $Z_{i,k} := \bar{Z}_{i,k} + \tilde{Z}_{i,k}$ and the volume measures $v_i := \bar{v}_i + \tilde{v}_i$. We also denote all other quantities of the sub-portfolios in the same way as the incremental losses and the volume measures.

Lemma. *For all $i, k \in \{1, \dots, n\}$ such that $i + k \geq n + 1$ there exists a constant $v_{i,k} > 0$ determined by the volume measures such that*

$$\bar{Z}_{i,k}^{\mathrm{AD}} + \tilde{Z}_{i,k}^{\mathrm{AD}} - Z_{i,k}^{\mathrm{AD}} = v_{i,k} \left(\frac{\bar{v}_i}{\sum_{j=0}^{n-k} \bar{v}_j} - \frac{\tilde{v}_i}{\sum_{j=0}^{n-k} \tilde{v}_j} \right) \left(\bar{\zeta}_k^{\mathrm{AD}} - \tilde{\zeta}_k^{\mathrm{AD}} \right)$$

This lemma provides a complete solution to the problem of additivity for the additive method (and even for the additive predictors of the individual future incremental losses). In particular, the additive method is always additive if there exists some c such that the identity

$$\bar{v}_i / \tilde{v}_i = c$$

holds for all $i \in \{0, 1, \dots, n\}$.

An analogon to the theorem on the additivity in the chain ladder method results immediately from the lemma:

Theorem.

(1) *If $\bar{\zeta}_k^{\mathrm{AD}} > \tilde{\zeta}_k^{\mathrm{AD}}$ and $\bar{v}_i / \sum_{j=0}^{n-k} \bar{v}_j > \tilde{v}_i / \sum_{j=0}^{n-k} \tilde{v}_j$* holds for all *$i, k \in \{1, \dots, n\}$ such that $i + k \geq n + 1$, then the inequality*

$$\bar{Z}_{i,k}^{\mathrm{AD}} + \tilde{Z}_{i,k}^{\mathrm{AD}} > Z_{i,k}^{\mathrm{AD}}$$

holds for all $i, k \in \{1, \dots, n\}$ such that $i + k \geq n + 1$.

(2) *If $\bar{\zeta}_k^{\mathrm{AD}} = \tilde{\zeta}_k^{\mathrm{AD}}$* or *$\bar{v}_i / \sum_{j=0}^{n-k} \bar{v}_j = \tilde{v}_i / \sum_{j=0}^{n-k} \tilde{v}_j$ holds for all $i, k \in \{1, \dots, n\}$* such that *$i + k \geq n + 1$, then the identity*

$$\bar{Z}_{i,k}^{\mathrm{AD}} + \tilde{Z}_{i,k}^{\mathrm{AD}} = Z_{i,k}^{\mathrm{AD}}$$

holds for all $i, k \in \{1, \dots, n\}$ such that *$i + k \geq n + 1$.*

(3) *If* $\bar{\zeta}_k^{\text{AD}} < \tilde{\zeta}_k^{\text{AD}}$ *and* $\bar{v}_i / \sum_{j=0}^{n-k} \bar{v}_j > \tilde{v}_i / \sum_{j=0}^{n-k} \tilde{v}_j$ *holds for all* $i, k \in \{1, \dots, n\}$ *such that* $i + k \geq n + 1$, *then the inequality*

$$\bar{Z}_{i,k}^{\text{AD}} + \tilde{Z}_{i,k}^{\text{AD}} < Z_{i,k}^{\text{AD}}$$

holds for all $i, k \in \{1, \dots, n\}$ *such that* $i + k \geq n + 1$.

By summation, the results of the theorem for the additive predictors of incremental losses yield corresponding results for the additive predictors of cumulative losses and for the additive reserves.

The additive method is based on the assumption of the existence of a development pattern for incremental loss ratios.

- If the existence of a development pattern for incremental loss ratios is assumed for each of the sub-portfolios, then there exist parameters $\bar{\zeta}_k$ and $\tilde{\zeta}_k$ such that

$$E[\bar{Z}_{i,k}] = \bar{v}_i \bar{\zeta}_k$$
$$E[\tilde{Z}_{i,k}] = \tilde{v}_i \tilde{\zeta}_k$$

  holds for all $k \in \{0, 1, \dots, n\}$ and $i \in \{0, 1, \dots, n\}$.
- If the existence of a development pattern for incremental loss ratios is assumed for the total portfolio, then there exist parameters ζ_k such that

$$E[Z_{i,k}] = v_i \zeta_k$$

  holds for all $k \in \{0, 1, \dots, n\}$ and $i \in \{0, 1, \dots, n\}$.

It thus follows that development patterns for incremental loss ratios exist for each of the sub-portfolios and also for the total portfolio if and only if there exists some c such that the identity

$$\bar{v}_i / \tilde{v}_i = c$$

holds for all $i \in \{0, 1, \dots, n\}$.

Example. Depending on the choice of the volume measure, different effects arise from the application of the additive method in *motor third party liability insurance*: If the number of contracts is chosen as the volume measure, the separation of bodily injury claims and pure property damage claims can be omitted, as the same volume measure is used for both types of losses and since the additive method is additive in this case.[1] By contrast, if the corresponding expected number of claims is chosen as the volume measure, then the volume measures for bodily injury claims and for pure property damage claims are usually not proportional and in this case the additive method is not additive in general.

[1] Let w_i denote the number of contracts in accident year i. Then one has $\bar{v}_i = w_i$ and $\tilde{v}_i = w_i$, and hence $v_i = 2w_i$. The additive method applied to either v_i and w_i produces the same results since scaling of the volume measures does not affect the predictors.

One possibility of a consistent modelling of the sub-portfolios and the total portfolio is provided by the *multivariate additive model*, which provides a justification of the *multivariate additive method*.

The remarks made on the multivariate chain ladder method also apply to the multivariate additive method.

Remarks

Assertion (1) of both theorems essentially states that, for every future incremental loss, the sum of the predictors from the sub-portfolios is always greater than the predictor from the total portfolio when one of the two sub-portfolios has at the same time a lower *development speed* and a higher *expansion speed* than the other. Similar interpretations can be given for assertions (2) and (3) of these theorems. The expansion over accident years is sometimes called *accident year inflation*.

The theoretical results of this article provide sufficient conditions for underestimation or overestimation of the reserves caused by the aggregation of sub-portfolios. Presumably, in actuarial practice these conditions will only be checked once the predictors have been computed and compared. If, however, it then turns out that the appropriate sufficient condition is fulfilled, then this check provides some useful information on the sub-portfolios.

Notes

Keywords: Additive Method, Chain Ladder Method (Basics), Development Patterns (Basics), Multivariate Methods, Volume Measures.

References: Ajne [1994], Barnett, Zehnwirth & Dubossarski [2005], Fuchs [2014], Klemmt [2005], Schmidt [2006b, 2012].

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