Chapter 2
The Plebański–Demiański Class of Black Hole Space-Times

Abstract The Plebański–Demiański class contains stationary, axially symmetric type D solutions of the Einstein–Maxwell equations with a cosmological constant. It covers many well-known black hole space-times like the Schwarzschild, Kerr or the Kottler space-time. The space-times are characterized by seven parameters: mass, spin, electric and magnetic charge, gravitomagnetic NUT charge, a so-called acceleration parameter and the cosmological constant. We review space-time properties like symmetries and isometries as well as the appearance of singularities as ring singularities or axial singularities. Furthermore, we discuss horizons, the ergoregion and a region with causality violation.

Keywords Plebanski-Demianski · Schwarzschild · Kerr · Kerr-Newman · Reissner-Nordstroem · NUT · C-metric · Metric tensor · Boyer-Lindquist coordinates · Space-time properties · Symmetries · Isometries · Singularities · Ring singularity · Axial singularity · Black hole horizon · Ergoregion · Causality violation · Conformal factor

We consider the general Plebański–Demiański class of stationary, axially symmetric type D solutions of the Einstein–Maxwell equations with a cosmological constant.[1] In fact, these solutions were first found by Debever (1971) but are better known in the form of Plebański and Demiański (1976). For the case without cosmological constant, these metrics can be traced back to Carter (1968) and in the Boyer–Lindquist coordinates, which we will use in the following, to Miller (1973). A detailed discussion of the Plebański–Demiański metrics can be found in the books by Griffiths and Podolský (2009) or Stephani et al. (2003). It is common to use units in which the speed of light and Newtons gravitational constant are normalized ($c = 1$, $G = 1$). With this rescaling, the Plebański–Demiański metric can be written in the Boyer–Lindquist coordinates $(r, \vartheta, \varphi, t)$ as

[1] The first four paragraphs as well as parts of Sect. 2.3 are based on my papers [1] and [3]. Section 2.4 contains expositions given in [3].

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$$g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{1}{\Omega^2}\Bigg(\Sigma\Big(\tfrac{1}{\Delta_r}\,\mathrm{d}r^2 + \tfrac{1}{\Delta_\vartheta}\,\mathrm{d}\vartheta^2\Big) + \frac{1}{\Sigma}\Big((\Sigma + a\chi)^2\Delta_\vartheta \sin^2\vartheta - \Delta_r\chi^2\Big)\,\mathrm{d}\varphi^2 + \frac{2}{\Sigma}\Big(\Delta_r\chi - a(\Sigma + a\chi)\Delta_\vartheta\sin^2\vartheta\Big)\,\mathrm{d}t\,\mathrm{d}\varphi - \frac{1}{\Sigma}\Big(\Delta_r - a^2\Delta_\vartheta\sin^2\vartheta\Big)\,\mathrm{d}t^2\Bigg), \tag{2.1}$$

see Griffiths and Podolský (2009, p. 311). Here, we use the abbreviations

$$\begin{aligned}\Omega &= 1 - \tfrac{\alpha}{\omega}(\ell + a\cos\vartheta)r, \\ \Sigma &= r^2 + (\ell + a\cos\vartheta)^2, \\ \chi &= a\sin^2\vartheta - 2\ell(\cos\vartheta + C),\end{aligned} \qquad \begin{aligned}\Delta_\vartheta &= 1 - a_3\cos\vartheta - a_4\cos^2\vartheta, \\ \Delta_r &= b_0 + b_1 r + b_2 r^2 + b_3 r^3 + b_4 r^4\end{aligned} \tag{2.2}$$

with the following coefficients of the polynomials Δ_ϑ and Δ_r

$$\begin{aligned} a_3 &= 2am\tfrac{\alpha}{\omega} - 4a\ell\big(\tfrac{\alpha^2}{\omega^2}(k+\beta) + \tfrac{\Lambda}{3}\big), \\ a_4 &= -a^2\big(\tfrac{\alpha^2}{\omega^2}(k+\beta) + \tfrac{\Lambda}{3}\big), \end{aligned} \tag{2.3}$$

$$\begin{aligned} b_0 &= k + \beta, \\ b_1 &= -2m, \\ b_2 &= \tfrac{k}{a^2-\ell^2} + 4\tfrac{\alpha}{\omega}\ell m - (a^2 + 3\ell^2)\big(\tfrac{\alpha^2}{\omega^2}(k+\beta) + \tfrac{\Lambda}{3}\big), \\ b_3 &= -2\tfrac{\alpha}{\omega}\Big(\tfrac{k\ell}{a^2-\ell^2} - (a^2-\ell^2)\Big(\tfrac{\alpha}{\omega}m - \ell\big(\tfrac{\alpha^2}{\omega^2}(k+\beta) + \tfrac{\Lambda}{3}\big)\Big)\Big), \\ b_4 &= -\big(\tfrac{\alpha^2}{\omega^2}k + \tfrac{\Lambda}{3}\big) \end{aligned} \tag{2.4}$$

and

$$k = \frac{1 + 2\frac{\alpha}{\omega}\ell m - 3\ell^2\big(\frac{\alpha^2}{\omega^2}\beta + \frac{\Lambda}{3}\big)}{1 + 3\frac{\alpha^2}{\omega^2}\ell^2(a^2-\ell^2)}(a^2 - \ell^2), \qquad \omega = \sqrt{a^2 + \ell^2}. \tag{2.5}$$

Basically, the coordinates t and r may range over all of $\mathbb{R}$ while ϑ and φ are standard coordinates on the two-sphere. Note, however, that for some values of the black-hole parameters r and ϑ have to be restricted, see Sect. 2.3. The Plebański–Demiański space-time depends on seven parameters (m, a, β, ℓ, α, Λ and C) which are to be interpreted in the following way: m is the mass of the black hole and a is its spin. β is a parameter that comprises electric and magnetic charge, $\beta = q_e^2 + q_m^2$ at least, if it is non-negative; for negative β, the metric cannot be interpreted as a solution to the Einstein–Maxwell equations because the electric or magnetic charge has to be imaginary then. Nonetheless, the case $\beta < 0$ is of interest because metrics of this form occur in some braneworld scenarios (Aliev and Gümrükçüoğlu 2005). The NUT parameter ℓ is to be interpreted as a gravitomagnetic charge (Griffiths and Podolský 2009, p. 219). The parameter α gives the acceleration of the black hole (Griffiths and Podolský 2009, p. 258) while Λ is the cosmological constant. The quantity C, which was introduced by Manko and Ruiz (2005), is relevant only if

$\ell \neq 0$. In this case, there is an *axial* singularity on the z axis and by choosing C appropriately this singularity can be distributed symmetrically or asymmetrically on the positive and the negative z axis. All the parameters, m, a, ℓ, β, Λ, α and C, may take arbitrary real values in principle, albeit not all possibilities are physically relevant.

If only the mass and the acceleration parameter are different from zero, we have the so-called C-metric[2] which describes a space-time with boost-rotation symmetry. This solution to the vacuum Einstein field equation was found by Levi-Civita (1919) and Weyl (1917, 1919). The name C-metric refers to the classification in the review of Ehlers and Kundt (1962). The rotating version of the C-metric was considered by Hong and Teo (2005) while a detailed discussion of accelerated space-times in general can be found in the book by Griffiths and Podolský (2009).

Commonly the C-metric is given in the form introduced by Hong and Teo (2003)

$$g^C_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = \frac{1}{\alpha^2(x+y)^2}\Big(-F(y)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}y^2}{F(y)} + \frac{\mathrm{d}x^2}{G(x)} + G(x)\,\mathrm{d}\varphi^2\Big) \tag{2.6}$$

with cubic functions $F(y) = -(1-y^2)(1-2\alpha m y)$ and $G(x) = (1-x^2)(1+2\alpha m x)$. The metric depends on two parameters, the mass m and the acceleration parameter α. The domain covered by the coordinates (τ, x, y, φ) actually contains *two* black holes accelerating away from each other with a conical singularity (a "strut") on the axis of rotational symmetry (Griffiths and Podolský 2009; Kinnersley and Walker 1970; Bonnor 1983; Bonnor and Davidson 1992). For our purposes, Boyer–Lindquist coordinates are more suitable, see Eq. (2.1), which cover only one of the two black holes.

The Plebański–Demiański class (2.1) covers many well-known space-times like the Schwarzschild, Kerr or Taub–NUT space-time; their charged versions ($\beta > 0$) and versions with non-vanishing cosmological constant Λ or acceleration α are also included. The non-accelerated space-times ($\alpha = 0$) are comprised in the Plebański or Kerr–Newman–NUT–(anti-)de Sitter class of metrics. Details about the covered space-times and the particular parameters of the space-times can be found in Table 2.1; a similar one is also presented in the book by Stephani et al. (2003, p. 325).

In some of these cases, the two polynomials Δ_ϑ and Δ_r, see (2.2), reduce to much simpler forms. For $\alpha = 0$, we find $k = (1-\ell^2\Lambda)(a^2-\ell^2)$, $\omega = \sqrt{a^2+\ell^2}$ and hence

$$\begin{aligned} \Delta_\vartheta &= 1 + \Lambda\big(\tfrac{4}{3}a\ell\cos\vartheta + \tfrac{1}{3}a^2\cos^2\vartheta\big),\\ \Delta_r &= \Delta - \Lambda\big((a^2-\ell^2)\ell^2 + (\tfrac{1}{3}a^2 + 2\ell^2)r^2 + \tfrac{1}{3}r^4\big), \end{aligned} \tag{2.7}$$

while $\ell = 0$ yields $k = a^2$, $\omega = \sqrt{a^2+\ell^2} = a$ and

[2] Note that this parameter C has nothing to do with the name "C-metric" for space-times of accelerated black hole(s).

Table 2.1 Metrics covered in the Plebański–Demiański class

a	β	ℓ	α	Λ	Space-time
×	×	×	×	×	Plebański–Demiański
					Schwarzschild
	×				Reissner–Nordström
×					Kerr
×	×				Kerr–Newman
		×			Taub–NUT
×		×			Kerr–NUT
×	×	×			Kerr–Newman–NUT
				×	Kottler or Schwarzschild–(anti-)de Sitter
×	×	×		×	Plebański or Kerr–Newman–NUT–(anti-)de Sitter
			×		C-metric or accelerated Schwarzschild
×			×		Rotating C-metric or accelerated Kerr

The × marks the particular black hole parameters of the space-time additional to the mass m

$$\begin{aligned} \Delta_\vartheta &= 1 - 2\alpha m \cos\vartheta + \left(\alpha^2(a^2+\beta) + \tfrac{\Lambda}{3}a^2\right)\cos^2\vartheta, \\ \Delta_r &= \Delta(1-\alpha^2 r^2) - \tfrac{\Lambda}{3}(a^2+r^2)r^2, \end{aligned} \tag{2.8}$$

where $\Delta = r^2 - 2mr + a^2 - \ell^2 + \beta$.

2.1 Symmetries

In the Plebański–Demiański class, all metric coefficients $g_{\mu\nu}$ noted in Eq. (2.1) are independent of t and φ which is why all space-times of this class stay invariant under translations of t and φ. Thus, the corresponding coordinate vector fields

$$\partial_t = \frac{\partial}{\partial t}, \qquad \partial_\varphi = \frac{\partial}{\partial \varphi} \tag{2.9}$$

are Killing vector fields that describe the symmetries of the space-time. Since their scalar products reproduce the t and φ metric coefficients

$$g_{tt} = g(\partial_t, \partial_t), \qquad g_{t\varphi} = g(\partial_t, \partial_\varphi), \qquad g_{\varphi\varphi} = g(\partial_\varphi, \partial_\varphi) \tag{2.10}$$

these coefficients have a geometric meaning, see Sect. 2.4.

2.2 Isometries

We have learned from the symmetries that it is not important at which time or at which angle φ we are looking at the black hole. The metric is the same and consequently also the described geometry. Of course, the translations of t and φ are isometries but there are more. In general, space-times which differ in one of the black hole parameters describe different geometric situations since the metric is changed. But for the Plebański–Demiański class there are globally isometric cases with opposite signs for some black hole parameters. Two are given by the coordinate transformations, see Appendix B

$$\begin{array}{ccc} (M_{[m,\ a,\beta,\ell,\ C,\alpha,\lambda]}, g) & & (M_{[m,a,\beta,\ \ell,\ C,\ \alpha,\lambda]}, g) \\ \Big\downarrow \binom{\vartheta}{\varphi}\mapsto\binom{\pi-\vartheta}{-\varphi} & & \Big\downarrow \binom{\vartheta}{\varphi}\mapsto\binom{\pi-\vartheta}{\varphi} \\ (M'_{[m,-a,\beta,\ell,-C,\alpha,\lambda]}, g') & & (M'_{[m,a,\beta,-\ell,-C,-\alpha,\lambda]}, g') \end{array} \tag{2.11}$$

These isometries tell us the following: Two black holes ($C = 0$) which differ in the rotation direction only describe the same geometry but space-times have to be mirrored at the equatorial plane ($\vartheta \mapsto \pi - \vartheta$) and at the plane defined by the rotation axis and the $\varphi = 0$ direction ($\varphi \mapsto -\varphi$). For black holes which differ in the sign of ℓ and α, one gets space-times mirrored at the equatorial plane.

2.3 Singularities

The metric (2.1) becomes singular at the roots of Ω, Σ, Δ_r, Δ_ϑ and $\sin\vartheta$. Some of them are mere coordinate singularities while others are true (curvature) singularities. As this issue is of some relevance for our purpose, we briefly discuss the different types of singularities in the following paragraphs.

Conformal factor. Ω becomes zero if

$$r = \frac{\sqrt{a^2+\ell^2}}{\alpha(\ell + a\cos\vartheta)}. \tag{2.12}$$

As the metric blows up if $\Omega \to 0$, Eq. (2.12) determines the boundary of the space-time. Because Ω enters as square into the metric (2.1), we restrict the space-time without loss of generality to the region where Ω is positive, see Fig. 2.4 on page 29. The allowed region is a half-space bounded by a plane ($\ell = 0$), a half-space bounded by one sheet of a two-sheeted hyperboloid ($\ell^2 < a^2$), a domain bounded by a paraboloid ($\ell^2 = a^2$), or a domain bounded by an ellipsoid ($\ell^2 > a^2$); see Fig. 2.1 for appropriate illustrations of these regions. For $\alpha = 0$ there is no restriction because $\Omega \equiv 1$. Note that Eq. (2.12) gives positive as well as negative r values depending on the signs of α, ℓ, a.

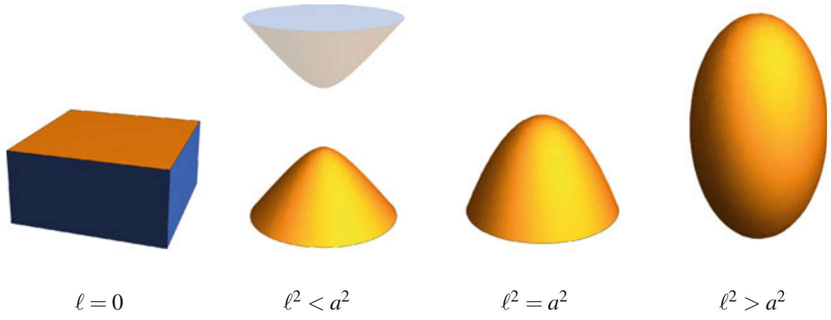

Fig. 2.1 Hyperboloids

Ring singularity. The equation $\Sigma = 0$ is equivalent to

$$r = 0 \quad \text{and} \quad \cos\vartheta = -\ell/a. \tag{2.13}$$

If $\ell^2 < a^2$, this condition is satisfied on a ring. The singularity on this ring turns out to be a true (curvature) singularity if $m \neq 0$. It is usually referred to as the *ring singularity*. Note that, apart from this singularity, the sphere $r = 0$ is regular. Hence, it is possible to travel through either of the two hemispheres ("throats") that are bounded by the ring singularity—from the region $r > 0$ to the region $r < 0$ or vice versa.

If $\ell^2 > a^2$, there is no ring singularity. Σ is different from zero everywhere and the entire sphere $r = 0$ is regular.

In the limiting case where $\ell^2 = a^2$, the ring singularity degenerates into a point on the axis. It becomes a point singularity for $\ell = a = 0$ that disconnects the space-time into the regions $r > 0$ and $r < 0$. The ring singularity is unaffected by α.

Axial singularity. The metric is singular on the z axis, i.e. for $\sin\vartheta = 0$, and this is always the case when using spherical polar coordinates. If $\alpha \neq 0$ or $\ell \neq 0$, this is not just a coordinate singularity but rather a true (conical) singularity on (at least a part of) the rotational axis. In the NUT case, the singularity depends on the Manko–Ruiz parameter C.

To demonstrate this, we observe that in the limit $\cos\vartheta \to \pm 1$ we have $\Sigma \to r^2 + (\ell \pm a)^2$ and $\chi \to -2\ell(\pm 1 + C)$. As a consequence, the metric coefficient

$$g^{tt} = \Omega^2 \left(\frac{\chi^2}{\Sigma \Delta_\vartheta \sin^2\vartheta} - \frac{(\Sigma + a\chi)^2}{\Sigma \Delta_r} \right) \tag{2.14}$$

diverges unless $C = \mp 1$. This divergent behavior indicates that either the coordinate function t or the metric g becomes pathological. It was shown by Misner (1963) that this singularity can be removed if one makes the time coordinate t periodic.

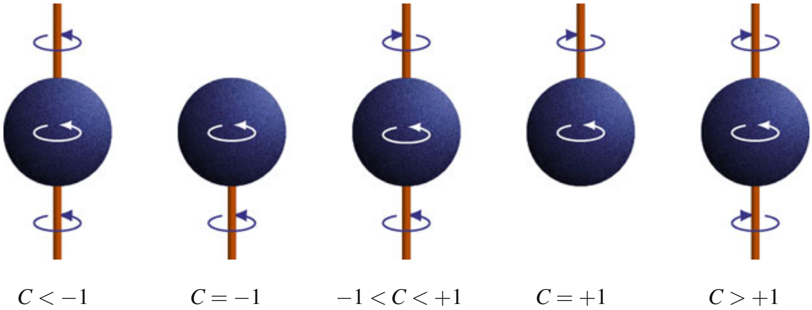

Fig. 2.2 Singularities on the z axis in Kerr–NUT space-times marked as (*red*) rotating rods

(Misner restricted himself to the Taub–NUT metric, $a = \beta = \Lambda = 0$, with $C = 1$ but his reasoning applies equally well to the general case.) We do *not* follow this suggestion because it leads to a space-time with closed timelike curves through *every* event. Instead, we adopt Bonnor's interpretation (Bonnor 1969, p. 145) of the axial singularity who viewed it as a "massless source of angular momentum", see also Stephani et al. (2003, p. 310). As pointed out by Manko and Ruiz (2005), this source term is splitted into two semi-infinite rotating rods with negative masses and infinite angular momenta where the rotation direction of the rods depends on C, see Fig. 2.2. The Manko–Ruiz parameter C is balancing the singularity. For $C = 1$, the singularity is on the half-axis $\vartheta = 0$, for $C = -1$ it is on the half-axis $\vartheta = \pi$ and for any other value of C it is on both half-axes. Thus, by choosing the Manko–Ruiz parameter C appropriately, one can decide on which part of the axis the singularity is situated. Note that each half-axis extends from $r = -\infty$ to $r = \infty$.

Metrics (2.1) with different values of C are *locally* isometric near all points off the axis. This follows from the fact that a coordinate transformation $t' = t - 2\ell\widetilde{C}\varphi$ yields, again, a metric (2.1) with $C' = C + \widetilde{C}$. For $\widetilde{C} = -C$ such a coordinate transformation eliminates the parameter C from the metric, see Kagramanova et al. (2010). Note, however, that this transformation does not work globally because φ is periodic and t is not, and it does not work near the axis because φ is pathological there. But there are globally isometric space-times as pointed out in Sect. 2.2.

Horizons. Straumann (2013, p. 471ff) explained in detail that horizons for the Kerr–Newman family are the null hypersurfaces

$$H = \{g(\xi, \xi) = 0\} \qquad \text{with } \xi = \partial_t - \frac{g_{t\varphi}}{g_{\varphi\varphi}}\partial_\varphi. \tag{2.15}$$

His argumentation applies equally well to the general Plebański–Demiański class. Therefore, the horizons of a Plebański–Demiański black hole are at

$$g(\xi, \xi) = 0 \qquad \Longleftrightarrow \qquad 0 = g_{\varphi t}^2 - g_{tt}g_{\varphi\varphi} = \frac{\Delta_r \Delta_\vartheta \sin^2\vartheta}{\Omega^4} \tag{2.16}$$

and can be found as real roots of Δ_r or Δ_ϑ which are coordinate singularities. In the following, we successively discuss both cases.

(i) A Plebański–Demiański space-time exhibits up to 4 horizons $r_1 > r_2 > \dots$ on spheres $r =$ constant since Δ_r is in general a polynomial of degree 4. If $\alpha = 0$ and $\Lambda = 0$, then Δ_r reduces to a second-degree polynomial

$$\Delta_r = r^2 - 2mr + a^2 - \ell^2 + \beta \tag{2.17}$$

and horizons can be found at

$$r_\pm = m \pm \sqrt{m^2 - a^2 + \ell^2 - \beta} \tag{2.18}$$

as long as $a^2 \leq a_{\max}^2 := m^2 + \ell^2 - \beta$. Then $r_+ (= r_1)$ is the outer (event) horizon of the black hole and $r_- (= r_2)$ is the inner horizon. For $a^2 > a_{\max}^2$ we would find, instead of a black hole, a naked singularity or a regular space-time. But we will not consider this possibility in the following because we are interested in the black hole case only. Then, the spin a is bounded by $a_{\max}$ and a maximally rotating black hole ($a^2 = a_{\max}^2$) is called *extremal black hole*. Since ∂_r is space like outside of the event horizon ($\Delta_r > 0$), communication is possible here. Therefore, this region is called *domain of outer communication* and we will place our observers for observing the shadow of the black hole within this region.

With cosmological constant (but $\alpha = 0$) we obtain for Δ_r, see Eq. (2.7)

$$\Delta_r = (r^2 - 2mr + a^2 - \ell^2 + \beta) - \Lambda\big((a^2 - \ell^2)\ell^2 + (\tfrac{1}{3}a^2 + 2\ell^2)r^2 + \tfrac{1}{3}r^4\big) \tag{2.19}$$

which has a strictly positive second derivative Δ_r'' if $\Lambda < 0$, as for $\Lambda = 0$. Hence, the number of zeros of Δ_r is either 2 or 0 and as above we have a black hole or a naked singularity or regular space-time. Again, the domain of outer communication around the black hole is the region between $r = \infty$ and the first horizon at r_1. If $\Lambda > 0$, the vector field ∂_r is timelike for big values of r. Therefore, the first horizon, if existing, is a cosmological horizon. We have a black hole if there are four horizons altogether. Then, the domain of outer communication is the region between the first and the second horizon. But in both cases the horizons could in general not be specified in a simple form because of the higher degree of Δ_r.

This applies also to an accelerated scenario with nonvanishing NUT charge ℓ or cosmological constant Λ. But if both are zero, the horizons can easily be determined. According to Eq. (2.8) Δ_r is factorized then

$$\Delta_r = (r^2 - 2mr + a^2 + \beta)(1 - \alpha^2 r^2); \tag{2.20}$$

therefore, we find the usual (Kerr–Newman) horizons at $r = r_\pm$ given by Eq. (2.18) with $\ell = 0$ and additional cosmological horizons at $r = \pm\frac{1}{\alpha}$. Of course, we must

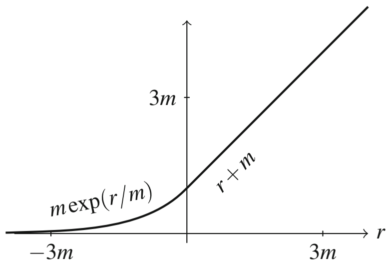

Fig. 2.3 Different scales for the r coordinate in extended polar plots

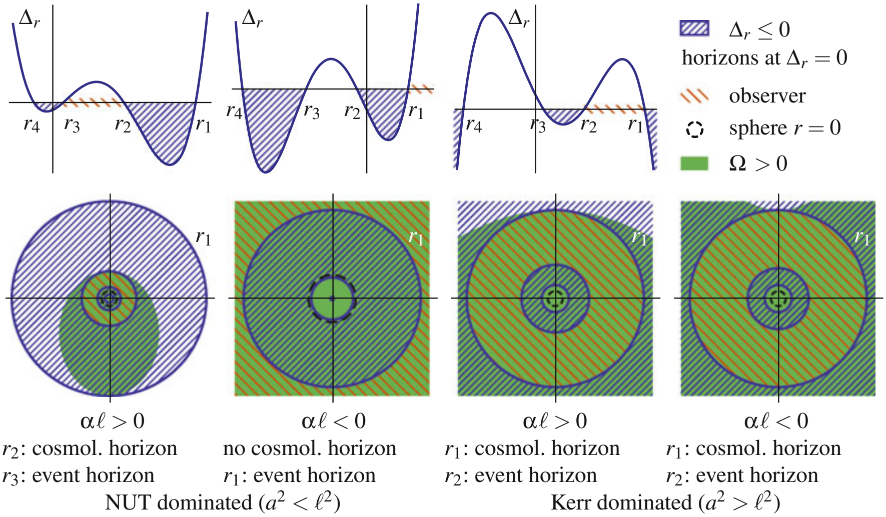

Fig. 2.4 Schematic illustrations of the graph of Δ_r (*upper* row) and extended polar plots of the region $\Omega > 0$ (*lower* row). Depending on the sign of its leading coefficient, Δ_r goes to $\pm\infty$ for big radii r; the sign changes if $a^2 \approx \ell^2$ (equality for $\Lambda = 0$) and with the change the space-time is no longer NUT but Kerr dominated. The space-time is restricted to that region where $\Omega > 0$ (green). Geometrically, the boundary of this region is an ellipsoid (*left*) or one sheet of a two-sheeted hyperboloid (*right*). In the NUT dominated case, the root r_1 of Δ_r is not contained in the ellipsoid $\Omega > 0$ for $\alpha\ell > 0$; thus, the event horizon is at r_3 instead. Interestingly, for $\alpha\ell < 0$ there is no cosmological horizon. The *red* hatched region (red hatching) marks the outer domain of communication ($\Delta_r > 0$) where observers are placed

have $|\alpha| < \frac{1}{r_+}$. For analyzing the general case, we have to take into account that only the regions where $\Omega > 0$, see Eq. (2.12), are allowed. Here, the vector field ∂_r could be timelike or space like for big values of r and this depends, of course, on the sign of the leading coefficient b_4 of Δ_r. Figure 2.4 shows in the first row schematic illustrations of the graph of Δ_r and in the second row *extended* polar plots of the region $\Omega > 0$.

Following a suggestion by O'Neill (1995), we show the entire range of the space-time, with the Boyer–Lindquist coordinate r increasing outward from the origin which corresponds to $r = -\infty$. But in order to not highlight the outer parts by a strong deformation, we use two different scales for the radial coordinate (see Fig. 2.3): In the inner region $r < 0$ (inside the sphere $r = 0$ marked by a dashed circle), the radial coordinate is plotted as $m \exp(r/m)$; this is continuously extended with $r + m$ in the outer region $r > 0$ (outside the sphere $r = 0$).

The sign of b_4 does not only define the causal character of ∂_r but also the character of the whole space-time since the sign changes at $a^2 \approx \ell^2$ (equality for $\Lambda = 0$), see Eq. (2.4). Thus, the space-time is NUT dominated with space like ∂_r ($b_4 > 0$) for big r if $a^2 < \ell^2$ and Kerr dominated with timelike ∂_r ($b_4 < 0$) for big r if $a^2 > \ell^2$.

In the NUT case (left columns in Fig. 2.4) where according to Eq. (2.13) we have no ring singularity, interesting things happen. For $\alpha\ell > 0$, the first root r_1 is not in the allowed region $\Omega > 0$. Hence, we have a cosmological horizon at r_2, the event horizon of the black hole at r_3 and in between the domain of outer communication. Different signs of α and ℓ, however, result in 3 negative roots of Δ_r; thus, the only positive root r_1 is the event horizon and the adjacent domain of outer communication is not bounded by a cosmological horizon. However, one can easily read of Eq. (2.12) that $\Omega < 0$ in the equatorial plane ($\vartheta = \frac{\pi}{2}$) is only possible for negative r values. The timelike case (right columns in Fig. 2.4) is similar to the non-accelerated space-times discussed before. Since all real roots of Δ_r are in the allowed region with $\Omega > 0$, the first root r_1 represents a cosmological horizon and the subsequent root r_2 is the black-hole horizon. Here, the domain of outer communication is the region between r_1 and r_2 where $\Delta_r > 0$.

(ii) As mentioned on page 28, the roots of Δ_ϑ are coordinate singularities, too; these indicate further horizons where the vector field ∂_ϑ changes the causal character from space like to timelike, just as the vector field ∂_r does at the roots of Δ_r. However, since these horizons lie on cones $\vartheta =$ constant instead of spheres $r =$ constant, such a situation would be hardly of any physical relevance. Therefore, we exclude it by limiting the parameters of the black hole appropriately: As $\Delta_\vartheta = 0$ implies

$$\cos\vartheta_\pm = \frac{-a_3 \pm \sqrt{a_3^2 + 4a_4}}{2a_4}, \tag{2.21}$$

$\Delta_\vartheta \neq 0$ is guaranteed for all real ϑ if the radicand in Eq. (2.21) is negative or if the absolute value of the entire right hand side of (2.21) is greater than 1. In all subsequently considered cases one of these sufficient conditions is fulfilled, see the corresponding tables in Appendix A.

For some special cases these conditions define simple constraints on the parameters. If $\alpha = 0$, Eq. (2.21) ends in $a\cos\vartheta_\pm = -2\ell \pm \sqrt{4\ell^2 - 3/\Lambda}$ and

$$a_3^2 + 4a_4 < 0 \qquad \Longleftrightarrow \qquad 4\ell^2\Lambda < 3. \tag{2.22}$$

For $\ell = 0$, $\Lambda = 0$ we find

$$\alpha \cos \vartheta_{\pm} = \frac{m \pm \sqrt{m^2 - \beta - a^2}}{a^2 + \beta} \geq \frac{1}{2m} \tag{2.23}$$

since $a^2 \leq a_{\text{max}}^2 = m^2 - \beta$ and $\beta < m^2$. Thus, $\Delta_\vartheta \neq 0$ is assured if $|\alpha| < \frac{1}{2m}$.

2.4 Ergoregion and Causality Violation

There are some other interesting regions around a black hole characterized by the change of the causal character of the Killing vector fields ∂_t and ∂_φ.

In the region where ∂_t becomes space like, i.e. $g_{tt} > 0$, no observer can move on a t-line. Thus, any observer in this region has to rotate (in φ direction). This region with $g_{tt} > 0$ is known as the *ergosphere* or the *ergoregion*,[3]

$$g_{tt} = -\frac{1}{\Omega^2 \Sigma}\left(\Delta_r - a^2 \Delta_\vartheta \sin^2 \vartheta\right); \tag{2.24}$$

its boundary, $g_{tt} = 0$, is called *static limit*. An ergoregion only exists if $a \neq 0$. Note that at the horizons, i.e., at the roots of Δ_r, the metric coefficient g_{tt} is positive, see Eq. (2.24). Hence, the horizons are always contained within the ergoregion. For $\alpha \neq 0$ or $\Lambda \neq 0$ there are cosmological horizons in addition to the black hole horizons; then the ergoregion consists of several connected components. At the poles ($\vartheta = 0, \pi$), the boundary of the ergoregion and the horizon share a common tangential plane. Since $\sin(\frac{\pi}{2} + \vartheta) = \sin(\frac{\pi}{2} - \vartheta)$, the ergoregion is symmetric with respect to the equatorial plane if $\alpha = 0$ and $\Lambda = 0$, cf. Eqs. (2.24) and (2.7). Actually, the ergoregion stays almost symmetric for small values of Λ, $\Lambda \leq 10^{-2}\text{m}^{-2}$, because then $\Delta_\vartheta \approx 1$. This behavior is lost with an acceleration $\alpha \neq 0$.

If $a \neq 0$ or $\ell \neq 0$, there are regions where the Killing field ∂_φ becomes timelike, $g_{\varphi\varphi} < 0$. This indicates causality violation because the φ-lines are closed timelike curves. For $\ell = 0$, the region where $g_{\varphi\varphi} = 0$ is completely contained in the domain where $r < 0$ and, thus, hidden behind the horizon for an observer in the domain of outer communication. In the case $\ell \neq 0$, however, there is a causality violating region in the domain of outer communication around the axial singularity.

By [1–3] I refer to my papers Grenzebach et al. (2014), Grenzebach (2015) and Grenzebach et al. (2015), respectively. Sentences marked with [i] can be found in total or only slightly modified in the ith paper

[3]Some authors call only the region between the horizon and the static limit *ergoregion*. This is that part of the region $g_{tt} > 0$ which an outside observer would be able to see.

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