

Chapter 2

Radiophysical and Optical Chaotic Oscillators Applicable for Information Protection

Let us proceed to the presentation of the original results of our studies. We begin from the description of the block-diagram of the chaotic oscillator of the radio-frequency range, introducing its mathematical model, simulation results and laboratory experiments. After that, we shall proceed to the discussion of the models and the properties of the nonlinear ring interferometers as the chaotic oscillators. We would like to draw the reader attention to their advantages owing to the two-dimensional delayed feedback. Let us consider the versions of the interferometer with one or two of such feedback loop circuits. At first, we will discuss the feasibility of the development of the nonlinear interferometer based on the optical fiber. We will “refract” the optical chaotic oscillators through the “prism” of the hypothesis suggested by S.R. Hameroff and R. Penrose considering the microtubule of the cytoskeleton as an intracellular calculator.

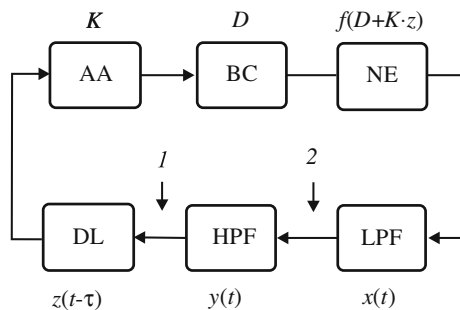
We now begin from the discussion of the suggested radio-electronic chaotic source.

2.1 The Radio-Electronic Oscillator of the Deterministic Chaos with Nonlinearity in the Form of Parabola Compositions

2.1.1 *The Structure and the Mathematical Model of the Oscillator*

Let us examine the self-oscillating system of the ring type with delay, its structure shown in Fig. 2.1. The oscillator consists of the following elements sequentially closed into the ring: a linear adjustable amplifier, a circuit of the DC components of the signal, a nonlinear element (NE), a low-frequency filter of the first order (LPF), a high frequency filter of the first order (HPF), a delay line. The delay line provides

Fig. 2.1 The structural diagram of the deterministic chaos oscillator. Accepted abbreviations: *AA* is a linear adjustable amplifier, *BC* is the bias circuit for DC signal component, *NE* is a nonlinear element, *LPF* is a low-pass filter of the first order, *HPF* is a high-pass filter of the first order, *DL* is a delay line



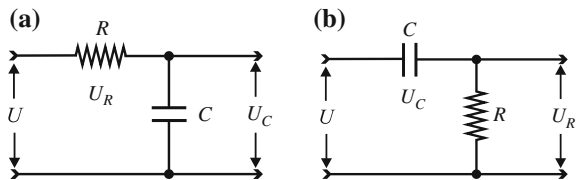
the high dimensionality of the phase space for the dynamic system (see discourse on the influence of the equivalence of the delay line and the inertial element upon dynamics at the end of Sect. 1.3.1). Due to this, the necessary condition for the chaotic mode implementation is fulfilled in the chaotic oscillator (as in case of the second-order (or higher order) filter performance, for example, in [1, 2]). In the aggregate, the adjustable amplifier, the bias circuit and the NE form the nonlinear amplifier.

In practice, nonlinear element is characterized by the certain persistence. But, we will assume that the nonlinear element is ideal, i.e. it immediately converts the input signal into the output one. Such an assumption is true when the characteristic time t_c of the signal transformation in the nonlinear element is much less than the time constant T_1 of LPF ($t_c \ll T_1$).

The low-frequency filter is meant for the limiting of the maximal oscillation frequency in the chaotic oscillator. The high-frequency filter limits the oscillating spectrum in the low-frequency region. Besides, since development of the deterministic chaos oscillator is directed to its application in the communication systems, then it is desired that the signal at transmitter outlet is set centralized (i.e. contains no DC component). This condition can also be achieved owing to this HPF. The signal centralization simplifies a circuit solution on the reception side, saving from the necessity of recovering the DC component.

Let us consider the signal passing through the feedback loop of the chaotic oscillator. The signal $z(t)$ (in the point 1 in Fig. 2.1) passes through the delay line and lags by τ time. The obtained signal $z(t - \tau)$ arrives at the nonlinear amplifier input. Its first block is the adjustable linear amplifier and it performs the signal scaling transformation. The bias circuit defines the operation point D at the transfer characteristic of the nonlinear element. The signal $D + Kz(t - \tau)$ passes to NE, where it is subject to nonlinear transformation of $f[D + Kz(t - \tau)]$. At LPF output (in the point 2 in Fig. 2.1) a certain signal $x(t)$ is produced. It passes to the input of HPF, at which output (in the point 1 in Fig. 2.1) the signal $z(t) = x(t) - y(t)$ is formed, $y(t)$ being the voltage drop in the HPF capacitor. The circle is closed.

Fig. 2.2 Electrical circuit
a of the LPF, **b** of the HPF of
 the first order



Let us proceed to the construction of the mathematical model of the chaotic oscillator, whose variants and development stages were approved at the Conferences [3–7] and described in papers [8–10].

The model of the deterministic chaos oscillator with delay. Let us establish the connection between input and output voltages of each elements of the oscillator and the whole circuit. Electrical circuits of the differential and integration RC—networks, which are LPF and HPF, are shown in Fig. 2.2. The dependence between the input voltage U and the LPF output voltage U_C can be obtained from the equality of currents flowing through the resistor R and the capacitor C :

$$I = I_C = I_R = C \frac{dU_C}{dt} = \frac{U_R}{R}.$$

From the equality $U = U_R + U_C$, we obtain the equation for LPF

$$\frac{dU_C}{dt} = \frac{U - U_C}{RC} \quad (2.1)$$

The dependence between the voltage at the input U and at the output U_R in HPF can be obtained from the equality of the currents flowing through R and C :

$$I = I_C = I_R = C \frac{dU_C}{dt} = \frac{U_R}{R}.$$

Taking into account the equalities $U = U_C + U_R$, we get the equation

$$\frac{dU_R}{dt} = \frac{dU}{dt} - \frac{U_R}{RC},$$

where $RC = T$ is a time constant of the filter. The HPF equation can be more conveniently written relating to the variable $U_C = U - U_R$

$$\frac{dU_C}{dt} = \frac{U - U_C}{RC}. \quad (2.2)$$

The account of delay can be performed by the introduction of the deviating argument:

$$U_n = U(t - \tau), \quad (2.3)$$

where U and U_n are input and output voltages of the delay line. Operations of scaling and determination of the operation point are carried out by the linear amplifier and the bias circuit and are described by the expression

$$U_k(t) = D + KU_n(t), \quad (2.4)$$

where U_n and U_k are input and output voltages of the adjustable amplifier, D —is the bias voltage, K is the gain coefficient. The nonlinear element (Fig. 2.1) transforms the signal according to the law of given transfer characteristic $f(U)$:

$$U_f = f(U_k), \quad (2.5)$$

where U_k and U_f are input and output voltages of NE. Now we do not dwell on the NE structure, as it will be discussed in detail later.

Equations (2.1)–(2.5) allow obtaining the system of the two differential equations describing the dynamics of the chaotic oscillator, which is formed by the series of elements connected according to the circuit in Fig. 2.2 [9]:

$$\begin{aligned} \frac{dx(t)}{dt} &= \frac{f(D + K(x(t-\tau) - y(t-\tau))) - x(t)}{T_1}, \\ \frac{dy(t)}{dt} &= \frac{x(t) - y(t)}{T_2}, \end{aligned} \quad (2.6)$$

where T_1 and T_2 are the time constants of LPF and HPF. We would like to remind that here $x(t)$ is the value of voltage at LPF output of the first order (the point 2 in Fig. 2.1); $y(t)$ is the voltage on the HPF capacitor; $z(t) = x(t) - y(t)$ is the voltage at the HPF output (the point 1); D is the bias DC voltage; K is the gain coefficient of adjustable amplifier; τ is the signal delay time in the delay lines.

Evidently, the obtained mathematical model describes the nonlinear dynamic system with the infinite-dimension phase space. It allows assumption that dynamics in this oscillator is chaotic. We note that the delay line (Fig. 2.1) is not the only way to increase the phase space dimension of the simple dynamic systems. We know [11–13] that the condition for the chaotic dynamics arising is the phase space dimension equal to three and more. In order to perform the transition to the finite analog of the chaotic oscillator, we may use the way of delay line replacement by the one or several transit-time elements. In this case, the dynamic system dimension is not large but equal to three or more, i.e. it is sufficient for the emergence of the chaotic dynamics in the oscillator. We have already mentioned (see Sect. 1.3.1) the parallels between the “hereditary” systems and the systems with inertia elements.

2.1.2 The Nonlinear Element: A Structure, a Mathematical Description

In addition to the literature review presented in Sect. 1.3.2, we have to mention several approaches to the synthesis of the radio-electronic nonlinear elements for the chaotic oscillators. The first approach consists in the integration of nonlinearity and amplitude limitation functions in the one separate device (or in the set of active and passive elements), whose volt-ampere characteristic has a segment with negative differential resistance. The tunnel diode, the IMPATT diode, the electronic tube (dynatron effect), the transistor (bipolar, single-junction, field effect etc.) can be attributed to such devices and elements. In the most cases, this segment is not large compared to the whole volt-ampere characteristic. The significant disadvantage (in the aspect of the data transmission application) of the discrete nonlinear elements is the parameter dispersion of different specimens of semiconductor devices, a variation of volt-ampere characteristic shape with a change of temperature during the period of device operation. Moreover, high resistance values, together with the spurious distributed capacitors, limit the operating frequency band.

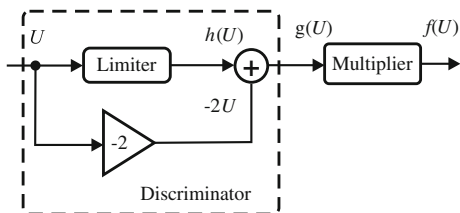
Under the second approach the functions of nonlinearity and limitations (at the ends of the transfer characteristics) are, on the contrary, separated. For instance, the nonlinear transfer function is formed by sequential connection of the amplifier, operating at the nonlinear segment, and the amplifier set in the limitation mode. This allows growing the operating frequency band and reducing requirements to the parameters of the nonlinear element parts (at the expense of the task sharing between the NE parts). However, the descending segment of the transfer characteristic of such an element is absent, or it is weakly expressed [14–16].

The third approach is similar to the second one. According to it, the nonlinear transfer characteristic can be created by the combination of the piecewise-linear and the nonlinear circuits performing the mathematical operations: addition, subtraction, involution, taking the logarithm, exponentiation [17–20]. Application of the operational amplifiers allows realization of the controllable transfer characteristics in the wide frequency bands. Moreover, it is expedient to apply the digital signal processors for the generation of the transfer characteristic. But, there are difficulties as well, for example, a high cost value, as well as the delay during signal processing.

Mentioned approaches (together with a variety of semiconductor discrete elements and the integrated circuits) allow synthesis of the desired transfer characteristic in the frequency band from the units of hertz to the tens of gigahertz.

In the context of chaotic oscillator creation, the choice is important of the transfer characteristic. Its shape (a local slope, a swing, location of the singular points along the curve) mainly determines the dynamic modes and scenarios of the transition to the chaos in the oscillator that can be realized. For instance, the value of the local slope and the transfer characteristic swing affect the conditions of the stability loss in the oscillator. The more a slope is, the less value of gain is required (in the oscillator) to excite the dynamic mode. When the deterministic chaos

Fig. 2.3 Structural circuit of the nonlinear element



oscillator is used in the structure of communication systems, the reduction of gain leads to the signal/noise ratio growth at the expense of the decrease of the additional noise influence. In turn, in the singular points vicinities the transfer characteristic changes its local slope. This allows the control of the oscillator dynamic mode. But growth of the singular point number, i.e. the presence of descending and ascending segments, increases a number and a variety of dynamic modes.

Therefore, we offered the deterministic chaotic oscillator, in which NE has a transfer characteristic in the form of parabola composition. In our case, the characteristic has five singular points: two maxima and three minima. The local slope of segments and a position of singular points can be controlled by NE parameters.

Figure 2.3 shows the circuit of the nonlinear element consisting of discriminator and multiplier. This discriminator realizes the inversed N -like characteristic. Indeed, having arrived to its input, the U signal divides and passes through the voltage limiter into the discriminator arms. In the first arm the signal passes through the voltage limiter thus obtaining the shape of $h(U)$ (Fig. 2.4a). Propagating along the second arm of the discriminator, the U signal undergoes the linear transformation and the signal of $-2U$ is obtained. After that, both signals are added, and the resulting signal $g(U)$ come out of the discriminator (Fig. 2.4b). The signal $g(U)$ from the discriminator passes to the both inputs of multiplier and the signal $g^2(U)$ arrives at its output (Fig. 2.4c). So, the NE transfer characteristic $f(U)$ is formed.

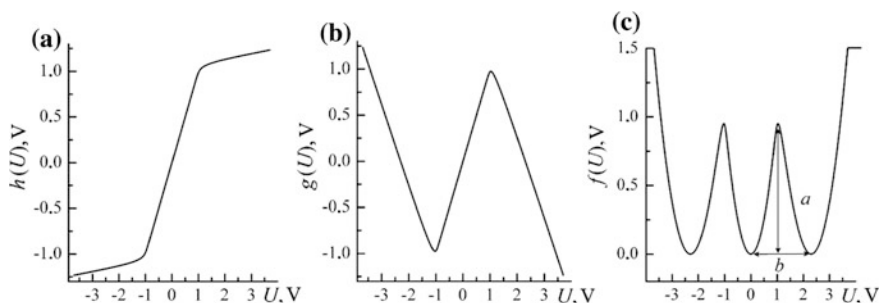


Fig. 2.4 Transfer characteristics: **a** of voltage limiter $h(U)$ in NE; **b** of the discriminator $g(U)$ in NE; **c** of the nonlinear element as a whole $f(U)$: $a = 0.95$ V, $b = 2.3$ V

The transfer characteristic of the limiter $h(U)$ (Fig. 2.4a) can be approximated by the exponential function. Then, using Fig. 2.3, we obtain

$$\begin{aligned} h(U) &= \operatorname{sgn}(U) \cdot M \cdot (1 - e^{\frac{-|U|}{V_t}}), \\ g(U) &= h(U) - 2 \cdot U, \\ f(U) &= g^2(U), \end{aligned} \quad (2.7)$$

where V_t, M are coefficients of approximation. The function (configuration of the chain of transposition 2.7) in the first approximation with accepted accuracy describes the transfer characteristic of the limiter and the nonlinear element as a whole [8].

Further, we shall call the $f(U)$ function as a transfer (volt-volt) characteristic, which has a shape of the parabola composition (Fig. 2.4c). The distance between maxima and minima of the transfer characteristic along the ordinate axis will be called as a swing a . The distance between the nearest minima along the abscissa axis will be called as a period b .

2.1.3 Analysis of Equilibrium State Stability in the Model of the Deterministic Chaos Oscillator

It is known that in mathematics there is a concept of the system roughness, or the structural stability. The roughness means that under the small variation of the system parameter the behavior of this system can change only in details, but it will not undergo the quantitative (structural) mode variation. For rough systems, the transition through the bifurcation point means the changing of the one structurally stable mode by another mode. In the very bifurcation point, the system ceases to be rough, i.e. the small change of the parameter leads to the replacement of the one mode by another [13, 21].

In the case of bifurcations of the equilibrium states, the points of bifurcation often correspond to the boundary areas of stability and instability of these states. The necessary condition for the existence of such boundaries is the equality to zero of the real part ($\operatorname{Re} \lambda = 0$) of the λ eigenvalues of the linearization matrix of the dynamic system. The last condition determines the points from the parameter space, in which the solution bifurcation occurs [22, 23].

Let us examine the equilibrium states stability analysis in the mathematical model (2.6) of the deterministic chaotic oscillator [7]. We linearize (2.6) by finding the partial derivatives with respect to the variables x, y and fulfilling the replacement $U = D + K[x(t - \tau) - y(t - \tau)]$:

$$\begin{aligned}\frac{\partial}{\partial x} \left(T_1 \frac{dx}{dt} \right) &= K \frac{\partial f(U)}{\partial U} \frac{dx(t-\tau)}{dx(t)} - 1, & \frac{\partial}{\partial y} \left(T_1 \frac{dx}{dt} \right) &= K \frac{\partial f(U)}{\partial U} \frac{dy(t-\tau)}{dy(t)}, \\ \frac{\partial}{\partial x} \left(T_2 \frac{dy}{dt} \right) &= 1, & \frac{\partial}{\partial y} \left(T_2 \frac{dy}{dt} \right) &= -1.\end{aligned}$$

Let us construct the equation for eigenvalues of λ

$$\begin{vmatrix} K \frac{\partial f(U)}{\partial U} \frac{dx(t-\tau)}{dx(t)} - 1 - \lambda & K \frac{\partial f(U)}{\partial U} \frac{dy(t-\tau)}{dy(t)} \\ 1 & -1 - \lambda \end{vmatrix} = 0.$$

Expanding the determinant, we obtain the characteristic equation

$$\left(\frac{\partial f(U)}{\partial U} K \frac{dx(t-\tau)}{dx(t)} - 1 - \lambda \right) (-1 - \lambda) - \frac{\partial f(U)}{\partial U} \left(K \frac{dy(t-\tau)}{dy(t)} \right) = 0.$$

In designations

$$\begin{aligned}S &= 2 - K \frac{\partial f(U)}{\partial U} \frac{dx(t-\tau)}{dx(t)}, \\ J &= 1 - K \frac{\partial f(U)}{\partial U} \left(\frac{dx(t-\tau)}{dx(t)} + \frac{dy(t-\tau)}{dy(t)} \right).\end{aligned}\tag{2.8}$$

The characteristic equation takes a form

$$\lambda^2 + S \cdot \lambda + J = 0.\tag{2.9}$$

It is known that in the small vicinity of solutions $x(t)$ and $y(t)$, the perturbed solution $X(t)$ and $Y(t)$ can be represented as $X(t) = x(t) + \exp(\lambda t/T_1)$ and $Y(t) = y(t) + \exp(\lambda t/T_2)$.

Then, for the *static state (state of equilibrium)*, differentials from $X(t)$ and $Y(t)$ are

$$\begin{aligned}dX(t) &= 0 + \frac{\lambda}{T_1} \exp\left(\lambda \frac{t}{T_1}\right) dt, & dX(t-\tau) &= 0 + \frac{\lambda}{T_1} \exp\left(\lambda \frac{t-\tau}{T_1}\right) dt, \\ dY(t) &= 0 + \frac{\lambda}{T_2} \exp\left(\lambda \frac{t}{T_2}\right) dt, & dY(t-\tau) &= 0 + \frac{\lambda}{T_2} \exp\left(\lambda \frac{t-\tau}{T_2}\right) dt,\end{aligned}$$

and ratios in (2.8) take the form of

$$\frac{dX(t-\tau)}{dX(t)} = \frac{\frac{\lambda}{T_1} \exp\left(\lambda \frac{t-\tau}{T_1}\right) dt}{\frac{\lambda}{T_1} \exp\left(\lambda \frac{t}{T_1}\right) dt} = \exp\left(-\lambda \frac{\tau}{T_1}\right),$$

$$\frac{dY(t - \tau)}{dY(t)} = \frac{\frac{\lambda}{T_2} \exp\left(\lambda \frac{t - \tau}{T_2}\right) dt}{\frac{\lambda}{T_2} \exp\left(\lambda \frac{t}{T_2}\right) dt} = \exp\left(-\lambda \frac{\tau}{T_2}\right).$$

We substitute the expressions obtained into (2.8) and, in the final form, write the characteristic equation (2.9) for equilibrium states as

$$\begin{aligned} \lambda^2 + S\lambda + J &= 0, \\ S &= 2 - K \frac{\partial f(U)}{\partial U} \exp\left(-\lambda \frac{\tau}{T_1}\right), \\ J &= 1 - K \frac{\partial f(U)}{\partial U} \left(\exp\left(-\lambda \frac{\tau}{T_1}\right) + \exp\left(-\lambda \frac{\tau}{T_2}\right) \right). \end{aligned} \quad (2.10)$$

Now we find the equilibrium states by equating the right parts of the equation system (2.6) to zero

$$\begin{aligned} \frac{\partial x(t)}{\partial t} &= \frac{f(D + K(x(t - \tau) - y(t - \tau))) - x(t)}{T_1} = 0, \\ \frac{\partial y(t)}{\partial t} &= \frac{x(t) - y(t)}{T_2} = 0. \end{aligned}$$

Hence,

$$f(D + K(x(t - \tau) - y(t - \tau))) - x(t) = 0, \quad x(t) - y(t) = 0.$$

Due to the time isotropy, i.e. the arbitrariness of t value, from the equation $x(t) - y(t) = 0$ follows the equation $x(t - \tau) - y(t - \tau) = 0$. The same result will be obtained, if to take into consideration that we consider the equilibrium states, i.e. $x(t) = x(t - \tau)$ and $y(t) = y(t - \tau)$. Therefore,

$$x(t) = y(t) = f(D).$$

We may conclude that in the equilibrium state $U = D$. The characteristic equation (2.10) does not contain the values of dynamic variables x, y that would vary independently. But this Eq. (2.10) contains U as their image, strictly equal to the D parameter. Therefore, if *several* equilibrium states take place in the system, they are all either stable or instable simultaneously, which is impossible. It means that the equilibrium state is *unique*. The relation $x = y = f(D)$ can serve as another proof of this conclusion.

2.2 Simulation of Static and Dynamic Modes of the Deterministic Chaos Oscillator

2.2.1 Stability of Equilibrium States

We know that under the loss of stability of the equilibrium states oscillations arise in the systems. Therefore, we can judge about the stability of these oscillations not only by referring to (2.10), but also by analyzing the conditions for the oscillation existence. Let consider them in detail. For this, it is necessary to provide the fulfillment of the conditions for the phase and amplitude balance. In the simplest case of our ring system, the amplitude balance means that the product of gain coefficient and the losses in the feedback loop should be equal to 1. Coincidence of phases at input and output of the linear amplifier is called the phase balance. To excite the sinusoidal oscillations, it is sufficient to fulfill the conditions for the amplitude and phase balance on the same frequency. On the contrary, to excite the multi-frequency oscillations, we must fulfill these conditions for each of the frequencies. The frequency selectivity in the chaotic oscillator can be performed by the given bandwidth of the pass-band filter of the first order (with characteristic times T_1, T_2), the delay time τ , the gain coefficient K and the value of the local slope of the transfer characteristic of the nonlinear element $df(U)/dU$.

Equations for the amplitude-frequency characteristic (AFC) and the phase-frequency characteristic of the pass-band filter, taking into account the delay line (τ), are represented by the following equations

$$\begin{aligned} k(\omega) &= \frac{1}{\sqrt{1 + (T_1\omega)^2}} \times \frac{T_2\omega}{\sqrt{1 + (T_2\omega)^2}}, \\ \varphi(\omega) &= \arctan\left(\frac{1}{T_2\omega}\right) - \arctan(T_1\omega) - \tau\omega. \end{aligned} \quad (2.11)$$

Based on (2.11), as for the neglecting of the phase delay in the nonlinear element and adjustable amplifier, we can draw a series of plots for AFC and the phase-frequency characteristics (Fig. 2.5). As it is known, with the help of the phase-frequency characteristic, it is rather convenient to graphically determine for which frequencies (and for what number of them) the condition for the phase balance is satisfied. And after that, referring to AFC, one can estimate the required gain coefficient K and the slope value $df(U)/dU$ of the NE transfer characteristic, necessary for the oscillation excitement at these frequencies in the oscillator. Let us give several examples of such an analysis.

Thus, in absence of delay ($\tau = 0$), when the operating point is on the ascending part of the NE transfer characteristic, it is necessary to use the curve for $\varphi(\omega)/\pi$ in Fig. 2.5a. According to its shape, we can easily conclude that the phase balance condition is satisfied on the radian frequency $\omega_0 = 0.45$ rad/s. Moreover, we can see from the AFC shape that this frequency ω_0 is near the AFC maximum, and

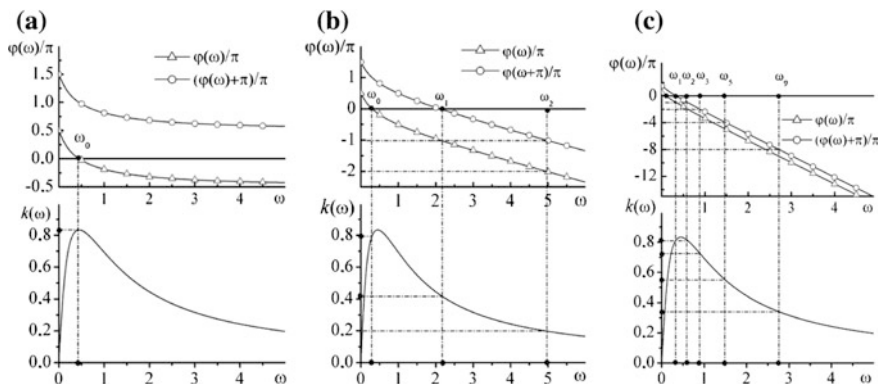


Fig. 2.5 The phase-frequency and amplitude-frequency characteristics of the pass-band filter of the first order with delay $\tau = 0$ (a), $\tau = 1$ (b) and $\tau = 10$ (c), calculated according formulas (2.11) at $T_1 = 1$ (LPF) and χ (HPF). Curves $\varphi(\omega)/\pi$ correspond to oscillator operation on the ascending part of the NE transfer characteristics, and curves $[\varphi(\omega) + \pi]/\pi$ —on the descending part

therefore, the amplitude balance condition fulfillment becomes easier. Let it now the operating point be on the descending part of the transfer characteristic (Fig. 2.5a, the curve $[\varphi(\omega) + \pi]/\pi$). Then, evidently, the oscillation of the arbitrary frequency arisen, for instance, at the output of the adjustable amplifier will return to its output at the same phase, which will lead to the decreasing of its amplitude and, as a consequence, to damping. In this case arising of the dynamic modes is impossible.

If $\tau \neq 0$, the phase balance condition will be satisfied for some countable frequency set, independent of the operating point position on the NE transfer characteristic. So, at $\tau = 1$ for the operating point, which is on the ascending part of the transfer characteristic (Fig. 2.5b, the curve $\varphi(\omega)/\pi$), this condition is satisfied not only for ω_0 , but for higher frequencies ω_{2n} , where $n \in 0, 1, \dots, N$. If the operating point is on the descending part (Fig. 2.5b, the curve $[\varphi(\omega) + \pi]/\pi$), the phase balance condition is satisfied for the frequency set with the odd indices ω_{2n+1} , where $n \in 0, 1, \dots, N$.

The amplitude balance condition requires that the total oscillation gain in the oscillator would be not less than 1. The filter AFC irregularity $k(\omega)$ for LPF and HPF for the case corresponding to Fig. 2.5b, when $k(\omega_0) = 0.79$, $k(\omega_1) = 0.41$, leads to the fact that for oscillation arising at the frequency ω_0 it is enough to have almost two times less gain than it is required for the frequency ω_1 . It is clear that we assume here that for chosen operating points on the transfer characteristic its modulus of the local slope $|df(U)/dU|$ is the same. In other words, in this example, oscillations at operating point position at the ascending part of the characteristic can be excited easier.

The growth of the delay time τ increases the number of frequencies, for which the phase and amplitude balance conditions are satisfied for the given gain K and $|df(U)/dU|$ (compare Fig. 2.5b, with Fig. 2.5c).

Fig. 2.6 The amplitude-frequency characteristic of the pass-band filter of 1st order and positions of spectral components $\omega_0, \dots, \omega_N$ at $T_1 = 1$ and different values of T_2 : 1— $T_2 = 1$; 2— $T_2 = 5$; 3— $T_2 = 10$; 4— $T_2 = 100$

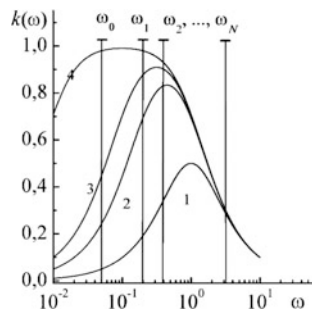


Figure 2.6 shows that, with the growth of HPF time constant value T_2 from 1 to 100, the AFC maximum increases from 0.5 to 1.0, and the pass-band filter bandwidth also increases. Hence, the greater is the T_2 time constant, the smaller values of the gain K and the local slope $|df(U)/dU|$ are enough to satisfy the amplitude balance condition (for frequencies $\omega_0, \dots, \omega_N$).

Let us return to the analysis of stability of equilibrium states on the basis of (2.10). It is known that, if the Lyapunov characteristic exponent is lower than zero ($\text{Re } \lambda < 0$), then such a state of the dynamic system is considered as stable, that is any arisen oscillations in the chaotic oscillator will disappear in time. If $\text{Re } \lambda > 0$, than the smallest fluctuation can turn the oscillator into the dynamic mode. Equation (2.10) is transcendental, therefore, the general (analytical) investigation of it can be labor-intensive. In the specific case, of course, we can get a solution analytically. For instance, at $K = 0$, or when $df(U)/dU = 0$, there is the single solution $\text{Re } \lambda = \lambda = -1$.

Figure 2.7 presents the calculation results for $\text{Re } \lambda$ on the plane of parameters $D - K$. The stable states ($\text{Re } \lambda < 0$) are marked by white color. Values of $\text{Re } \lambda$, which are greater or lower than zero, are marked by grey color tins. Maps in Fig. 2.7 are drawn in the order of delay time τ increase in the feedback loop. Figure 2.7a illustrates the case when $\tau = 0$. On it the areas corresponding to the three descending parts of the NE transfer characteristic ($df(U)/dU < 0$) have a shape of the horizontal white bands, i.e. equilibrium states at such bias D are stable. Their stability loss (dark bands in the figure) is possible only for bias D corresponding to the ascending parts ($df(U)/dU > 0$) with the proper combination of values K and D . These conclusions agree with the results of phase and amplitude balance condition analysis for the case 2.5, a.

The delay τ in the feedback loop τ leads to the cardinal change in the structure of instability regions (as well as of stability regions) of equilibrium states. These dark areas grow with τ increase, and their asymmetry with respect to horizontal lines corresponding to values of $df(U)/dU = 0$ decreases (Fig. 2.7b) and practically disappears (Fig. 2.7c, d).

So, for delay $\tau = 0.05$ (Fig. 2.7b) the instability regions fill the larger part of the plane $D - K$. Nevertheless, areas of both dark and white color remain asymmetric with respect to horizontal lines corresponding to values $df(U)/dU = 0$. Regions of

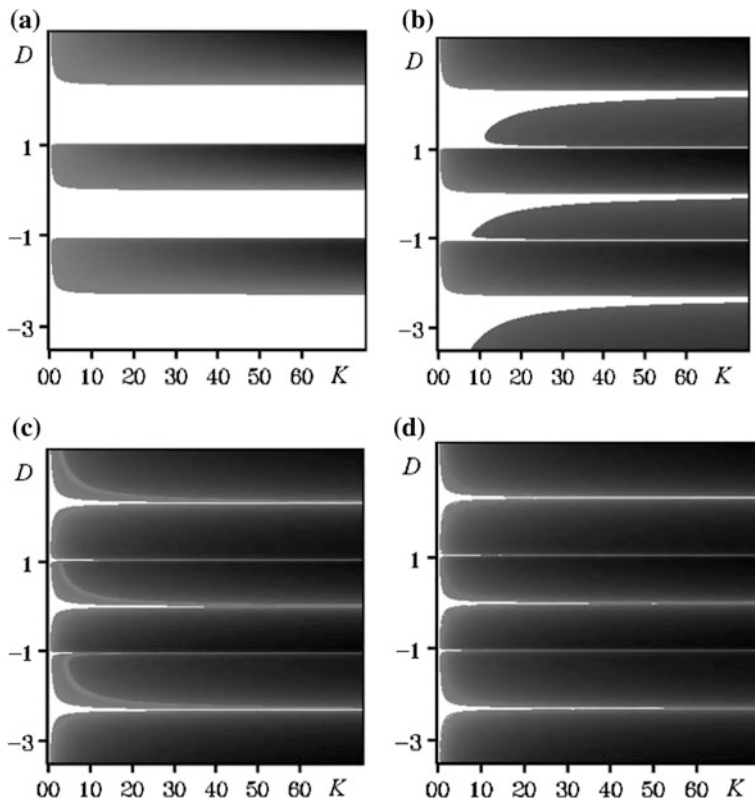


Fig. 2.7 Maps of $\text{Re}\lambda$ distribution on the plane D — K for equilibrium states in the model of chaotic oscillator (2.6) for $T_1 = 1$ (LPF), $T_2 = 7$ (HPF), $a - \tau = 0.05$, $b - \tau = 0.5$, $c - \tau = 30$; $d - \tau = 1000$

stability corresponding to operating points on the descending parts of the NE transfer characteristic have greater area than areas of operation on the ascending part.

This agrees with the results of analysis of the phase and amplitude balance conditions (Fig. 2.5b). Indeed, in the first case the phase balance conditions are satisfied for frequencies $\omega_1, \omega_3, \dots, \omega_{2n-1}$, but the amplitude balance conditions are achieved only at the rather great values of the gain ($K > 10$). And in the second case, even for small values of K the amplitude balance condition is fulfilled for ω_0 , and for this frequency the phase balance condition is already set (Fig. 2.5b). It is clear, that with gain K growth the amplitude balance condition is satisfied for greater number of frequencies, and thus, the multi-frequency oscillations are possible.

These calculations conform with experimental results. As an example, we introduce Fig. 2.8, presenting the signal for the output of the chaotic oscillator, where the delay time is by one order less than the time constant of HPF ($\tau \ll T_2$),

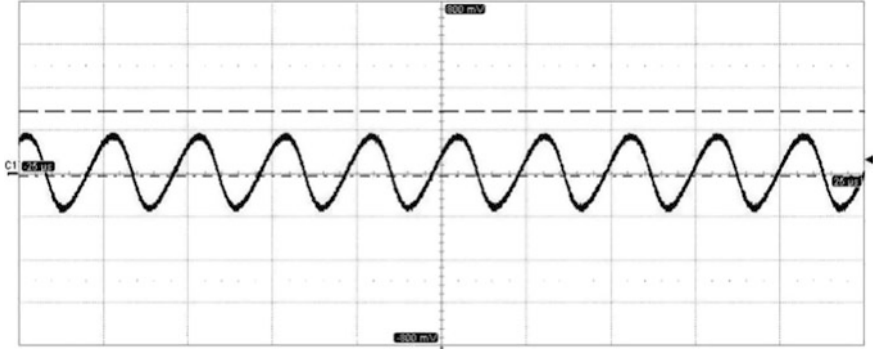


Fig. 2.8 Oscilloscope patterns for oscillations in the oscillator implemented on the base of circuit in Fig. 2.1, when $K = 2.5$, $D = -1.2$ V, $T_1 = 2.7 \cdot 10^{-8}$ s, $T_2 = 1.2 \cdot 10^{-4}$ s, $\tau = 1.5 \cdot 10^{-8}$ s

the bias voltage $D = 1.2$ V corresponds to operating point on the ascending part of the NE transfer characteristic, and the gain $K = 2.5$. The main oscillation occurs at the cyclic frequency $\omega_0 = 2\pi \cdot 196$ kHz = $1.23 \cdot 10^6$ rad/s. When the operating point is on the descending part of the NE transfer characteristic, there are no oscillations in chaotic oscillator.

Figures 2.9 and 2.10 show calculation results for $\text{Re}\lambda$ on the plane of parameters $T_2 - \tau$ for cases when $df(U)/dD > 0$ and $df(U)/dD < 0$, respectively. Inside the maps $\text{Re}\lambda(T_2, \tau)$ we can conditionally extract three zones. The zone I is on the left and higher than the ascending diagonal of the figure. More precisely, the zone I corresponds to the case when the signal delay τ in the feedback loop is lower, or

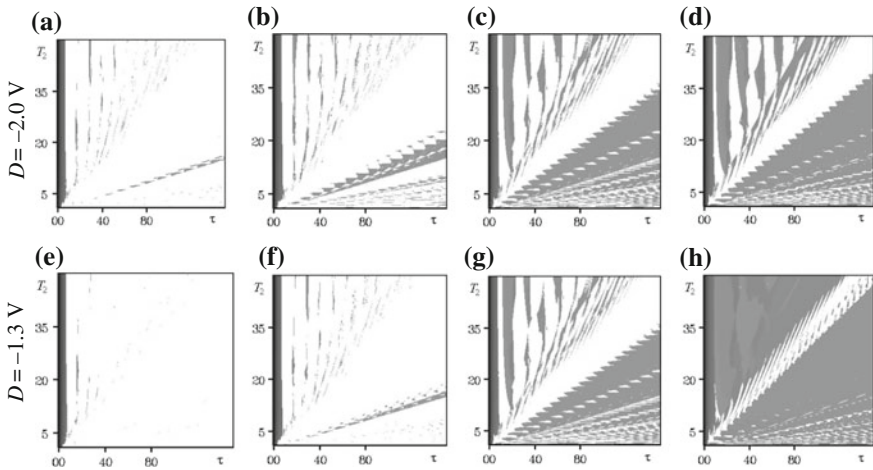


Fig. 2.9 Maps of $\text{Re}\lambda$ distribution on the plane $T_2 - \tau$ for equilibrium states in the chaotic oscillator model (2.6) for $T_1 = 1$; **a** $K = 1.3$; **b** $K = 1.5$; **c** $K = 1.7$; **d** $K = 2.1$; **e** $K = 0.4$; **f** $K = 0.5$; **g** $K = 0.7$; **h** $K = 1.0$. $D = -2.0$ V (**a-d**) and $D = -1.3$ V (**e-h**)

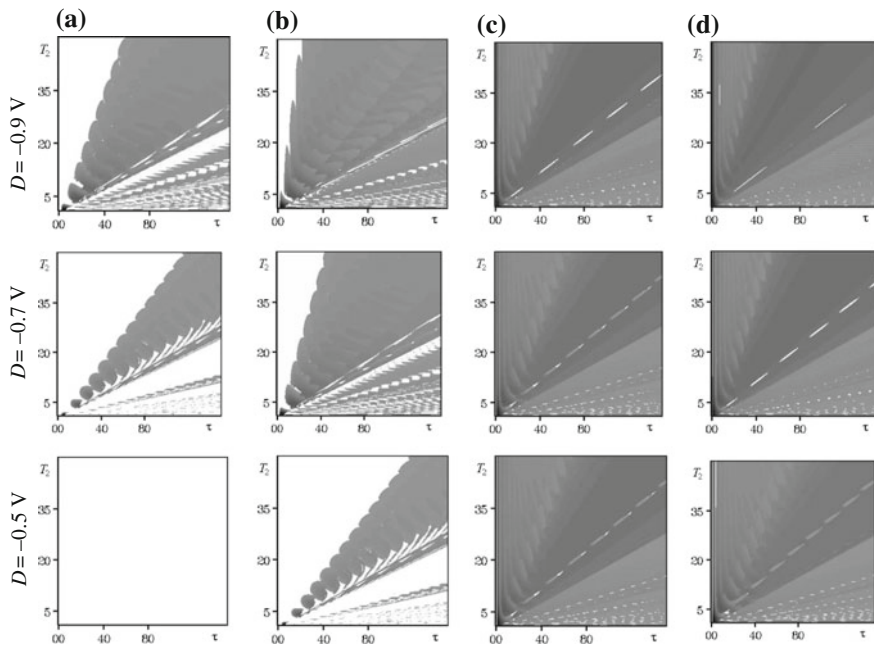


Fig. 2.10 Maps of $\text{Re}\lambda$ distribution on the plane $T_2 - \tau$ for equilibrium states in the chaotic oscillator model (2.6) for $T_1 = 1$; **a** $K = 0.5$; **b** $K = 0.7$; **c** $K = 1.9$; **d** $K = 2.3$

equal to the doubled time constant T_2 of HPF ($\tau/T_2 < 2$). Regions of stability and instability of equilibrium states in the zone 1 are oriented vertically: instability regions have look like irregular spots of different sizes and shapes, but they are elongated along the OT_2 axis. In this zone, for all values of T_2 , the presence of the instability region of the equilibrium states for $\tau < 5$ (Fig. 2.9) is typical, and that agrees with the map structures on the plane $D - K$ (Fig. 2.7a, b). Instability regions increase in sizes with the gain K growth, while stability regions decrease (Fig. 2.9a–d, e–g). When the value of K is great enough, only the unstable equilibrium states correspond to zone 1 ($\text{Re}\lambda > 0$). See, for example, Fig. 2.9g.

The zone 2 is situated more right-hand and below of the ascending diagonal (Fig. 2.9). It corresponds to the case, at which the signal delay value τ in the feedback loop is more or equal to four value of the time constant T_2 of HPF ($\tau/T_2 > 4$). In contrast to the zone 1, areas of stability and instability of equilibrium states in the zone 2 are situated along the beams of various slope, which comes from the origin.

The zone 3 belongs to the diagonal in the area of parameters $2 < \tau/T_2 < 4$ (Fig. 2.9). In it the region of stable equilibrium states has a shape of a triangle with the apex near the origin. The stability region becomes narrower with the growth of the gain K .

2.2.2 Operating Modes in the Deterministic Chaos Oscillator

Dynamic modes in the deterministic chaos oscillator breadboard are specified by the oscillator construction and by the properties of its components. They are set in its model by the form of the evolution operator of the dynamic system and by the parameter values in the operator, in our case by the values of the vector (D, K, T_1, T_2, τ) . Depending on the variation of its controlling parameters, the oscillator demonstrates different dynamic properties. For example, with the growth of gain K the complexification of the dynamic mode takes place. The simple attractors in the phase space change into the complex ones, and that can finally lead to the rise of the deterministic chaos. In this regard, the gain K of the linear amplifier (in the structure of nonlinear amplifier) becomes a measure of the system nonlinearity.

Variation of the position of the operating point (D, K, T_1, T_2, τ) in the parameter space and set along some direction allows observing the appropriate sequence of bifurcations resulting in the chaotic attractor formation. Typical bifurcation sequences are generalized by the concept of bifurcation mechanisms, i.e. the scenarios of the transition to the chaos (development of chaos) [24]. In practice, it is convenient to represent the chaos development scenarios in the bifurcation diagrams.

For our oscillator model (2.6) with NE having the transfer characteristic, shown in Fig. 2.4c, the phase portraits are presented in Fig. 2.11 for different values of controlling parameters. On this figure, along the abscissa axis the values of the dynamic variable x are marked, and along the ordinate axis—the values of the variable y , while the maps are drawn in the order of gain K increase within the range from 1 to 25. To simplify their understanding, the figures are drawn in grey color gradation technique. The color of the phase portrait area reflects the frequency (duration of stay) of the dynamic system in the appropriate condition: darker areas correspond to the long-term and the light areas—to the short-term system stay in the given area.

Under $K = 1$ we have the stable limit cycle. With K growth up to the value of 1.25, there develops the aggravation of anharmonicity of dynamic variable oscillations, and it leads to the growth in the number of maxima during the period of these oscillations. At the bifurcation diagram (Fig. 2.12) this event *looks like* the first act of period doubling. We can easily distinguish the typical divarication of the previously existed branches; moreover, pairs of branches can arise from the single point.

However, the classical bifurcation of the period doubling takes place during the further growth of K : from 1.25 to 1.29 (Figs. 2.11 and 2.12a). A cascade of such bifurcations in the value range $K \in (1.29; 1.31)$ leads to the deterministic chaos mode (Fig. 2.11, $K = 1.31$) at some critical value of K_{cr} . The bifurcation diagrams (Fig. 2.12a, b) graphically demonstrate the transition from the limit cycle to the chaotic attractor. In the specialized literature, this scenario is called the transition to

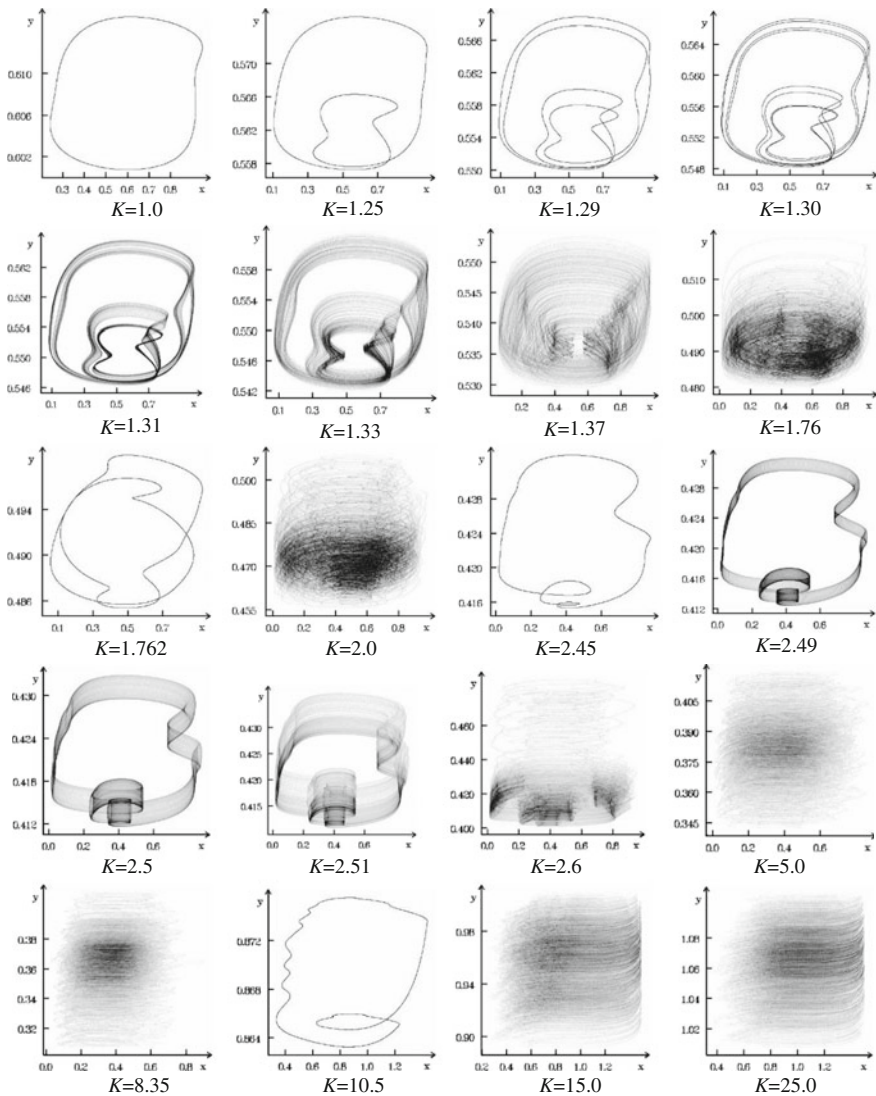


Fig. 2.11 Phase portraits demonstrating a transition from the periodic motion to the chaos through the period doubling bifurcation, as well as through an intermittency in the model (2.6) for gain K changing from 1 to 25, when $D = -0.8$ V, $T_1 = 1$, $T_2 = 100$, $\tau = 5$; $a = 0.95$ V, $b = 2.3$ V

the chaos through a sequence of period doubling bifurcations (by sub-harmonic cascade). We know that the deterministic chaos mode is aperiodic (i.e. the period is infinite) and unstable according to Lyapunov, and is also characterized by the continuous frequency spectrum. The strange attractor in the system phase space corresponds to these oscillations. Its properties are clearly seen in Fig. 2.12 for $K = 1.37$ and $K = 1.76$.

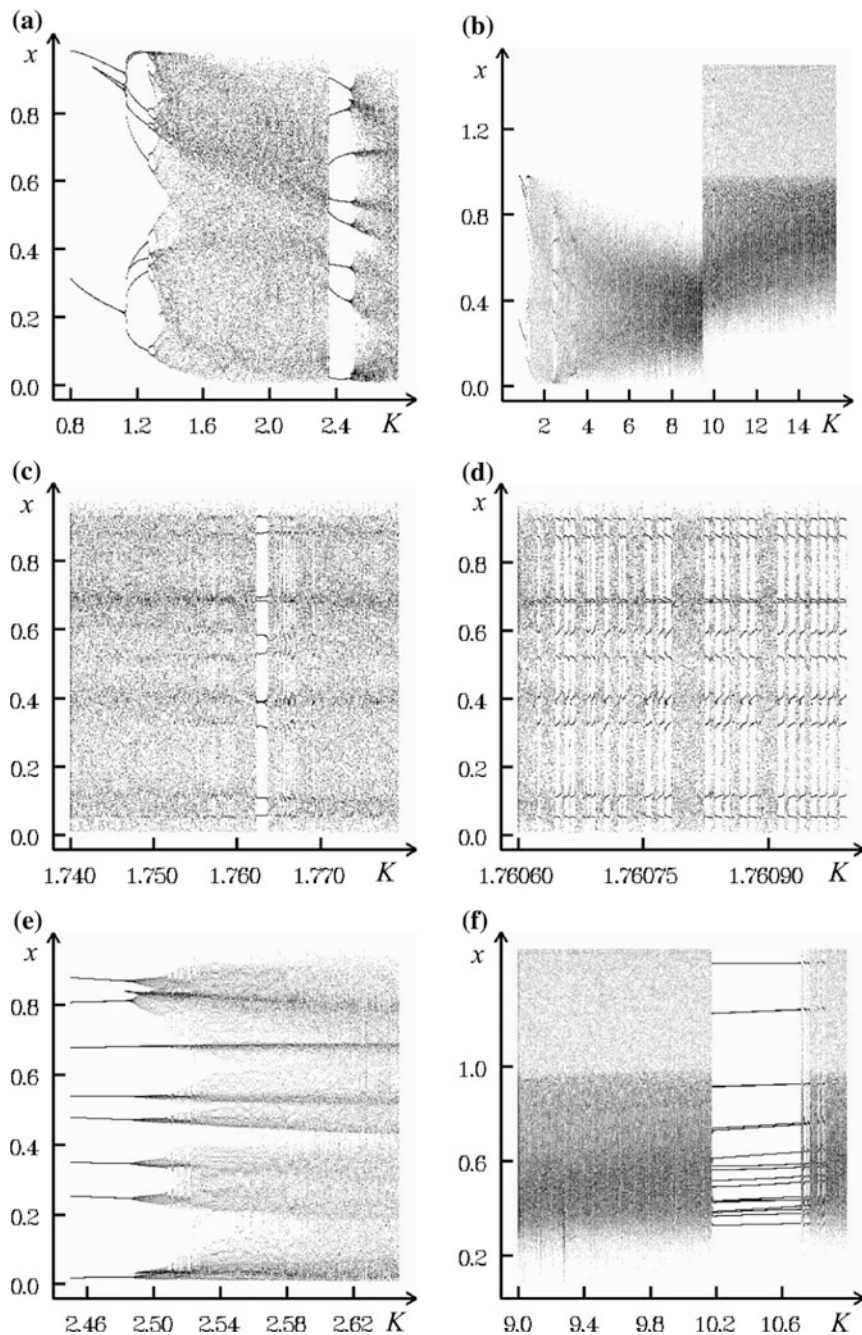


Fig. 2.12 Bifurcation diagrams in the chaotic oscillator model (2.6) in the plane: gain K —extreme values of x voltage at LPF output, when the bias $D = -0.8$ V, $T_1 = 1$, $T_2 = 100$, $\tau = 5$, $a = 0.95$ V, $b = 2.3$ V. Diagrams **a**, **c–e** are zoomed fragments of Figure **b**

Phase portraits for $K = 1.762$, $K = 2.45$, $K = 10.5$ in Fig. 2.11, and also the structure of the bifurcation diagrams in Fig. 2.12c, a, e respectively, justify the existence of “periodicity windows”. In the literature, they call so the areas of the parameter space corresponding to periodic motions in the system, if, with a change of the certain parameter, these areas are surrounded by the areas of such parameters, under which the chaos arises. Periodicity windows in Fig. 2.12a, e (for $K \in [2.39; 2.48]$) can be explained by a presence of slightly sloping part near the extreme of the NE transfer characteristic (the same can be observed in logistic representation). If this characteristic were, for instance, piecewise-linear (say, of the triangle, or of the pyramid type), then periodicity windows would disappear, as well as the sequence of the period doubling bifurcation [24].

“Bandy” shape of the phase portraits in Fig. 2.11 for $K = 2.5$ and 2.51 corresponds to the scenario of transition to the chaos through the collapse of the double-frequency oscillating mode (collapse of the two-dimensional torus) [13, p. 205–217; 25, 26, p. 247–256; 27, p. 209–216]. The similar attractor has been observed in the our model of the modified radio-frequency prototype of the nonlinear ring interferometer [28, 29] (under the title “the chaos of the displaced limit cycle”). In the last model, the intermittency is observed as well. Both aspects of this similarity are probably caused by the fact that there are common circumstances in all these cases:

- (1) the presence of LPF and HPF in circuits of both chaotic oscillators, at that, $T_2 \gg T_1$;
- (2) the (quasi)-periodic shape of the nonlinear transfer characteristic (the harmonic function and a composition of the type Fig. 2.4c);
- (3) the presence of delay.

It is known that the transition from the periodic oscillations may happen by a jump resulting from the only one bifurcation. Such a mechanism is called rigid. It is accompanied by the *intermittency* phenomenon. Intermittency is the alternation mode of the almost-regular oscillations (the laminar phase) with intervals of the chaotic behavior (the turbulent phase), which is observed just after the threshold of the chaos arising. Exactly such a structure is typical for the signal (Fig. 2.13) at LPF output in the model of the chaotic oscillator. The intermittence mode corresponds to the phase portraits in Fig. 2.11 for $K = 1.76$, $K = 2$ and $K = 2.6$. Figure 2.12c demonstrates the rigid mechanism: “periodicity window” ($K \in [1.760; 1.765]$) has the sharp boundaries.

Properties of intermittency are distinguishable at the bifurcation diagrams (Fig. 2.12c, d) due to the grey gradation and to the peculiarities of the figure drawing approach. This approach differs by the fact that this diagram contains the extremes only of the dynamic variables that can be met at the time segment several times less than the characteristic duration of intermittence stages. Because of the random distribution (on the time axis t) of these stages, they are represented by the same random manner in the structure of the bifurcation diagram with K growth (Fig. 2.12d). Therefore, as a whole, the diagram looks like as a set of alternating small-scaling areas, whose view is typical for the diagram depicting the chaotic or periodic mode.

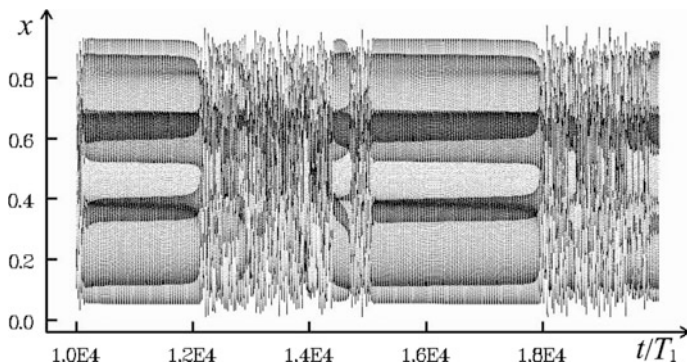


Fig. 2.13 A signal at LPF output in the model (2.6) for $K = 1.76075$, $D = -0.8$ V, $T_1 = 1$, $T_2 = 100$, $\tau = 5$, $a = 0.95$ V, $b = 2.3$ V

We must note in Fig. 2.12b two typical areas separated by the straight line $K \approx 9.5$. For the first area ($K < 9.5$) the dynamic of the x variable is typical, which is limited by the values 0.0 V and 1.0 V. The second area ($K > 9.5$) is characterized by dynamics of x variable within the limits from 0.2 V to 1.5 V.

For a series of bifurcation diagrams in Fig. 2.14, the bias voltage D is a bifurcation parameter. A series is drawn for different values of gain K : 1.3 (*a*, *b*); 5.0 (*c*, *d*);

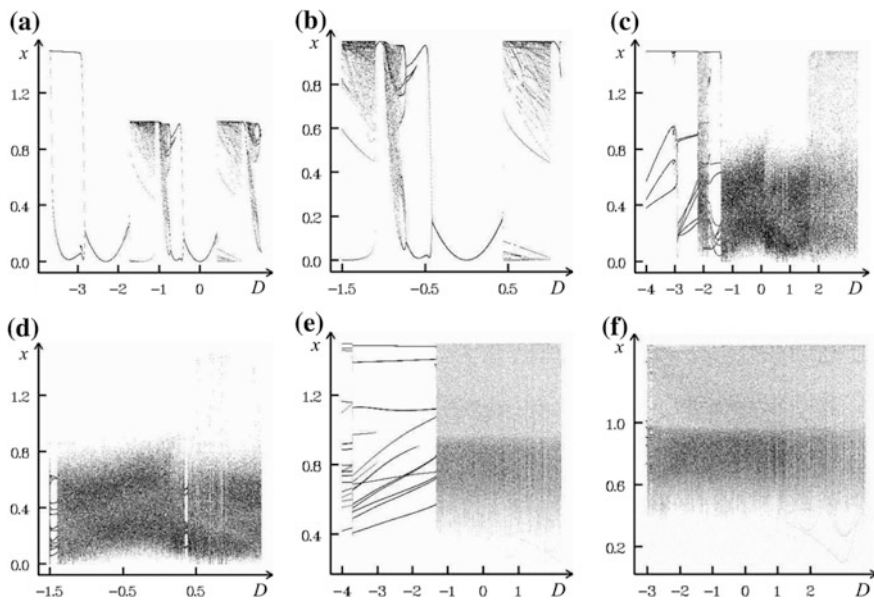


Fig. 2.14 Bifurcation diagrams in the model of the chaotic oscillator (2.6) in the plane xOD , when $T_1 = 1$, $T_2 = 100$, $\tau = 5$, $a = 0.95$ V, $b = 2.3$ V: $K = 1.3$ (*a*, *b*); $K = 5.0$ (*c*, *d*); $K = 25.0$ (*e*, *f*). Figure 2.14e versus f is drawn at decrease of the value D

25 (*e, f*). Figure 2.14b, d are enlargement of the area $D \in [-1.5 \text{ V}; 1.5 \text{ V}]$ on Fig. 2.14a, c, but these diagrams are drawn for another initial conditions than the diagrams in Fig. 2.14a, c. The diagram in Fig. 2.14g, in contrast to the other diagrams (primarily, to Fig. 2.14e), is drawn for decrease of the value D .

According to the view of Fig. 2.14a, b, for small values of K ($K = 1.3$) the static and periodic modes are dominating. The chaotic mode is realized near maxima of the transfer characteristic ($D = \pm 1.2 \text{ V}$) only in narrow areas ($D \in [-1.5 \text{ V}; -0.7 \text{ V}]$ and $D \in [0.7 \text{ V}; 1.5 \text{ V}]$), dissected by even narrower intervals of regular dynamics near the same maxima. At gain K growth (Fig. 2.14c, d, e), areas of chaotic dynamics widen. On bifurcation diagrams in Fig. 2.14c, d ($K = 5$) areas of periodic and chaotic dynamics are easily distinguished. We would like to note two types of chaotic modes: the first one arises (Fig. 2.14d) in the area $D \in [-1.5 \text{ V}; 1.5 \text{ V}]$ and is characterized by variations of x within the limits $[0.0; 1.0]$; the second area takes place for $D \approx -2.0 \text{ V}$ and $D \in [1.5 \text{ V}; 4.0 \text{ V}]$ (Fig. 2.14c) and differs by larger range of values $x \in [0.0; 1.5]$. The same behavior is observed in Fig. 2.12b.

At higher gain ($K = 25$) the periodic mode and the chaotic one of the second type (Fig. 2.12e, f) are observed in the chaotic oscillator mode. Besides it should be noted that, in the last case, x varies within the limits $x \in [0.4; 1.5]$, and it realizes in the area $D \in [-1.5 \text{ V}; 4.0 \text{ V}]$ for quasi-stationary increase of D from -4.0 V to 4.0 V , and in the area $D \in [-3.0 \text{ V}; 4.0 \text{ V}]$ at decrease of D from 4.0 V to -4.0 V .

The area of stable dynamics on the diagram (Fig. 2.14e) is located within the boundaries $D \in [-4.0 \text{ V}; -1.5 \text{ V}]$ or $D \in [-4.0 \text{ V}; -3.0 \text{ V}]$ for quasi-stationary increase, or decrease of the bifurcation parameter D . Mentioned differences depending on the D variation direction can be explained by the hysteresis presence (multi-stability). The essential width of its “loop” corresponds to the noticeable difference between these boundaries on D . Due to hysteresis, we should observe the dependence of the diagram shape upon the initial conditions in the chaotic oscillator (in the moment of diagram construction beginning). It would principally become apparent as the differences in the diagram structure in Fig. 2.14a, c and b, d.

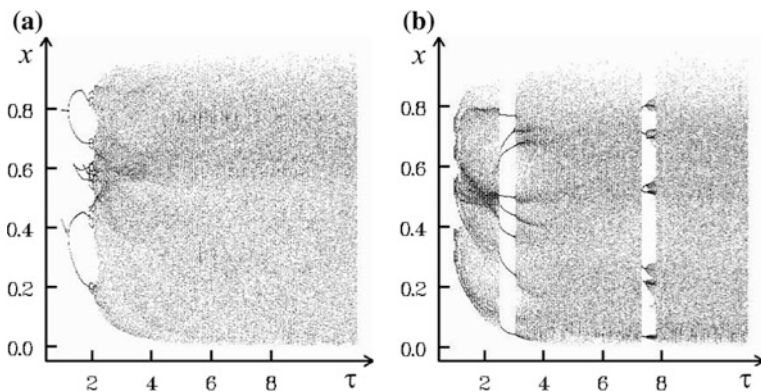


Fig. 2.15 Bifurcation diagrams in the chaotic oscillator model (2.6) in the plane $xO\tau$, when $D = -0.8 \text{ V}$, $T_1 = 1$, $T_2 = 100$, $a = 0.95 \text{ V}$, $b = 2.3 \text{ V}$: $K = 2.0$ (a); $K = 3.0$ (b)

Figure 2.15 represents bifurcation diagrams for the case when the controlling parameter is the delay time. We have clearly distinguishable areas of periodicity (stability) in this Figure, and there are two such areas in Fig. 2.15b.

Let us be limited the analysis of calculation experiments by this.

2.3 Modes and Scenarios of Transitions to Chaotic Oscillations in the Radio-Frequency Oscillator of Deterministic Chaos

Let us proceed to the discussion of the processes in the chaotic oscillator breadboard realized in accordance with the model (2.6) and the structural circuit in Fig. 2.1. It is developed and designed for experimental research and comparison with mathematical simulation results. But firstly we must briefly describe the breadboard.

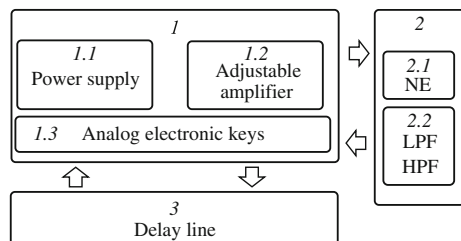
2.3.1 The Breadboard of Deterministic Chaos Oscillator

The structural circuit of the breadboard is shown in Fig. 2.16.

Design of the chaotic oscillator breadboard has been carried out with reference to the requirement of the module implementation. Together with the application of the unified units, this allows maximal simplification of the development of the data transmission system breadboard.

The module 1 (Fig. 2.16) ensures matching and commutation of signals in the chaotic oscillator, as well as the power supply of its units. It consists of the four main sub-modules. The sub-module 1.1 (a power source) provides the regulated voltage for all modules 1 and 2. The sub-module 1.2 (an adjustable amplifier) performs linear transformations of signals (amplification with gain K and bias by the value D). The sub-module 1.3 (analogous electronic gates) is intended for the signal commutation and specification of the initial conditions in the breadboard. The module 2 ensures nonlinear transformation of signals passing from the module 1, with their subsequent return. It includes sub-modules: 2.1 (the nonlinear element)

Fig. 2.16 The structural diagram of the laboratory breadboard of the chaotic oscillator



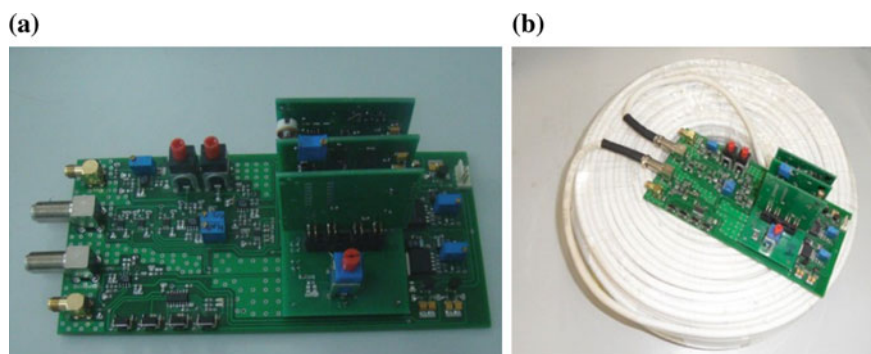


Fig. 2.17 The external view of the chaotic oscillator without the delay

and 2.2 (filters of high and low frequencies of the first order). The module 3 (the delay line) provides signal delay, thus, increases the dimension of the phase space of the dynamic system (the chaotic oscillator).

Figure 2.17 shows the laboratory breadboard of the deterministic chaos oscillator without the delay line (a) and with it (b). The delay line 9 represents the segment of the coaxial cable (RG-6) with wave impedance 75Ω . The cable with a length from 0.1 to 100 m in this case performs the lag in the time range from 15 to 300 ns (the minimal unavoidable lag arises as the signal passes via electrical circuits of the oscillator). The time lag increase due to the coaxial cable application is limited mainly by the bulkiness of the design.

If the experiments require longer signal time lags, one can apply the delay line based on the fiber optics (Fig. 2.18). To match the electrical and optical sections of the breadboard, we can use the optical-electronic transceiver. In our case, it allows operation even when the length of the optical cable is up to 3 km.

Application of the waveguide elements as the delay lines is justified by the following circumstances. Due to small damping, AFC regularity and linearity of the

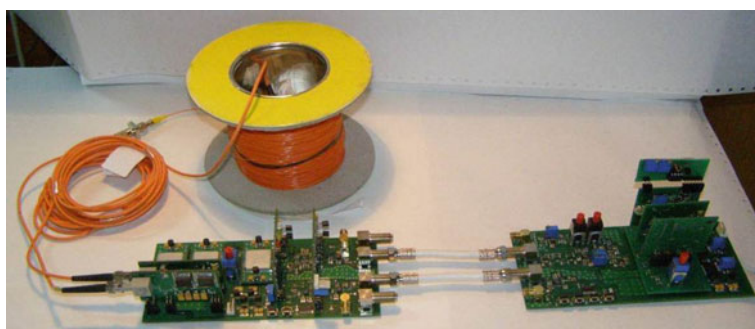


Fig. 2.18 The laboratory breadboard of the deterministic chaos oscillator with the fiber-optical delay line

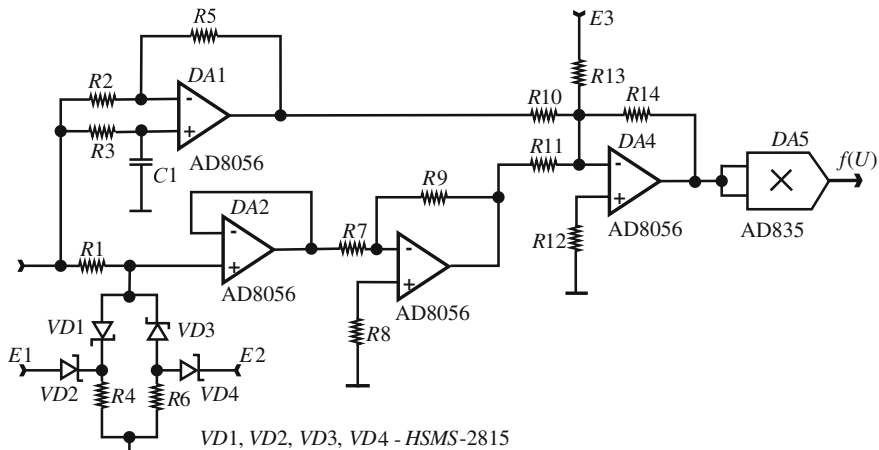


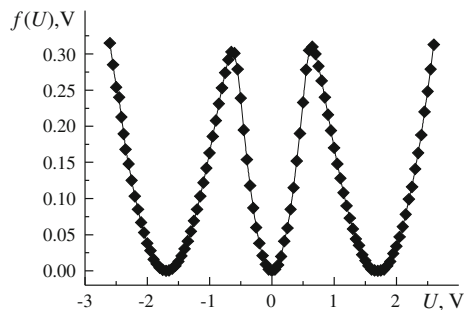
Fig. 2.19 The electrical circuit of the nonlinear element

phase-frequency characteristic of the optical fiber in the band from 0 to 100 MHz, the lag time up to 15 μ s is achievable. The further increase of the length of the fiber-optical line is limited by the necessity of multiple connections of the fiber hanks. This attenuates the signal and leads to the low signal/noise ratio at the output of the transceiver.

From all the developed modules and sub-modules of the deterministic chaotic oscillator breadboard (Fig. 2.16) we will now present only the electrical circuit of the nonlinear element (Fig. 2.19). Its transfer characteristic has a form of parabola composition (Fig. 2.20).

Proceeding to the discussion of the processes in the laboratory breadboard of the chaotic oscillator, we shall structure description according scenario types of transition to the chaos.

Fig. 2.20 The transfer characteristic of the nonlinear element in Fig. 2.19 for parameters $E_1 = 667$ mV, $E_2 = -667$ mV: $a = 0.31$ V, $b = 1.7$ V



2.3.2 Transition to the Chaos Through the Period Doubling Bifurcation

As we know, the graphic representation of the dynamic modes and their direct analysis during the experiment is realized through the construction of the phase portraits. Figure 2.21 shows a series of phase portraits plotted for the case of the controlling parameter K growth, the voltage value at the delay line input is used for

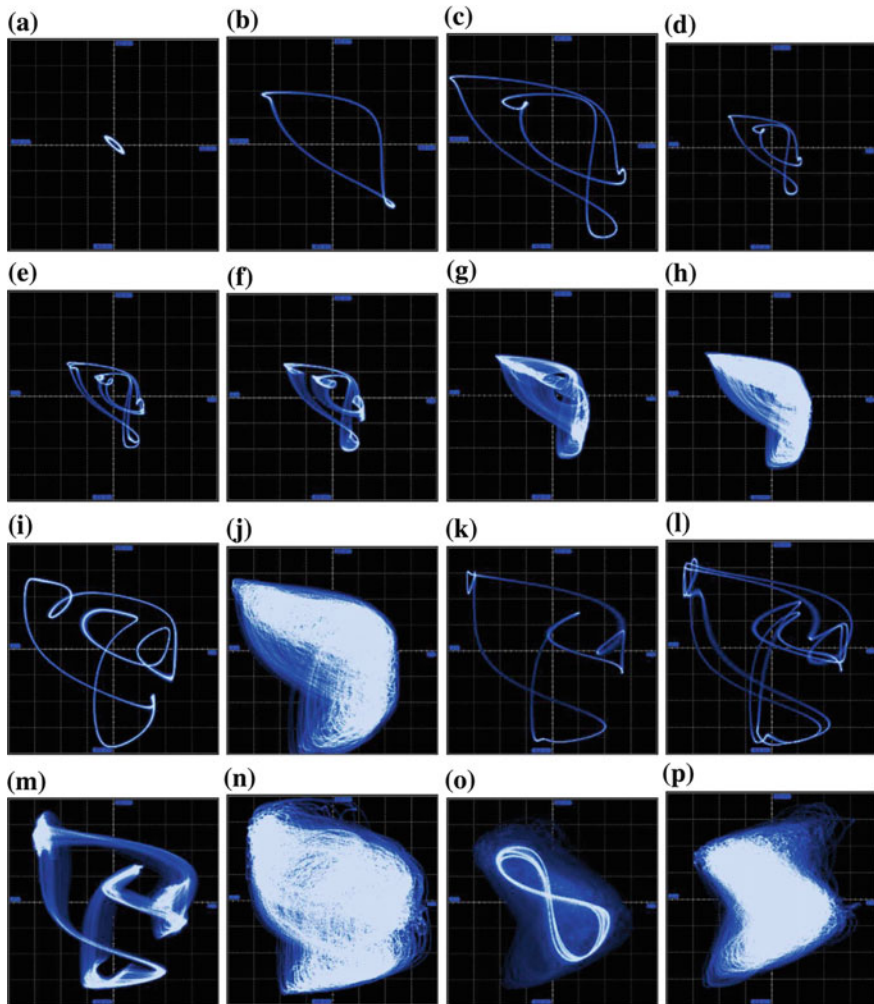


Fig. 2.21 Phase portraits obtained in the experiment with the chaotic oscillator: the signal $z(t)$ is on the abscissa axis, the signal $z(t-\tau)$ is on the ordinate axis. The gain K increases from 1 (a) to 8 (p), the bias voltage $D = -0.57$ V

ordinate axis, and for the abscissa axis—the output delay line voltage is used. There are modes in the chaotic oscillator, for which the dynamics performs near the frequency, given by the inversed time delay ($f_1 = 1/\tau$). Therefore, representation of signals on the phase portraits in such coordinates simplifies distinguishing between the dynamic modes and the scenarios of transition to them.

Figure 2.21 is plotted in the afterglow mode of the oscilloscope screen: the brighter are the separate places of the screen, the oftener the signal “falls” into these places. In other words, such method of the portrait plotting allows evaluating the frequency of the “visits” the dynamic system pays to a certain area of its phase space. The similar possibility is provided by the phase portraits in the Sect. 2.2 (for instance, in Fig. 2.11) and by the two-dimension histograms.

The ellipse in the center of the oscilloscope screen in Fig. 2.21a, confirms the presence of the periodic oscillation mode in the chaotic oscillator. There is one dominating component in the power spectrum of these oscillations, and there are also components, which are less by 40 dBm (Fig. 2.22a). In this case, the phase and amplitude balance conditions in the oscillator are satisfied at the frequency $f_1 = 1.036$ MHz. With K gain growth, the stable periodic process is realized in the chaotic oscillator with the period $T(K) = 0.959$ μ s (the limit cycle C). At the same time, both main spectral component amplitude ($f_1 = 1.043$ MHz) and higher harmonics increase. In the power spectrum there are both even (2nd, 4th, 6th) and odd (3rd, 5th, 7th) harmonics present, whose values are commensurable with the level of the main oscillation (Fig. 2.22b). With further increase of K , the period doubling bifurcation occurs, leading to the emerging of the stable limit cycle $2C$ with the period $2T(K) = 1.918$ μ s (Fig. 2.21c, d). This is accompanied by the signal with half-frequency harmonics appearing in the spectrum $f_{1/2} = 0.52$ MHz of the main oscillation (Fig. 2.22c). The further increase of the gain is accompanied by the regular bifurcation of the period doubling (Figs. 2.21e and 2.22d) and by the appearance of the stable limit cycle $4C$ with the period $4T(K) = 3.836$ μ s.

The similar set of period doubling bifurcations (caused by K growth) gives birth to sub-harmonics $f_{1/2}, f_{1/4}, f_{1/8} \dots f_{1/(2^n)}$ in the signal power spectrum leading to the dynamic chaos arising (Fig. 2.21f–i and 2.22f–j). The uniform attractor filling the phase portraits, as well as the continuous spectrum of the chaotic signal in some frequency band, confirms the applicability of the deterministic chaotic oscillator for the data transmission in the mode with the chaotic carrier (Table 2.1).

2.3.3 Transition to the Chaos Through Intermittency

Transition to the chaos through the intermittency phenomenon was investigated for the first time in the studies by Pomeau and Manneville [24, 30]. Mathematical modeling shows that in the chaotic oscillator the scenario of transition to the chaos is possible by the exactly mentioned way. Thus, it can be distinguished on the bifurcation diagrams (Figs. 2.12c, d and 2.13) for the chaotic oscillators with the operating point on the descending part of the NE transfer characteristic in the

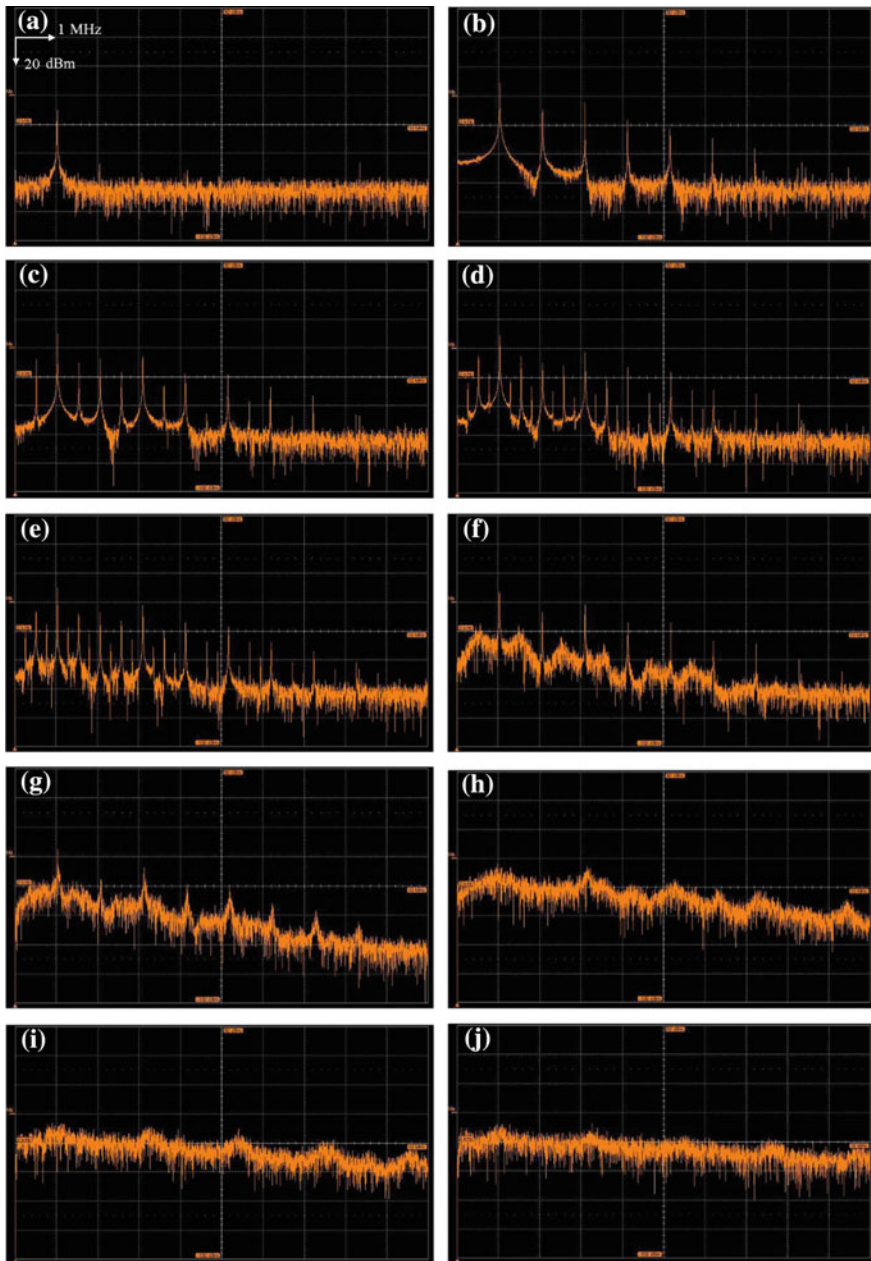


Fig. 2.22 Variation of the power spectrum of the $z(t)$ signal on the HPF output in the experiment with the chaotic oscillator at gain K increase from 1 (a) to 8 (j), the bias voltage $D = -0.57$ V. One cell of the axes is 1 MHz and 20 DB. Correspondence of the power spectra in Fig. 2.22 to the phase portraits in Fig. 2.21 see in Table 2.1

Table 2.1 Correspondence of the power spectra in Fig. 2.22 to phase portraits in Fig. 2.21

Figure 2.22	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>
Figure 2.21	<i>a</i>	<i>b</i>	<i>c, d</i>	<i>d</i>	<i>e</i>
Figure 2.22	<i>f</i>	<i>g</i>	<i>H</i>	<i>h</i>	<i>j</i>
Figure 2.21	<i>i</i>	<i>i</i>	<i>J</i>	<i>n</i>	<i>p</i>

vicinity of the stable area. At the photo picture from the oscilloscope screen at HPF output we clearly see areas of the quasi-periodic (the laminar phase) and chaotic (the turbulent phase) dynamics. In the experiment with the chaotic oscillator, for parameters $D = -570$ mV and $K = 3.62$, the periodic more is realized (Fig. 2.23a). In the vicinity of this point on the parameter plane DOK the transition to the chaos occurs through intermittency: (1) with K growth to the critical value $K = 3.64$, the regular oscillations begin to be interrupted by chaotic bursts (Fig. 2.23b); (2) the bursts duration grows (Fig. 2.23c, d) with growth of controlling parameter K until oscillations become completely irregular.

Figure 2.24 shows the signal $z(t)$ and its wavelet-transformation with the base function Morlet wavelet (Gabor wavelet). Time realizations are obtained on the digital oscilloscope Lecroy Wave Surfer $\times 62$ with embedded ADC with discretization period $\Delta t = 1$ ns. However, to create the wavelet-spectrograms the samples have been resampled down: each 50th sample is used, i.e. for the

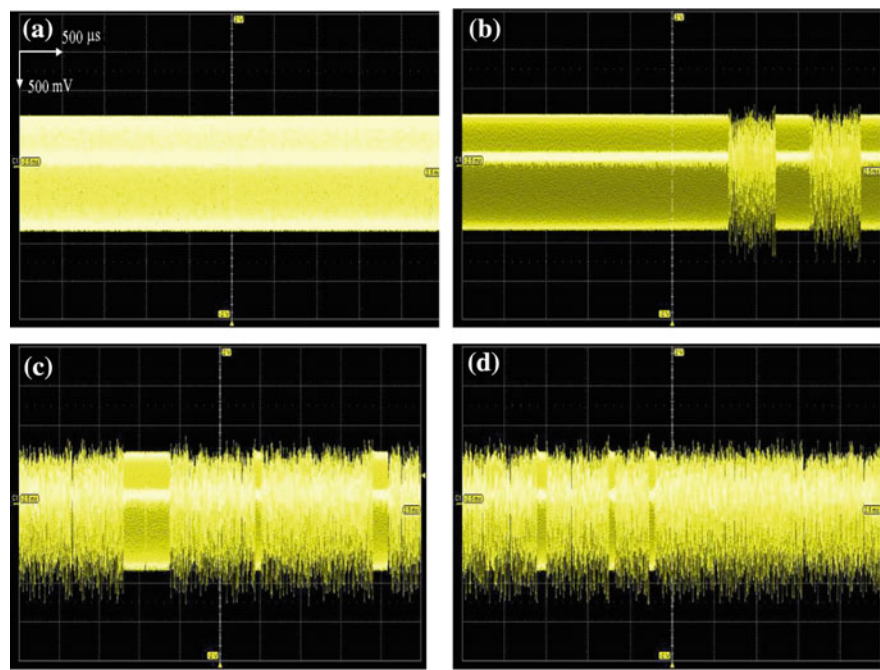


Fig. 2.23 The photo picture form the oscilloscope screen: the signal $z(t)$ in the chaotic oscillator for $D = -0.57$ B: **a** $K = 3.62$; **b** 3.64 ; **c** 3.7 ; **d** 3.71

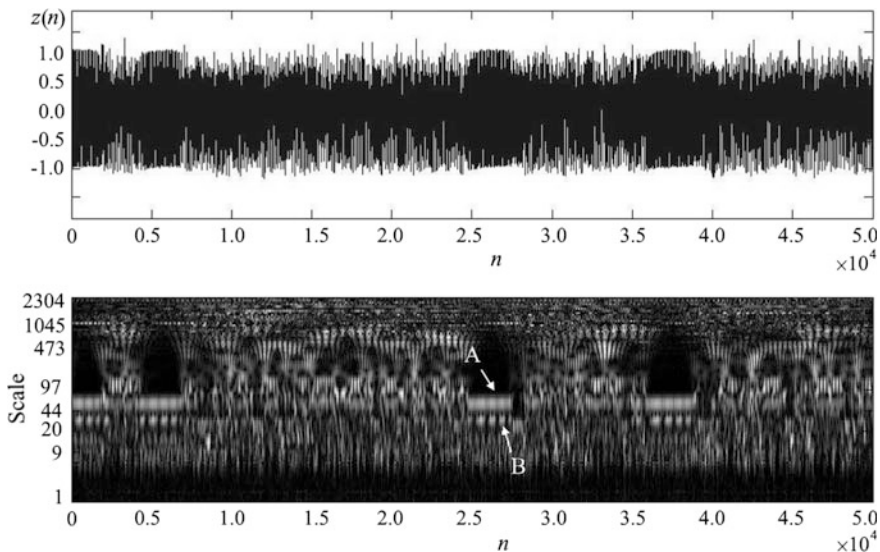


Fig. 2.24 Time realization of $z(t)$ in the chaotic oscillator breadboard and its wavelet-spectrogram in the intermittence mode for $D = -0.572$ V

wavelet-spectrograms $\Delta t = 50$ nsec. The typical structures, corresponding to the laminar (regular) and turbulent phases, are clearly seen on the wavelet plane. The regular phase is mainly characterized by two global maxima on the scales $1/f_1 \approx 60$, $1/f_2 \approx 30$. Two light areas A and B (oriented parallel to time axis) consisting of alternating light and dark bands, correspond to them. The presence of these areas and its almost periodic structure relates, first of all, to the shape of the regular signal (with its Fourier spectrum).

After the dynamics in the chaotic oscillator transfers to the turbulent phase, the shape of the wavelet-spectrum changes: the scale of the splitting observed is $1/f_1$, $1/f_2$, and there is the “burst” of oscillating phenomena with different time scaling. The excitement (irregular-alternative), as well as the oscillation suppression in the chaotic oscillator with the various scaling, occurs during the whole turbulent phase. It is known [30], that in such cases the main oscillation energy fit into this very scaling range, within which the two light areas are located before the “burst”. Nevertheless, the energy scale distribution considerably changes during the turbulent phase.

2.3.4 Transition to the Chaos Through a Collapse of Two-Frequency Oscillating Mode

For the value of the parameter $D = -570$ mV, $K = 5.56$ the periodic mode (the limit cycle in Fig. 2.25a) is realized in the laboratory breadboard of the chaotic oscillator.

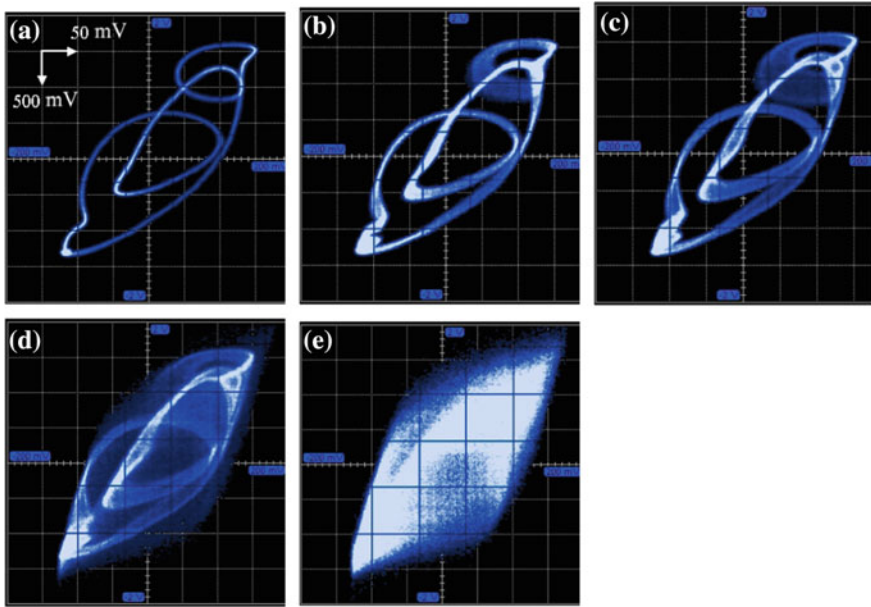


Fig. 2.25 Phase portraits obtained in the experiment: signals $x(t)$ and $y(t)$ are marked on the axes of abscissa and ordinate. Here the bias $D = -570$ mV, the gain K has values: **a** 5.56; **b** 5.63; **c** 5.71; **d** 5.74; **e** 5.83

If you start increasing the value of the K parameter, the attractor shape changes (Fig. 2.25a–e): the line thickness increases and the trajectories on the attractor are dispersed (dithered), which means the chaos arising.

Figure 2.26a–e shows the power spectra corresponding to this transition from the periodic mode to the chaos with K growth. The limit cycle (Fig. 2.25a, $K = 5.56$) is characterized by the presence of harmonics $f_2 = 1.316$ MHz, $f_3 = 1/\tau = 1.974$ MHz etc. ($f_n = n \times 658$ kHz) together with the main frequency $f_1 = 658$ kHz. Growth of the gain to values $K = 5.63$ leads to appearance the components on frequencies $f_s = 110$ kHz and $2f_s = 220$ kHz in the power spectrum. In contrast to the Feigenbaum scenario, new harmonics mf_s appear at frequencies aliquant to the half of the fundamental frequency f_1 (Fig. 2.26b). Nonlinear interaction of the last harmonics and the harmonics f_n of the “main series” leads to appearance of oscillations on the frequencies $f_{n1} = (n \times 658) \pm 110$ kHz in the signal spectrum. The presence of the low-frequency components leads to the dithering of the limit cycle trajectories (Fig. 2.25b).

The further growth of K enriches the spectrum of the new low-frequency components. Their nonlinear interaction becomes a reason for the appearance of the new harmonics and sub-harmonics in the high-frequency part of the power spectrum. The further growth of K leads to the increase in the number of the components

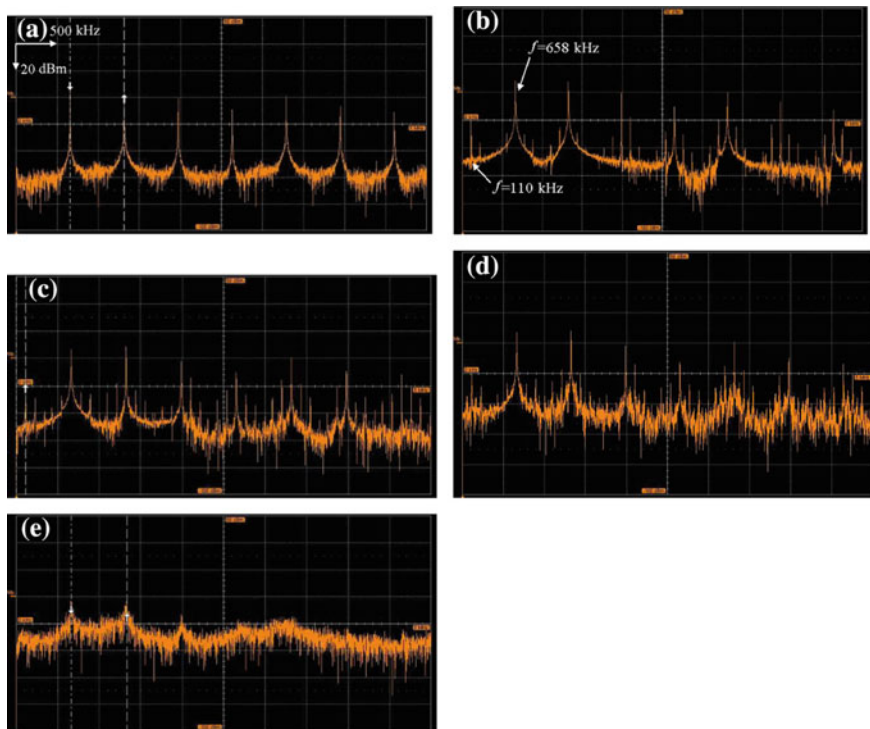


Fig. 2.26 Variation of the Fourier spectrum of the $y(t)$ signal at HPF output in the experimental set up at the gain K growth

in the power spectrum (Fig. 2.26c, d). After exceeding some threshold value K one recognizes the oscillation mode as chaotic (Fig. 2.26e).

2.3.5 Transition to the Chaos Through a “Semi-Torus” Collapse

In [2, 31] there is described the transition to the chaotic motion type developing in accordance with the auto-parametric scenario in the autonomous dissipative nonlinear dynamic system. Let its state be instable, owing to which the system performs the periodic motion $\varepsilon(t)$ with the frequency ω_1 . Let the system nonlinearity be described by the function $f[\varepsilon(t)]$. At $\varepsilon(t)$ evolution, the local slope of nonlinearity $df[\varepsilon(t)]/d\varepsilon(t)$ changes influencing to stability conditions of this or that possible states of the system under consideration. Let properties of the given system be such that with the exceeding by the intensity of the process $\varepsilon(t)$ of some threshold ξ the conditions for self-excitation of the second process $\theta(t)$ with the frequency

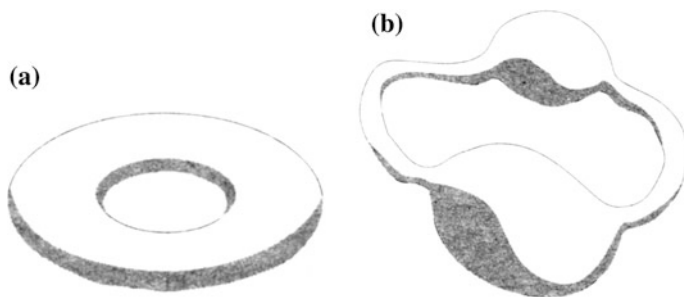


Fig. 2.27 An example of 2D-torus (a) and of a set diffeomorphic to 2D-torus (b) [21]

$\omega_2 > \omega_1$ are created. The second (fast) process exists only during the part of the period of the first process $\Delta t \in T_\varepsilon = 2\pi/\omega_1$, for which the intensity of the process $\varepsilon(t)$ exceeds ξ . During the remaining part of the period $T_\varepsilon - \Delta t$, the conditions of existence for $\theta(t)$ are absent, and, therefore, the second process is damping. In this case in the phase space there is a *semi-torus*—object representing the limit cycle, in whose jumps the torus fragments arise [2, 31]. Because of the presence of internal fluctuations, the torus fragments appear each time in various initial conditions. Therefore, the transition to chaos is possible either by the collapse of half-torus as a whole, or by the “winding” of the phase trajectory over it. The transition to chaos has been called *auto-parametric*, since in this case the system changes its parameter—the local slope $df[\varepsilon(t)]/d\varepsilon(t)$ of nonlinearity—“independently”.

Let us try to consider the semi-torus (as an object in the phase space involved into the complex behavior of the dynamic systems) in a more *general* way. To do so, it is useful, firstly, to take into account that very fact that in nonlinear dynamics and in the context of the attractor typology it is normal to speak about attractors of the torus type and about motion on the torus, etc. Under two-dimensional torus in the three-dimension phase space we here assume not only the ideal circumference rotating around some axis, but not intersecting it (the torus in its traditional understanding—Fig. 2.27a), but also the so-called set, which is diffeomorphic in relation to the 2D-torus (Fig. 2.27b).

Less strictly speaking, we may say that a radius of the rotation circle can vary greatly during rotation, but the period of this variation may be equal to the rotation period. Moreover, it is the arbitrary closed line that may rotate instead the circle! Simultaneously, we can change the distance between this closed curve and the rotation axis. In other words, the attractor of the torus type (or simply—torus) is similar to the highly deformed (inflated or flattened here and there) hollow hoop, but without breaks on the hoop surface.

Under such representation of these attractors, the half-torus looks like as the limiting case of the torus, when some of its parts in their transverse size are compressed to the size of a line. We put it as “looks like”, since when the intensity of the process $\varepsilon(t)$ does not exceed the threshold ξ and the conditions of existence for $\theta(t)$ are absent, then it means that earlier existed fast motion $\theta(t)$ is damping

and, sooner or later, this damping will go asymptotically and exponentially. In other words, the motion $\theta(t)$ does not disappear completely and, therefore, the part of the torus does not degenerate into a line. Hence, the inaccuracy of the interpretation of the half-torus as a limit cycle, in whose “jumps” there are fragments of the torus, is obvious.

Secondly, the slow process $\varepsilon(t)$ controlling the excitation conditions for the fast motion $\theta(t)$ can be a product of the external system. That is, $\varepsilon(t)$ can be external modulation influence. Then, due to the specific choice of the dynamic system, its parameters and impact characteristics, a certain half-torus can be realized in the phase space [32]. Such an approach requires the presence of the external system, but, owing to this, it is more flexible concerning the possibility of generating attractors with arbitrary number of the sectors of the fast motion and of the types of this fast motion. Besides, it is expedient to draw the bifurcation and stability diagrams of the static states of the controllable system and to interpret them as the modulation characteristics of this modulator system [33].

Let us return to the consideration of the modes of our radio-frequency chaotic oscillator. Now we are to find out, if this scenario of the transition to chaos could be realized. Figure 2.28 presents phase portraits, time realizations and signal power spectra in the deterministic chaotic oscillator, when its operating point lays on the *ascending* part of the NE transfer characteristic (Fig. 2.20) in contrast with the above discussed case. The attractor shape corresponds to the shape of the half-torus, as we can see, for instance, from [2, p. 71].

Probably, we have not observed the semi-torus earlier, due to the choice of the operating point on the *descending* part. This assumption, evidently, could be substantiated by involving the maps of stability of equilibrium states (Figs. 2.7, 2.9 and 2.10) and by referring to the amplitude-phase conditions for oscillation excitement (Fig. 2.5). We must remind that they confirm the essential differences in excitement conditions for the chaotic oscillator—depending on at which part of the NE transfer characteristic the operating point is situated—on ascending, or on descending one.

The transition to the deterministic chaos mode will now be considered in detail. Under the value of parameter $K = 1.76$, the periodic oscillation (the limit cycle) with the fundamental frequency $f = 97.1$ kHz arises in the chaotic oscillator. With K growth, the period of this oscillation in the chaotic oscillator increases (under $K = 2.27$ the fundamental oscillation occurs at the frequency $f = 87.2$ kHz), due to the dependence of the amplitude-phase condition upon K , as well as to the increasing role of the nonlinearity in the oscillator. Under $K = 3.37$ the higher frequency oscillations appear (the torus) at the time periods corresponding to the extremes of the dynamic variable $x(t)$ (that also corresponds to the signal extremes in the input of the nonlinear element). The further growth of gain ($K = 4.43$) leads to the increase of the high-frequency oscillation amplitude.

After that, the phase trajectory intermixing (destruction) takes place and the strange attractor appears in the phase space ($K = 4.48$ and $K = 4.69$). To such a behavior in the oscillator the continuous Fourier spectrum explicablely corresponds (Fig. 2.28c).

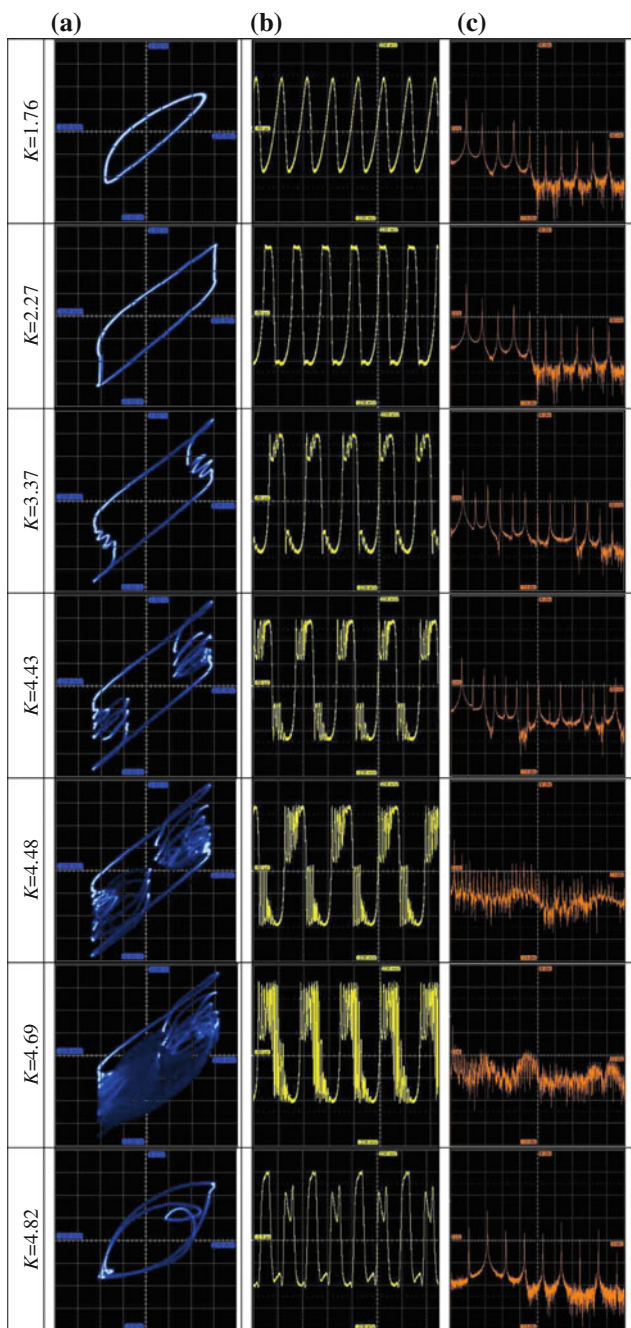


Fig. 2.28 The phase portraits (a), the signals (b), the power spectra (c) for $D = 300$ mV

If to continue growing the gain, then the destruction of the strange attractor occurs, and it is replaced by the limit cycle with a shape different from the earlier existed one and the fundamental frequency turns to be $f = 1.061$ MHz ($K = 4.82$, Fig. 2.28a). The phase portrait has a shape of the doubled loop typical for the attractor that has gone through the period doubling bifurcation. This allows suspecting that here are the presence of a *hysteresis* and the possibility (with K reduction) of observing “the single” limit cycle with the fundamental frequency $f = 2.122$ MHz. Then, in Fig. 2.28a ($K = 4.82$), we can consider the spectrum component with $f = 1.061$ MHz as a sub-harmonic for the frequency 2.122 MHz. The fact that the further growth of K leads to the period doubling bifurcation cascade and to the appearance of the chaotic mode confirms this assumption.

In addition to the phase portraits and spectra, we are now to handle the bifurcation diagrams. They will also allow revealing both static and dynamic modes studied in the experiment.

2.3.6 Bifurcation Diagrams

Let us draw the bifurcation diagrams for the case when the bias voltage D (Fig. 2.29a), or the nonlinearity coefficient K (Fig. 2.30), serves as the bifurcation parameter. The variation of bias voltage D affects the local slope of the transfer characteristic (Fig. 2.29c), accompanied by the mode changes during both the physical experiment (Fig. 2.29a) and the modeling phase (Fig. 2.29b). The values of K have been chosen in order to ensure the dynamic mode operability under the maximal number of D bias values, varying in the range from -1.9 V to 1.9 V with a step of 0.05 V. It has made 76 measurements possible.

Because of the noise in experimental set up, the false extremes of the signals appear, and therefore, their representation on the bifurcation diagram (Fig. 2.29a) lead to larger blur compared to the calculated one. Nevertheless, comparison of

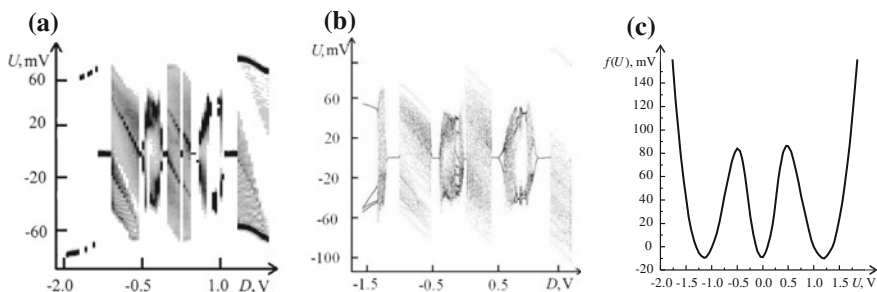


Fig. 2.29 The bifurcation diagram with the D bifurcation parameter for $K = 11.3$: **a** experiment, **b** calculation; **c** experimental NE transfer characteristic ($a = 86$ mV, $b = 1.2$ V), included in the model (2.6) structure. Darkening degree on the diagrams correspond to the falling frequency of the signal local extreme in the given point

Figs. 2.29a, b confirms qualitative (structural and morphological) and even quantitative similarity of calculation and laboratory results. We can clearly distinguish zones with periodic and chaotic dynamics, as well as static modes. Lack of coincidence in the amplitude values is the result of the presence of the amplifier at the output of the chaotic oscillator.

The static mode, i.e. the absence of oscillation in the chaotic oscillator is represented on the bifurcation diagram (Fig. 2.29) by the single line. Static modes correspond to the slightly sloping parts on the transfer characteristic, where the derivative $df(U)/dU$ is small or equal to zero. The static mode by itself is not suitable for the data transmission by the chaotic carrier, but the knowledge of parameter areas in the chaotic oscillator corresponding to the static modes is necessary for the reliable implementation of dynamic modes in the transmitter.

Periodic modes are interpreted on the bifurcation diagram as pairs of lines (or several pairs of lines), and areas with the chaotic mode are represented by the continuum of points having different tints of grey color.

During the plotting of the experimental bifurcation diagrams with bifurcation parameter K (Fig. 2.30) ten values of the D controlling parameter from -0.8 V to 0.8 V have been chosen within the range of D , the ones corresponding to the different values of the local slope of NE transfer characteristic in Fig. 2.29c.

These experiments demonstrate various scenarios of transition to chaos. The sequence of period doubling bifurcations is observed both on the descending and the ascending parts of the transfer characteristic (Fig. 2.30a–g, j). The transition from the limit cycle mode to chaos through the collapse (by jump as a result of the only one bifurcation) in operating point ($D = 0.8$ V) on the descending part (Fig. 2.30j). Moreover, the collapse is observed at $D = -0.5$ V (Fig. 2.30c) and $D = 0.5$ V (Fig. 2.30i). The mentioned operating points correspond to the extremes on the NE transfer characteristic (Fig. 2.29c). Initially, the dynamic system is in the equilibrium state, but with the K growth up to the critical value ($K \approx 13$) it abruptly transmits to the deterministic chaos mode. Near the value $K = 12$ on the bifurcation diagrams (Fig. 2.30b, d, f, j) there are stability areas present. According to the modeling results (Fig. 2.12), with parameter K growth and while approaching the right boundary of the area, the scenario of transition to chaos is realized through the collapse of the two-frequency self-oscillations mode.

Thus, we get the mathematical model of the chaotic oscillator and the appropriate radio-electronic breadboard. We also have investigated the conditions for both theoretical and experimental researches, allowing the dynamic chaos appearance. Besides we have revealed the similarity between the results of computer modeling and laboratory experiments. This allows application of the oscillator in the confidential communication systems using chaos.

Now we are to proceed to the analysis of the chaotic oscillators of the optical range, which are applicable for the information protection also [34].

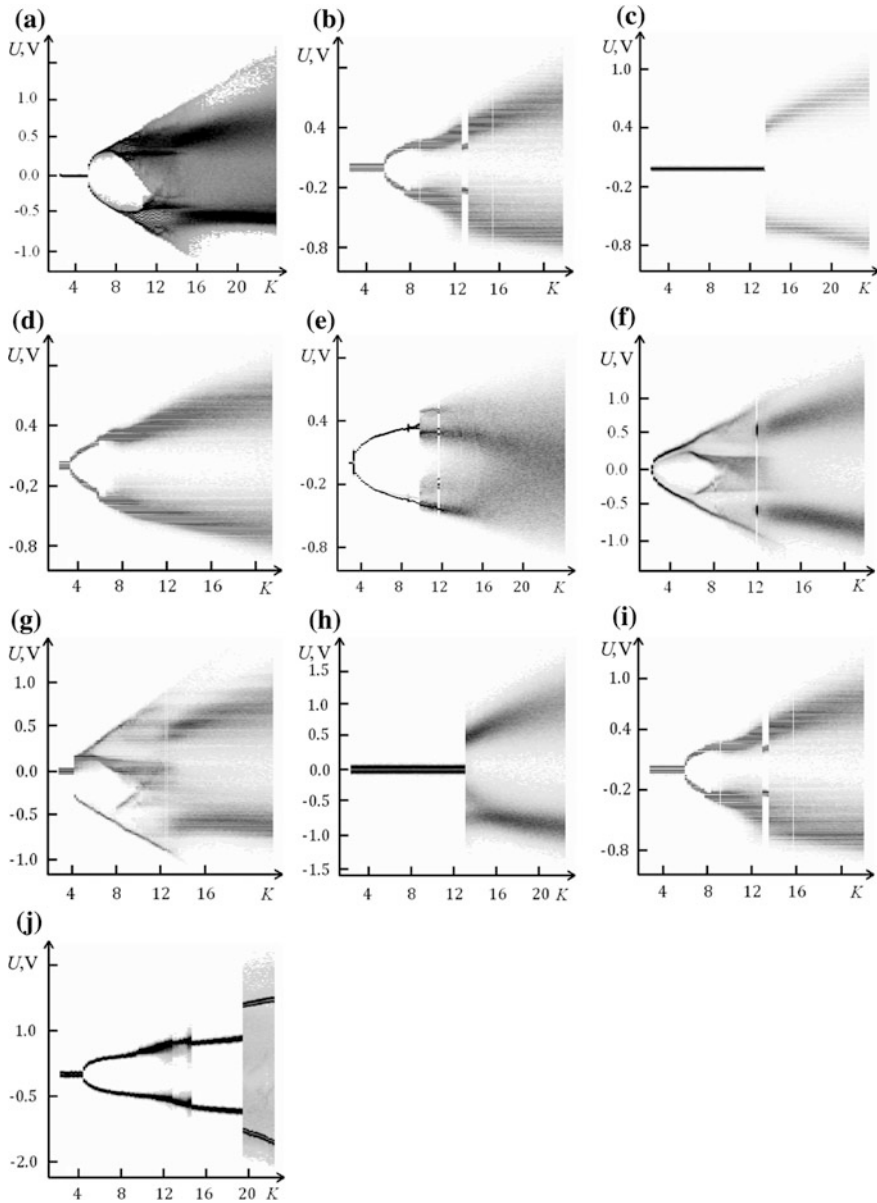


Fig. 2.30 Experimental bifurcation diagrams upon the parameter K for $a = 86$ mV, $b = 1.2$ V: **a** $D = -0.8$ V; **b** $D = -0.65$ V; **c** $D = -0.5$ V; **d** $D = -0.4$ V; **e** $D = -0.3$ V; **f** $D = 0.3$ V; **g** $D = 0.4$ V; **h** $D = 0.5$ V; **i** $D = 0.65$ V; **j** $D = 0.8$ V

2.4 The Ring Interferometer with the Kerr Nonlinear Medium and Its Modifications as the Deterministic Chaos Oscillators

We rely on ideas of K. Ikeda and S.A. Akhmanov and M.A. Vorontsov and choose the nonlinear ring interferometer as an oscillator of the deterministic chaos in the optical range. It differs from the Ikeda system by a possibility of two-dimensional transformation of the light field of the laser beam in its transverse plane, and from the Akhmanov and Vorontsov interferometer by a presence of many bypasses of radiation through the feedback loop. Moreover, we will offer two-circuit fiber-optical modification of the interferometer, as well as structurally related set of interferometers.

Naturally, we start with producing the mathematical model [34–38].

2.4.1 Mathematical Models of Processes in the Nonlinear Ring Interferometer

Relationship between the fields at input and output of the Kerr medium with diffusion, and the linear element in the interferometer. G. Kerr showed [39–42] that many isotropic substances (gases, liquids, glasses) placed in the static electrical field behave themselves as the single-axis crystals whose optical axes are directed along the force lines of the field. If n_0 is the refraction index of the substance when the field is absent, and $n_{||}$ and $n_{\perp}$ are refraction indices for extraordinary and ordinary beams, then Kerr and Havelock laws are true:

$$(n_{||} - n_{\perp}) = \lambda B \mathbf{E}^2, \quad (n_{||} - n_{\perp}) = 2(n_{\perp} - n_0). \quad (2.12)$$

Here B is the Kerr constant, λ is the wavelength of the light wave, for which the refraction index is measured, $\mathbf{E}$ is a vector of the electrical field intensity, which causes anisotropy. From (2.12) it is easy to obtain:

$$n_{\perp}(\lambda) = n_0 + n_2 \mathbf{E}^2, \quad n_{||}(\lambda) = n_0 + 2n_2 \mathbf{E}^2, \quad (2.13)$$

where $n_2 = \lambda_0 B$. If the field alternate in time and with invariable direction of the electrical vector intensity act upon the medium, then the uniform molecule orientation, and, hence, refraction indices $n_{||}$ and $n_{\perp}$, will be established asymptotically. We designate the appropriate alternative refraction indices in the given time moment as $n_{||}(t)$ and $n_{\perp}(t)$. Let us suppose that the asymptotic transient process is carried out in accordance with the time constant τ_n . Then, for alternative fields the following relations will act:

$$\frac{dn_{\perp}(t)}{dt} = \frac{[n_{\perp} - n_{\perp}(t)]}{\tau_n}; \quad \frac{dn_{\parallel}(t)}{dt} = \frac{[n_{\parallel} - n_{\parallel}(t)]}{\tau_n}.$$

Substituting (2.13) into this equation, we now generalize these two equations into the form:

$$\frac{dn(t)}{dt} = \frac{n_0 + an \cdot n_2 \mathbf{E}^2(t) - n(t)}{\tau_n}. \quad (2.14)$$

Here $an = 1$, if $n = n_{\perp}$; $an = 2$, if $n = n_{\parallel}$. The refraction index of the nonlinear medium (NM) varies due to the diffusion (with coefficient D) of its molecules or the charge carriers. This leads to the appearance of local (small-scale) transverse coupling of the fields. The diffusion effect can be taken into account phenomenologically [43, 44] by introducing the Laplace operator into Eq. (2.14). In approximation of the thin medium (when there is no diffusion along z —axis) and after designation $\mathbf{r} \equiv (x, y)$ we obtain:

$$\frac{\partial n(\mathbf{r}, t)}{\partial t} = \frac{n_0(\mathbf{r}) + an \cdot n_2(\mathbf{r}) \mathbf{E}^2(\mathbf{r}, t) - n(\mathbf{r}, t)}{\tau_n(\mathbf{r})} + D\Delta[n(\mathbf{r}, t) - n_0(\mathbf{r})]. \quad (2.15)$$

Let us introduce the effective diffusion coefficient that is the coefficient normalized in accordance with relaxation time for NM and the square of some typical size r_0 : $D_e(\mathbf{r}) = \frac{\tau_n(\mathbf{r})D}{r_0^2}$. Then, from (2.15) and designating $\mathbf{r} := \mathbf{r}/r_0$, we obtain the equation describing the evolution of the refraction index of the thin (along z axis) spatially irregular (in the plane xOy) Kerr medium under the impact of the spatially non-uniform (in the plane xOy) and time-varying electric field. The specific condition here is that the diffusion of NM molecules should be taken into account:

$$\begin{aligned} \tau_n(\mathbf{r}) \frac{\partial n(\mathbf{r}, t)}{\partial t} = & an \cdot (\mathbf{r}) \mathbf{E}(\mathbf{r}, t) - [n(\mathbf{r}, t) - n_0(\mathbf{r})] \\ & + D_e(\mathbf{r}) \Delta[n(\mathbf{r}, t) - n_0(\mathbf{r})] \end{aligned} \quad (2.16)$$

Let the NM expansion l (Fig. 2.31) along the propagation direction (Oz axis) be so small that we can neglect the beams curvature during their passing through the medium and operate with some l -average value of the refraction index. Then we can also neglect the variation of the Δt_n time field propagation in the medium t_n for the time interval t_n , i.e. $\omega \Delta t_n \ll 2\pi$. In this case, having considered energy transformation effects due to the NM polarization and relaxation, as well as dispersion and diffraction, negligible, we obtain the relationship between electrical field intensity (or its some component) at input and output of the nonlinear medium:

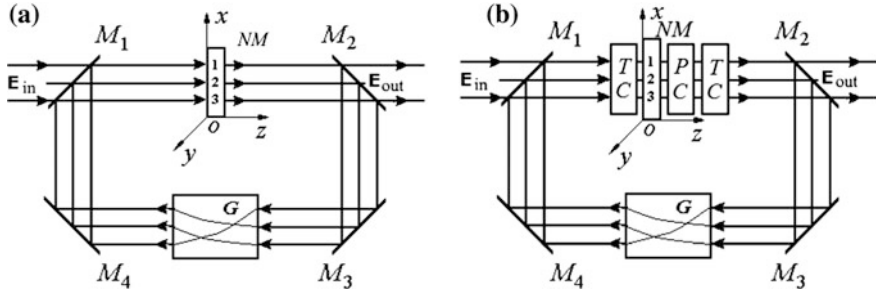


Fig. 2.31 Schematic diagram of two versions of the nonlinear ring interferometer. The beam course is shown for the case of the beam rotation (by the element G) in the plane xOy by 120° . We accept the following designations: NM is the nonlinear medium with the length l ; G is a linear element performing large-scale field transformation; mirrors M_1, M_2 have the reflection factor $R_i[\Theta(\mathbf{r}, t)]$ in intensity, and M_3, M_4 have the factor equaled to 1, TC are the transparent conductors, on which the DC voltage is applied; PC is the photo conductor (its conductance depends on light intensity, which passes through it)

$$\mathbf{E}_{out,nm}(\mathbf{r}, t) = [\alpha(\mathbf{r}, t)]^{0.5} C_n(\mathbf{r}) \mathbf{E}_{in,nm}[\mathbf{r}, t - t_n(\mathbf{r}, t)], \quad (2.17)$$

where $C_n(\mathbf{r})$ are losses in the nonlinear medium, for instance the Bouguer $[\exp(-\alpha_b l/2)]$; $t_n(\mathbf{r}, t) = n(\mathbf{r}, t)l/c$ is the total time of propagation of the light wave (or its component) that, by the time moment t , arrives to the point $\mathbf{r}$ of the NM output plane from the point $\mathbf{r}$ of the NM input plane; c is the light speed; $n(\mathbf{r}, t)$ is the refraction index for this wave (component); $\alpha(\mathbf{r}, t) = 1/\{1 + dt_n(\mathbf{r}, t)/dt\}$ is a coefficient showing how many times (at first approximation) the volumetric density of the wave energy will increase as a result of the variation of the NM optical length [35]. From inequality $\omega \Delta t_n \ll 2\pi$ and with the assumption that $\Delta t_n \approx t_n(\mathbf{r}, t) \frac{dn(\mathbf{r}, t)}{dt}$, we obtain another formulation of the above-mentioned approximation: $\frac{dn(\mathbf{r}, t)}{dt} \ll \frac{\lambda}{n(\mathbf{r}, t)l} < 1$. It means that $\alpha(\mathbf{r}, t) \approx 1$ and in (2.16) the variation speed of the refraction index is limited: $\frac{d[n(\mathbf{r}, t)l]}{cdt} \ll \frac{2\pi c}{\omega n(\mathbf{r}, t)l}$ or $\frac{dn(\mathbf{r}, t)}{dt} \ll \frac{2\pi c^2}{\omega n(\mathbf{r}, t)l^2}$.

The impact of the linear optical element (G in Fig. 2.31) upon the large-scale field transformation is that the beam having arrived to the point $\mathbf{r}' \equiv (x', y')$ of the transverse section of the laser beam then transfers to the point $\mathbf{r} \equiv (x, y)$. As a result, the beam is capable of undergoing a series of consecutive transformations:

- (1) M_x (or M_y) is its mirror representation with respect to the straight line parallel to Ox axis (or Oy) passing through the beam center (optical axis of the interferometer);
- (2) it is rotated by the Δ angle;
- (3) stretching of its transverse section by σ_x times along Ox axis (or by σ_y times along the Oy axis);

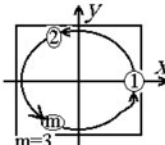
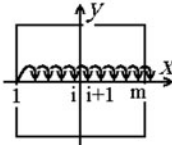
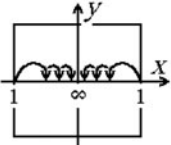
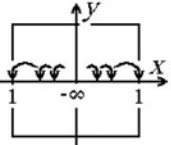
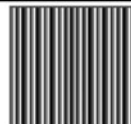
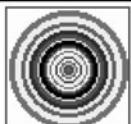
Simplest types of beam transformations by the element G			
Rotation ($\Delta=2\pi M/m$)	Shift (δ)	Compression ($1/\sigma$)	Stretching (σ)
			
Type of the chain of the transposition points (CTP)			
Closed finite	Open-ended finite	Open-ended infinite	
		($m=\infty$)	($m=-\infty$)
Forming idealized structures [45]			
			

Fig. 2.32 The relation between simplest types of transformations of the laser beam in the NRI feedback loop (which gives the configuration of the chain of transposition points) and the view of formed optical structure $U(\mathbf{r}, t)$ in the transverse beam section

- (4) it is shifted by the value δ_x along the Ox axis, or by δ_y along the Oy axis (Fig. 2.32). The beam shift $|\mathbf{r} - \mathbf{r}'|$ can then be congruent to the size of the light beam.

We assume that the vector $\mathbf{E}$ direction does not change here, as well as the input beam center coincides with the origin of the plane xOy . Then we can calculate the result of the field transformation by the element of the large-scale transfiguration in the case when light diffraction and dispersion can be set as negligible by the application of the equations system:

$$\begin{aligned}
 \mathbf{E}_{out}(\mathbf{r}, t) &= \kappa[\mathbf{r}', t, \Theta(\mathbf{r}', t)] \frac{\mathbf{E}_{in}\{\mathbf{r}', t - t_G[\Theta(\mathbf{r}', t)]\}}{(\sigma_x \sigma_y)^{0.5}}, \\
 x' &= M_x \left[(x - \delta_x) \cos \frac{(\Delta)}{\sigma_x} + (y - \delta_y) \sin \frac{(\Delta)}{\sigma_y} \right], \\
 y' &= M_y \left[(y - \delta_y) \cos \frac{(\Delta)}{\sigma_y} - (x - \delta_x) \sin \frac{(\Delta)}{\sigma_x} \right],
 \end{aligned} \tag{2.18}$$

where $M_x = 1$ when mirror image is absent with respect to the straight line parallel to the Ox axis; $M_x = -1$ when mirror image is present; similarly, $M_y = 1$ when mirror image is absent with respect to the straight line parallel to the Oy axis, and $M_y = -1$ when mirror image is present; $\Theta(\mathbf{r}', t)$ is the angle between the Ox axis and the vector $\mathbf{E}(\mathbf{r}', t)$; $t_G[\Theta(\mathbf{r}', t)]$ is a time for field propagation in the element G ; $\kappa[\mathbf{r}', t, \Theta(\mathbf{r}', t)]$ is the loss coefficient in amplitude [36].

The model of processes in the nonlinear ring interferometer. Equations (2.16), (2.17), (2.18) allow obtaining the mathematical description of the processes in the nonlinear ring interferometers (NRI), which are realized according to the various circuits (for example, Fig. 2.31a, b). Let us discuss in detail their differences.

In the first of them (Fig. 2.31a) the high-intensive optical field in the NM causes the anisotropy of the Kerr medium. Due to this, the refraction index changes. As (2.16) does not take into account the feature of the vector $\mathbf{E}$ orientation change, we must assume that this light wave is plane-polarized. Then the refraction index is defined by the Eq. (2.16) for $an = 2$, $n = n_{\parallel}$, as the NM optical axis is codirectional with the vector $\mathbf{E}$.

In the second interferometer (Fig. 2.31b), the optical field intensity is low enough, and we can neglect its direct influence upon the nonlinear medium. The light arriving in the photo-conductor changes its conductivity, and then the value of electrical (not optical) field (applied to NM and causing the anisotropy) changes. The similar physical principles were used in researches [43, 44]. The photo-conductor reacts to the incident light intensity, therefore, the optical wave polarization can be arbitrary. The refraction index obeys to the Eq. (2.16) for $an = 1$, $n = n_{\perp}$ (the optical axis of NM is co-directional with the direction of light propagation). Here under $\mathbf{E}(\mathbf{r}, t)$ we should understand the alternate component of the intensity of the non-optical field that is applied to NM, and under D_e —the equivalent diffusion coefficient taking into consideration the diffusion both of the NM molecules and of the charge carriers of the photo-conductor. For simplicity, we assume that the equality $\Delta \mathbf{E}_{nonoptic}^2 = \mathbf{E}_{optic}^2$ is true.

Let us reveal some relationships referring to the linear-polarized quasi-monochrome component of the field propagating in the interferometer. In this case, we presuppose that there is neither rotation of the polarization plane, no changing of its view. We use index “ p ” meaning the presence of the appropriate dependence of the certain value upon the orientation of the vector of the electric field intensity. We also use index “ w ” meaning frequency dependence.

Let the “delay time” $t_{0p}(\mathbf{r}', t)$ be the name for the time of the propagation of the field arriving from the point $\mathbf{r}'$ of the NM output plane (through the feedback loop) to the point $\mathbf{r}$ of the NM input plane by the time moment t . The time $t_{0p}(\mathbf{r}', t)$ also includes time t_{Gp} . Then, evidently,

$$\begin{aligned} \mathbf{E}_{in,nm,pw}(\mathbf{r}, t) &= \mathbf{E}_{fb,pw}(\mathbf{r}, t) + [1 - R_{1p}]^{0.5} \mathbf{E}_{in,pw}(\mathbf{r}, t), \\ \mathbf{E}_{fb,pw}(\mathbf{r}, t) &= R_{1p}^{0.5} \mathbf{E}_{out,G,pw}(\mathbf{r}, t) = R_{1p}^{0.5} \kappa_p(\mathbf{r}', t) E_{in,G,pw}(\mathbf{r}', t - t_{Gp}) / \sigma, \\ \mathbf{E}_{in,G,pw}(\mathbf{r}', t - t_{Gp}) &= \left\{ \frac{R_{2p}}{1 + \frac{dt_{0p}(\mathbf{r}', t)}{dt}} \right\}^{0.5} E_{out,nm,pw}[\mathbf{r}', t - t_{0p}(\mathbf{r}', t)], \\ \mathbf{E}_{out,nm,pw}[\mathbf{r}', t - t_{0p}(\mathbf{r}', t)] &= \{ \alpha_w[\mathbf{r}', t - t_{0p}(\mathbf{r}', t)] \}^{0.5} \\ &\quad \times C_n(\mathbf{r}') \mathbf{E}_{in,nm,pw} \{ \mathbf{r}', t - t_{nw}[\mathbf{r}', t - t_{0p}(\mathbf{r}', t)] - t_{0p}(\mathbf{r}', t) \}. \end{aligned}$$

Here $\mathbf{E}_{fb,pw}$ is the field having arrived to the NM input from the feedback loop (FBL); $\mathbf{E}_{in,pw}$ is the field at the interferometer input; $t_{nm}(\mathbf{r}, t) = l \cdot n_w(\mathbf{r}, t)/c$ is the light wave propagation time with the wavelength λ_w in the medium; $n_w(\mathbf{r}, t) = n_0 + \frac{[n(\mathbf{r}, t) - n_0]\lambda_w}{\lambda_0}$; $\kappa(\mathbf{r}', t)$ are the losses in FBL of NRI, i.e. the amplitude transfer coefficient of FBL in contrast to the losses $\kappa[\mathbf{r}', t, \Theta(\mathbf{r}', t)]$ in the element G in (2.18). Hence,

$$\begin{aligned} \mathbf{E}_{in,nm,pw}(\mathbf{r}, t) &= (1 - R_{1p})^{0.5} \mathbf{E}_{in,pw}(\mathbf{r}, t) \\ &\quad + C_n(\mathbf{r}') (R_{1p} R_{2p})^{0.5} \frac{\kappa_p(\mathbf{r}', t)}{\sigma} \eta_{pw}(\mathbf{r}', t) \\ &\quad \times \mathbf{E}_{in,nm,pw}[\mathbf{r}', t - t_{nw}(\mathbf{r}', t - t_{0p}(\mathbf{r}', t)) - t_{0p}(\mathbf{r}', t)], \end{aligned}$$

where $n_{pw}(\mathbf{r}', t) = \frac{1}{\left[1 + \frac{d[t_{0p}(\mathbf{r}', t) + t_{nw}(\mathbf{r}', t - t_{0p}(\mathbf{r}', t))]}{dt}\right]^{0.5}}$ (implicitly η_{pw}^2 is similar to α_w).

Similarly to [35, 43, 44], we introduce the loss characteristic $\gamma_p(\mathbf{r}', t) \equiv 2\kappa_p(\mathbf{r}', t)C_n(\mathbf{r}') (R_{2p}R_{1p})^{0.5}$. Obviously, $\gamma \in [0; 2]$, but for the real systems, without limitation of generality, we can assume that $\gamma \in (0; 2)$. Keeping in mind that the field $\mathbf{E}$ can be represented as the sum of the two components $\mathbf{E}_x, \mathbf{E}_y$, each of which in turn is the sum of the set of the quasi-monochrome components $\mathbf{E}_{xw}, \mathbf{E}_{yw}$ with the frequency w , we obtain the model of the processes in the interferometer:

$$\begin{aligned} \tau_n(\mathbf{r}) \frac{\partial n(\mathbf{r}, t)}{\partial t} &= an \cdot n_2(\mathbf{r}) \left[\mathbf{E}_{in,nm,x}^2(\mathbf{r}, t) + \mathbf{E}_{in,nm,y}^2(\mathbf{r}, t) \right] \\ &\quad - [n(\mathbf{r}, t) - n_0(\mathbf{r})] + D_e(\mathbf{r}) \Delta [n(\mathbf{r}, t) - n_0(\mathbf{r})], \\ \mathbf{E}_{in,nm,pw}(\mathbf{r}, t) &= [1 - R_{1p}]^{0.5} \mathbf{E}_{in,pw}(\mathbf{r}, t) + \frac{0.5\gamma_p(\mathbf{r}', t)}{\sigma} \eta_{pw}(\mathbf{r}', t) \\ &\quad \times \mathbf{E}_{in,nm,pw}[\mathbf{r}', t - t_{nw}[\mathbf{r}', t - t_{0p}(\mathbf{r}', t)] - t_{0p}(\mathbf{r}', t)]. \end{aligned} \tag{2.19}$$

Here $\mathbf{E}_{in,nm,x}(\mathbf{r}, t) = 0$, $p = \text{"y"}$ or $\mathbf{E}_{in,nm,y}(\mathbf{r}, t) = 0$, $p = \text{"x"}$, $an = 2$ for the optical scheme in Fig. 2.31a; $an = 1$, $p \in \{\text{"x"}, \text{"y"}\}$ for the scheme in Fig. 2.31b.

Due to the equivalence of the influence of the components of the refraction index n and the delay time t_0 upon the dynamics of the processes in the interferometer [35], we merge $t_{0p}(\mathbf{r}, t)$ and $n_0(\mathbf{r})l/c$ into the single variable $t_{ep}(\mathbf{r}, t) \equiv n_0(\mathbf{r})l/c + t_{0p}(\mathbf{r}, t)$. The value $t_{ep}(\mathbf{r}, t)$ means the fraction of the time of the light field propagation. It is conditioned by the presence of the linear part $n_0(\mathbf{r})$ of the NM refraction index, as well as the time $t_{0p}(\mathbf{r}, t)$, while $t_{ep}(\mathbf{r}, t)$ can be referred to as the equivalent delay time in NRI. We are to use the value of delay time conditioned by the non-linear part of the refraction index $t_u(\mathbf{r}, t) = l[n(\mathbf{r}, t) - n_0(\mathbf{r})]/c$. Then, the processes modeling allowing for the many field bypasses in NRI (2.19) can be rewritten as:

$$\begin{aligned}
\tau_n(\mathbf{r}) \frac{\partial t_u(\mathbf{r}, t)}{\partial t} &= an \cdot n_2(\mathbf{r}) \frac{l}{c} \left[\mathbf{E}_{in, nm, x}^2(\mathbf{r}, t) + \mathbf{E}_{in, nm, y}^2(\mathbf{r}, t) \right] \\
&\quad - t_u(\mathbf{r}, t) + D_e(\mathbf{r}) \Delta t_u(\mathbf{r}, t), \\
\mathbf{E}_{in, nm, pw}(\mathbf{r}, t) &= (1 - R_{1p})^{0.5} \mathbf{E}_{in, pw}(\mathbf{r}, t) \\
&\quad + \frac{0.5 \gamma_p(r', t)}{\sigma} \eta_{pw}(r', t) E_{in, nm, pw}(r', t - \tau_{pw}).
\end{aligned} \tag{2.20}$$

Here $\tau_p \equiv \tau_p(r', t) \equiv t_{ep}(r', t) + t_u[r', t - t_{op}(r', t)] \approx t_{ep}(r', t) + t_u[r', t - t_{ep}(r', t)]$ is the propagation time of the p component of the light field with a wavelength λ_0 that, by the time moment t , has arrived (through the feedback loop) to the point $\mathbf{r}$ of the NM input plane from the point $\mathbf{r}'$ of the same plane (full time delay, a time of the complete bypass of the interferometer);

$$\begin{aligned}
\tau_{pw} &\equiv \tau_{pw}(r', t) = t_{ep}(r', t) + t_{uw}[r', t - t_{0p}(r', t)], \\
t_{uw}(\mathbf{r}, t) &\equiv \frac{\lambda_w t_u(\mathbf{r}, t)}{\lambda_0} = \frac{\omega_0 t_u(\mathbf{r}, t)}{\omega_w}; \\
\eta_{pw}(r', t) &= \frac{1}{\left[1 + \frac{d\tau_{pw}(r', t)}{dt} \right]^{0.5}}.
\end{aligned}$$

The mathematical model (2.20) is true for the field of arbitrary type at the input of the interferometer (excluding the polarization type limitations in the system in Fig. 2.31a). Therefore, it can serve as a foundation for further developments producing new different variants of the models.

The process model in approximation of the slowly-varying amplitudes, phases, modulations of the polarization plane position, delay time and electrical field losses. In Eq. (2.20) the electrical field intensity appears, which can vary with a period much less than the variation time of the optical field intensity and the value of the refraction index. Therefore, it is reasonable to transform it in approximation of slowly-varying amplitudes, phases, modulations of the polarization plane position, delay time and field energy losses. For this, it is necessary to specify the field type $\mathbf{E}(\mathbf{r}, t)$ at the NRI input.

In literature we can find the cases, when $\mathbf{E}(\mathbf{r}, t)$ is the linear-polarized monochrome non-modulated field [43–48]. It is interesting for us to study a more general case. We would like to note that optimization of the interference interaction of elliptically-polarized beams requires the homothetic polarization ellipses [49]. The circular and linear polarization types are universal, as far as they satisfy the mentioned condition.

Let the field in the NRI input consists of the two components with circular polarization (Fig. 2.33):

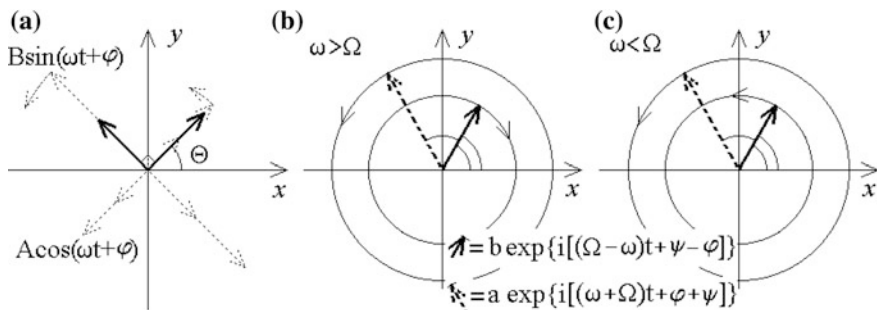


Fig. 2.33 The structure of bi-chromatic optical radiation $\mathbf{E}(\mathbf{r}, t)$: *a*—for writing in the form (2.21), *b*, *c*—for writing in the form (2.22). *Thick lines* in figure *a* correspond to instantaneous state of the intensity vectors for $\omega t + \varphi(\mathbf{r}, t) = 30^\circ$, $\theta(\mathbf{r}, t) = 45^\circ$, $A/B = 2/3$, the *dashed lines* illustrate possible states of these vectors at arbitrary $\omega t + \varphi(\mathbf{r}, t)$, and its rotation direction is shown for $\Omega > 0$. In figures *b*, *c* the *thick lines* corresponding to instantaneous state of intensity vectors for $\omega t + \varphi(\mathbf{r}, t) = 30^\circ$, $\theta(\mathbf{r}, t) = 90^\circ$, $A/B = 1/5$, rotation directions (opposite to *b* when $\omega > \Omega$ and equal to *c* when $\omega < \Omega$) are shown for $\omega > 0$, $\Omega > 0$. The *dashed line* corresponds to the field component with the frequency $\omega + \Omega$ and the amplitude $a(\mathbf{r}, t)$, and the *solid thick lines*—to the component with $\omega - \Omega$ and $b(\mathbf{r}, t)$

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) = & \mathbf{e}[\Theta(\mathbf{r}, t)]A(\mathbf{r}, t) \cos[\omega t + \varphi(\mathbf{r}, t)] \\ & + \mathbf{e}\left[\Theta(\mathbf{r}, t) + \frac{\pi}{2}\right]B(\mathbf{r}, t) \sin[\omega t + \varphi(\mathbf{r}, t)], \end{aligned} \quad (2.21)$$

where ω is the fundamental frequency of the light field, $\Theta(\mathbf{r}, t) = \psi(\mathbf{r}, t) + \Omega t$ is the angle between the $\mathbf{e}(\Theta)$ vector, which gives the polarization direction, and the Ox axis that lies in the (xOy) plane of the beam transverse section (Ω can be commensurable with ω), Ω is the frequency of synchronous rotation of the polarization vectors $\mathbf{e}$ lying in the plane that we call the polarization plane; $A(\mathbf{r}, t)$, $B(\mathbf{r}, t)$, $\varphi(\mathbf{r}, t)$, $\psi(\mathbf{r}, t)$ are the amplitude, phases, the polarization plane position of the light field insignificantly varying during the time $T = 2\pi/\omega$. In contrast to [35], there is here the second member, i.e. $B \neq 0$. The sign Ω characterizes the rotation direction of the polarization vectors $\mathbf{e}$.

If to introduce designations $a(\mathbf{r}, t) \equiv \frac{A(\mathbf{r}, t) + B(\mathbf{r}, t)}{2}$, $b(\mathbf{r}, t) \equiv \frac{A(\mathbf{r}, t) - B(\mathbf{r}, t)}{2}$, then (2.21) will be expressed via projections of $\mathbf{E}(\mathbf{r}, t)$:

$$\begin{aligned} E_x(\mathbf{r}, t) = & a(\mathbf{r}, t) \cos[(\omega + \Omega)t + \varphi(\mathbf{r}, t) + \psi(\mathbf{r}, t)] \\ & + b(\mathbf{r}, t) \cos[(\omega - \Omega)t + \varphi(\mathbf{r}, t) - \psi(\mathbf{r}, t)], \\ E_y(\mathbf{r}, t) = & a(\mathbf{r}, t) \sin[(\omega + \Omega)t + \varphi(\mathbf{r}, t) + \psi(\mathbf{r}, t)] \\ & - b(\mathbf{r}, t) \sin[(\omega - \Omega)t + \varphi(\mathbf{r}, t) - \psi(\mathbf{r}, t)]. \end{aligned} \quad (2.22)$$

Thus, at interferometer input there arrives a sum of quasi-monochrome fields with amplitudes $a(\mathbf{r}, t)$, $b(\mathbf{r}, t)$ and with frequencies $\omega + \Omega$ of the circular polarizations of the different (for $\omega > \Omega$), or of the identical (for $\omega < \Omega$), rotation

directions (Fig. 2.33b, c). Here ω (or Ω for $\omega < \Omega$) means average frequency, and $2\Omega(2\omega$ for $\omega < \Omega$) is the frequency interval between field components. To reflect the specificity of the optical field under consideration, we deal with $q \equiv \Omega/\omega$ that is bi-chromaticity parameter, or non-monochromaticity parameter, according to [36].

We assume that using the specific engineering approaches we have achieved the fulfillment of the conditions: $t_{ex}(\mathbf{r}, t) = t_{ey}(\mathbf{r}, t) = t_e(\mathbf{r}, t)$, $R_{1x} = R_{1y} = R_{2x} = R_{2y} = R$, $\gamma_x(\mathbf{r}, t) = \gamma_y(\mathbf{r}, t) = \gamma(\mathbf{r}, t)$, $\eta_{xw}(\mathbf{r}, t) = \eta_{yw}(\mathbf{r}, t) = \eta_w(\mathbf{r}, t)$, $\gamma(\mathbf{r}, t) \equiv 2R\kappa(\mathbf{r}, t)C_n(\mathbf{r})$. Let us agree that we do not take into account the dependence of the field propagation time in NM upon frequency (in this case we may designate $\eta_w(\mathbf{r}, t) = \eta(\mathbf{r}, t)$, $t_{uw}(\mathbf{r}, t) = t_u(\mathbf{r}, t)$ etc., and $\tau_p \equiv \tau_p(\mathbf{r}', t) = t_{ep}(\mathbf{r}', t) + t_u[\mathbf{r}', t - t_{ep}(\mathbf{r}', t)] = \tau$, $\tau_{pw} \equiv \tau_{pw}(\mathbf{r}', t) \equiv t_{ep}(\mathbf{r}', t) + t_{uw}[\mathbf{r}', t - t_{0p}(\mathbf{r}, t)] = \tau$, where $\tau \equiv \tau(\mathbf{r}', t) \equiv t_e(\mathbf{r}', t) + t_u[\mathbf{r}', t - t_e(\mathbf{r}', t)]$. These approximations essentially simplify examination because all fields $\mathbf{E}$ in (2.20) become presentable in the form (2.21).

In typical case, the relaxation time of the nonlinear part of the refraction index $\tau_n(\mathbf{r}) \gg T$ (or, more strictly, $-\omega T \frac{dt_u}{dt} \ll 2\pi$ and $\Omega T \frac{dt_u}{dt} \ll 2\pi$), where $T \equiv \frac{2\pi}{\omega}$ is a period of the light oscillations. Taking into account that during the time T the values A, B and φ vary by values much less than A, B and 2π , we carry out averaging of the Kerr medium equation (in the model (2.20)) over the time interval $[t; t + T]$:

$$\begin{aligned} \tau_n(\mathbf{r}) \frac{\partial t_u(\mathbf{r}, t)}{\partial t} &= D_e(\mathbf{r}) \Delta t_u(\mathbf{r}, t) - t_u(\mathbf{r}, t) \\ &\quad + an \cdot n_2(\mathbf{r}) \frac{l}{c} \frac{1}{T} \int_t^{t+T} \left\{ \mathbf{E}_{in, nm, p'}^2(\mathbf{r}, t') + \mathbf{E}_{in, nm, p}^2(\mathbf{r}, t') \right\} dt' \\ &= D_e(\mathbf{r}) \Delta t_u(\mathbf{r}, t) - t_u(\mathbf{r}, t) + an \cdot n_2(\mathbf{r}) \frac{l}{c} \left[a_{in, nm}^2(\mathbf{r}, t) + b_{in, nm}^2(\mathbf{r}, t) \right]. \end{aligned} \quad (2.23)$$

This equation describes the dynamics of the medium refraction index dependence [through $t_u(\mathbf{r}, t)$] upon the amplitude square of the incident field, i.e. the high-frequency Kerr effect [39]. The similar procedure of “averaging” is equivalent to the action of the low-frequency filter, which excludes the fast varying processes (not interesting for us) from the consideration.

Let us estimate the magnitude $\eta(\mathbf{r}', t) = \frac{1}{\left\{ 1 + \frac{d\tau(\mathbf{r}', t)}{dt} \right\}^{0.5}}$. Proceeding from the accepted approximation $\omega \Delta \tau \ll 2\pi$ and assuming that $\Delta \tau \approx T \frac{d\tau(\mathbf{r}', t)}{dt}$, we obtain the inequality $\frac{d\tau(\mathbf{r}', t)}{dt} \ll \frac{2\pi}{T\omega} = 1$, from which we get $\eta(\mathbf{r}', t) \approx 1$. Further omitting the index “in”, and replacing the index “in.nm” by “nm” and using (2.23) from (2.20), we get the model for the processes in NRI:

$$\begin{aligned}
\tau_n(\mathbf{r}) \frac{\partial U(\mathbf{r}, t)}{\partial t} &= D_e(\mathbf{r}) \Delta U(\mathbf{r}, t) - U(\mathbf{r}, t) + f(\mathbf{r}, t), \\
f(r, t) &= n_2(\mathbf{r}) l k \cdot an \langle \mathbf{E}_{nm}^2(\mathbf{r}, t) \rangle_T = an \cdot n_2(\mathbf{r}) l k [a_{nm}^2(\mathbf{r}, t) + b_{nm}^2(\mathbf{r}, t)], \\
\mathbf{E}_{nm}(\mathbf{r}, t) &= (1 - R)^{0.5} \mathbf{E}(\mathbf{r}, t) + \frac{\gamma(\mathbf{r}', t)}{2\sigma} \mathbf{E}_{nm}(\mathbf{r}', t - \tau).
\end{aligned} \tag{2.24}$$

Here $k = \omega/c$; $U(\mathbf{r}, t) \equiv \omega t_u(\mathbf{r}, t)$ is a nonlinear phase shift; $\tau \equiv \tau(\mathbf{r}', t) = t_e(\mathbf{r}', t) + \frac{U[\mathbf{r}', t - t_e(\mathbf{r}', t)]}{\omega}$; $\gamma(\mathbf{r}', t) \equiv 2R\kappa(\mathbf{r}', t)C_n(\mathbf{r}')$; $an = 1$ (Fig. 2.31b); $an = 2$, $\Omega = 0$, $\psi = \text{const}$ (Fig. 2.31a).

Here the formulas for $a(\mathbf{r}, t)$ and $\varphi(\mathbf{r}, t)$ are useful:

if

$$\begin{aligned}
a(\mathbf{r}, t) \exp\{j[\omega t + \varphi(\mathbf{r}, t)]\} &= a_1(\mathbf{r}, t) \exp\{j[\omega t + \varphi_1(\mathbf{r}, t)]\} \\
&+ a_2(\mathbf{r}, t) \exp\{j[\omega t + \varphi_2(\mathbf{r}, t)]\},
\end{aligned}$$

then $a(\mathbf{r}, t) = (Ac^2 + As^2)^{0.5} = [a_1^2 + a_2^2 + 2a_1a_2 \cos(\varphi_1 - \varphi_2)]^{0.5} \geq 0$,

$$\varphi(\mathbf{r}, t) = \arg(Ac, As) \equiv \begin{cases} \frac{\pi}{2} - \arctan \frac{Ac}{As}, & As > 0 \\ \frac{3\pi}{2} - \arctan \frac{Ac}{As}, & As < 0 \\ 0, & As = 0, Ac \geq 0 \\ \pi, & As = 0, Ac < 0 \end{cases} \in [0, 2\pi), \tag{2.25}$$

where $Ac \equiv a_1 \cos \varphi_1 + a_2 \cos \varphi_2$, $As \equiv a_1 \sin \varphi_1 + a_2 \sin \varphi_2$.

We would like to remind that the field $\mathbf{E}(\mathbf{r}, t)$ has a view (2.22). Then (2.25) should be used independently for the components with frequencies $(1 + q)\omega$ and $(1 - q)\omega$. Hence,

$$a_{nm}(\mathbf{r}, t) = (Ac^2 + As^2)^{0.5}, \quad \varphi_{nm}(\mathbf{r}, t) + \psi_{nm}(\mathbf{r}, t) = \arg(Ac, As), \tag{2.26}$$

where

$$\begin{aligned}
Ac &= (1 - R)^{0.5} a(\mathbf{r}, t) \cos[\varphi(\mathbf{r}, t) + \psi(\mathbf{r}, t)] \\
&+ \frac{\gamma(\mathbf{r}', t)}{2\sigma} a_{nm}(\mathbf{r}', t - \tau) \cos[\varphi_{nm}(\mathbf{r}', t - \tau) + \psi_{nm}(\mathbf{r}', t - \tau) - (1 + q)\omega\tau],
\end{aligned}$$

$$\begin{aligned}
As &= (1 - R)^{0.5} a(\mathbf{r}, t) \sin[\varphi(\mathbf{r}, t) + \psi(\mathbf{r}, t)] \\
&+ \frac{\gamma(\mathbf{r}', t)}{2\sigma} a_{nm}(\mathbf{r}', t - \tau) \sin[\varphi_{nm}(\mathbf{r}', t - \tau) + \psi_{nm}(\mathbf{r}', t - \tau) - (1 + q)\omega\tau],
\end{aligned}$$

$$q \equiv \frac{\Omega}{\omega}, \quad \tau \equiv \tau(\mathbf{r}', t) = t_e(\mathbf{r}', t) + \frac{U[\mathbf{r}', t - t_e(\mathbf{r}', t)]}{\omega}.$$

Similarly,

$$b_{nm}(\mathbf{r}, t) = (Ac^2 + As^2)^{0.5}, \quad \varphi_{nm}(\mathbf{r}, t) - \psi_{nm}(\mathbf{r}, t) = \arg(Ac, As) \quad (2.27)$$

where

$$\begin{aligned} Ac &= (1 - R)^{0.5} b(\mathbf{r}, t) \cos[\varphi(\mathbf{r}, t) - \psi(\mathbf{r}, t)] \\ &+ \frac{\gamma(\mathbf{r}', t)}{2\sigma} b_{nm}(\mathbf{r}', t - \tau) \cos[\varphi_{nm}(\mathbf{r}', t - \tau) - \psi_{nm}(\mathbf{r}', t) - (1 - q)\omega\tau], \\ As &= (1 - R)^{0.5} b(\mathbf{r}', t) \sin[\varphi(\mathbf{r}, t) - \psi(\mathbf{r}, t)] \\ &+ \frac{\gamma(\mathbf{r}', t)}{2\sigma} b_{nm}(\mathbf{r}', t - \tau) \sin[\varphi_{nm}(\mathbf{r}', t - \tau) - \psi_{nm}(\mathbf{r}', t) - (1 - q)\omega\tau]. \end{aligned}$$

Let us rewrite (2.24) in the form

$$\begin{aligned} \tau_n(\mathbf{r}) \frac{\partial U(\mathbf{r}, t)}{\partial t} &= D_e(\mathbf{r}) \Delta U(\mathbf{r}, t) - U(\mathbf{r}, t) + f(\mathbf{r}, t), \\ f(\mathbf{r}, t) &= an \cdot n_2(\mathbf{r}) lk [a_{nm}^2(\mathbf{r}, t) + b_{nm}^2(\mathbf{r}, t)]. \end{aligned} \quad (2.28)$$

Formulas (2.26)–(2.28) form the mathematical model of processes in NRI, the one taking into account many bypasses of the field equivalent to (2.24), but of a more understandable form. Using (2.26) and (2.27), we can prove that the following relation is true:

$$\begin{aligned} a_{nm}^2(\mathbf{r}, t) + b_{nm}^2(\mathbf{r}, t) &= (1 - R) [a^2(\mathbf{r}, t) + b^2(\mathbf{r}, t)] \\ &+ \left[\frac{\gamma(\mathbf{r}', t)}{2\sigma} \right]^2 [a_{nm}^2(\mathbf{r}', t - \tau) + b_{nm}^2(\mathbf{r}', t - \tau)] \\ &+ \frac{(1 - R)^{0.5} \gamma(\mathbf{r}', t)}{\sigma} \times \{ a(\mathbf{r}, t) a_{nm}(\mathbf{r}', t - \tau) \\ &\times \cos[(1 + q)\omega\tau + \varphi(\mathbf{r}, t) - \varphi_{nm}(\mathbf{r}', t - \tau) + \psi(\mathbf{r}, t) - \psi_{nm}(\mathbf{r}', t - \tau)] \\ &+ b(\mathbf{r}, t) b_{nm}(\mathbf{r}', t - \tau) \\ &\times \cos[(1 - q)\omega\tau + \varphi(\mathbf{r}, t) - \varphi_{nm}(\mathbf{r}', t - \tau) - \psi(\mathbf{r}, t) + \psi_{nm}(\mathbf{r}', t - \tau)] \}. \end{aligned} \quad (2.29)$$

Let us introduce nonlinearity coefficients corresponding to the usual scheme. For this, we rewrite $f(\mathbf{r}, t)$ using (2.29) into the form

$$\begin{aligned}
f(\mathbf{r}, t) = & an \cdot n_2(\mathbf{r})lk(1-R)[a^2(\mathbf{r}, t) + b^2(\mathbf{r}, t)] \\
& + an \cdot n_2(\mathbf{r})lk(1-R) \left[\frac{\gamma(\mathbf{r}, t)}{2\sigma} \right]^2 \frac{a_{nm}^2(\mathbf{r}', t - \tau) + b_{nm}^2(\mathbf{r}', t - \tau)}{1-R} \\
& + an \cdot n_2(\mathbf{r})lk(1-R)^{0.5} \left[(1-R)^{0.5} \frac{\gamma(\mathbf{r}', t)}{\sigma} \right] \\
& \times \{ a(\mathbf{r}, t) a(\mathbf{r}', t - \tau) \frac{a_{nm}(\mathbf{r}', t - \tau)}{a(\mathbf{r}', t - \tau)(1-R)^{0.5}} \\
& \times \cos[(1+q)\omega\tau + \varphi(\mathbf{r}, t) - \varphi_{nm}(\mathbf{r}', t - \tau) + \psi(\mathbf{r}, t) - \psi_{nm}(\mathbf{r}', t - \tau)] \\
& + b(\mathbf{r}, t) b(\mathbf{r}', t - \tau) \frac{b_{nm}(\mathbf{r}', t - \tau)}{b(\mathbf{r}', t - \tau)(1-R)^{0.5}} \\
& \times \cos[(1-q)\omega\tau + \varphi(\mathbf{r}', t) - \varphi_{nm}(\mathbf{r}', t - \tau) - \psi(\mathbf{r}, t) + \psi_{nm}(\mathbf{r}', t - \tau)] \}.
\end{aligned}$$

Taking into consideration that the second summand can be transformed as

$$\begin{aligned}
& an \cdot n_2(\mathbf{r})lk(1-R) \left[\frac{\gamma(\mathbf{r}', t)}{2\sigma} \right]^2 \frac{a_{nm}^2(\mathbf{r}', t - \tau) + b_{nm}^2(\mathbf{r}', t - \tau)}{1-R} \\
& = an \cdot n_2(\mathbf{r})lk(1-R) \left[\frac{\gamma(\mathbf{r}', t)}{2\sigma} \right]^2 \frac{a^2(\mathbf{r}', t - \tau) a_{nm}^2(\mathbf{r}', t - \tau)}{a^2(\mathbf{r}', t - \tau)(1-R)} \\
& + an \cdot n_2(\mathbf{r})lk(1-R) \left[\frac{\gamma(\mathbf{r}', t)}{2\sigma} \right]^2 \frac{b^2(\mathbf{r}', t - \tau) b_{nm}^2(\mathbf{r}', t - \tau)}{b^2(\mathbf{r}', t - \tau)(1-R)} \\
& = \left[\frac{\gamma(\mathbf{r}', t)}{2\sigma} \right]^2 Ka(\mathbf{r}, \mathbf{r}', t - \tau, \mathbf{r}', t - \tau) a_{nm,r}^2(\mathbf{r}', t - \tau) \\
& + \left[\frac{\gamma(\mathbf{r}', t)}{2\sigma} \right]^2 Kb(\mathbf{r}, \mathbf{r}', t - \tau, \mathbf{r}', t - \tau) b_{nm,r}^2(\mathbf{r}', t - \tau),
\end{aligned}$$

where

$$\begin{aligned}
a_{nm,r}(\mathbf{r}, t) & \equiv \frac{a_{nm}(\mathbf{r}, t)}{(1-R)^{0.5} a(\mathbf{r}, t)}, \\
b_{nm}(\mathbf{r}, t) & \equiv \frac{b_{nm}(\mathbf{r}, t)}{(1-R)^{0.5} b(\mathbf{r}, t)}, \\
Ka(\mathbf{r}_n, \mathbf{r}, t, \mathbf{r}', t - \tau) & \equiv (1-R)an \cdot n_2(\mathbf{r}_n)lka(\mathbf{r}, t)a(\mathbf{r}', t - \tau), \\
Kb(\mathbf{r}_n, \mathbf{r}, t, \mathbf{r}', t - \tau) & \equiv (1-R)an \cdot n_2(\mathbf{r}_n)lkb(\mathbf{r}, t)b(\mathbf{r}', t - \tau),
\end{aligned} \tag{2.30}$$

we get

$$\begin{aligned}
 \tau_n(\mathbf{r}) \frac{\partial U(\mathbf{r}, t)}{\partial t} &= D_e(\mathbf{r}) \Delta U(\mathbf{r}, t) - U(\mathbf{r}, t) + f(\mathbf{r}, t), \\
 f(\mathbf{r}, t) &= Kab(\mathbf{r}, t, \mathbf{r}) + \left[\frac{\gamma(\mathbf{r}', t)}{2\sigma} \right]^2 Ka(\mathbf{r}, \mathbf{r}', t - \tau, \mathbf{r}', t - \tau) a_{nm,r}^2(\mathbf{r}', t - \tau) \\
 &\quad + \left[\frac{\gamma(\mathbf{r}', t)}{2\sigma} \right]^2 Kb(\mathbf{r}, \mathbf{r}', t - \tau, \mathbf{r}', t - \tau) b_{nm,r}^2(\mathbf{r}', t - \tau) \\
 &\quad + \frac{\gamma(\mathbf{r}', t)}{\sigma} \times \{ Ka(\mathbf{r}, \mathbf{r}, t, \mathbf{r}', t - \tau) a_{nm,r}(\mathbf{r}', t - \tau) \\
 &\quad \times \cos[(1 + q)\omega\tau + \varphi(\mathbf{r}, t) - \varphi_{nm}(\mathbf{r}', t - \tau) + \psi(\mathbf{r}, t) - \psi_{nm}(\mathbf{r}', t - \tau)] \\
 &\quad + Kb(\mathbf{r}, \mathbf{r}, t, \mathbf{r}', t - \tau) b_{nm,r}(\mathbf{r}', t - \tau) \\
 &\quad \times \cos[(1 - q)\omega\tau + \varphi(\mathbf{r}, t) - \varphi_{nm}(\mathbf{r}', t - \tau) - \psi(\mathbf{r}, t) + \psi_{nm}(\mathbf{r}', t - \tau)] \}.
 \end{aligned} \tag{2.31}$$

Here $Kab(\mathbf{r}, t, \mathbf{r}_n) \equiv (1 - R)an \cdot n_2(\mathbf{r}_n)lk[a^2(\mathbf{r}, t) + b^2(\mathbf{r}, t)]$, $\tau \equiv \tau(\mathbf{r}', t) = t_e(\mathbf{r}', t) + U[\mathbf{r}', t - t_e(\mathbf{r}', t)]/\omega$. Formulas (2.26), (2.27), (2.30) and (2.31) form the mathematical model of the processes in the multipass NRI. This model is equivalent to the formulas (2.26)–(2.28), but they also contain the nonlinearity coefficients.

If to use approximation of the single field pass in NRI $\{a_{nm}(\mathbf{r}', t - \tau) = (1 - R)^{0.5}a(\mathbf{r}', t - \tau), \quad b_{nm}(\mathbf{r}', t - \tau) = (1 - R)^{0.5}b(\mathbf{r}', t - \tau), \quad \varphi_{nm}(\mathbf{r}', t - \tau) = \varphi(\mathbf{r}', t - \tau), \quad \psi_{nm}(\mathbf{r}', t - \tau) = \psi(\mathbf{r}', t - \tau)\}$, or of large losses $\left\{ \left[\frac{\gamma(\mathbf{r}', t)}{2\sigma} \right]^2 \approx 0, \quad \varphi_{nm}(\mathbf{r}', t - \tau) = \varphi(\mathbf{r}', t - \tau), \quad \psi_{nm}(\mathbf{r}', t - \tau) = \psi(\mathbf{r}', t - \tau) \right\}$, and if to accept: $B_{in} = 0$, $A_{in} = const$, $\psi = const$, $\Omega = 0$, $p = 0$, $an = 2$, $\sigma = 1$, $t_e = 0$, $\gamma(\mathbf{r}', t) = \gamma$, $n_2(\mathbf{r}) = n_2$, $I_0 = A_{in}^2$, then we obtain from (2.31) the result coincident with the mathematical model described in [43, 44, 46] and presented in Chap. 1. As this fact demonstrates the absence of contradictions between (2.31) and the model in [43, 44, 46], we can evaluate it as a verification.

The point model of processes in the interferometer. We have described above the optical density of the nonlinear medium by the differential equation in the partial derivatives, supposing that medium molecules or the charge carriers are included into diffusion motion. Examining such systems for solution stability and for bifurcation arising is a more complicated problem than the study of the systems describing by the ordinary differential equations. Therefore, in (2.31) we can neglect diffusion ($D_e = 0$) and, thus, to undergo to the so-called point model.

The name “point model” is justified by the fact that the total set of transverse section points of the light beam (depending on the type of large-scale field transformation by the G element in the feedback loop) is divided into the infinite number of subsets, each independent from one another. Under independence of subsets we understand that the processes (nonlinear and interference) in *different* subsets will run independently. In other words, between the field and the parts of the nonlinear medium (as well as between fields themselves), the physical interaction is absent, if they correspond to the *different* subsets.

Vice versa, such interactions exist, if fields and medium parts correspond to the points of the *same* subset. These subsets represent the chains of points, “coupled” by the sequential displacement of the light beam in the transverse beam plane (due to the action of the G element (Fig. 2.31)).

In other words, the light beam passing through the nonlinear medium and the feedback loop of NRI in the point i (for instance, in Fig. 2.31 $i = 1, 2, 3$) acquires the phase shift U_i and feels the time delay t_{e1} . Because of the presence of the G element, the beam gets to the point $i + 1$. Here, “adding” with the one of the input beams of the interferometer, it, in accordance with the model (2.24), affects the rate of variation of the nonlinear phase shift value U_{i+1} . We note that due to the beam closing in Fig. 2.31, the index value $i + 1 = 4$ should be accepted as equal to $i + 1 = 1$. Exactly so the U_i shift in the point i influences on the U_{i+1} shift in the point $i + 1$. If the point number in these subsets is finite and equal to m , then we speak about the **degenerated** two-dimensional feedback of m -order [44]. The given types of points, according to terminology of [47], are called the transposition points, and m is the order of transposition (from Latin *trans*—through + *positio*—position). Accordingly, we shall speak about chains of the transposition points (CTP). The illustration to the CTP concept is given in Fig. 2.32. With such feedback structure, the **trajectory** of the beam is closed after m passes of NRI. It is easy to see that, according to the accepted approach of numeration of the transposition points, we imply the operation $(i \bmod m) + 1$ under writing $i + 1$, where the symbol $i \bmod m$ means the remainder after division of i by m . Physically it means that the beam from the m -th point will get to the 1st one.

The point approximation, understood in the mentioned way, makes it possible that from the model (2.26), (2.27), (2.30) and (2.31), we can obtain the system of ordinary differential equations (ODE), which describes dynamics of the nonlinear phase shift for the single CTP

$$\begin{aligned}
 \tau_{ni} \frac{dU_i(t)}{dt} &= -U_i(t) + f_i, \\
 f_i \equiv f_i(t) &= an \cdot n_{2i} k [a_{nm}^2(t) + b_{nm}^2(t)] = Kab_{i,i}(t) \\
 &+ \left[\frac{\gamma_{i-1}(t)}{2\sigma} \right]^2 Ka_{i,i-1,i-1}(t - \tau, t - \tau) a_{nm,r,i-1}^2(t - \tau) \\
 &+ \left[\frac{\gamma_{i-1}(t)}{\sigma} \right]^2 Kb_{i,i-1,i-1}(t - \tau, t - \tau) b_{nm,r,i-1}^2(t - \tau) \\
 &+ \frac{\gamma_{i-1}(t)}{2\sigma} \{ Ka_{i,i,i-1}(t, t - \tau) a_{nm,r,i-1}(t - \tau) \\
 &\times \cos[(1 + q)\omega\tau + \varphi_i(t) - \varphi_{nm,i-1}(t - \tau) + \psi_i(t) - \psi_{nm,i-1}(t - \tau)] \\
 &+ Kb_{i,i,i-1}(t, t - \tau) b_{nm,r,i-1}(t - \tau) \\
 &\times \cos[(1 - q)\omega\tau + \varphi_i(t) - \varphi_{nm,i-1}(t - \tau) - \psi_i(t) + \psi_{nm,i-1}(t - \tau)] \}.
 \end{aligned} \tag{2.32}$$

Here

$$\begin{aligned}
a_{nm,r,i}(t) &\equiv \frac{a_{nm,i}(t)}{(1-R)^{0.5}a_i(t)}, \\
b_{nm,r,i}(t) &\equiv \frac{b_{nm,i}(t)}{(1-R)^{0.5}b_i(t)}, \\
Ka_{i,j,k}(t, t-\tau) &\equiv (1-R)an \cdot n_{2i}lka_j(t)a_k(t), \\
Kb_{i,j,k}(t, t-\tau) &\equiv (1-R)an \cdot n_{2i}lkb_j(t)b_k(t), \\
Kab_{i,j}(t) &\equiv (1-R)an \cdot n_{2j}lk[a_i^2(t) + b_i^2(t)], \\
\tau &\equiv \tau_{i-1}(t) = t_{e,i-1}(t) + U_{i-1}[t - t_{e,i-1}(t)]/\omega, \\
\gamma_{i-1}(t) &\equiv 2R\kappa_{i-1}(t)C_{n,i-1}, \\
a_{nm,i}(t) &= (Ac_a^2 + As_a^2)^{0.5}, \quad \varphi_{nm,i}(t) + \psi_{nm,i}(t) = \arg(Ac_a, As_a), \\
b_{nm,i}(t) &= (Ac_b^2 + As_b^2)^{0.5}, \quad \varphi_{nm,i}(t) - \psi_{nm,i}(t) = \arg(Ac_b, As_b), \\
Ac_a &= (1-R)^{0.5}a_i(t) \cos[\varphi_i(t) + \psi_i(t)] \\
&\quad + \frac{\gamma_{i-1}(t)}{2\sigma}a_{nm,i-1}(t-\tau) \cos[\varphi_{nm,i-1}(t-\tau) + \psi_{nm,i-1}(t-\tau) - (1+q)\omega\tau], \\
As_a &= (1-R)^{0.5}a_i(t) \sin[\varphi_i(t) + \psi_i(t)] \\
&\quad + \frac{\gamma_{i-1}(t)}{2\sigma}a_{nm,i-1}(t-\tau) \sin[\varphi_{nm,i-1}(t-\tau) + \psi_{nm,i-1}(t-\tau) - (1+q)\omega\tau], \\
Ac_b &= (1-R)^{0.5}b_i(t) \cos[\varphi_i(t) - \psi_i(t)] \\
&\quad + \frac{\gamma_{i-1}(t)}{2\sigma}b_{nm,i-1}(t-\tau) \cos[\varphi_{nm,i-1}(t-\tau) - \psi_{nm,i-1}(t-\tau) - (1-q)\omega\tau], \\
As_b &= (1-R)^{0.5}b_i(t) \sin[\varphi_i(t) - \psi_i(t)] \\
&\quad + \frac{\gamma_{i-1}(t)}{2\sigma}b_{nm,i-1}(t-\tau) \sin[\varphi_{nm,i-1}(t-\tau) - \psi_{nm,i-1}(t-\tau) - (1-q)\omega\tau].
\end{aligned}$$

Depending on the type of the field transformation into the G range, the variation of i range is different. Under beam compression $i \in [0, +\infty]$; under stretching $i \in [-\infty, 0]$; with rotation by the angle $\Delta \neq 2\pi N/m$ $i \in (-\infty, +\infty)$; with rotation by the angle $\Delta = 2\pi N/m$ $i \in [1, m]$; when there is a shift by the δ value $i \in [1, r_b/\delta]$ (m, N are the mutually prime integer numbers, r_b is the beam radius). Under rotation by the angle $\Delta = 2\pi N/m$, the chain of transposition points is closed, i.e. the field from the last CTP point m after passing the feedback loop arrives into the first one (U_m affects on U_1). During the other field transformations, the unclosed CTPs are probably formed. In case of the closed chain of the transposition points, the

m parameter defines a number of independent dynamic variables, i.e. the phase space dimension of the dynamic system equals to $4m$. This gives exclusiveness to this parameter: variation of its value performs the transition to another dynamic system.

Mathematical models (2.24), (2.32) of the processes are universal, with respect to the view of spatial-temporal functions, whose laws determine the modulation of the field characteristics and interferometer parameters (excluding limitation on maximal rate of modulation), and, therefore, they represent wide opportunities for the development of the theoretical and applied aspects. Molecules diffusion of the nonlinear medium and the optical field diffraction occurring in laboratory conditions do not allow us to consider the processes taking place in the near points of the beam's transverse section plane (in neighbor chains of the transposition points) as independents ones. This circumstance applies restriction on the point model (2.33) application.

Together with the ODE point models, the discrete mapping models are widely used in nonlinear dynamics. Their advantages are the simplification of calculations and the calculation time reduction with keeping the main regulations in system behavior intact. Let us construct such a model and, continuing our selected strategy, reveal point model of the processes in the interferometer. We note that the ODE point model transfers into this model when we use the approximation of the instantaneous response, or if we use the static mode of the NRI operation, i.e. in the absence of variations in the field amplitude time and in phases in NRI and, hence, without the phase shift ($dU/dt = 0$).

The model of optical field dynamics in the nonlinear ring interferometer (in terms of discrete mapping). In NRI static operation mode, the point model (2.32) is reduced to the recurrent relation. We assume that the static mode takes place when the values a_i , b_i , φ_i , ψ_i , γ_i , t_{e1} are constant in time. Then, from (2.32) we have:

$$\begin{aligned}
 U_i = & an \cdot n_{2i} l k \left[a_{nm,i}^2 + b_{nm,i}^2 \right] = Kab_{i,i} + \left[\frac{\gamma_{i-1}}{2\sigma} \right]^2 \\
 & \times \left[Ka_{i,i-1,i-1} a_{nm,t,i-1}^2 + Kb_{i,i-1,i-1} b_{nm,r,i-1}^2 \right] \\
 & + \frac{\gamma_{i-1}}{\sigma} \{ Ka_{i,i,i-1} a_{nm,r,i-1} \\
 & \times \cos[(1+q)(\Phi_{i-1} + U_{i-1}) + \varphi_i + \psi_i - \varphi_{nm,i-1} - \psi_{nm,i-1}] \\
 & + Kb_{i,i,i-1} b_{nm,r,i-1} \\
 & \times \cos[(1-q)(\Phi_{i-1} + U_{i-1}) + \varphi_i - \psi_i - \varphi_{nm,i-1} + \psi_{nm,i-1}] \}.
 \end{aligned} \tag{2.33}$$

Here

$$\begin{aligned}
a_{nm,r,i} &\equiv \frac{a_{nm,i}}{(1-R)^{0.5}a_i}, \\
b_{nm,r,i} &\equiv \frac{b_{nm,i}}{(1-R)^{0.5}b_i}, \\
Ka_{i,j,k} &\equiv (1-R)an \cdot n_{2i}lka_ja_k, \\
Kb_{i,j,k} &\equiv (1-R)an \cdot n_{2i}lkb_jb_k, \\
Kab_{i,j} &\equiv (1-R)an \cdot n_{2i}lk[a_i^2 + b_i^2], \\
\gamma_{i-1} &\equiv 2R\kappa_{i-1}C_{n,i-1}, \quad \Phi_i \equiv \omega t_{e,i}, \\
a_{nm,i} &= (Ac_a^2 + As_a^2)^{0.5}, \quad \varphi_{nm,i} + \psi_{nm,i} = \arg(Ac_a, As_a), \\
b_{nm,i} &= (Ac_b^2 + As_b^2)^{0.5}, \quad \varphi_{nm,i} - \psi_{nm,i} = \arg(Ac_b, As_b), \\
Ac_a &= (1-R)^{0.5}a_i \cos(\varphi_i + \psi_i) + \frac{\gamma_{i-1}}{2\sigma}a_{nm,i-1} \\
&\quad \times \cos[\varphi_{nm,i-1} + \psi_{nm,i-1} - (1+q)(\Phi_{i-1} + U_{i-1})], \\
As_a &= (1-R)^{0.5}a_i \sin(\varphi_i + \psi_i) + \frac{\gamma_{i-1}}{2\sigma}a_{nm,i-1} \\
&\quad \times \sin[\varphi_{nm,i-1} + \psi_{nm,i-1} - (1+q)(\Phi_{i-1} + U_{i-1})], \\
Ac_b &= (1-R)^{0.5}b_i \cos(\varphi_i - \psi_i) + \frac{\gamma_{i-1}}{2\sigma}b_{nm,i-1} \\
&\quad \times \cos[\varphi_{nm,i-1} - \psi_{nm,i-1} - (1-q)(\Phi_{i-1} + U_{i-1})], \\
As_b &= (1-R)^{0.5}b_i \sin(\varphi_i - \psi_i) + \frac{\gamma_{i-1}}{2\sigma}b_{nm,i-1} \\
&\quad \times \sin[\varphi_{nm,i-1} - \psi_{nm,i-1} - (1-q)(\Phi_{i-1} + U_{i-1})].
\end{aligned}$$

In case of the uniform (within the limits of the single CTP) optical properties of NM in NRI ($n_2 = n_{2j}$) and input field amplitudes ($a = a_i$, $b = b_i$), the following relations are true: $Kab = Ka + Kb$, where $Ka \equiv (1-R)an \cdot n_{2i}lk \cdot a^2$, $Kb \equiv (1-R)an \cdot n_{2i}lk \cdot b^2$, $Kab \equiv (1-R)an \cdot n_{2i}lk(a^2 + b^2)$. It is convenient to introduce the total nonlinearity parameter K and a part Q_a of intensity of the component with a the frequency $(1+q)\omega$ according to the rule: $K \equiv Kab = (Ka + Kb)$, $Q_a \equiv Ka/K$. Then $Ka = KQ_a$, $Kb = K(1 - Q_a)$. In case of the uniformity of the other optical properties of NRI ($\Phi = \Phi_i$, $\gamma = \gamma_i$) and the input field ($\psi_i = 0$, $\varphi_i = 0$), it is easy to obtain the discrete map (DM) from (2.33)

$$\begin{aligned}
U_i &= an \cdot n_{2i}lk(a_{nm,i}^2 + b_{nm,i}^2) = K\{1 + (\frac{\gamma}{2\sigma})^2[Q_a a_{nm,r,i-1}^2 + (1 - Q_a)b_{nm,r,i-1}^2] \\
&\quad + \frac{\gamma}{\sigma}\{Q_a a_{nm,r,i-1} \cos[(1+q)(\Phi + U_{i-1}) + \Psi - \Psi_{nm,i-1}] \\
&\quad + (1 - Q_a)b_{nm,r,i-1} \cos[(1-q)(\Phi + U_{i-1}) + \Theta - \Theta_{nm,i-1}]\}\}.
\end{aligned} \tag{2.34}$$

Here

$$\begin{aligned}
 a_{nm,r,i} &\equiv \frac{a_{nm,i}}{(1-R)^{0.5}a}, \\
 b_{nm,r,i} &\equiv \frac{b_{nm,i}}{(1-R)^{0.5}b}, \\
 \gamma &\equiv 2R\kappa C_n, \Phi \equiv \omega t_e, \\
 a_{nm,i} &= (Ac_a^2 + As_a^2)^{0.5}, \\
 \Psi_{nm,i} &= \arg(Ac_a, As_a), \\
 b_{nm,i} &= (Ac_b^2 + As_b^2)^{0.5}, \\
 \Theta_{nm,i} &= \arg(Ac_b, As_b),
 \end{aligned}$$

$$\begin{aligned}
 Ac_a &= (1-R)^{0.5}a \cos \Psi + \frac{\gamma}{2\sigma} a_{nm,i-1} \cos[\Psi_{nm,i-1} - (1+q)(\Phi + U_{i-1})], \\
 As_a &= (1-R)^{0.5}a \sin \Psi + \frac{\gamma}{2\sigma} a_{nm,i-1} \sin[\Psi_{nm,i-1} - (1+q)(\Phi + U_{i-1})], \\
 Ac_b &= (1-R)^{0.5}b \cos \Theta + \frac{\gamma}{2\sigma} b_{nm,i-1} \cos[\Theta_{nm,i-1} - (1-q)(\Phi + U_{i-1})], \\
 As_b &= (1-R)^{0.5}b \sin \Theta + \frac{\gamma}{2\sigma} b_{nm,i-1} \sin[\Theta_{nm,i-1} - (1-q)(\Phi + U_{i-1})],
 \end{aligned}$$

and for the simplification of formula writing, the total and differential phases are introduced:

$$\Psi = \varphi + \psi, \Theta = \varphi - \psi, \Psi_{nmi} = \varphi_{nm,i} + \psi_{nm,i}, \Theta_{nm,i} = \varphi_{nm,i} - \psi_{nm,i}.$$

$$\text{Obviously, } U_i = an \cdot n_2 l k \cdot \left(a_{nm,i}^2 + b_{nm,i}^2 \right) = K \left[Q_a a_{nm,r,i}^2 + (1 - Q_a) b_{nm,r,i}^2 \right].$$

Then we can rewrite (2.34) as

$$\begin{aligned}
 a_{nm,i} &= \left(Ac_{a,i}^2 + As_{a,i}^2 \right), \Psi_{nm,i} = \arg(Ac_{a,i}, As_{a,i}), \\
 b_{nm,i} &= \left(Ac_{b,i}^2 + As_{b,i}^2 \right), \Theta_{nm,i} = \arg(Ac_{b,i}, As_{b,i}),
 \end{aligned} \tag{2.35}$$

where

$$\begin{aligned}
 Ac_{a,i} &= (1-R)^{0.5}a \cos \Psi + \frac{\gamma}{2\sigma} a_{nm,i-1} \cos[\Psi_{nm,i-1} - (1+q)(\Phi + U_{i-1})], \\
 As_{a,i} &= (1-R)^{0.5}a \sin \Psi + \frac{\gamma}{2\sigma} a_{nm,i-1} \sin[\Psi_{nm,i-1} - (1+q)(\Phi + U_{i-1})], \\
 Ac_{b,i} &= (1-R)^{0.5}b \cos \Theta + \frac{\gamma}{2\sigma} b_{nm,i-1} \cos[\Theta_{nm,i-1} - (1-q)(\Phi + U_{i-1})], \\
 As_{b,i} &= (1-R)^{0.5}b \sin \Theta + \frac{\gamma}{2\sigma} b_{nm,i-1} \sin[\Theta_{nm,i-1} - (1-q)(\Phi + U_{i-1})]
 \end{aligned}$$

$$\begin{aligned}
U_{i-1} &= K \left[Q_a a_{nm,r,i-1}^2 + (1 - Q_a) b_{nm,r,i-1}^2 \right] \\
&= \frac{K}{1-R} \left\{ Q_a \left[\frac{a_{nm,i-1}}{a} \right]^2 + (1 - Q_a) \left[\frac{b_{nm,i-1}}{b} \right]^2 \right\}, \\
a_{nm,r,i} &\equiv \frac{a_{nm,i}}{(1-R)^{0.5} a}, \quad b_{nm,r,i} \equiv \frac{b_{nm,i}}{(1-R)^{0.5} b}, \quad \gamma \equiv 2R\kappa C_n, \quad \Phi \equiv \omega t_e.
\end{aligned}$$

We divide $a_{nm,i}$, $Ac_{a,i}$, $As_{a,i}$ by $[(1-R)^{0.5}a]$, and $b_{nm,i}$, $Ac_{b,i}$, $As_{b,i}$ —by $[(1-R)^{0.5}b]$, then (2.35) will transform to the form

$$\begin{aligned}
a_{nm,r,i} &= \left(Ac_{a,r,i}^2 + As_{a,r,i}^2 \right)^{0.5}, \quad \Psi_{nm,i} = \arg(Ac_{a,i}, As_{a,i}), \\
b_{nm,r,i} &= \left(Ac_{b,r,i}^2 + As_{b,r,i}^2 \right)^{0.5}, \quad \Theta_{nm,i} = \arg(Ac_{b,i}, As_{b,i}),
\end{aligned} \tag{2.36}$$

where

$$\begin{aligned}
Ac_{a,r,i} &= \cos \Psi + \frac{\gamma}{2\sigma} a_{nm,r,i-1} \cos[\Psi_{nm,i-1} - (1+q)(\Phi + U_{i-1})], \\
As_{a,r,i} &= \sin \Psi + \frac{\gamma}{2\sigma} a_{nm,r,i-1} \sin[\Psi_{nm,i-1} - (1+q)(\Phi + U_{i-1})], \\
Ac_{b,r,i} &= \cos \Theta + \frac{\gamma}{2\sigma} b_{nm,r,i-1} \cos[\Theta_{nm,i-1} - (1-q)(\Phi + U_{i-1})], \\
As_{b,r,i} &= \sin \Theta + \frac{\gamma}{2\sigma} b_{nm,r,i-1} \sin[\Theta_{nm,i-1} - (1-q)(\Phi + U_{i-1})], \\
U_{i-1} &= K \left[Q_a a_{nm,r,i-1}^2 + (1 - Q_a) b_{nm,r,i-1}^2 \right].
\end{aligned}$$

As we know, the maximal dimension of the phase space corresponding to the dynamic system is determined by the minimal number of independent variables necessary for the complete description of the system behavior. Since the model (2.36) is composed with respect to the four variables: $a_{nm,r,i}$, $b_{nm,r,i}$, $\varphi_{nm,i}$, $\psi_{nm,i}$, the maximal value of the phase space dimension equals to four.

We note that having introduced the change $\mathbf{a}_{nm,r,i} = a_{nm,r,i} \exp(j\Psi_{nm,i})$, $\mathbf{b}_{nm,r,i} = b_{nm,r,i} \exp(j\Theta_{nm,i})$, $\mathbf{a} = \exp(j\Psi)$, $\mathbf{b} = \exp(j\Theta)$, we can rewrite (2.36) in the more compact complex form:

$$\begin{aligned}
\mathbf{a}_{nm,r,i} &= \mathbf{a} + \frac{\gamma}{2\sigma} \mathbf{a}_{nm,r,i-1} \exp[-j(1+q)(\Phi + U_{i-1})], \\
\mathbf{b}_{nm,r,i} &= \mathbf{b} + \frac{\gamma}{2\sigma} \mathbf{b}_{nm,r,i-1} \exp[-j(1-q)(\Phi + U_{i-1})],
\end{aligned} \tag{2.37}$$

where $U_{i-1} = K \left\{ Q_a |\mathbf{a}_{nm,r,i-1}|^2 + (1 - Q_a) |\mathbf{b}_{nm,r,i-1}|^2 \right\}$.

Let us proceed to the two-circuit modification of the nonlinear ring interferometer. It has structurally more complicated (sometimes, even—“elaborate”) configuration of chains of the transposition points. And for this very reason, this modification is more promising for the construction of the optical systems for confidential communication.

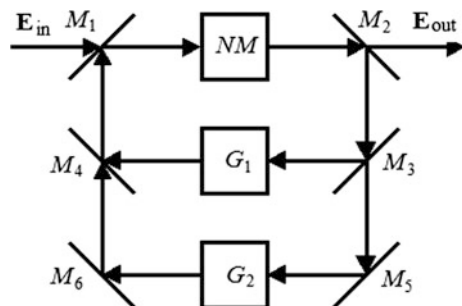
2.4.2 Double-Circuit Nonlinear Ring Interferometer and Models of Processes in It

As in the case of single-circuit ring interferometer, we construct a hierarchy of processes description in the double-circuit device starting from the model composed of equations in the partial derivatives and finishing by description in terms of discrete mapping.

The diagram of the double-circuit nonlinear ring interferometer and the dynamic model of the optical field in it (equation in partial derivatives). We have already mentioned above that the nonlinear ring interferometer (Fig. 2.31) is an example of the system capable of generating both regular optical structures and the deterministic chaos. It is natural to think that traditional NRI construction is the specific case of ring optical systems. Say, such systems can have several feedback loops (FBL), as well as the unequal nonlinear media. Developing this assumption, we examine (as the first stage) the double-circuit ring interferometer (DNRI), shown in Fig. 2.34. It is equipped with the additional FBL, in which one more large-scale transformation (the same or different) of the light field is produced. We can expect that increase in the number of device parameters makes it more difficult for the enemy to “crack” the cryptosystem, which may have DNRI as its basis, owing to the key length growth (see Sect. 1.3).

In approximations used during the construction of the NRI model (2.24), the model of processes in DNRI, evidently, takes a view

Fig. 2.34 The diagram of the double-circuit nonlinear ring interferometer (2.38)



$$\begin{aligned}
\tau_n(\mathbf{r}) \frac{\partial U(\mathbf{r}, t)}{\partial t} &= D_e(\mathbf{r}) \Delta U(\mathbf{r}, t) - U(\mathbf{r}, t) + f(\mathbf{r}, t), \\
f(\mathbf{r}, t) &= n_2(\mathbf{r}) l k n \langle \mathbf{E}_{nm}^2(\mathbf{r}, t) \rangle_T = a n \cdot n_2(\mathbf{r}) l k [a_{nm}^2(\mathbf{r}, t) + b_{nm}^2(\mathbf{r}, t)], \\
\mathbf{E}_{nm}(\mathbf{r}, t) &= (1 - R_1)^{0.5} \mathbf{E}(\mathbf{r}, t) + \mathbf{E}_{FBL}(\mathbf{r}, t).
\end{aligned} \tag{2.38}$$

Here $\mathbf{E}_{nm}(\mathbf{r}, t)$ is the field at nonlinear medium input, $\mathbf{E}(\mathbf{r}, t)$ is the field at DNRI input, $\mathbf{E}_{fbl}(\mathbf{r}, t)$ is the field component arriving at NM input from the “united” feedback loop. It is easy to show that

$$\mathbf{E}_{FBL}(\mathbf{r}, t) = \frac{\gamma_1(\mathbf{r}', t)}{2\sigma_1} \mathbf{E}_{nm}(\mathbf{r}', t - \tau_1) + \frac{\gamma_2(\mathbf{r}_2', t)}{2\sigma_2} \mathbf{E}_{nm}(\mathbf{r}_2', t - \tau_2), \tag{2.39}$$

where $\gamma_1(\mathbf{r}', t) \equiv 2C_n(\mathbf{r}')\kappa(\mathbf{r}', t)R_I$ and $\gamma_2(\mathbf{r}_2', t) \equiv 2C_n(\mathbf{r}_2')\kappa_2(\mathbf{r}_2', t)R_{II}$ is doubled coefficient of loss/transfer in the first and second FBL; $R_I \equiv (R_2R_3R_4R_1)^{0.5}$, $R_{II} \equiv [R_2(1 - R_3)(1 - R_4)R_1]^{0.5}$; R_i are reflection coefficients of the appropriate mirrors; $C_n(\mathbf{r})$ are losses in the nonlinear medium; $\kappa_i(\mathbf{r}_i', t)$ are losses in the elements (excluding mirrors) of i -th FBL of DNRI; σ_i is a stretching beam coefficient in i -th FBL; $\tau_i \equiv \tau_i(\mathbf{r}_i', t) = t_{ei}(\mathbf{r}_i', t) + U[\mathbf{r}', t - t_{e,i}(\mathbf{r}_i', t)]/\omega$ is the propagation time for the optical field component arrived (through i -th FBL) to the time moment t in the point $\mathbf{r}$ of the NM input plane from the point $\mathbf{r}_i'$ of the same plane (the total delay time, the total time of the interferometer bypass through i -th FBL). Expressions (2.38) and (2.39) form the process model in DNRI.

Let us remind that if the amplitude $a_i(\mathbf{r}, t)$ and phase $\varphi_i(\mathbf{r}, t)$ of the interfering components are known, the amplitude $a(\mathbf{r}, t)$ and phase $\varphi(\mathbf{r}, t)$ of the total field $\mathbf{ia}(\mathbf{r}, t) \cos[\omega t + \varphi(\mathbf{r}, t)] = \mathbf{i} \sum_i a_i(\mathbf{r}, t) \cos[\omega t + \varphi_i(\mathbf{r}, t)]$ can be found according to the formulas

$$a(\mathbf{r}, t) = (Ac^2 + As^2)^{0.5}; \quad \varphi(\mathbf{r}, t) = \arg(Ac, As), \tag{2.40}$$

where $Ac \equiv a(\mathbf{r}, t) \cos[\varphi(\mathbf{r}, t)] = \sum_i a_i(\mathbf{r}, t) \cos[\varphi_i(\mathbf{r}, t)]$, $As \equiv a(\mathbf{r}, t) \sin[\varphi(\mathbf{r}, t)] = \sum_i a_i(\mathbf{r}, t) \sin[\varphi_i(\mathbf{r}, t)]$. Besides, let us remind that the field of intensity varies in time according to the law (2.22) and is characterized by amplitudes $a(\mathbf{r}, t)$, $b(\mathbf{r}, t)$, phases $\Psi(\mathbf{r}, t) = \varphi(\mathbf{r}, t) + \psi(\mathbf{r}, t)$, $\Theta(\mathbf{r}, t) = \varphi(\mathbf{r}, t) - \psi(\mathbf{r}, t)$, of two components with frequencies $\omega + \Omega$, $\omega - \Omega$.

Hence, to calculate $\mathbf{E}_{nm}(\mathbf{r}, t)$ in the case of DNRI $i = 1, \dots, 3$, and we must use formulas (2.40) for $a(\mathbf{r}, t)$ and $\varphi(\mathbf{r}, t)$ independently for components with frequency $\omega + \Omega$ and $\omega - \Omega$. In other words, for the description of the component with frequency $\omega + \Omega$ we should accept $a_1 = (1 - R_1)^{0.5} a(\mathbf{r}, t)$, $a_2 = \frac{\gamma_1(\mathbf{r}_1', t) a_{nm}[\mathbf{r}_1', t - \tau_1]}{2\sigma_1}$,

$a_3 = \frac{\gamma_2(\mathbf{r}'_2, t) a_{nm}[\mathbf{r}'_2, t - \tau_2]}{2\sigma_2}$, $\varphi_1 = \Psi(\mathbf{r}, t)$, $\varphi_2 = \Psi_{nm}[\mathbf{r}'_1, t - \tau_1] - (\omega + \Omega)\tau_1$,
 $\varphi_3 = \Psi_{nm}[\mathbf{r}'_2, t - \tau_2] - (\omega + \Omega\tau_2)$, and for the components with frequency $\omega - \Omega$,
 similarly, $a_1 = (1 - R_1)^{0.5} b(\mathbf{r}, t)$, $a_2 = \frac{\gamma_1(\mathbf{r}'_1, t) b_{nm}[\mathbf{r}'_1, t - \tau_1]}{2\sigma_1}$, $a_3 = \frac{\gamma_2(\mathbf{r}'_2, t) b_{nm}[\mathbf{r}'_2, t - \tau_2]}{2\sigma_2}$,
 $\varphi_1 = \Theta(\mathbf{r}, t)$, $\varphi_2 = \Theta_{nm}[\mathbf{r}'_1, t - \tau_1] - (\omega - \Omega)\tau_1$, $\varphi_3 = \Theta_{nm}[\mathbf{r}'_2, t - \tau_2] - (\omega - \Omega\tau_2)$. Then, the model (2.38), (2.39) of processes in DNRI can be rewrite in the form

$$\begin{aligned}
 \tau_n(\mathbf{r}) \frac{\partial U(\mathbf{r}, t)}{\partial t} &= D_e(\mathbf{r}) \Delta U(\mathbf{r}, t) - U(\mathbf{r}, t) + f(\mathbf{r}, t), \\
 f(\mathbf{r}, t) &= n_2(\mathbf{r}) l k a n \cdot \langle \mathbf{E}_{nm}^2(\mathbf{r}, t) \rangle_T = a n \cdot n_2(\mathbf{r}) l k [a_{nm}^2(\mathbf{r}, t) + b_{nm}^2(\mathbf{r}, t)], \\
 \mathbf{E}_{nm}(\mathbf{r}, t) &= (1 - R_1)^{0.5} \mathbf{E}(\mathbf{r}, t) + \frac{\gamma_1(\mathbf{r}'_1, t)}{2\sigma_1} \mathbf{E}_{nm}(\mathbf{r}'_1, t - \tau_1) \\
 &+ \frac{\gamma_2(\mathbf{r}'_2, t)}{2\sigma_2} \mathbf{E}_{nm}(\mathbf{r}'_2, t - \tau_2), \\
 a_{nm}(\mathbf{r}, t) &= (Ac_a^2 + As_a^2)^{0.5}, \quad \Psi_{nm}(\mathbf{r}, t) = \text{Arg}(Ac_a, As_a), \\
 b_{nm}(\mathbf{r}, t) &= (Ac_b^2 + As_b^2)^{0.5}, \quad \Theta_{nm}(\mathbf{r}, t) = \text{Arg}(Ac_b, As_b), \\
 Ac_a &= (1 - R_1)^{0.5} a(\mathbf{r}, t) \cos[\Psi(\mathbf{r}, t)] \\
 &+ \frac{\gamma_1(\mathbf{r}'_1, t)}{2\sigma_1} a_{nm}(\mathbf{r}'_1, t - \tau_1) \cos[\Psi_{nm}(\mathbf{r}'_1, t - \tau_1) - (1 + q)\omega\tau_1] \\
 &+ \frac{\gamma_2(\mathbf{r}'_2, t)}{2\sigma_2} a_{nm}(\mathbf{r}'_2, t - \tau_2) \cos[\Psi_{nm}(\mathbf{r}'_2, t - \tau_2) - (1 + q)\omega\tau_2], \\
 As_a &= (1 - R_1)^{0.5} a(\mathbf{r}, t) \sin[\Psi(\mathbf{r}, t)] \\
 &+ \frac{\gamma_1(\mathbf{r}'_1, t)}{2\sigma_1} a_{nm}(\mathbf{r}'_1, t - \tau_1) \sin[\Psi_{nm}(\mathbf{r}'_1, t - \tau_1) - (1 + q)\omega\tau_1] \\
 &+ \frac{\gamma_2(\mathbf{r}'_2, t)}{2\sigma_2} a_{nm}(\mathbf{r}'_2, t - \tau_2) \sin[\Psi_{nm}(\mathbf{r}'_2, t - \tau_2) - (1 + q)\omega\tau_2], \\
 Ac_b &= (1 - R_1)^{0.5} b(\mathbf{r}, t) \cos[\Theta(\mathbf{r}, t)] \\
 &+ \frac{\gamma_1(\mathbf{r}'_1, t)}{2\sigma_1} b_{nm}(\mathbf{r}'_1, t - \tau_1) \cos[\Theta_{nm}(\mathbf{r}'_1, t - \tau_1) - (1 - q)\omega\tau_1] \\
 &+ \frac{\gamma_2(\mathbf{r}'_2, t)}{2\sigma_2} b_{nm}(\mathbf{r}'_2, t - \tau_2) \cos[\Theta_{nm}(\mathbf{r}'_2, t - \tau_2) - (1 - q)\omega\tau_2], \\
 As_b &= (1 - R_1)^{0.5} b(\mathbf{r}, t) \sin[\Theta(\mathbf{r}, t)] \\
 &+ \frac{\gamma_1(\mathbf{r}'_1, t)}{2\sigma_1} b_{nm}(\mathbf{r}'_1, t - \tau_1) \sin[\Theta_{nm}(\mathbf{r}'_1, t - \tau_1) - (1 - q)\omega\tau_1] \\
 &+ \frac{\gamma_2(\mathbf{r}'_2, t)}{2\sigma_2} b_{nm}(\mathbf{r}'_2, t - \tau_2) \sin[\Theta_{nm}(\mathbf{r}'_2, t - \tau_2) - (1 - q)\omega\tau_2],
 \end{aligned} \tag{2.41}$$

where $\tau_i \equiv \tau_i(r'_i, t) = t_{ei}(r'_i, t) + U[r'_i, t - t_{ei}(r'_i, t)]/\omega$.

Let us introduce designations

$$\begin{aligned}
 a_{nm,r}(\mathbf{r}, t) &\equiv \frac{a_{nm}(\mathbf{r}, t)}{(1 - R_1)^{0.5} a(\mathbf{r}, t)}, \\
 b_{nm,r}(\mathbf{r}, t) &\equiv \frac{b_{nm}(\mathbf{r}, t)}{(1 - R_1)^{0.5} b(\mathbf{r}, t)}, \\
 Kab(\mathbf{r}, t, \mathbf{r}_n) &\equiv (1 - R_1) an \cdot n_2(r_n) lk[a^2(r, t) + b^2(r, t)], \\
 Ka(\mathbf{r}_n, \mathbf{r}, t, \mathbf{r}', t - \tau) &\equiv (1 - R_1) an \cdot n_2(\mathbf{r}_n) lka(\mathbf{r}, t) a(\mathbf{r}', t - \tau), \\
 Kb(\mathbf{r}_n, \mathbf{r}, t, \mathbf{r}', t - \tau) &\equiv (1 - R_1) an \cdot n_2(\mathbf{r}_n) lkb(\mathbf{r}, t) b(\mathbf{r}', t - \tau).
 \end{aligned} \tag{2.42}$$

Than, the model (2.41) can be rewritten as

$$\begin{aligned}
 \tau_n(\mathbf{r}) \frac{\partial U(\mathbf{r}, t)}{\partial t} &= D_e(\mathbf{r}) \Delta U(\mathbf{r}, t) - U(\mathbf{r}, t) + f(\mathbf{r}, t), \\
 f(\mathbf{r}, t) &= Kab(\mathbf{r}, t, \mathbf{r}) \\
 &+ \frac{\gamma_1^2(\mathbf{r}'_1, t)}{4\sigma_1^2} [Ka[\mathbf{r}, \mathbf{r}'_1, t - \tau_1, \mathbf{r}'_1, t - \tau_1] a_{nm,r}^2(r'_1, t - \tau_1) \\
 &+ Kb[\mathbf{r}, \mathbf{r}'_1, t - \tau_1, \mathbf{r}'_1, t - \tau_1] b_{nm,r}^2(r'_1, t - \tau_1)] \\
 &+ \frac{\gamma_2^2(\mathbf{r}'_2, t)}{4\sigma_2^2} [Ka[\mathbf{r}, \mathbf{r}'_2, t - \tau_2, \mathbf{r}'_2, t - \tau_2] a_{nm,r}^2(r'_2, t - \tau_2) \\
 &+ Kb[\mathbf{r}, \mathbf{r}'_2, t - \tau_2, \mathbf{r}'_2, t - \tau_2] b_{nm,r}^2(r'_2, t - \tau_2)] \\
 &+ \frac{\gamma_1(\mathbf{r}'_1, t)}{\sigma_1} Ka[\mathbf{r}, \mathbf{r}, t, \mathbf{r}'_1, t - \tau_1] a_{nm,r}(\mathbf{r}'_1, t - \tau_1) \\
 &\times \cos[\Psi(\mathbf{r}, t) - \Psi_{nm}(\mathbf{r}'_1, t - \tau_1) + (1 + q)\omega\tau_1] \\
 &+ \frac{\gamma_2(\mathbf{r}'_2, t)}{\sigma_2} Ka[\mathbf{r}, \mathbf{r}, t, \mathbf{r}'_2, t - \tau_1] a_{nm,r}(\mathbf{r}'_2, t - \tau_1) \\
 &\times \cos[\Psi(\mathbf{r}, t) - \Psi_{nm}(\mathbf{r}'_2, t - \tau_2) + (1 + q)\omega\tau_2] \\
 &+ \gamma_1(\mathbf{r}'_1, t) Kb(\mathbf{r}_n, \mathbf{r}, t, \mathbf{r}'_1, t - \tau_1) b_{nmr}(\mathbf{r}'_1, t - \tau_1) \\
 &\times \cos[\Theta(\mathbf{r}, t) - \Theta_{nm}(r'_1, t - \tau_1) + (1 - q)\omega\tau_1] / \sigma_1 \\
 &+ \gamma_2(\mathbf{r}'_2, t) Kb(\mathbf{r}_n, \mathbf{r}, t, \mathbf{r}'_2, t - \tau_2) b_{nmr}(\mathbf{r}'_2, t - \tau_2) \\
 &\times \cos[\Theta(\mathbf{r}, t) - \Theta_{nm}(r'_2, t - \tau_2) + (1 - q)\omega\tau_2] / \sigma_2
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \gamma_1(\mathbf{r}'_1, t) \gamma_2(\mathbf{r}'_2, t) K a(\mathbf{r}_n, \mathbf{r}'_1, t - \tau_1, \mathbf{r}'_2, t - \tau_2) \\
& \times a_{nm,r}(\mathbf{r}'_1, t - \tau_1) a_{nm,r}(\mathbf{r}'_2, t - \tau_2) \\
& \times \cos[(1+q)(\tau_2 - \tau_1)\omega + \Psi_{nm}(\mathbf{r}'_1, t - \tau_1) - \Psi_{nm}(\mathbf{r}'_2, t - \tau_2)] / (\sigma_1 \sigma_2) \\
& + \frac{1}{2} \gamma_1(\mathbf{r}'_1, t) \gamma_2(\mathbf{r}'_2, t) K b(\mathbf{r}_n, \mathbf{r}'_1, t - \tau_1, \mathbf{r}'_2, t - \tau_2) \\
& \times b_{nm,r}(\mathbf{r}'_1, t - \tau_1) b_{nm,r}(\mathbf{r}'_2, t - \tau_2) \\
& \times \cos[(1-q)(\tau_2 - \tau_1)\omega + \Theta_{nm}(\mathbf{r}'_1, t - \tau_1) - \Theta_{nm}(\mathbf{r}'_2, t - \tau_2)] / (\sigma_1 \sigma_2),
\end{aligned}$$

$$\begin{aligned}
a_{nm}(\mathbf{r}, t) &= (Ac_a^2 + As_a^2)^{0.5}, \quad \Psi_{nm}(\mathbf{r}, t) = \text{Arg}(Ac_a, As_a), \\
b_{nm}(\mathbf{r}, t) &= (Ac_b^2 + As_b^2)^{0.5}, \quad \Theta_{nm}(\mathbf{r}, t) = \text{Arg}(Ac_b, As_b),
\end{aligned}$$

$$\begin{aligned}
Ac_a &= (1 - R_1)^{0.5} a(\mathbf{r}, t) \cos[\Psi(\mathbf{r}, t)] \\
& + \frac{\gamma_1(\mathbf{r}'_1, t)}{2\sigma_1} a_{nm}(\mathbf{r}'_1, t - \tau_1) \cos[\Psi_{nm}(\mathbf{r}'_1, t - \tau_1) - (1+q)\omega\tau_1] \\
& + \frac{\gamma_2(\mathbf{r}'_2, t)}{2\sigma_2} a_{nm}(\mathbf{r}'_2, t - \tau_2) \cos[\Psi_{nm}(\mathbf{r}'_2, t - \tau_2) - (1-q)\omega\tau_2], \\
As_a &= (1 - R_1)^{0.5} a(\mathbf{r}, t) \sin[\Psi(\mathbf{r}, t)] \\
& + \frac{\gamma_1(\mathbf{r}'_1, t)}{2\sigma_1} a_{nm}(\mathbf{r}'_1, t - \tau_1) \sin[\Psi_{nm}(\mathbf{r}'_1, t - \tau_1) - (1+q)\omega\tau_1] \\
& + \frac{\gamma_2(\mathbf{r}'_2, t)}{2\sigma_2} a_{nm}(\mathbf{r}'_2, t - \tau_2) \sin[\Psi_{nm}(\mathbf{r}'_2, t - \tau_2) - (1-q)\omega\tau_2], \\
Ac_b &= (1 - R_1)^{0.5} b(\mathbf{r}, t) \cos[\Theta(\mathbf{r}, t)] \\
& + \frac{\gamma_1(\mathbf{r}'_1, t)}{2\sigma_1} b_{nm}(\mathbf{r}'_1, t - \tau_1) \cos[\Theta_{nm}(\mathbf{r}'_1, t - \tau_1) - (1-q)\omega\tau_1] \\
& + \frac{\gamma_2(\mathbf{r}'_2, t)}{2\sigma_2} b_{nm}(\mathbf{r}'_2, t - \tau_2) \cos[\Theta_{nm}(\mathbf{r}'_2, t - \tau_2) - (1-q)\omega\tau_2], \\
As_b &= (1 - R_1)^{0.5} b(\mathbf{r}, t) \sin[\Theta(\mathbf{r}, t)] \\
& + \frac{\gamma_1(\mathbf{r}'_1, t)}{2\sigma_1} b_{nm}(\mathbf{r}'_1, t - \tau_1) \sin[\Theta_{nm}(\mathbf{r}'_1, t - \tau_1) - (1-q)\omega\tau_1] \\
& + \frac{\gamma_2(\mathbf{r}'_2, t)}{2\sigma_2} b_{nm}(\mathbf{r}'_2, t - \tau_2) \sin[\Theta_{nm}(\mathbf{r}'_2, t - \tau_2) - (1-q)\omega\tau_2].
\end{aligned} \tag{2.43}$$

In practical modeling, instead of (2.42), it is more convenient to use the designation:

$$\begin{aligned}
 a_{nm,n}(\mathbf{r}, t) &= \frac{a_{nm}(\mathbf{r}, t)}{(1 - R_1)^{0.5} a_{in, \max\{\mathbf{r}, t\}}}, \\
 b_{nm,n}(\mathbf{r}, t) &= \frac{b_{nm}(\mathbf{r}, t)}{(1 - R_1)^{0.5} b_{in, \max\{\mathbf{r}, t\}}}, \\
 a_n(\mathbf{r}, t) &= \frac{a(\mathbf{r}, t)}{a_{in, \max\{\mathbf{r}, t\}}}, \\
 b_n(\mathbf{r}, t) &= \frac{b(\mathbf{r}, t)}{b_{in, \max\{\mathbf{r}, t\}}}, \\
 n_{2n}(\mathbf{r}) &= \frac{n_2(\mathbf{r})}{n_{2\max\{\mathbf{r}\}}}, \\
 K_a &= (1 - R_1) a n \cdot n_{2\max\{\mathbf{r}\}} l k (a_{in, \max\{\mathbf{r}, t\}})^2, \\
 K_b &= (1 - R_1) a n \cdot n_{2\max\{\mathbf{r}\}} l k (b_{in, \max\{\mathbf{r}, t\}})^2, \\
 K &= K_a + K_b, \quad Q_a = K_a / K,
 \end{aligned}$$

where $a_{in \max\{\mathbf{r}, t\}}$, $b_{in \max\{\mathbf{r}, t\}}$ are maximal values of the input field amplitude, $n_{2\max\{\mathbf{r}\}}$ is the maximal value of the nonlinear refraction parameter. Then, instead of (2.43) we finally obtain from the model (2.38)

$$\begin{aligned}
 \tau_n(\mathbf{r}) \frac{\partial U(\mathbf{r}, t)}{\partial t} &= D_e(\mathbf{r}) \Delta U(\mathbf{r}, t) - U(\mathbf{r}, t) + f(\mathbf{r}, t), \\
 f(\mathbf{r}, t) &= Q_a K n_{2n}(\mathbf{r}) a_{nm,n}^2(\mathbf{r}, t) + K(1 - Q_a) n_{2n}(\mathbf{r}) b_{nm,n}^2(\mathbf{r}, t), \\
 a_{nm,n}(\mathbf{r}, t) &= (Ac_a^2 + As_a^2)^{0.5}, \quad \Psi_{nm}(\mathbf{r}, t) = \text{Arg}(Ac_a, As_a), \\
 b_{nm,n}(\mathbf{r}, t) &= (Ac_b^2 + As_b^2)^{0.5}, \quad \Theta_{nm}(\mathbf{r}, t) = \text{Arg}(Ac_b, As_b), \\
 Ac_a &= a_n(\mathbf{r}, t) \cos[\Psi(\mathbf{r}, t)] \\
 &+ \frac{\gamma_1(\mathbf{r}'_1, t)}{2\sigma_1} a_{nm,n}(\mathbf{r}'_1, t - \tau_1) \cos[\Psi_{nm}(\mathbf{r}'_1, t - \tau_1) - (1 + q)\omega\tau_1] \\
 &+ \frac{\gamma_2(\mathbf{r}'_2, t)}{2\sigma_2} a_{nm,n}(\mathbf{r}'_2, t - \tau_2) \cos[\Psi_{nm}(\mathbf{r}'_2, t - \tau_2) - (1 + q)\omega\tau_2], \\
 As_a &= a_n(\mathbf{r}, t) \sin[\Psi(\mathbf{r}, t)] \\
 &+ \frac{\gamma_1(\mathbf{r}'_1, t)}{2\sigma_1} a_{nm,n}(\mathbf{r}'_1, t - \tau_1) \sin[\Psi_{nm}(\mathbf{r}'_1, t - \tau_1) - (1 + q)\omega\tau_1] \\
 &+ \frac{\gamma_2(\mathbf{r}'_2, t)}{2\sigma_2} a_{nm,n}(\mathbf{r}'_2, t - \tau_2) \sin[\Psi_{nm}(\mathbf{r}'_2, t - \tau_2) - (1 + q)\omega\tau_2],
 \end{aligned}$$

$$\begin{aligned}
Ac_b &= b_n(\mathbf{r}, t) \cos[\Theta(\mathbf{r}, t)] \\
&+ \frac{\gamma_1(\mathbf{r}'_1, t)}{2\sigma_1} b_{nm,n}(\mathbf{r}'_1, t - \tau_1) \cos[\Theta_{nm}(\mathbf{r}'_1, t - \tau_1) - (1 - q)\omega\tau_1] \\
&+ \frac{\gamma_2(\mathbf{r}'_2, t)}{2\sigma_2} b_{nm,n}(\mathbf{r}'_2, t - \tau_2) \cos[\Theta_{nm}(\mathbf{r}'_2, t - \tau_2) - (1 - q)\omega\tau_2], \\
As_b &= b_n(\mathbf{r}, t) \sin[\Theta(\mathbf{r}, t)] \\
&+ \frac{\gamma_1(\mathbf{r}'_1, t)}{2\sigma_1} b_{nm,n}(\mathbf{r}'_1, t - \tau_1) \sin[\Theta_{nm}(\mathbf{r}'_1, t - \tau_1) - (1 - q)\omega\tau_1] \\
&+ \frac{\gamma_2(\mathbf{r}'_2, t)}{2\sigma_2} b_{nm,n}(\mathbf{r}'_2, t - \tau_2) \sin[\Theta_{nm}(\mathbf{r}'_2, t - \tau_2) - (1 - q)\omega\tau_2].
\end{aligned} \tag{2.44}$$

Similar to this, as it was made above for NRI, let us perform the transition from (2.44) to the DNRI point model (i.e. those in which $D_e = 0$).

The model of optical field dynamics in the double-circuit nonlinear ring interferometer (“point” model, in terms of discrete mapping). Under rotation of the light beam by the angle $\Delta_j = 2\pi M_j/m$ (or for the beam shift by $\delta_j = M_j\delta$ or when its compression is by $\sigma_j = M_j\sigma$), the point model composed for single CPT has a form

$$\begin{aligned}
\tau_{ni} \frac{dU(t)}{dt} &= -U_i(t) + Q_a K n_{2n,i} a_{nm,n,i}^2(t) + K(1 - Q_a) n_{2n,i} b_{nm,n,i}^2(t), \\
a_{nm,n,i}(t) &= (Ac_{ai}^2 + As_{ai}^2)^{0.5}, \quad \Psi_{nm,i}(t) \text{Arg}(Ac_{ai}, As_{ai}), \\
b_{nm,n,i}(t) &= (Ac_{bi}^2 + As_{bi}^2)^{0.5}, \quad \Theta_{nm,i}(t) \text{Arg}(Ac_{bi}, As_{bi}), \\
Ac_{ai} &= a_{ni}(t) \cos[\Psi_i(t)] \\
&+ \frac{\gamma_{1,i-M_1}(t)}{2\sigma_1} a_{nm,n,i-M_1}(t - \tau_1) \cos[\Psi_{nm,i-M_1}(t - \tau_1) - (1 + q)\omega\tau_1] \\
&+ \frac{\gamma_{2,i-M_2}(t)}{2\sigma_2} a_{nm,n,i-M_2}(t - \tau_2) \cos[\Psi_{nm,i-M_2}(t - \tau_2) - (1 + q)\omega\tau_2],
\end{aligned}$$

$$\begin{aligned}
As_{ai} &= a_{ni}(t) \sin[\Psi_i(t)] \\
&+ \frac{\gamma_{1,i-M_1}(t)}{2\sigma_1} a_{nm,n,i-M_1}(t - \tau_1) \sin[\Psi_{nm,i-M_1}(t - \tau_1) - (1+q)\omega\tau_1] \\
&+ \frac{\gamma_{2,i-M_2}(t)}{2\sigma_2} a_{nm,n,i-M_2}(t - \tau_2) \sin[\Psi_{nm,i-M_2}(t - \tau_2) - (1+q)\omega\tau_2], \\
Ac_{bi} &= b_{ni}(t) \cos[\Theta_i(t)] \\
&+ \frac{\gamma_{1,i-M_1}(t)}{2\sigma_1} b_{nm,n,i-M_1}(t - \tau_1) \cos[\Theta_{nm,i-M_1}(t - \tau_1) - (1-q)\omega\tau_1] \\
&+ \frac{\gamma_{2,i-M_2}(t)}{2\sigma_2} b_{nm,n,i-M_2}(t - \tau_2) \cos[\Theta_{nm,i-M_2}(t - \tau_2) - (1-q)\omega\tau_2], \\
As_{bi} &= b_{ni}(t) \sin[\Theta_i(t)] \\
&+ \frac{\gamma_{1,i-M_1}(t)}{2\sigma_1} b_{nm,n,i-M_1}(t - \tau_1) \sin[\Theta_{nm,i-M_1}(t - \tau_1) - (1-q)\omega\tau_1] \\
&+ \frac{\gamma_{2,i-M_2}(t)}{2\sigma_2} b_{nm,n,i-M_2}(t - \tau_2) \sin[\Theta_{nm,i-M_2}(t - \tau_2) - (1-q)\omega\tau_2],
\end{aligned} \tag{2.45}$$

where $\tau_1 \equiv \tau_{1,i-M_1}(t) = t_{e1,i-M_1}(t) + U_{i-M_1}[t - t_{e1,i-M_1}(t)]/\omega$, $\tau_2 \equiv \tau_{2,i-M_2}(t) = t_{e2,i-M_2}(t) + U_{i-M_2}[t - t_{e2,i-M_2}(t)]/\omega$.

Let us ask the question: how great is the number of points in CTP (what is the range for the variation of the point numbers)? It is clear that this number is conditioned by the CTP structure type. This type, in turn, is determined by the character of the large-scale spatial transformation of the light field. In contrast with the NRI case, in DNRI such transformation is performed by two linear elements $G_i()$ and not by only one of them. We remind that the main types of CTP for NRI are demonstrated in Fig. 2.32. They are the following: single-type beam transformations in two FBL $i \in [0, +\infty]$ with the beam compression; $i \in [-\infty, 0]$ with stretching; $i \in [1, r_b/\delta]$ when there is a shift by the value $\delta_j = M_j\delta$ (r_b is the beam radius); $i \in [1, m]$ under rotation by angles $\Delta_j = 2\pi M_j/m$ (M_1 and m , M_2 and m are mutually prime integer numbers). Here, (in terms of CTP) M_i is a value of displacement step within CTP. For instance, when there is a field rotation, M_i is a numerical expression (in units of $2\pi/m$) of the field rotation angle Δ_i . We shall speak later in detail about CTP types for DNRI.

As we already mentioned in the preamble to the dynamics U model construction in terms of discrete mapping (2.36) and (2.37), together with the ODE point models, it is expedient to research these mapping procedures. To reveal them in the case of DNRI, we assume that the latter operates in the static mode, i.e. values

$a_{nm,n,i}$, $b_{nm,n,i}$, $a_{n,i}$, $b_{n,i}$, $\Psi_{nm,i}$, $\Theta_{nm,i}$, Ψ_i , Θ_i , $\gamma_{j,i}$, $t_{ej,i}$ are constants in time. Then, instead of (2.45), we have:

$$\begin{aligned}
 a_{nm,n,i} &= \left(Ac_{a,i}^2 + As_{a,i}^2 \right)^{0.5}, \quad \Psi_{nm,i} = \text{Arg}(Ac_{a,i}, As_{a,i}), \\
 b_{nm,n,i} &= \left(Ac_{b,i}^2 + As_{b,i}^2 \right)^{0.5}, \quad \Theta_{nm,i} = \text{Arg}(Ac_{b,i}, As_{b,i}), \\
 Ac_{ai} &= a_{ni}(t) \cos[\Psi_i] \\
 &\quad + \frac{\gamma_{1,i-M_1}}{2\sigma_1} a_{nm,n,i-M_1} \cos[\Psi_{nm,i-M_1} - (1+q)\omega\tau_1] \\
 &\quad + \frac{\gamma_{2,i-M_2}}{2\sigma_2} a_{nm,n,i-M_2} \cos[\Psi_{nm,i-M_2} - (1+q)\omega\tau_2], \\
 As_{ai} &= a_{ni} \sin[\Psi_i] \\
 &\quad + \frac{\gamma_{1,i-M_1}}{2\sigma_1} a_{nm,n,i-M_1} \sin[\Psi_{nm,i-M_1} - (1+q)\omega\tau_1] \\
 &\quad + \frac{\gamma_{2,i-M_2}}{2\sigma_2} a_{nm,n,i-M_2} \sin[\Psi_{nm,i-M_2} - (1+q)\omega\tau_2], \\
 Ac_{bi} &= b_{ni} \cos[\Theta_i] \\
 &\quad + \frac{\gamma_{1,i-M_1}}{2\sigma_1} b_{nm,n,i-M_1} \cos[\Theta_{nm,i-M_1} - (1-q)\omega\tau_1] \\
 &\quad + \frac{\gamma_{2,i-M_2}}{2\sigma_2} b_{nm,n,i-M_2} \cos[\Theta_{nm,i-M_2} - (1-q)\omega\tau_2], \\
 As_{bi} &= b_{ni} \sin[\Theta_i] \\
 &\quad + \frac{\gamma_{1,i-M_1}}{2\sigma_1} b_{nm,n,i-M_1} \sin[\Theta_{nm,i-M_1} - (1-q)\omega\tau_1] \\
 &\quad + \frac{\gamma_{2,i-M_2}}{2\sigma_2} b_{nm,n,i-M_2} \sin[\Theta_{nm,i-M_2} - (1-q)\omega\tau_2],
 \end{aligned} \tag{2.46}$$

where i is a point number in the chain of the transposition points (in the transverse plane of the laser beam), $\tau_1 \equiv \tau_{1,i-M_1} = t_{e1,i-M_1} + U_{i-M_1}/\omega$, $\tau_2 \equiv \tau_{2,i-M_2} = t_{e2,i-M_2} + U_{i-M_2}/\omega$, $U_i = Q_a K n_{2n,i} a_{nm,n,i}^2 + K(1 - Q_a) n_{2n,i} b_{nm,n,i}^2$.

Evidently, obtained Eq. (2.46) can be interpreted as the vector function of the vector argument of type $\mathbf{E}_i = \mathbf{F}(\mathbf{E}_{i-M_1}, \mathbf{E}_{i-M_2})$, where the designation $\mathbf{E}_i \equiv (a_{nm,n,i}, \Psi_{nm,i}, b_{nm,n,i}, \Theta_{nm,i})$ is used. Let $M_1 \geq M_2$. Then, DM serves in DNRI as a model of the *spatial* variation of amplitudes $a_{nm,n,i}$, $b_{nm,n,i}$ and phases $\Theta_{nm,i}$ of fields, and it is composed of the M_1 of vector equalities through the function $\mathbf{E}_i$ (here DM has a dimension $4M_1 \leq 4m$):

$$\begin{aligned}
\mathbf{E}_i &= \mathbf{F}(\mathbf{E}_{i-M_1}, \mathbf{E}_{i-M_2}), \\
\mathbf{E}_{i+1} &= \mathbf{F}(\mathbf{E}_{i+1-M_1}, \mathbf{E}_{i+1-M_2}), \\
&\dots\dots\dots \\
\mathbf{E}_{i+M_2} &= \mathbf{F}[\mathbf{E}_{i+M_2-M_1}, \mathbf{F}(\mathbf{E}_{i+M_2-M_2+M_1}, \mathbf{E}_{i+M_2-M_2-M_2})], \\
\mathbf{E}_{i+M_2+1} &= \mathbf{F}[\mathbf{E}_{i+M_2+1-M_1}, \mathbf{F}(\mathbf{E}_{i+M_2+1-M_2-M_1}, \mathbf{E}_{i+M_2+1-M_2-M_2})], \\
&\dots\dots\dots \\
\mathbf{E}_{i+2M_2} &= \mathbf{F}\{\mathbf{E}_{i+2M_2-M_1}, \mathbf{F}[\mathbf{E}_{i+2M_2-M_2-M_1}, \mathbf{F}(\mathbf{E}_{i+2M_2-2M_2-M_1}, \mathbf{E}_{i+2M_2-3M_2})]\}, \\
\mathbf{E}_{i+2M_2+1} &= \mathbf{F}\{\mathbf{E}_{i+2M_2+1-M_1}, \mathbf{F}[\mathbf{E}_{i+2M_2+1-M_2-M_1}, \\
&\mathbf{F}(\mathbf{E}_{i+2M_2+1-2M_2-M_1}, \mathbf{E}_{i+2M_2+1-3M_2})]\}, \\
&\dots\dots\dots \\
\mathbf{E}_{i+M_1-1} &= \dots
\end{aligned}$$

Obtained DM should be rewritten into the traditional view after extracting two indices from the only one: the first one i responding to the iteration number, the second l —to the equation number in DM, where $l = nM_2 + j$, $l = [1; M_1]$; $j = [1; M_2 - 1]$:

$$\mathbf{E}_{i+1,l} = \mathbf{F}\left\{\mathbf{E}_{i,l-0M_2}, \mathbf{F}\left[\mathbf{E}_{i,l-1M_2}, \mathbf{F}(\mathbf{E}_{i,l-2M_2}, \dots, \mathbf{F}(\mathbf{E}_{i,l-nM_2}, \mathbf{E}_{i,M_1-M_2+j}) \dots)\right]\right\}. \quad (2.47)$$

The essential DNRI peculiarity is the rich variety of values of spatial field transformation angles in i -th FBL and, to a considerable degree, by the variety of their combinations. This gives birth to the variety of the geometric structures of CTP, which requires separate consideration.

Features of the field transformation in DNRI feedback loops and a concept of isodynamics. We have already mentioned above the availability of DNRI application for the problem of information protection in the optical range. So, even within the framework of the point model, for the case of optical field rotation in feedback loops by angles $\Delta_i = 2\pi M_i/m$ ($i \in \{1, 2\}$), a plenty of combinations of integer numbers M_1, M_2, m arises. This indicates the increase of the power of the key set compared to the NRI case. In its turn, the question arises about the presence of *equivalent* (isodynamic) combinations of the mentioned numbers. Let us specify in what terms we must understand here the term *isodynamics*. The idea of isodynamics can be illustrated with the help of the Ohm law. If some specific values of currents i_i and resistances r_i satisfy the relations $i_1 r_1 = i_2 r_2$, $i_1 \neq i_2$, $r_1 \neq r_2$, then the values (i_1, r_1) and (i_2, r_2) are isodynamic in terms of the influence on the voltage u at each of the two different conductors, since $u = i_1 r_1 = i_2 r_2$.

Now we generalize these interpretations for the case of the two abstract dynamic systems with dynamical variables (variables of system states) $U_{ij}(\mathbf{r}, t) \in \mathbf{U}_i(\mathbf{r}, t) \equiv \{U_{ij}(\mathbf{r}, t)\}$ and parameters $\mathbf{p}_i(\mathbf{r}, t) \equiv \{p_{i,k}\}$, where $i \in \{1; 2\}$ responds to the system number; $j \in \{1; \dots; m\}$, $k \in \{1; \dots; N_i\}$; m_i and N_1 is a number of dynamic

variables and parameters for the i -th system. We will state that dynamics (evolutions) of the two system copies are isodynamic in terms of $\mathbf{F}$, if the relation of isodynamics of evolutions is satisfied

$$\mathbf{F}[\mathbf{r}, t, \mathbf{U}_1(\mathbf{r}, t), \mathbf{U}_1(\mathbf{r}, 0), \mathbf{U}_2(\mathbf{r}, t), \mathbf{U}_2(\mathbf{r}, 0), \mathbf{p}_1(\mathbf{r}, t), \mathbf{p}_2(\mathbf{r}, t)] \approx 0, \quad (2.48)$$

where $\mathbf{F}[\dots] \equiv \{F_1[\dots]; \dots; F_N[\dots]\}$ is the some vector-function; $\mathbf{U}_1(\mathbf{r}, 0)$, $\mathbf{U}_2(\mathbf{r}, 0)$ are initial conditions. In the general case, in the relation (2.48) under arguments $\{\mathbf{U}_1(\mathbf{r}, t), \mathbf{U}_1(\mathbf{r}, 0), \mathbf{U}_2(\mathbf{r}, t), \mathbf{U}_2(\mathbf{r}, 0), \mathbf{p}_1(\mathbf{r}, t), \mathbf{p}_2(\mathbf{r}, t)\}$ we should understand the complete spatial-temporal realization of the functions, but not their separate values $\mathbf{U}_1, \mathbf{U}_2, \mathbf{p}_1, \mathbf{p}_2$ in the single point $(\mathbf{r}, t)$. The relation (2.48) in the specific case can take a meaning of the relation of identity evolutions: $\mathbf{U}_1(\mathbf{r}, t) - \mathbf{U}_2(\mathbf{r}, t) = 0$. We will put that values $[\mathbf{U}_1(\mathbf{r}, 0), \mathbf{p}_1(\mathbf{r}, t)]$ and $[\mathbf{U}_2(\mathbf{r}, 0), \mathbf{p}_2(\mathbf{r}, t)]$ are equivalent in terms of the chosen relation $\mathbf{F}[\dots] \approx 0$, if the condition (2.48) is satisfied [50, 51]. Naturally, the ascertainment of isodynamic properties is relevant for the solution of the problem about unicity of the cryptographic key [50, 51].

Let us return to the analysis of the structure of the chain of the transposition points in the transverse section of the laser beam. For the simplest *types* of field transformations in the element G of the single-circuit NRI, the CTP *types* are shown in Fig. 2.32. It its turn, the CTP type determines the structure of the mutual dependence of dynamic variable values, i.e. the dependences of U_{i+1} from U_i , in combination with the characteristics of closeness and non-closeness and finiteness (infiniteness).

In the case of DNRI, the complication of CTP structure is possible not only due to the inevitable extension of typology (because of increase in the number of combinations of the field transformation types), but because of the complication of CTP structure inside each of its type. As before, we are limited by rotation of the light field in i -th FBL by the angle $\Delta_i = 2\pi M_i/m$. Then, depending on combination of numbers M_i and m values, the CTP structure essentially changes. For three numbers (three indices): m, M_1, M_2 we can consider as the natural “coding” of the structure. In Tables 2.2, 2.3 and 2.4, where various configurations of transitions between points in CTP (various structures of chains of the transposition points), and in further description we use exactly this coding. In figures and tables, the lines represent transitions between points in CTP: the dashed lines—transitions in FBL₁, the solid ones—in FBL₂.

From Tables 2.2, 2.3 and 2.4 we clearly see how manifold the CTP structure begins to complicate with the number m growth at the expense of the second FBL presence. In case of the single FBL, each of the Tables 2.3 and 2.4 would contain only the one cell corresponding to the three numbers $m11$. But the representation way used in Tables 2.3 and 2.4 is redundant. For instance, CTP with indices 221 is *isodynamic* to CTP with indices 212, since DNRI with CTP 221 and DNRI 212 are isodynamic (in terms of a variety of the possible types of modes in the interferometer [51]). Moreover, CTP 222 is divided into a pair of independent CTP 111, i.e., strictly speaking, CTP 222 is not the chain of the transposition points. Similarly, CTP 333 is divided into three CTP 111. In the last two cases, the DNRI with CTP mmm is identical to the DNRI with CTP 111 [52].

Table 2.2 Transition configuration between points in CTP in case of the “single-point” ($m = 1$) process model in DNRI—double-circuit nonlinear ring interferometer (DNRI)

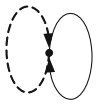
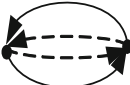
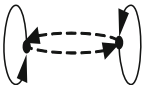
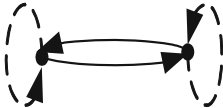
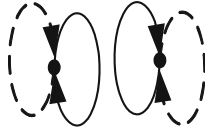
$m=1$		M_2
		1
M_1	1	 111

Table 2.3 Transition configuration between point in CTP in case of the “double-point” ($m = 2$) process model in DNRI—double-circuit nonlinear ring interferometer (DNRI)

$m=2$		M_2	
		1	2
M_1	1	 211	 212
	2	 221=212	 222=2(111)

If m is a prime number, the only first line is relevant in Tables 2.2, 2.3 and 2.4, all other transition configurations can be represented through configurations, which are present in it. Otherwise, one should choose values of triple indices in tables according to the following rule: $m = M_1N_1$, $m = M_2N_2$, where M_1 and M_2 are mutually prime numbers.

For example, for CTP composed of six points, the $M_1 = 3$ copies of $N_1 = 2$ (double) point structures and $M_2 = 2$ copies of $N_2 = 3$ (triple) point structures are realized. This case is represented in Fig. 2.35.

The case presented in Fig. 2.35, can be interpreted as a system of *three* oscillators (double-point NRI) related to six communication channels, but we can interpret it as a system of *two* oscillators (three-point NRI) related to the same six communication channels. On the contrary, the case CTP 511 and others, seemingly, should be interpreted as an appearance of the system of five additional connections in the one oscillator (five-point NRI), due to the arrangement of the second FBL [52].

Table 2.4 Transition configuration between points in CTP in case of the “triple-point” ($m = 3$) process model in DNRI—double-circuit nonlinear ring interferometer (DNRI)

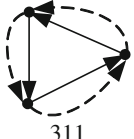
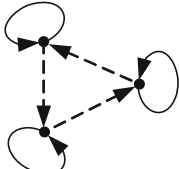
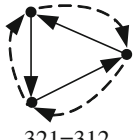
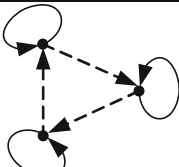
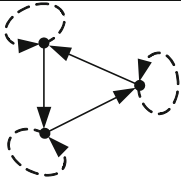
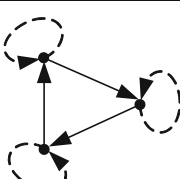
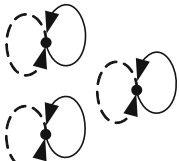
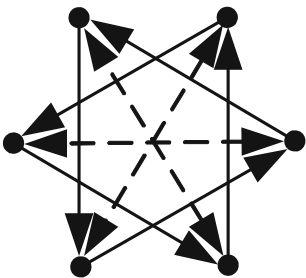
$m=3$		M_2		
		1	2	3
M_1	1	 311	 312	 313
	2	 321=312	 322=311	 323=313
	3	 331=313	 332=323	 333=3(111)

Fig. 2.35 Possible structures of transitions between six points forming CTP. *Dashed lines* represent transition between CTP points in the first FBL, *solid lines*—in the second FBL



Now we have mathematical models for dynamics of amplitude and phase of the light fields in ordinary and modified NRIs. Besides, for DNRI, the connection of CTP structure with possible spatial field transformations is discovered with the help of linear elements G_i in FBL, and the isodynamic combinations of these transformations are also revealed. We do not aim at the systematic of all the aspects of

nonlinear ring interferometers behavior, we, however, now proceed to the demonstration of their properties relevant for the cryptologic applications specifically. Involving the numerical methods for this purpose, we are to show that NRI and DNRI are capable of functioning in the chaotic mode.

2.4.3 Dynamics in the Ring Interferometer Models

Regular and chaotic behavior in the NRI point model and in NRI DM model.

A concept of the spatial deterministic chaos. For instance, the computing experiment, whose results Figs. 2.36 and 2.37 demonstrate, indicates the complicated dynamics in the point model (2.32) of NRI. These figures show, for cases of absence and presence of amplitude modulation of the input field for two different nonlinearity coefficients, the time realization, phase portraits and Fourier spectra of the wave amplitude A_i at the output, where i is a number of transposition points in CTP. The figures allow us to see: the modulation action changes the amplitude A_i spectrum so greatly that under regular (especially, chaotic) behavior of the model the problem of recovery of the modulation type at NRI input is non-trivial.

Let us proceed to the discrete mapping (DM) model. We remind that DM (2.36) and (2.47) are obtained in approximation of the static operation mode of NRI. We

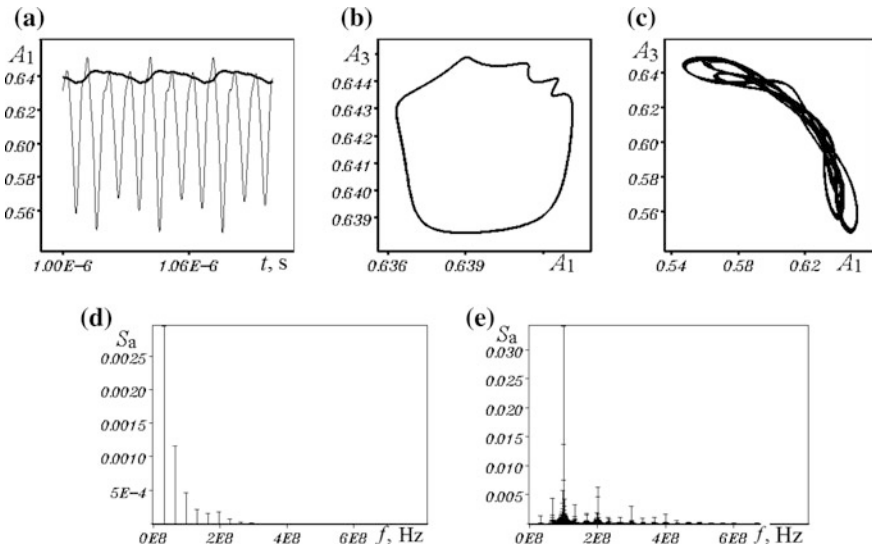


Fig. 2.36 Time realization A_1 (a), phase portraits (b, c), Fourier spectra S_a of output amplitude A_i of the wave (d, e) in the case of autonomous mode [a (thick curve), b, d] and the amplitude modulation mode [a (thin curve), c, d]. $A_1 = A(\mathbf{r}, t)$, where $\mathbf{r} = (0.5; 0)$; $A_3 = A(\mathbf{r}, t)$, where $\mathbf{r} = (0; 0.5)$. $K = 4.55$. The modulation law: $A_{in}(r, t) = [1 + 0.01 \cos(2\pi f_1 t)/1.01]$, where $f_1 = \frac{1}{30.618 \cdot 10^{-9}} \approx 0.3266 \cdot 10^8$ corresponds to the frequency of the Fourier spectrum harmonic, which has the maximal amplitude (e)

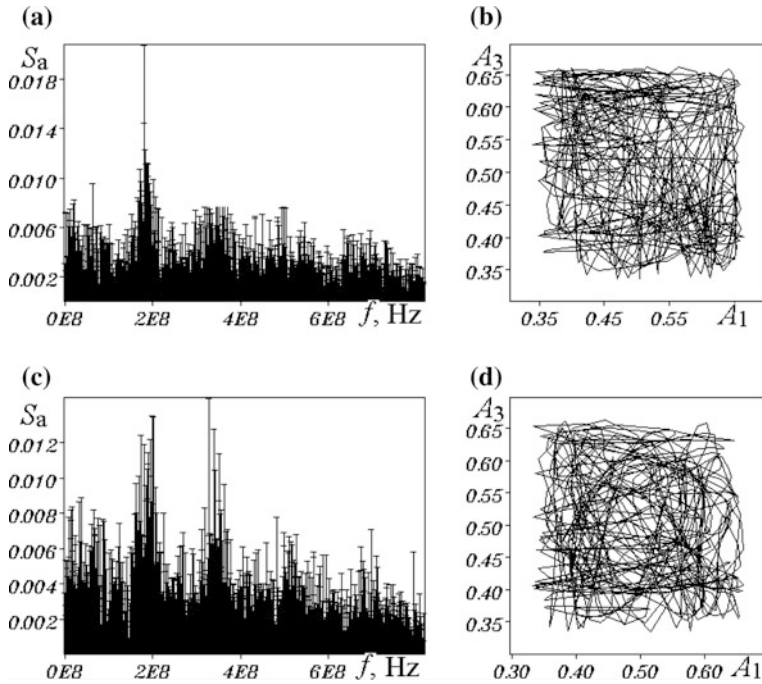


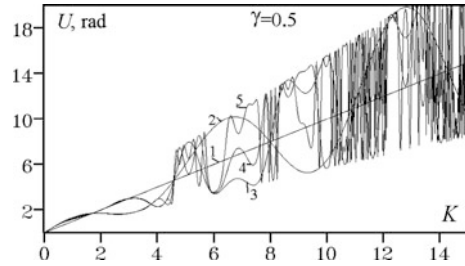
Fig. 2.37 Fourier spectra S_a of the output wave power A_1 (a, c) and phase portraits (b, d) in the case of the autonomous mode (a, b) and the amplitude modulation mode (b, d), which are the same as in Fig. 2.36. $K = 10$

can understand that this circumstance may cause confusion, as the question arises: what non-trivial behavior is possible in DM describing the NRI in its *static* mode? Here two answers suggest themselves. Firstly, the ODE point model transfers into the DM model, if we use the approximation of the instantaneous response. Secondly, the static mode, while excluding all time dynamics, by no means exclude the complex spatial distributions of the system dynamic variables. In the last case it is rightful to speak about the “spatial” deterministic chaos [53, 54]. Then, the variation of the discrete evolution variable i in DM formally corresponds to the examination of the space points (according to some algorithm) by the *observer*, who registers, for example, the values of $a_{nm,r,i}$, $b_{nm,r,i}$, $\varphi_{nm,i}$, $\psi_{nm,i}$, revealed in (2.36).

Here the term “deterministic” indicates that the disarray obeys some regulation built into the mathematical model and not caused by some random factor. Such a word usage is intended to specify the existing similarity/contrast with the widely known temporal, i.e. dynamic (deterministic), chaos in the single-dimensional system models, and with spatial-temporal chaos, or the turbulence, in the multi-dimensional models. We would like to note that systems, in which the spatial deterministic chaos is possible, *are not practically studied*.

For instance, Fig. 2.38 shows the complicated spatial dependence of the non-linear phase shift from the number of the transposition point. The DM $U_{i+1} =$

Fig. 2.38 Dependence of values of nonlinear phase shift U upon the nonlinearity parameter K in the first points of non-closed CTP for the static NRI mode at shift by the element G of the optical field (along the x -axis in the plane xOy). Numbers are the point numbers in CTP



$K[1 + \gamma \cos(U_i + \Phi)]$ serves as its base, describing the spatial variation U_i in NRI in great losses approximation [55]. Then the complex distribution of the nonlinear phase shift $U(\mathbf{r})$ (see Fig. 2.32) is formed in the transverse plane of the laser beam in NRI.

Taking into consideration the cryptologic orientation of investigations, the phase portraits (Figs. 2.39 and 2.40) are constructed and the fractal dimension (capacity) of D_0 attractor is calculated for different sets of the DM model parameters (2.36). The example of attractor in four-dimensional dynamic system is shown in Fig. 2.40, where it is projected on the plane and is drawn in the three-dimensional space. As expected, increase of the fractal dimension is caused by the complication of the attractor structure and, accordingly, by the complication of NRI mode functioning. Usually, this is caused by the chaotic dynamics, necessary for the development of the confidential communication systems.

More complete picture of the influence of the parameters of the model upon the character of dynamics it demonstrates is achieved by the construction of the so-called Lyapunov characteristic exponents Λ (Fig. 2.41) and the maps of the fractal dimension on the plane of parameters under consideration (Fig. 2.42). Among them the most important are the nonlinearity coefficient K , the loss characteristic γ and the fraction Q_a of the component intensity with the frequency $\omega(1 + q)$, since their set determines the nonlinearity degree in the system. The map $\Lambda(K, \gamma)$ in Fig. 2.41 reveals the possibility of both the regular ($\Lambda \leq 0$) and the chaotic dynamics ($\Lambda > 0$) in the model. A variety of the fractal dimension map structures $D_0(K, \gamma)$, depending on the parameters of input radiation spectrum, is illustrated by the series in Fig. 2.42. Maps are represented by grey color tins. The white color corresponds to the minimal value of the fractal dimension ($D_0 = 0$), the black color – to the maximal one. Since the attractor of zero dimension corresponds to the chaotic attractor, the structure of maps $\Lambda(K, \gamma)$ and $D_0(K, \gamma)$ should be similar. Indeed, maps in Figs. 2.41 and 2.42, *a* are similar in the structure, in spite of some difference in spectral characteristics of radiation q, Q_a .

As we know, there are no methodical recommendations in the specialized literature referring to the comparable analysis of maps of such a type. Therefore, it is difficult to present the impartial estimation of the variation degree in the map structure. In order to simplify the formation of the integral representation of the dynamic system properties, it is expedient to construct the plots of dimension density for these maps (Fig. 2.43), together with the maps of the fractal dimension

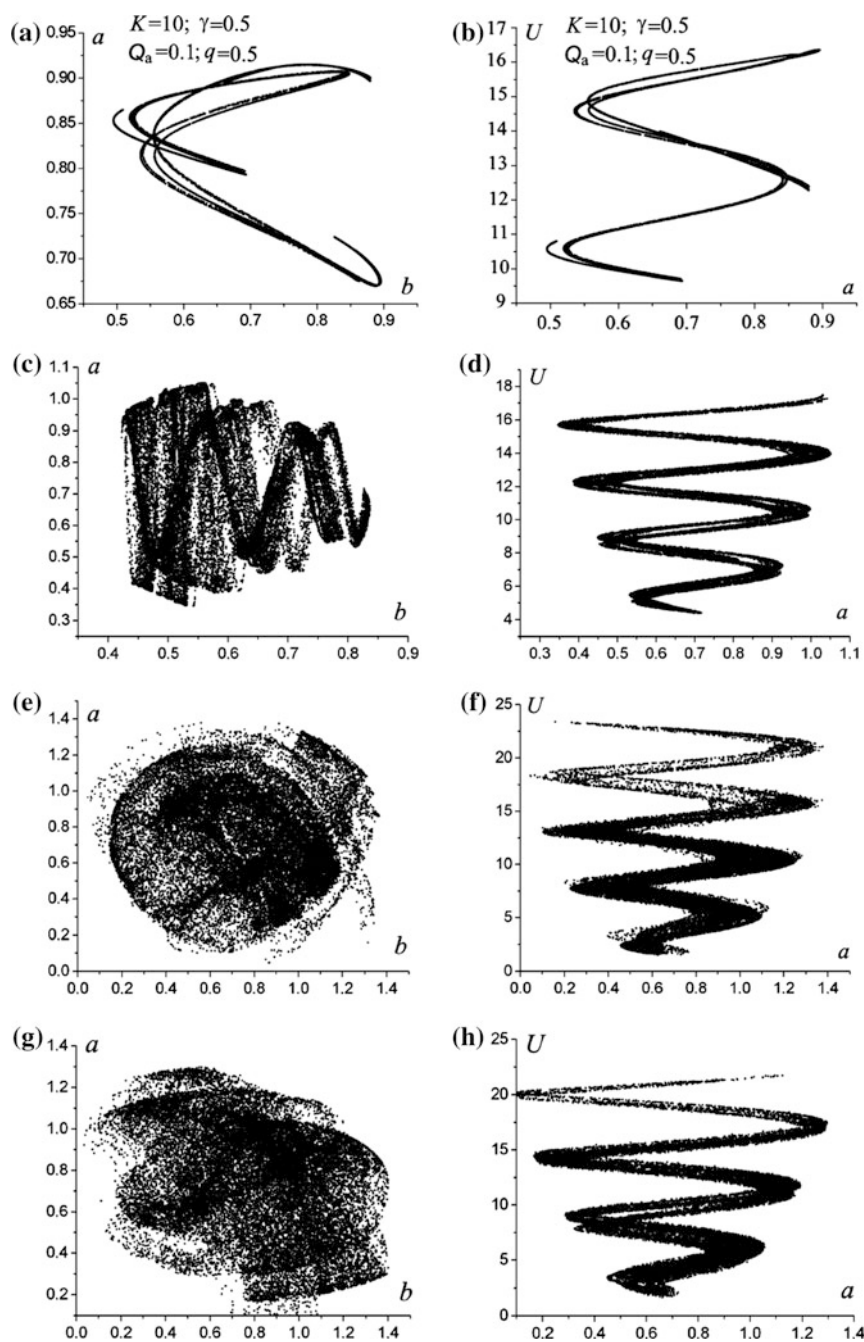


Fig. 2.39 Phase portraits of attractors with fractal dimension: $D_0 = 1.18496$ (a, b); $D_0 = 1.62791$ (c, d); $D_0 = 2.03342$ (e, f); $D_0 = 2.13637$ (g, h)

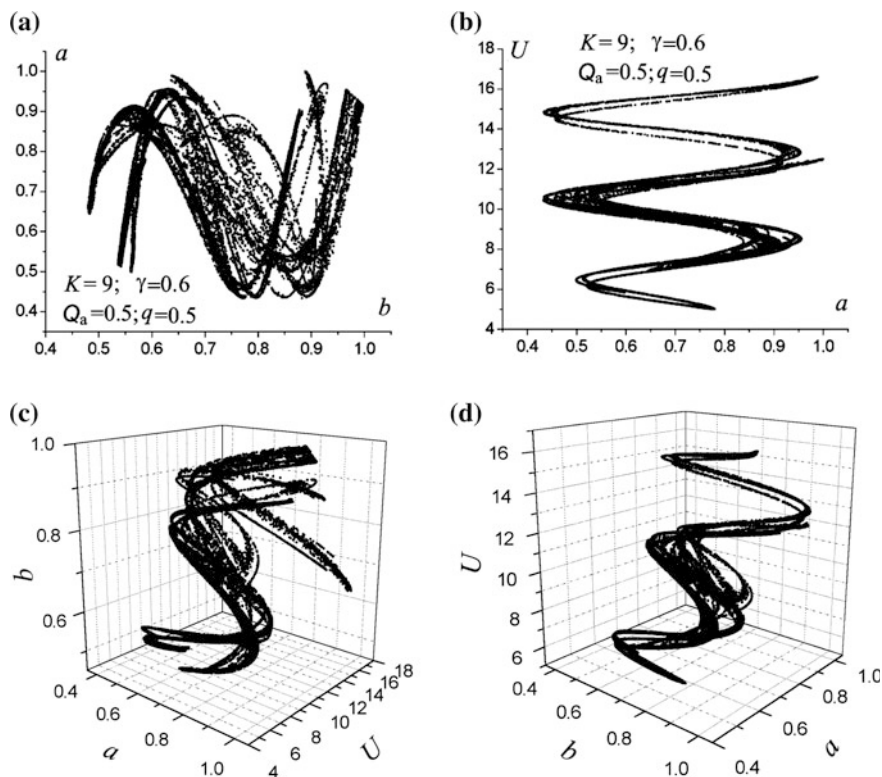


Fig. 2.40 2D-phase portraits (a, b) and 3D-phase portraits (c, d) of the attractor with fractal dimension $D_0 = 1.53023$ of the 4D-system

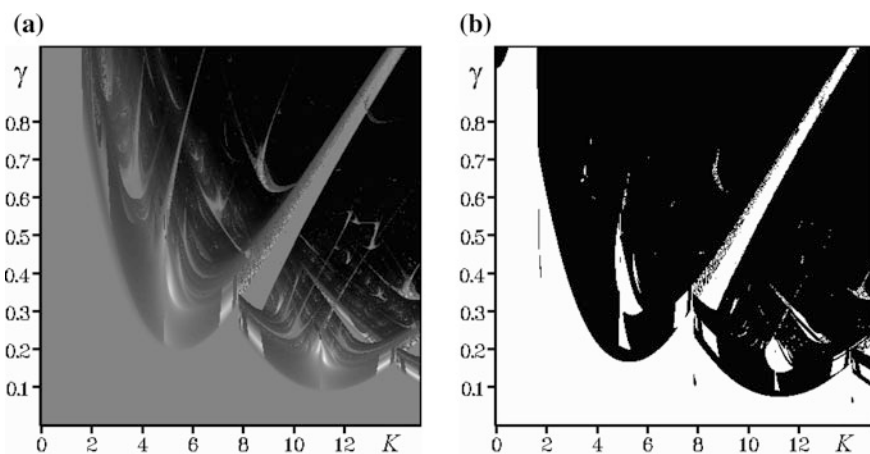


Fig. 2.41 Maps (in sense of charts) of Lyapunov's exponents $\Lambda(K, \gamma)$ (a) and the contrast map sign $[\Lambda(K, \gamma)]$ (b) for NRI model in the form of DM (2.36) for $q=0$, $\Phi=0$, $Q_a=1$. The white color corresponds to the minimal value of Λ , the black color corresponds to maximal value

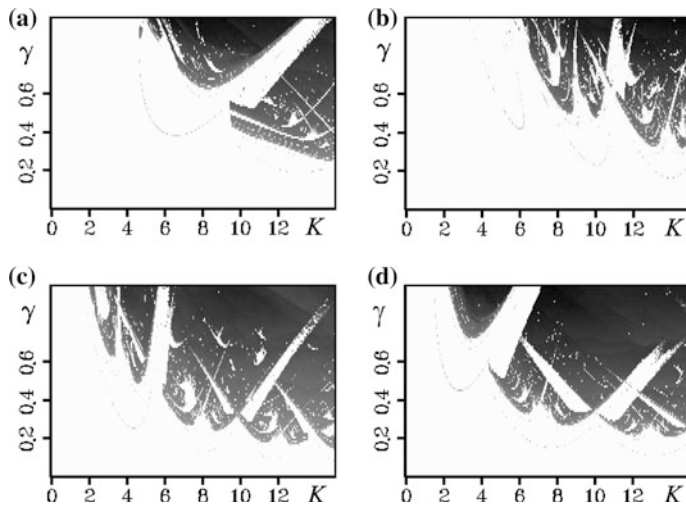


Fig. 2.42 Maps (in sense of charts) of fractal dimension of the attractor in the process model in NRI (2.36) at variation of bi-harmonic parameters: $q = 0.5$, $Q_a = 0.1$ (a); $q = 0.9$, $Q_a = 0.2$ (b); $q = 0.8$, $Q_a = 0.5$ (c); $q = 0.1$, $Q_a = 0.8$ (d)

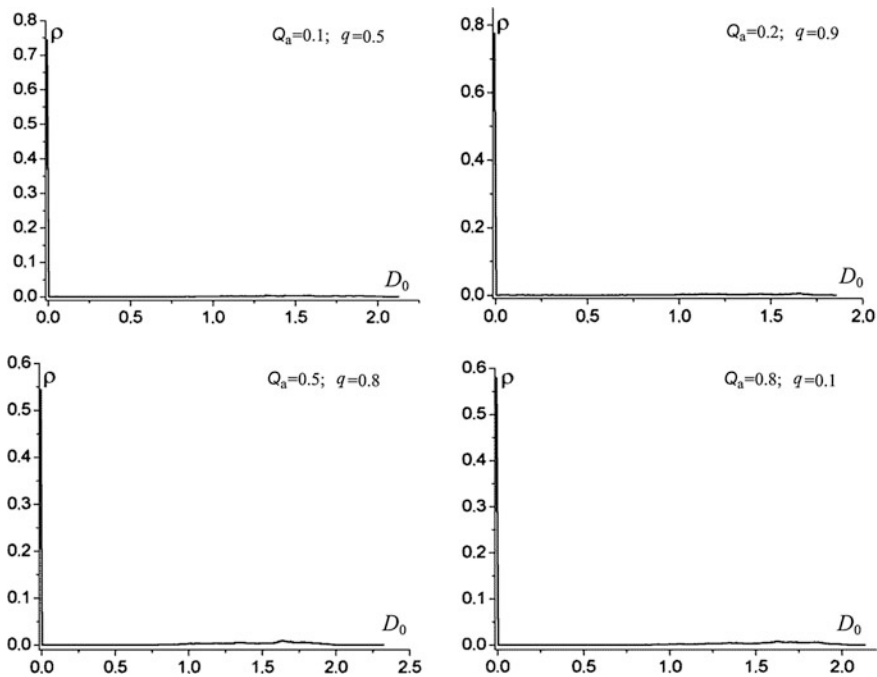


Fig. 2.43 Distribution density of the fractal dimension D_0 corresponding to the map $D_0(K, \gamma)$ corresponding to the appropriate model (2.36) at different values of bi-harmonic parameters: q and Q_a . $K \in [0; 15]$, $\gamma \in [0; 1]$

Table 2.5 A fraction P of map area of the fractal dimension $D_0(K, \gamma)$ corresponding to the interval of values D_0 , $K \in [0; 15]$, $\gamma \in [0; 1]$

Characteristics of bi-chromatic spectrum					
$Q_a = 0.5; q = 0$ (monochrome)		$Q_a = 0.8; q = 0.1$		$Q_a = 0.5; q = 0.2$	
Interval of values D_0	P (%)	Interval of values D_0	P (%)	Interval of values D_0	P (%)
[0; 0.1]	54.93	[0; 0.1]	58.27	[0; 0.1]	66.51
[0.9; 1.53]	42.63	[0.9; 2.14]	39.94	[0.9; 2.49]	31.75
[0.1; 0.9]	2.45	[0.1; 0.9]	1.8	[0.1; 0.9]	1.74
Characteristics of bi-chromatic spectrum					
$Q_a = 0.1; q = 0.5$			$Q_a = 0.5; q = 0.5$		
Interval of values D_0	P (%)	Interval of values D_0	P (%)	Interval of values D_0	P (%)
[0; 0.1]	74.75	[0; 0.1]	59.53	[0; 0.1]	66.51
[0.9; 2.13]	23.73	[0.9; 2.92]	38.38	[0.9; 2.49]	31.75
[0.1; 0.9]	1.52	[0.1; 0.9]	2.11	[0.1; 0.9]	1.74
Characteristics of bi-chromatic spectrum					
$Q_a = 0.5; q = 0.8$			$Q_a = 0.2; q = 0.9$		
Interval of values D_0	P (%)	Interval of values D_0	P (%)	Interval of values D_0	P (%)
[0; 0.1]	54.69	[0; 0.1]	77.89	[0; 0.1]	66.51
[0.9; 2.33]	42.79	[0.9; 1.86]	19.96	[0.9; 2.49]	31.75
[0.1; 0.9]	2.53	[0.1; 0.9]	2.53	[0.1; 0.9]	1.74

of the attractor values D_0 . Besides, it is also reasonable in this case to calculate the part P of the map area corresponding to this or that interval of D_0 values (Table 2.5). From the joint examination of Fig. 2.43 and Table 2.5, we can deduce how the percentages for $D_0 \in [0; 0.1]$, $D_0 \in [0.1; 0.9]$ and $D_0 \in [0.9; D_{0\max}]$ depend upon the set of values of the model parameters. As a whole, all these representations of the dynamic system behavior allow us to predict the most favorable ranges of parameters for the cryptosystem operation.

Let us proceed to the peculiarities of the DNRI model behavior.

Peculiarities of the process dynamics in the NRI point model with field rotation. Computer experiments show that the choice of DNRI parameters (the nonlinearity coefficient K , the transmission coefficient in FBL γ_i , delay time $t_{e,i}$ in FBL₁ and FBL₂, the field rotation angle Δ_i in each of the FBL) allows controlling the dynamics character. Thus, the control can be carried out by:

- (1) the U mean values and the phase shift between U oscillations in (transposition) points of the transverse section of the laser beam;
- (2) type and final state of the setting process whose possible form is the transient process, the transition being from the periodic mode to the quasi-periodic one (Fig. 2.44a); from periodic one to chaotic one (Fig. 2.44b); from static one to periodic one (Fig. 2.44c); from the periodic in-phase (at the two transposition points) mode to the periodic out-of-phase mode at the same points (Fig. 2.44d); from chaotic mode to periodic mode (Fig. 2.44e) [38].

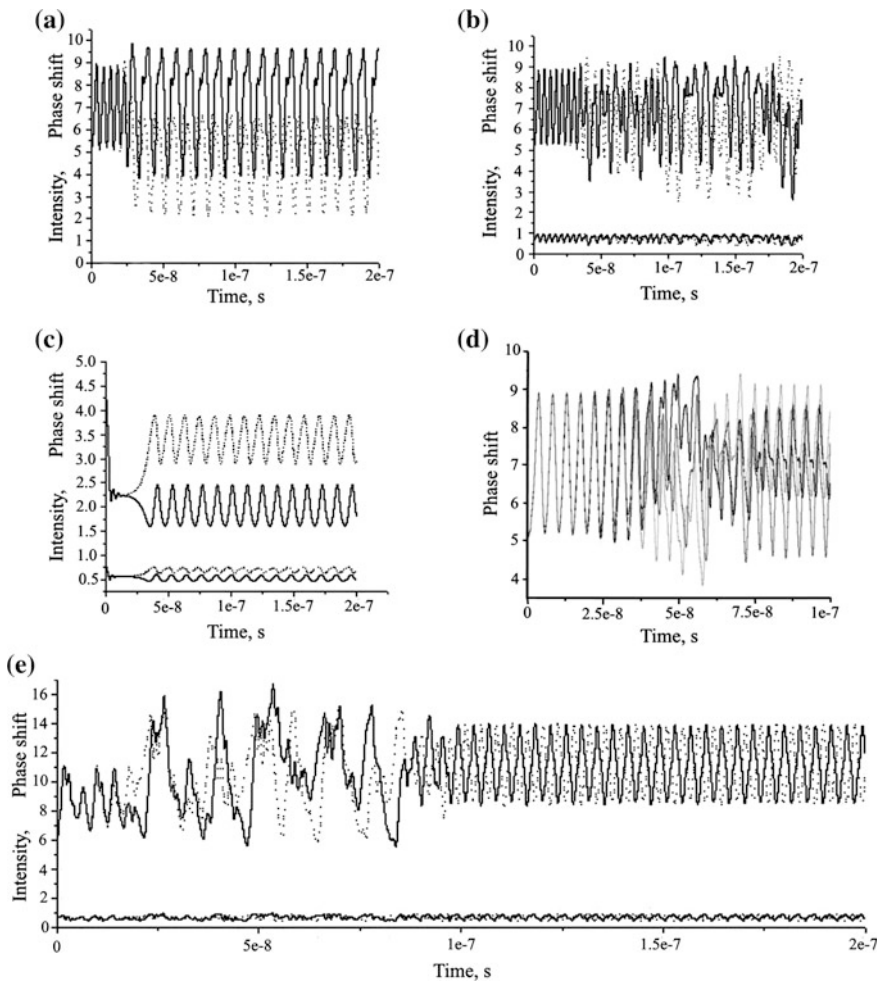


Fig. 2.44 Time realizations of the nonlinear phase shift in DNRI demonstrating the control (at the expense of the relation choice of parameter values) of character and final transient process for $\gamma_1 = (0.5 - \gamma_2^{0.5})$: $m = 2$, $\Delta_1 = \Delta_2 = 180^\circ$, $t_{e1} = 1$, $t_{e2} = 2$ (a–c, e); $K = 5.5$, $\gamma_1 = 0.64$, $\gamma_2 = 0.3$ (a); $K = 5.5$, $\gamma_1 = 0.678$, $\gamma_2 = 0.2$ (b); $K = 3.5$, $\gamma_1 = 0.58$, $\gamma_2 = 0.4$ (c); $K = 10$, $\gamma_1 = 0.678$, $\gamma_2 = 0.2$ (e); $m = 3$, $\Delta_1 = 120^\circ$, $\Delta_2 = 240^\circ$, $K = 5.5$, $t_{e1} = 1$, $t_{e2} = 2$, $\gamma_1 = 0.66$, $\gamma_2 = 0.25$ (d)

From the structure of the phase portraits on the plane $A_{i+1}OA_i$, with varying parameters of the each of FBLs, we can see that for some model parameters the typical symmetric forms are possible (Figs. 2.45 and 2.46), along with such configurations, to which (quasi) periodic (Fig. 2.46) or aperiodic (Fig. 2.47) motions correspond. During the construction of the phase portraits, the transient process, which is capable of adding distortions to their structure, is eliminated.

For better understanding of how model parameters influence the character of the dynamic mode, we do the mapping of the fractal dimension $D_0(t_{e1}, t_{e2})$ of the

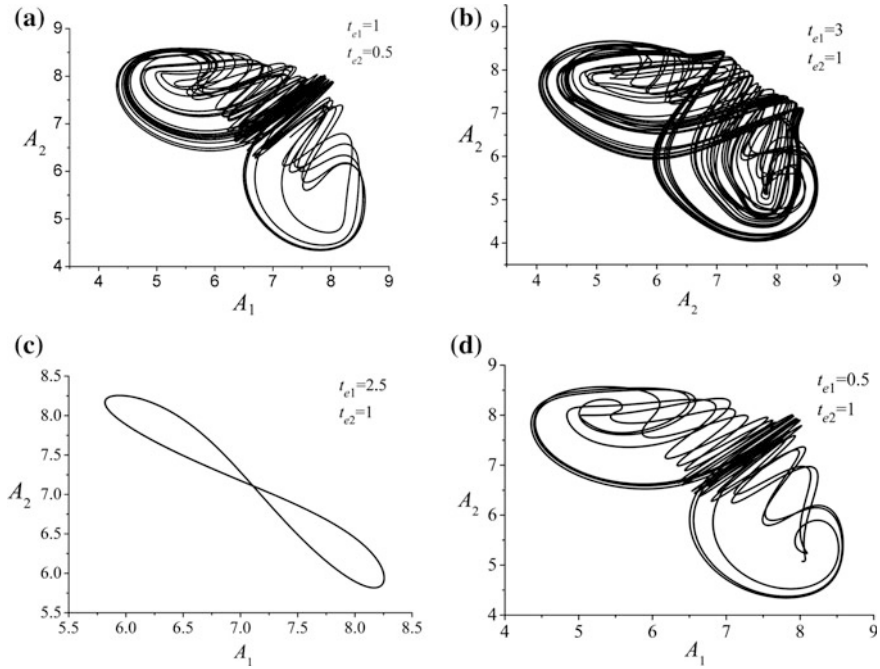


Fig. 2.45 Phase portraits of symmetric form; rotations $\Delta_1 = 180^\circ$, $\Delta_2 = 180^\circ$ (a, b, d); rotations $\Delta_1 = 180^\circ$, $\Delta_2 = 0^\circ$ (c)

attractor in the model (Fig. 2.48). As it is demonstrated above, the DNRI peculiarity is a diversity of geometric CTP structures. To find out how relations and angle values of field rotation Δ_1 , Δ_2 in FBL_1 and FBL_2 affect the mapping structure of the fractal dimension $D_0(t_{e1}, t_{e2})$ and the fractional maps of the attractors $[[D_0 - \text{round}(D_0)]]$, it is necessary to calculate and construct its series. Here “round” means the rounding operation. Since these calculations are rather resource-capacious, it is expedient to decrease a number of calculated points. All maps (Figs. 2.49 and 2.50) are drawn under condition that FBL_1 , FBL_2 in DNRI are isodynamic according to their influence on dynamics (i.e. double transmission coefficients γ_i are the same: $\gamma_1 = \gamma_2 = 0.25$).

Analysis of the map structure allows several generalizations. If the field rotation angle $\Delta \neq 0$ in both FBLs, the attractor fractal dimension does not decrease (as a rule, it is essentially increased) compared to the case when one of the angles $\Delta_i = 0$ [56]. The parameter areas where the attractor fractal dimension is high are themselves widening. Obviously, on the D_0 maps and on the fractional maps there are areas with the same “strangeness degrees” of attractors (the identical modes) that correspond to the various sets of time delay values t_{e1}, t_{e2} in FBL. These conclusions agree with the observations (Figs. 2.45, 2.46 and 2.47) of U behavior in

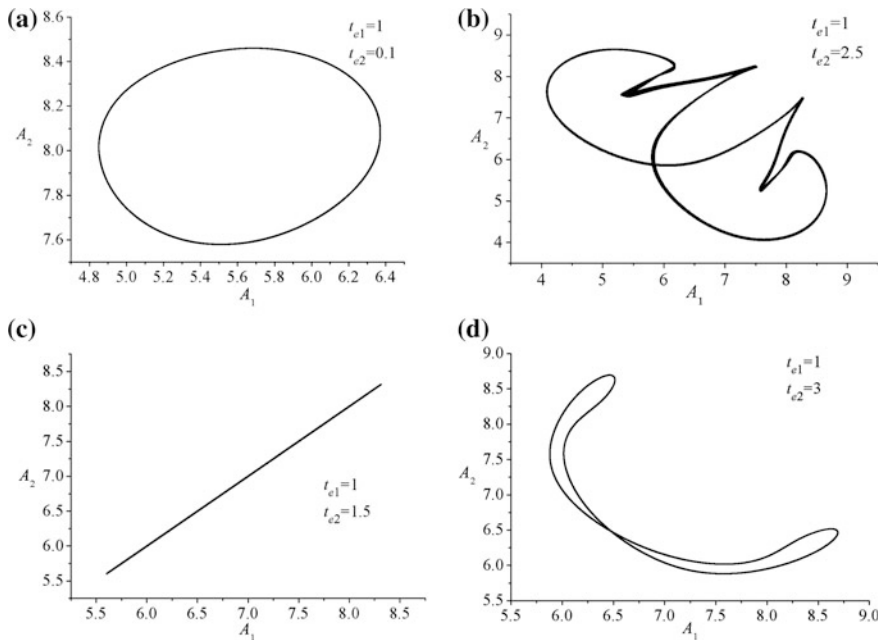


Fig. 2.46 Phase portraits responsible for periodic motions; rotations $\Delta_1 = 180^\circ$, $\Delta_2 = 180^\circ$ (a, b, c); rotations $\Delta_1 = 180^\circ$, $\Delta_2 = 0^\circ$ (d)

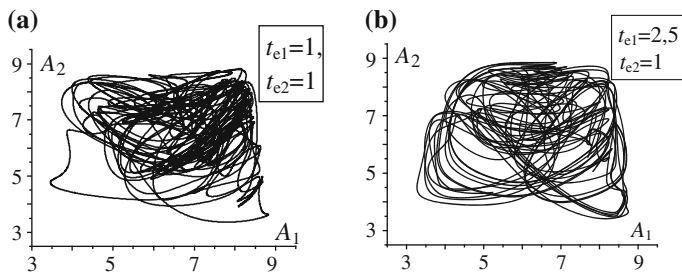


Fig. 2.47 Phase portraits responsible aperiodic motions; rotations $\Delta_1 = 120^\circ$, $\Delta_2 = 120^\circ$ for different values of t_{e1} , t_{e2}

various points of the beam transverse section: dynamics of U becomes complicated compared to the U behavior, when the one of the field rotation angles $\Delta_i = 0$. It is appropriate to remind that in accordance with publication [57], the great values of the fractal dimension $D_0(t_{e1}, t_{e2})$ are considered as a precondition for the high secretiveness of messages in the system of confidential communications.

Construction of the D_0 fractional map can serve as the reference point for the choice of DNRI parameters favorable for its functioning as an encoder applying the

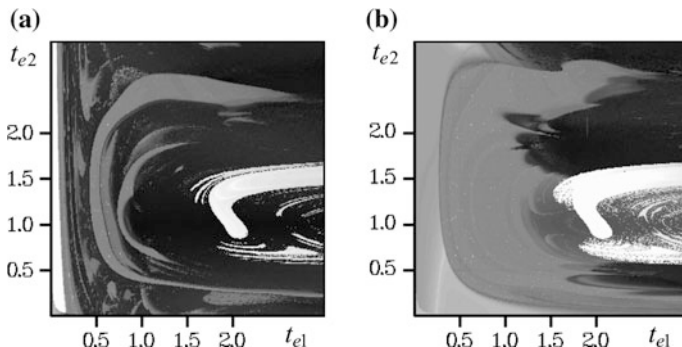


Fig. 2.48 Dependence of the fractal dimension $D_0(t_{e1}, t_{e2})$ upon the ratio of time delay in the feedback loops. The darker areas correspond to larger values of D_0 . $m = 2$, $\Delta_1 = 180^\circ$, $K = 5.5$, $\Delta_2 = 180^\circ$ (a), $\Delta_2 = 0^\circ$ (b)

deterministic chaos mode. We can control these areas sizes by choosing different combinations of DNRI parameters [34].

We have already considered the interferometers with two-dimensional feedback, often neglecting the molecule diffusion of the nonlinear medium, as well as the light diffraction, while transiting to the point models. Interferometers based on the optical fiber can be regarded as close to such models. However, the light dispersion and the field mode structure may have to be taken into account in this case.

Besides, having focused on such models, we can design radio electronic prototypes of the optical interferometers. With their aid—i.e., examining their properties, we can test the experimental modeling results, when construction of precision optical system is not expedient. Oscillators realizing the principle of these radio electronic analogs, are rather unusual in traditional radio electronics. The reason for this is the absence of the amplifier with the positive feedback. Oscillation generation, provided at the expense of interference amplification, occurs in the space of the signal parameters with some a priori given form—sinusoidal, for instance. Operation in this paradigm allows us to qualitatively confirm the modeling results and to receive evidence that chaos generation in these devices and their development for the diversification of the set of radio-frequency oscillators are possible [28, 29, 58].

For example, we will consider later in brief the simplest version of the fiber-optical interferometer and its description, while the radio-frequency analogs of interferometers will not be described to save time and space.

2.4.4 The Nonlinear Fiber-Optical Interferometer

The diagram of the nonlinear ring interferometer based on the optical fiber with nonlinear properties is shown in Fig. 2.51 [59, 60].

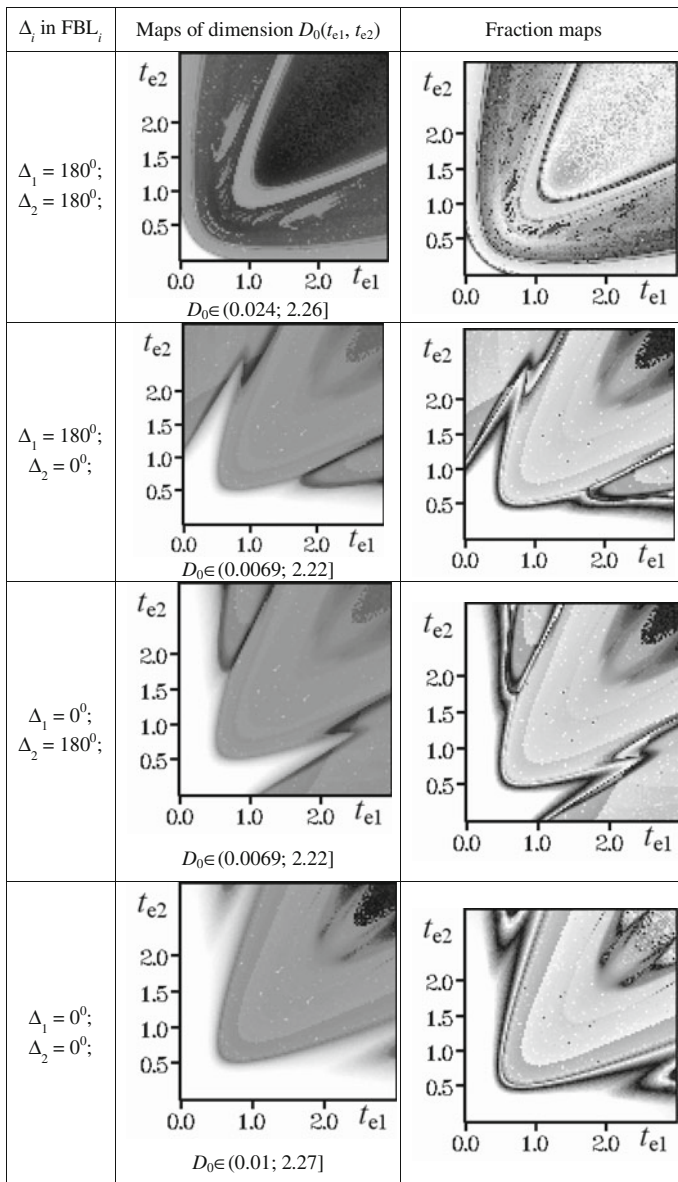


Fig. 2.49 Dependence of the fractal dimension maps structure $D_0(t_{e1}, t_{e2})$ and the fractional maps of attractors in the model (2.45) versus combination of angle values Δ_i of the light field rotation in the transverse plane of the laser beam in i -th FBL of DNRI, when the nonlinearity parameter $K = 5.5$, a point number in CTP $m = 2$, $\gamma_1 = \gamma_2 = 0.25$. On maps $D_0(t_{e1}, t_{e2})$ gradations of the grey tins correspond to increase of the value of $0 \leq D_0 \leq D_{0\max}$. On fractional maps the light areas correspond to integer dimension, and the grey tins symbolize the growth of D_0 deviation from the integer

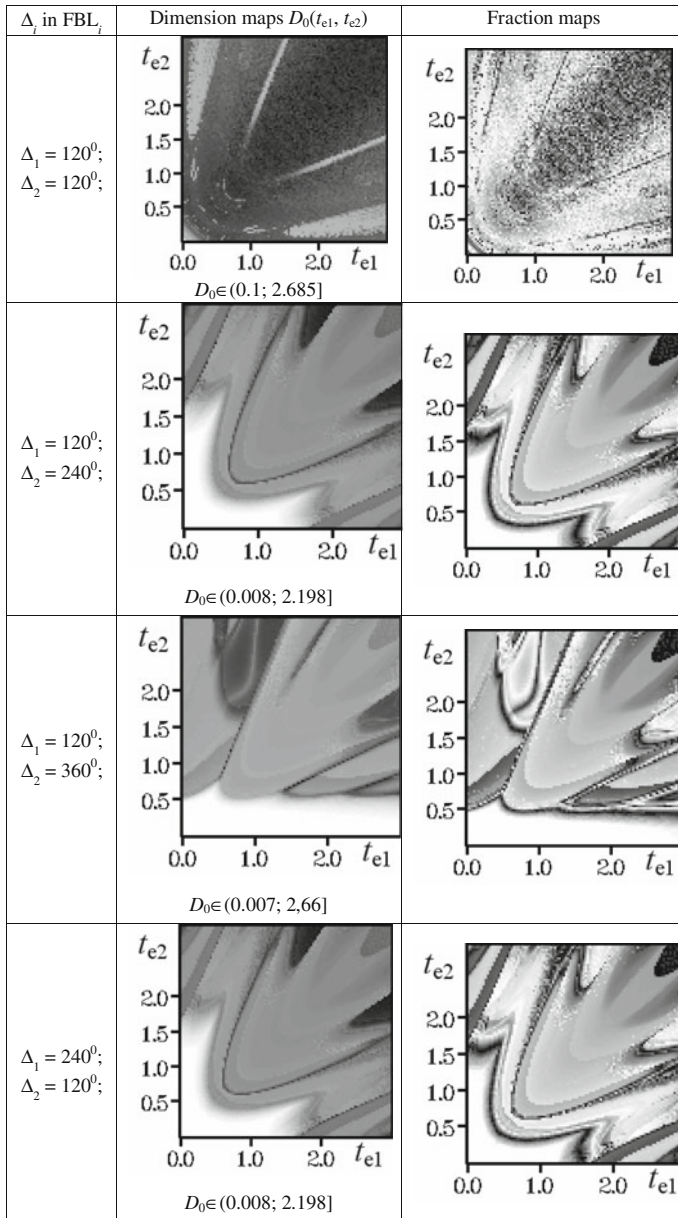
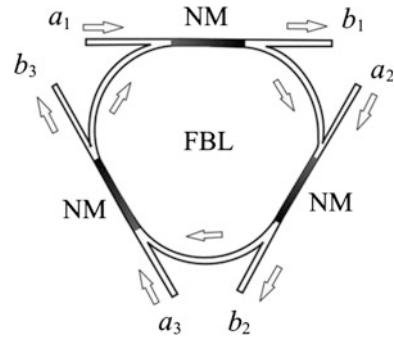


Fig. 2.50 Dependence of the fractal dimension maps structure $D_0(t_{e1}, t_{e2})$ and the fractional maps of attractors in the model (2.45) versus the combination of angle values Δ_i of the light field rotation in the transverse plane of the laser beam in i -th FBL on DNRI, when the nonlinearity parameter $K = 5.5$, a point number in CTP $m = 3$, $\gamma_1 = \gamma_2 = 0.25$. On maps $D_0(t_{e1}, t_{e2})$ the grey tin gradation corresponds to growth of the value of $0 \leq D_0 \leq D_{0\max}$. On fractional maps the light areas correspond to integer dimension, and the grey gradations symbolize the growth of D_0 deviation from the integer

Fig. 2.51 The diagram of the nonlinear ring interferometer on the base of the optical fiber: a_i, b_i are inputs and outputs of three fibers, NM is the nonlinear medium, FBL is the feedback loop of the interferometer (marked by three arrows near a center)



The nonlinear fiber-optical interferometer represents several optical fibers connected in such a manner that a part of light field from the output of the one of fibers b_i would return to the input a_i of another fiber (Fig. 2.51). A number of such fibers may be large enough. The interferometer constructively consists of the two parts formed by the optical fiber: the nonlinear medium and feedback loop.

Optical fibers are made of quartz, which has very small nonlinear refraction index of the order $n_2 = 5 \times 10^{-13} \text{ cm}^2/\text{kW}$ [44]. Therefore, in usual situations, this material is not suited for the application as the nonlinear element. However, the optical fiber may have a huge extension and in the wavelength region ($\lambda = 1.5 \mu\text{m}$), where the quartz glass has a low absorption factor, the relative powerful laser are applicable. The opportunities for obtaining deterministic chaos based on optical glass are presented in publication [61]. This makes us capable of realizing the suggested scheme. Of no little importance is the fact that glass doping can increase the nonlinear part of the refraction index. Thus, the glass doped by semiconductor CdHgTe has a parameter determining the nonlinear part of refraction index of $n_2 \approx 10^{-6} \text{ cm}^2/\text{kW}$. Under the nonlinear fiber we imply the optical one from the material with relaxation time of more than 10^{-12} s . There can be different polymer materials, or some kind of glass with the implanted impurities. The feedback loop is made of the fiber based on the quartz glass.

While designing the models for the dynamics of phase and amplitude of the optical field and for the refraction index of NM in nonlinear fiber-optical interferometer, we accept the following approximations:

- The fiber is single-mode and weakly directing, i.e. the difference between refraction indices of a core and an envelope is small;
- The fiber core and the envelope for the fiber with the stepping profile that forms the feedback loop are made of the materials with the identical dispersion, independent from the passing signal intensity;
- The nonlinear medium (a nonlinear element) represents, for instance, the optical fiber, in which the refraction index of the core and the envelope depends (linearly) on the intensity of the passing light signal (the Kerr nonlinearity);
- The chaotic signal spectrum (in the deterministic chaos mode) falls into the “transparent window” of the fiber;

- Damping in the feedback loop, the polarization dispersion, light reflection from the fiber interfaces and the light scattering are low, i.e. the model anticipates many single-directed light passes through the feedback loop;
- Losses at interfaces depend neither on the frequency, nor on the light amplitude in the fiber;
- The polarized molecule diffusion in the optical glass can be neglected;
- The relaxation time of the nonlinear medium is small ($\tau \approx 10^{-12}$ s) compared to the bypass time of the feedback loop (signal delay).

Then we can obtain the following mathematical model [59, 60]:

$$\begin{aligned}
 \tau \frac{du_{1,nel}(t)_i}{dt} &= u_{10,nel,i} - u_{1,nel}(t)_i + \rho_1(t)_{i-1} K_{0,core}(t)_i + \rho_1(t)_{i-1} \gamma K_{core}(t - t_k)_{i-1} \\
 &\quad + 2\rho_1(t)_{i-1} \sqrt{K_{0,core}(t)_i \gamma K_{core}(t - t_k)_{i-1}} \\
 &\quad \times \cos[\varphi_{nel}(t - t_k)_{i-1} + \varphi_{lin,i-1} + \varphi(t - t_k)_{i-1} - \varphi_{0,i}(t)], \\
 \tau \frac{du_{2,nel}(t)_i}{dt} &+ u_{2,nel}(t)_i = u_{20,nel,i} + [1 - \rho_1(t)_{i-1}] K_{0,fac}(t)_i \\
 &\quad + [1 - \rho_1(t)_{i-1}] \gamma K_{fac}(t - t_k)_{i-1} \\
 &\quad + 2[1 - \rho_1(t)_{i-1}] \sqrt{K_{0,fac}(t)_i \gamma K_{fac}(t - t_k)_{i-1}} \\
 &\quad \times \cos[\varphi_{nel}(t - t_k)_{i-1} + \varphi_{lin,i-1} + \varphi(t - t_k)_{i-1} - \varphi_{0,i}(t)], \\
 \varphi_{nel}(t) &= \left[u_{2,nel}(t) + \{u_{1,nel}(t) - u_{2,nel}\} \frac{d(V_{lin} B_{lin})}{dV_{lin}} \right], \\
 \varphi_{lin} &= u_{2,lin} + (u_{1,lin} - u_{2,lin}) \frac{d(V_{lin} B_{lin})}{dV_{lin}}. \\
 \phi(t)_i &= \begin{cases} (\pi/2) - \arctan(A_{Re}/A_{Im}), & A_{Im} < 0 \\ (3\pi/2) - \arctan(A_{Re}/A_{Im}), & A_{Im} > 0 \\ 0, & A_{Im} = 0, A_{Re} \geq 0 \\ 3\pi/2, & A_{Im} = 0, A_{Re} < 0 \end{cases} \\
 A_{Re} &= A_0(t)_i \cos[\varphi_0(t)_i] + \gamma A(t - t_k)_{i-1} \\
 &\quad \times \cos[\varphi_{nel}(t - t_k)_{i-1} + \varphi_{lin,i-1} + \varphi(t - t_k)_{i-1}], \\
 A_{Im} &= A_0(t)_i \sin[\varphi_0(t)_i] + \gamma A(t - t_k)_{i-1} \\
 &\quad \times \sin[\varphi_{nel}(t - t_k)_{i-1} + \varphi_{lin,i-1} + \varphi(t - t_k)_{i-1}].
 \end{aligned} \tag{2.49}$$

There i is the numeration of the optical fiber, t_k is the light field propagation time in the feedback loop, $\gamma = \theta \cdot \exp(\alpha l_f)$, θ is the coefficient of radiation losses at interfaces and due to its transition into modes of higher orders, α and l_f is the Bouguer damping exponent and the fiber length, $V = \frac{2\pi(n_1^2 - n_2^2) \cdot a}{\lambda}$ is the reduced spatial frequency, B is the phase parameter [62, 63], or reduced phase, φ is the

phase of the sun of the input field and the field passed through the feedback loop, φ_0 is the field phase at the interferometer input, φ_{nel} and φ_{lin} are phase shifts in the nonlinear element and in the feedback loop, A_{Re} and A_{Im} are the real and imaginary parts of the complex amplitude of the optical field, $u_{1,nel}$, $u_{2,nel}$ are field phase variations during passing the core and the envelope of the nonlinear fiber, $I(t)$ is the intensity of the optical field, coefficients ρ_1 and $(1 - \rho_1)$ determine which part of power propagates in the core and in the envelope, $K_{0,core}(t)$, $K_{0,fac}(t)$ and $K_{core}(t - t_k)$, $K_{fac}(t - t_k)$ are nonlinearity coefficients at the input and at the output of the interferometer; indices “core” and “fac” correspond to the core and the envelope of the fiber, and indices “lin” and “nel” correspond to the linear and nonlinear (because of the Kerr optical effect) parts of the refraction index.

Examples of the results of the system modeling (2.49), without taking into account the material dispersion, are shown in Fig. 2.52. In the model, the core diameter of the optical fiber varies so that the V parameter does not exceed 2.405. At core diameter $a = 2 \mu\text{m}$ the optical field propagates mainly in the fiber core. At $a = 4.5 \mu\text{m}$ when $V \approx 2.405$, the light portion propagating in the fiber envelope achieves 20 % in power. This power portion is enough to change the refraction index, whose value thus makes the general change in the conditions of the light propagation in the optical fiber possible. Therefore, dynamics in the system complicates judging by character of the time realization and the Fourier spectra (Fig. 2.52a, b).

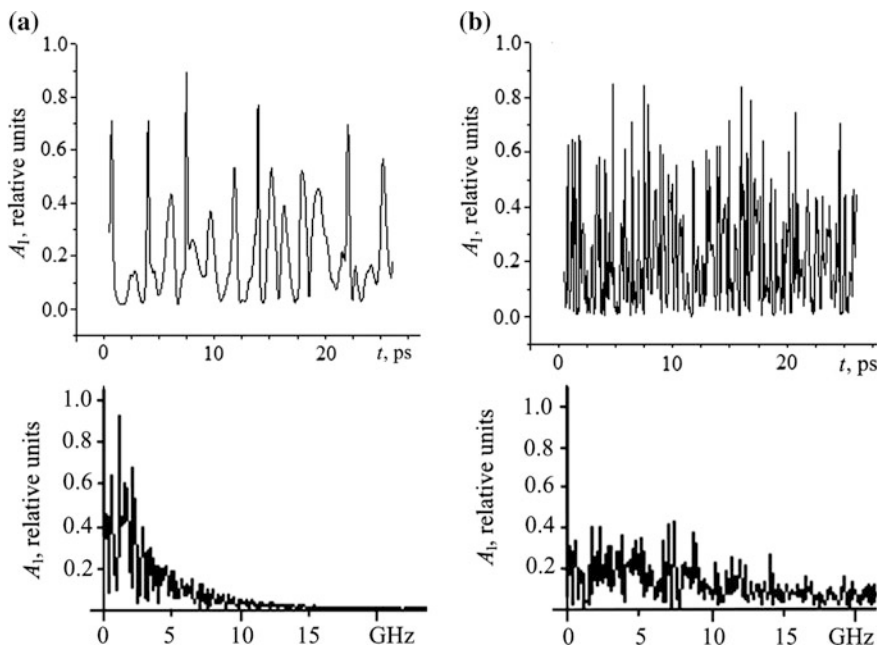


Fig. 2.52 The optical field amplitude $A_1(t)$ at interferometer output and its spectrum. The core diameter: a — $2 \mu\text{m}$ b — $4.5 \mu\text{m}$

We now finish the discussion of the optical chaotic oscillators by addressing some exotic object, which may also be the source for chaotic processes. Although we will not use it later in constructing the models of the confidential communication, but it is, probably, still promising both for these purposes and for data processing.

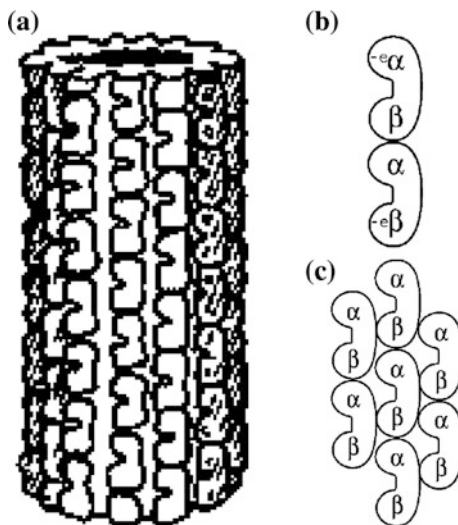
2.4.5 *The Double-Circuit NRI and Structurally Connected NRIs: Prospects for Chaos Generating and Data Processing*

In our discussion of the nonlinear interferometers demonstrating the mono-, bi- and multi-stability, the nano-bio-photon object—the microtubule of the cytoskeleton (Fig. 2.53)—is unexpectedly but reasonably included. Each microtubule represents the albuminous polymer consisting of sub-units called tubulins. Tubulin dimers can exist in two different conformers (geometric configurations) and are capable of transfer from the one state to the other. In other words, tubulin molecules are *bistable*, and in this respect, they are similar to triggers in computers.

According to the premise of neurophysiologist Hameroff, the astrophysicist and mathematician Penrose, the microtubule is the hypothetic quantum “computing device” at the intracellular level [64–66].

The scientists came to such conclusion in 1996 discussing the possibility of relationships between the conscientious human behavior and the quantum phenomena in the brain cells. They introduced ideas prompting for the synthesis of quantum-mechanic and information approaches [65]. Experiments with isolated

Fig. 2.53 The scheme of the molecule structure of cytoskeleton microtubule (a); tubulin molecules in different conformer with opposite dipole moments (b); the nearest vicinity of the tubulin molecule (c) [65, 67]



nervous cell confirm this approach [68]. They proved that even the isolated neuron being closed by the artificial feedback manifests complex and interesting behavior, and can learn and store results of obtained behavior tactics. Evidently, this fact contradicts wide-spread point of view referring to the role of neuron network in information processing.

Somewhat later, the interest to the properties of cytoskeleton microtubule leads to construction of its quantum-mechanic model by Slyadnikov. He considered the cytoskeleton microtubule as the object with the properties similar to those of ferroelectric and the spin glass. Thus, the transition of tubule molecule from the one conformer to the other is considered as the electronic barrier overcoming between two potential well [69–73].

Taking the above-mentioned into consideration in terms of abstract-system approach, the cytoskeleton microtubule presents itself as the *periodic sequence of bistable structure elements* (Fig. 2.53). Since the frequency values of the electron tunneling in the tubuline molecule are close to the optical range, the idea arises to start the search for the cytoskeleton microtubule analogs in optics. In our opinion, the double-circuit nonlinear ring interferometer can serve as the one of the possible structure analogs (Fig. 2.34) [38, p. 222; 52, 74–78].

Depending on the interferometer parameter combination as the dynamic system, the double-circuit NRI (Fig. 2.34) can be mono-, bi-, and multi-stable system, even when the field rotation in the feedback loop is absent (see [38, p. 208–210; 79]). In this case, the chain of the transposition points consists of only one transposition point (Fig. 2.54a), while the whole set of chains of the transposition points symbolizes a set of not-connected (between themselves) bi-stable elements. The latter we are to consider as the *functional analogs of tubulin molecules*.

The presence of the other field transformation in the second feedback loop causes the connection between these already formed (due to a presence of G_1 in the first circuit) elements (Fig. 2.54b). The similar connection can be caused by the polarized molecule diffusion in the nonlinear medium and/or by the light diffraction. It means that the double-circuit can be considered as the structural analog of the cytoskeleton microtubule.

It is easy to see that the purpose of the second FBL in the double-circuit NRI is the integration of the single-circuit “point” NRI into some integrity. We can achieve

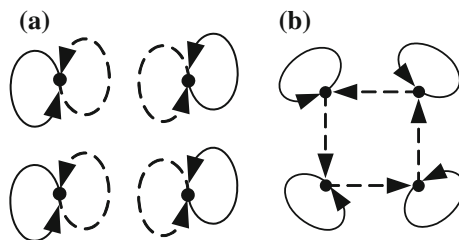
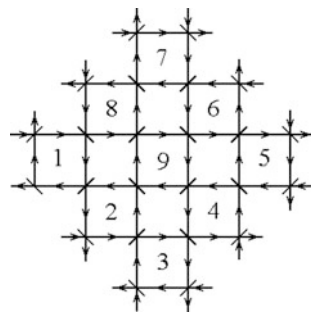


Fig. 2.54 CTP configuration corresponding to the angles of rotation Δ in the feedback loops of the double-circuit NRI: 0° and 0° (a); 0° and 90° (b). *Solid lines* indicate transitions between CTP points in the first FBL, *dashed lines* indicate in the second

Fig. 2.55 The planar structure from NRI: *arrows* indicate the possible ways of input/output, and radiation propagation, numbers are the conditional NRI numbers [38]



such integration by connecting a series of NRI without addressing FBL for assistance, thus designing the structure reminding honeycombs or lattices, as shown in Fig. 2.55 [38]. Now the various 2D-surfaces: cylinder, torus, sphere etc. can be covered by interferometers (like polymers from tubulin molecule forming the microtubule. In this case one “elementary” NRI has not less than three neighbors (for the three-mirror interferometer) optically coupled with it by the one NRI.

We should improve this analog of bio-photon information technologies in accordance with the notions concerning the importance of self-organization in cognitive processes [80–82]. Obviously, the planar technology in Fig. 2.55 is easy to realize with the help of the integrated and fiber optics technologies, and, in prospect, with the support from the molecule technologies [83, p. 135–183, 261–293; 84, p. 279–281], keeping in mind nano-structure optics regulations [85].

Let us approximate the NRI composition in Fig. 2.55 to the cytoskeleton microtubule structure (Fig. 2.53). In the example in Fig. 2.55, each “internal” structural element (NRI) in the network (the planar structure) of NRI has four “neighbors”—NRIs—optically connected with it. Nevertheless, the structure of cytoskeleton microtubule is similar to the honeycomb (Fig. 2.53c). Therefore, it is logical to construct the network of NRIs, where each “internal” NRI would have six “neighbors” (Fig. 2.56a) [86].

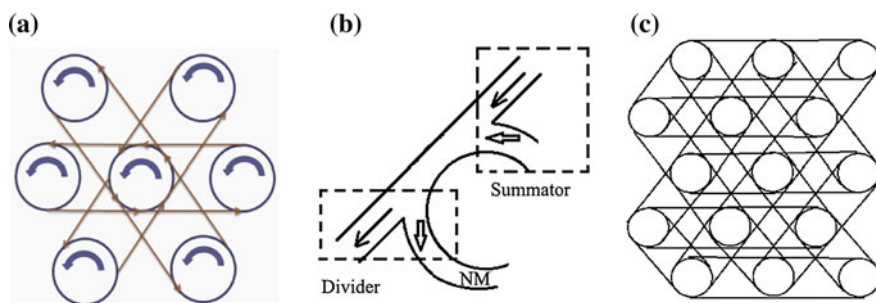


Fig. 2.56 The offered structural network component (an analog of Fig. 2.53, c) from seven coupled NRIs, where *arrows* indicate the propagation direction of the optical field; tangents to the circle correspond to optical fibers connecting NRIs are shown in (a). The construction of the one from six NRI links is shown in (b). The network from 5 lines and 6 columns ($i = 1 \dots 5, j = 1 \dots 6$) is shown in (c) [86]

Then, each of NRIs possessing the form of a circle (Fig. 2.56a) can be represented by six similar links formed by summator, divider and the nonlinear (Kerr) medium (Fig. 2.56b). For the internal NRI processes description, six differential equations are required with referring to the nonlinear phase shift of the optical field in the nonlinear medium of the one link.

In the simplest case, these equations can be derived from the same fundamental principles as the point models of NRI (2.32) or DNRI (2.45), designed earlier, if to neglect the waveguide effects in the fiber and the dispersion. For this, by analogy with the previous assumptions, we presuppose that the nonlinear medium in each link has a length l and the optical Kerr effect is demonstrated in it with nonlinear refraction index n_2 . Moreover, we assume that the optical field is single-frequency but modulated. Let us designate the value of nonlinear phase shift in the k -th link of the interferometer of the i -th line, j -th column (Fig. 2.56c), as $U_{ijk}(t)$, and let $\varphi_{ijk}(t)$ and $a_{ijk}(t)$ be the designations for the phase and the light field amplitude normalized to $a_{\max}$ in the point in the middle between the adder and the coupler in this link ($a_{\max}$ is the maximal value of the light field amplitude at inputs of the network). Then, it appears that dynamics of these variables obeys to the following equations and relations

$$\begin{aligned}
 \tau_n \frac{dU_{ijk}(t)}{dt} &= -U_{ijk}(t) + KR_{ijk}a_{ijk}^2(t), \\
 a_{ijk}(t) &= (Ac^2 + As^2)^{0.5}, \quad \varphi_{ijk}(t) = \arg(Ac, As), \\
 Ac &\equiv \frac{\gamma_{bi}}{2} (1 - R_{i'j'k'})^{0.5} a_{i'j'k'}(t - \tau_{bi}) \cos[\varphi_{i'j'k'}(t - \tau_{bi}) - U_{ijk'}(t - \tau_{bi}) - \varphi_{0bi}] \\
 &\quad + \frac{\gamma_{NM}}{2} (R_{ijk'})^{0.5} a_{ijk'}(t - \tau_{bs}) \cos[\varphi_{i'j'k'}(t - \tau_{bs}) - U_{ijk'}(t - \tau_{bs}) - \varphi_{0bs}], \\
 As &\equiv \frac{\gamma_{bi}}{2} (1 - R_{i'j'k'})^{0.5} a_{i'j'k'}(t - \tau_{bi}) \sin[\varphi_{i'j'k'}(t - \tau_{bi}) - U_{ijk'}(t - \tau_{bi}) - \varphi_{0bi}] \\
 &\quad + \frac{\gamma_{NM}}{2} (R_{ijk'})^{0.5} a_{ijk'}(t - \tau_{bs}) \sin[\varphi_{i'j'k'}(t - \tau_{bs}) - U_{ijk'}(t - \tau_{bs}) - \varphi_{0bs}], \\
 k' &= k - 1 \text{ for } k > 1, \text{ otherwise } k' = 6, \\
 i' = i, j' = j + 2 &\text{ for } k = 1, i' = i - 1, j' = j + 1 \text{ for } k = 2, \\
 i' = i - 1, j' = j - 1 &\text{ for } k = 3, \\
 i' = i, j' = j - 2 &\text{ for } k = 4, \\
 i' = i + 1, j' = j - 1 &\text{ for } k = 5, j' = j + 1, j' = j + 1 \text{ for } k = 6.
 \end{aligned} \tag{2.50}$$

Here τ_n is relaxation time of NM; $K = l \cdot n_2 a_{\max}^2 / 2$ is the nonlinearity coefficient; R_{ijk} is the reflection coefficient of the “effective mirror” imitating the coupler operation in the k -th link of the interferometer of i -th line, j -th column; γ_{nm} and γ_{bi} are doubled loss coefficients in amplitude: in the nonlinear medium and at propagation between interferometers; τ_{bi} and τ_{bs} are light propagation times between interferometers and the time constant of the light propagation of DC component from the one link to the next in some particular interferometer (“intra-interferometer” and “intra-link” delay); $\varphi_{0,bi}$, $\varphi_{0,bs}$ are the phase shifts arising because of the presence of τ_{bi} and τ_{bs} .

Pilot simulations based on (2.50) demonstrate the periodic oscillations of the nonlinear phase shift $U_{ijk}(t)$, the field amplitude $a_{ijk}(t)$ and phase $\varphi_{ijk}(t)$. Rather

high dimensionality of the phase space of the given dynamic system and the similarity between its nonlinearity type and the nonlinearity in NRI and DNRI allow expecting chaos generating in it [86].

Hereinafter, it is expedient to determine (by the means of the computer experiment and by the examination of the breadboard) the boundaries of analogy under investigation between the network from NRIs and the microtubule of the cytoskeleton, as well as a possibility of data processing with the help of the suggested networks made of NRIs. First of all, we must test its operability in the tasks of patterns recognition and clusterization [38].

Post factum, the reader may, probably, discover a parallel between our (Figs. 2.54, 2.55 and 2.56) structural analogs of cytoskeleton microtubule and the earlier synergetic models. Namely, those models describing the processes in closed and open chains (single-directing and double-directing) [87–91], two-dimension lattices [92–95] and other assemblies [96–100] of oscillators. This list of publications is, of course, incomplete, and comparison and discussion of their proximity to the phenomena in the microtubule of the cytoskeleton forms the separate problem to be solved. Here, we believe that the axiomatic approach to its modeling would be productive (see, for instance, [93, pp. 169–171; 101, pp. 256–258].

In the following chapters, the above-considered chaotic oscillators will allow designing the systems of nonlinear-dynamic cryptology based on them.

2.5 Conclusions

This chapter provides the theoretical examination of the radio-frequency oscillator of the deterministic chaos with the nonlinearity in the form of parabolas composition. The structure and mathematic model of the chaotic oscillator consisting of nonlinear element, amplifier, low-frequency filter and high-frequency filter of the first order, forming one closed ring, are described. The stability analysis of the equilibrium states is performed together with modeling of dynamic modes, as well as the bifurcation behavior in the chaotic oscillator. Results are presented in the graphical forms.

The experimental set up and the structural diagram of the breadboard deterministic chaos oscillator of the radio-frequency range is demonstrated. Experimental research results of the modes in the oscillator are examined. The dynamic modes are identified, together with the scenarios of transition to chaos in the oscillator.

Different variants of the nonlinear ring interferometers (with the Kerr nonlinearity) performing as the deterministic chaotic oscillators are considered theoretically. The appropriate mathematical models of the processes in these devices are constructed in the form of differential equations and the discrete maps. A presence of the second feedback loop, spatial transformation of the laser beam in its transverse plane, a presence of multiple beam passes through the feedback loop of the interferometer, all these aspects are accounted for during modeling. Stability,

bifurcation behavior and the types of dynamic modes are also analyzed. The concept of the “spatial” deterministic chaos is introduced. Advantages of the double-circuit interferometer as a cipherer in the system of protected communication are predicted. Possibility for the construction of the radio-frequency analogs for the optical interferometers as the promising chaos sources is discussed.

The nonlinear interferometer based on optical fiber is suggested, its mathematical model is made, the possibility for the chaotic mode in it is shown.

In order to develop the themes referring to the optical chaotic oscillator, the Hameroff–Penrose hypothesis is involved suggesting the cytoskeleton microtubule as the intra-cellular computer. The functional microtubule analog in the form of the “fabric” composed of the structurally-coupled nonlinear ring interferometers is presented. Such “fabric” makes it possible to cover surfaces of various shapes. The pilot variant of the mathematical model is produced.

Obtained maps of fractal dimension (capacity) of attractors, maps of Lyapunov characteristic exponents and other representations of chaotic oscillator properties allow prediction of parameter value ranges, the ones that appear to be most favorable for the cryptosystem operation.

The next chapter is devoted to the theoretical analysis and experimental examination of the radio-physical systems of data transmission applying the considered chaotic oscillators.

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