Chapter 2
Steady-State Conduction

In one dimension in the x-direction the rate of heat transfer or heat flux is expressed according to *Fourier's law* as outlined in Sect. 1.1.

$$\dot{q}''_x = -k \cdot \frac{dT}{dx} \tag{2.1}$$

where k is the thermal conductivity. For simplicity the mathematical presentation of the heat transfer phenomena is here in general made for one-dimensional cases only. Corresponding presentations in two and three dimensions can be found in several textbooks such as [1, 2].

Under steady-state conditions the heat flux is independent of x, i.e. the derivative of $\dot{q}''_x$ is zero and we get

$$\frac{d}{dx}\left(k \cdot \frac{dT}{dx}\right) = 0 \tag{2.2}$$

The corresponding equation for cylinders with temperature gradients in the radial direction only is

$$\frac{1}{r}\frac{d}{dr}\left(k \cdot r\frac{dT}{dr}\right) = 0 \tag{2.3}$$

where r is the radius. Solutions for steady-state cases are found in Sect. 2.2.

U. Wickström, *Temperature Calculation in Fire Safety Engineering*,
DOI 10.1007/978-3-319-30172-3_2

2.1 Plane Walls

Consider a plane wall having surface temperatures T_1 and T_2. Figure 2.1 shows the temperature distribution under steady-state conditions which means the heat flux is constant across the plate. Figure 2.1a shows the temperature distribution when the heat conductivity is constant, i.e. the second derivative of the temperature is zero according to Eq. 2.2 and thus the temperature distribution becomes linear. Figure 2.1b shows the temperature distributions in structure with two layers of materials with different conductivities. The material to the left has the lower conductivity. Figure 2.1c indicates the temperature distribution when the conductivity is increasing with temperature. The temperature gradient is higher where the temperatures are lower and thereby the conductivity. This is particular the case for insulating materials where the conductivity increases considerably at elevated temperatures.

The rate of heat conducted per unit area $\dot{q}''$ through a wall, see Fig. 2.2, is proportional to the thermal conductivity of the wall material times the temperature difference ΔT between the wall surfaces divided by the wall thickness L, and according to *Fourier's law* (c.f. Eq. 2.1)

$$\dot{q}'' = k \cdot \frac{\Delta T}{L} = k \cdot \frac{(T_1 - T_2)}{L} \tag{2.4}$$

In an electric circuit analogy, this case can be illustrated according to Fig. 2.2. The heat flow through the wall over an area A may then be written as

$$\dot{q}'' = (T_1 - T_2)/R_k \tag{2.5}$$

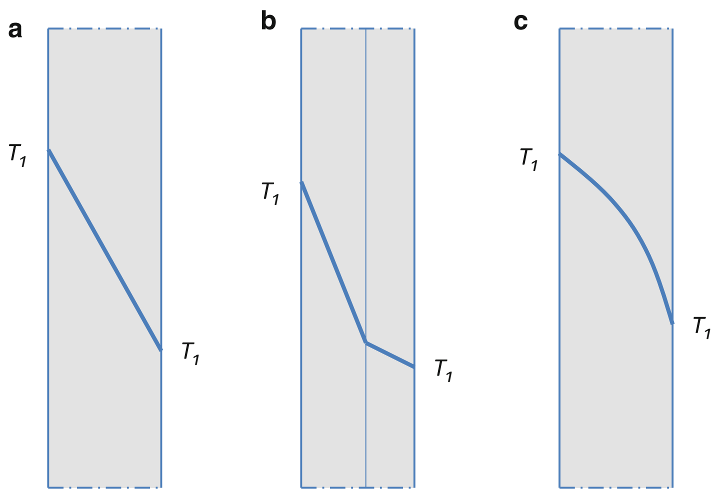


Fig. 2.1 Steady-state temperature distribution in a plane wall. (**a**) Constant conductivity, (**b**) two materials with a low and high conductivity and (**c**) conductivity increasing with temperature

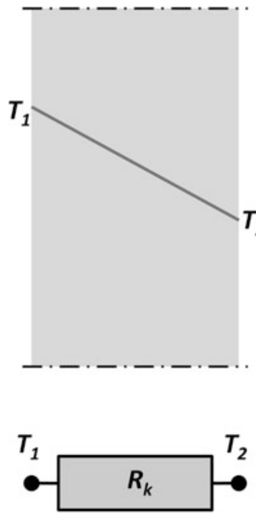


Fig. 2.2 One-dimensional steady-state thermal conduction. Linear temperature distribution across a wall and an electric analogy of one-dimensional heat flux. The thermal resistance $R = L/k$

where the *thermal resistance of the solid* then can be identified as

$$R_k = \frac{L}{k} \tag{2.6}$$

The electric analogy may also be used for more complex problems involving both series and parallel thermal resistance. A typical problem is a wall consisting of several layers, see Fig. 2.3.

The total thermal resistance R_{tot} between the inside and outside surfaces may then be written as:

$$R_{tot} = R_1 + R_2 + R_3 = \frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{L_3}{k_3} \tag{2.7}$$

and the heat flux $\dot{q}''$ through the assembly from the inside to the outside may be written as:

$$\dot{q}'' = \frac{\Delta T}{R_{tot}} = \frac{T_0 - T_\mathrm{i}}{R_{tot}} \tag{2.8}$$

The temperature T_{1-2} at the interface between material 1 and 2 may be written as

$$T_{1-2} = T_i - \frac{R_1\,(T_i - T_o)}{R_{tot}} = \frac{(R_2 + R_3)\cdot\ T_i + R_1 \cdot T_o}{R_{tot}} \tag{2.9}$$

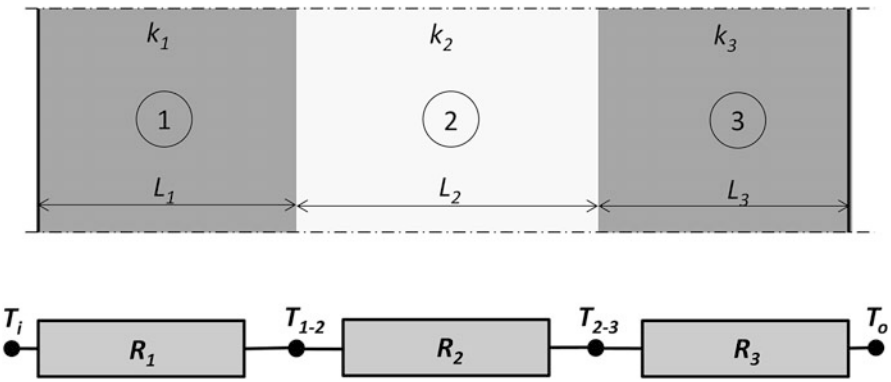


Fig. 2.3 Electric circuit analogy of one-dimensional heat transfer across a wall consisting of three layers

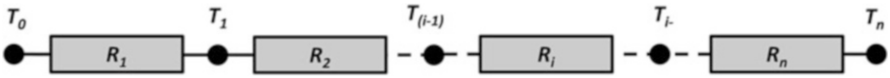


Fig. 2.4 Electric circuit analogy of one-dimensional heat transfer across a wall consisting of several layers

In a general way the total thermal resistance of an assembly thermally modelled as shown in Fig. 2.4 may be obtained as the sum of the components

$$R_{tot} = \sum\nolimits_{j=1}^{n} R_j \tag{2.10}$$

and the heat flux $\dot{q}''$ through can be calculated as

$$\dot{q}'' = \frac{T_0 - T_n}{R_{tot}} \tag{2.11}$$

and the temperature at an interface i as shown in Fig. 2.4 may be calculated as

$$T_i = T_0 + \frac{\sum\nolimits_{j=1}^{i} R_j}{R_{tot}} (T_n - T_0) = \frac{T_n \cdot \sum\nolimits_{j=1}^{i} R_j + T_0 \cdot \sum\nolimits_{j=i+1}^{n} R_j}{R_{tot}} \tag{2.12}$$

Example 2.1 A wall consists of 20 mm wood panel, 100 mm fibre insulation and 12 mm gypsum board with conductivities equal to 0.14, 0.04 and 0.5 W/(m^2 K), respectively. The wood outer surface has a constant temperature of 75 °C and the inner gypsum board surface a temperature of 15 °C. Calculate the temperatures at the insulation interface surfaces (T_1 and T_2).

Solution $R_{tot} = 0.020/0.14 + 0.1/0.04 + 0.012/0.5 = 0.143 + 2.5 + 0.024 = 2.67\text{W/K}$. Then $T_1 = 75 + \frac{0.143}{2.67} \cdot (15 - 75) = 71.8\,°\text{C}$ and $T_2 = 75 + \frac{0.143+2.5}{2.67} \cdot (15 - 75) = 15.6\,°\text{C}$.

The presentation so far includes heat transfer in solids only with boundary conditions of the first kind, i.e. prescribed surface temperatures. In most cases in fire protection engineering, however, the boundary condition between a surrounding fluid/environment and a solid surface is specified as a boundary condition of the third kind. In the simplest form the boundary condition is then described by the *Newton's law of cooling*. It may be seen as a heat transfer condition for convection and it states that the heat transfer to a surface is directly proportional to the difference between the surrounding gas temperature T_g and the surface temperature T_s:

$$\dot{q}'' = h \cdot (T_g - T_s) \tag{2.13}$$

where the constant of proportionality factor h is the heat transfer coefficient. The *surface thermal resistance* R_h between the gas phase and the solid phase can then be written as

$$R_h = \frac{1}{h} \tag{2.14}$$

Thus for the case illustrated in Fig. 2.5, the total resistance between the gas phase on the left side and the surface on the right side of the wall may be written as

$$R_{tot} = \left(\frac{1}{h} + \frac{L}{k}\right) = R_h + R_k \tag{2.15}$$

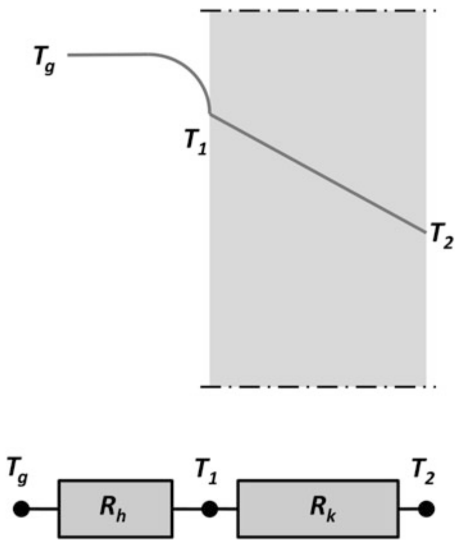


Fig. 2.5 An electric circuit analogy of one-dimensional heat transfer to a surface and through a wall with surface and solid thermal resistances

and the surface temperature T_1 may be written as a function of the gas temperature T_g and the temperature T_2 as

$$T_1 = \frac{R_h \cdot T_2 + R_k \cdot T_g}{R_h + R_k} \tag{2.16}$$

Example 2.2 Calculate the surface temperature T_1 of a 12 mm wooden board if the gas temperature on the exposed side $T_g = 100$ °C and the temperature on the non-exposed side is $T_2 = 20$ °C. Assume the conductivity of the wood k = 0.2 W/(m K) and heat transfer coefficient $h = 5$ W/(m^2 K).

Solution Equation 2.15 yields $R = R_h + R_k = \left(\frac{1}{5} + \frac{0.012}{0.2}\right)(\mathrm{m}^2\,\mathrm{K})/\mathrm{W} = 0.2 + 0.06$ $(\mathrm{m}^2\,\mathrm{K})/\mathrm{W}$ and Eq. 2.16 yields $T_1 = \frac{0.2 \cdot 20 + 0.06 \cdot 100}{0.26}\,°\mathrm{C} = 38\,°\mathrm{C}$.

2.2 Cylinders

Cylinders often experience temperature gradients in the radial direction only, and may therefore be treated as one dimensional. The solid thermal resistance between the inner radius r_i and an arbitrary radius r in a cylinder (see Fig. 2.6) assuming constant heat conductivity may then be written as

$$R_k = \frac{\ln\left(\frac{r}{r_i}\right)}{2\pi k} \tag{2.17}$$

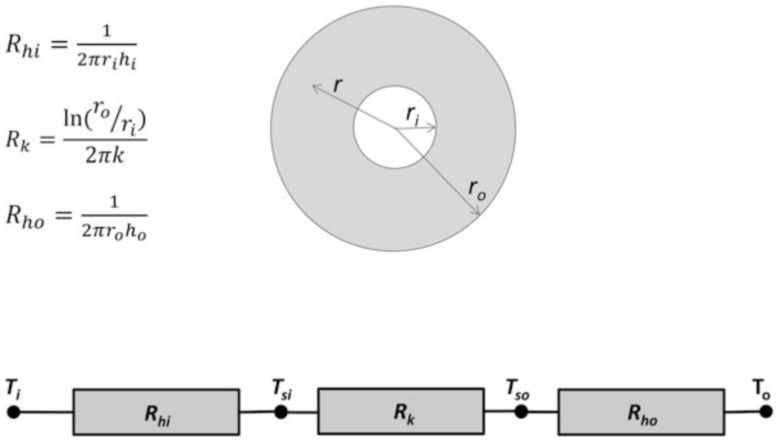


Fig. 2.6 Thermal resistances between the media with a temperature T_i inside a cylindrical pipe and the outside gas with a temperature T_o

where k is the thermal conductivity. The surface thermal resistance may be written as

$$R_h = \frac{1}{2\pi r h} \tag{2.18}$$

Hence the thermal resistance between the inner and outer gases or liquids of a pipe is obtained by summarizing the surface and solid resistances as indicated in Fig. 2.6, i.e. the total thermal resistance over a unit length is

$$R_{tot} = R_{hi} + R_k + R_{ho} = \frac{1}{2\pi} \cdot \left(\frac{1}{r h_i} + \frac{\ln\left(r_o/r_i\right)}{k} + \frac{1}{r_o h_o} \right) \tag{2.19}$$

A uniform heat flux over a unit length of a pipe may then be calculated as

$$\dot{q}_l' = \frac{T_i - T_o}{R_{tot}} = \frac{2\pi\,(T_i - T_o)}{\frac{1}{r_i \cdot h_i} + \frac{\ln\left(r_o/r_i\right)}{k} + \frac{1}{r_o \cdot h_o}} \tag{2.20}$$

The temperatures T_{is} at the inner surface can be obtained as

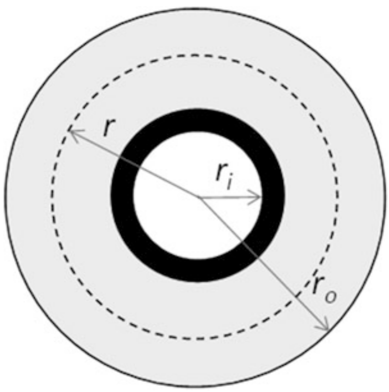
$$T_{is} = \frac{R_{hi} T_o + (R_k + R_{ho})\, T_i}{R_{tot}} \tag{2.21}$$

and T_{os} at the outer surface as

$$T_{os} = \frac{(R_{hi} + R_k)\, T_o + R_{ho}\, T_i}{R_{tot}} \tag{2.22}$$

Example 2.3 Consider an insulated steel pipe with an outer coating as shown in Fig. 2.7 exposed to fire with a constant temperature of 800 °C. The temperature of

Fig. 2.7 Insulated steel pipe with an outer coating



the inside medium/fluid is 100 °C. The inner and outer radii of the steel pipe are 30 and 28 mm, respectively. The insulation is 50 mm thick and has a conductivity of 0.5 W/(m K). The inner heat transfer coefficient is 100 W/(m^2 K) and the outer 50 W/(m^2 K). Calculate the temperature of the steel pipe which is assumed to be constant along the radius.

Solution Calculating for a unit length. Equation 2.19 yields $R_{tot} = 0.057 + 0.312 + 0.040 = 0.409$ (m K)/W. Thus the inner (steel) temperature $T_i = \frac{0.057 \cdot 800 + (0.312 + 0.04) \cdot 100}{0.409} = 198\,^\circ\mathrm{C}$.

Springer

http://www.springer.com/978-3-319-30170-9

Temperature Calculation in Fire Safety Engineering
Wickström, U.
2016, XIII, 243 p. 148 illus., 24 illus. in color., Hardcover
ISBN: 978-3-319-30170-9