

Chapter 2

Using Singular Value Decomposition to Estimate Frequency Response Functions

Kevin L. Napolitano

Abstract This paper uses singular value decomposition as an estimator to calculate frequency response functions (FRFs). Both reference and response spectral data are used to estimate a set of orthogonal basis vectors. The basis vectors are used to calculate estimates of the reference and response spectra, which are then used to calculate FRF matrices. It is shown that a singular value decomposition based FRF technique is equivalent to the H_V method for a single response and that a subset of both reference and response vectors can be used to generate the basis vectors. A generalized version of the H_2 method can be derived if only response data is used to generate the basis vectors. The calculation of multiple coherence with respect to different FRF estimators is discussed. A comparison between the different FRF estimators is presented using test data.

Keywords Frequency response function • Singular value decomposition • Eigenvalue analysis • Multiple coherence • Total least squares

2.1 Introduction

A typical modal test involves measuring frequency response functions (FRFs) from inputs that are typically excitation forces and outputs that are typically accelerometers. Inputs are defined here as the controlled inputs into a structure, and output as uncontrolled response due to the inputs. Responses and references are respectively defined as the numerator and denominator terms in the FRF. The references are usually the input measurements, and the responses are usually the output measurements. In some cases, such as transmissibility measurements, outputs are used as references.

The use of linear algebra to estimate FRFs is well established [1, 2]. Least-squares techniques are used to derive the FRF estimators H_1 and H_2 , and eigenvalue analysis is used to derive the H_V FRF estimator. The original goal of this internal research project was to determine whether singular value decomposition (SVD), a linear algebra technique that has gained wide acceptance in structural dynamics [3, 4], can also be used to estimate FRFs.

This paper uses singular value decomposition to calculate FRFs. Both reference and response spectral data are used to estimate a set of orthogonal basis vectors equal to the number of independent inputs. The basis vectors are used to calculate estimates of the reference and response spectra, which are then used to calculate FRF matrices.

First, a review of H_1 and H_2 is presented, followed by a review of singular value decomposition. Then an SVD-based FRF estimator, H_{SVD} , is derived and is shown to be equivalent to a generalized version of the H_V estimator. The SVD methodology is derived for the most generalized case where a partial set of inputs and outputs is used to calculate the basis vectors. An equation for multiple coherence that is valid for all FRF estimators is presented. A comparison between H_1 , H_{SVD} , H_V , and a generalized version of H_2 is presented showing that obtaining high-quality FRFs depends upon the selection of a set of channels that can adequately represent the proper basis vectors.

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2.2 Review of FRF Estimators

An FRF matrix is a linear estimate of several responses at a given frequency due to several inputs at that same frequency. The equation for an FRF at a single frequency is

$$\begin{bmatrix} H(\omega) \\ N_{RES} \times N_{REF} \end{bmatrix} \begin{bmatrix} X(\omega) \\ N_{REF} \times N_S \end{bmatrix} = \begin{bmatrix} Y(\omega) \\ N_{RES} \times N_S \end{bmatrix} \quad (2.1)$$

The matrix $[H]$ is the FRF matrix at a given frequency ω . Hereafter, the frequency notation is removed and it should be assumed that all matrices presented in this paper correspond to a single frequency. The matrices $[X]$ and $[Y]$ contain the spectral data, usually from a Fourier transform of time-history data, for all the reference and response channels, respectively. The number of rows and columns in $[H]$ are equal to the number of response (N_{RES}) and reference (N_{REF}) channels, respectively, and the number of rows in $[X]$ and $[Y]$ are equal to N_{REF} and N_{RES} , respectively. The number of columns in $[X]$ and $[Y]$ are equal to the number of samples (N_s) used to estimate the FRF matrix. Note that N_s is defined here as the number of events used to estimate the FRF. If the Welch method [5] is used, then N_s is the number of frames of time-history data converted to the frequency domain. If the Daniell method [6] is used, then N_s is the number of spectral lines near the frequency of interest that are used to estimate the FRF.

Equation (2.1) can be reordered by taking the conjugate transpose (i.e., Hermitian), which leads to the standard format for a least-squares solution [7].

$$[X]^H [H]^H = [Y]^H \quad (2.2)$$

Every FRF estimator based on linear algebra follows the same procedure:

1. Define a set of basis vectors that can adequately represent the columns of $[X]^H$ and $[Y]^H$.
2. Use the basis vectors to estimate references and responses. These estimates are defined here as $[\tilde{X}]^H$ and $[\tilde{Y}]^H$, respectively.
3. Solve for the FRF matrix using estimated reference and response channels as inputs such that

$$[\tilde{X}]^H [H]^H = [\tilde{Y}]^H \quad (2.3)$$

2.2.1 Review of H_I

The traditional description of the H_I method assumes that there is no error on the references and derives an expression to minimize the error on the responses in a least-squares sense. Another way to consider H_I is that the columns of $[X]^H$ are the basis vectors that are used to describe both the reference and response spectral matrices. Thus the estimated values are a linear combination of the basis vectors such that

$$[\tilde{X}]^H = [X]^H [C_X] \quad \text{and} \quad [\tilde{Y}]^H = [X]^H [C_Y] \quad (2.4)$$

The constant matrices $[C_X]$ and $[C_Y]$ are solved for by substituting the measured values for the estimated values. The matrix $[C_Y]$ is solved for in a least-squares sense such that $[C_Y] = ([X][X]^H)^{-1} [X][Y]^H$. The matrix $[C_X]$ is the identity matrix since the estimated references are equal to the measured references. Therefore

$$[\tilde{X}]^H = [X]^H \quad \text{and} \quad [\tilde{Y}]^H = [X]^H ([X][X]^H)^{-1} [X][Y]^H \quad (2.5)$$

Substituting Eq. (2.5) into Eq. (2.3) leads to

$$[X]^H [H_I]^H = [X]^H ([X][X]^H)^{-1} [X][Y]^H \quad (2.6)$$

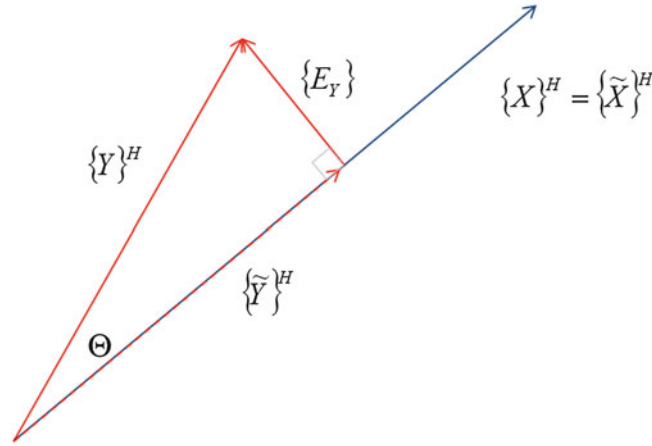


Fig. 2.1 Geometric representation of H_1 : single input. The column vector $\{Y\}^H$ is projected onto the basis vector $\{X\}^H$

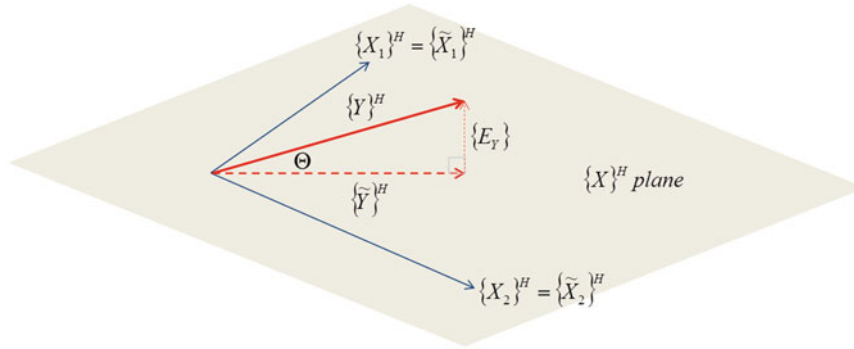


Fig. 2.2 Geometric representation of H_1 : multiple input. Each column vector in $\{Y\}^H$ is projected onto the plane formed by the basis vectors in $\{X\}^H$

After pre-multiplying by the pseudo-inverse of $\{X\}^H$, the conjugate transpose can be taken to obtain the familiar equation for $[H_1]$ as

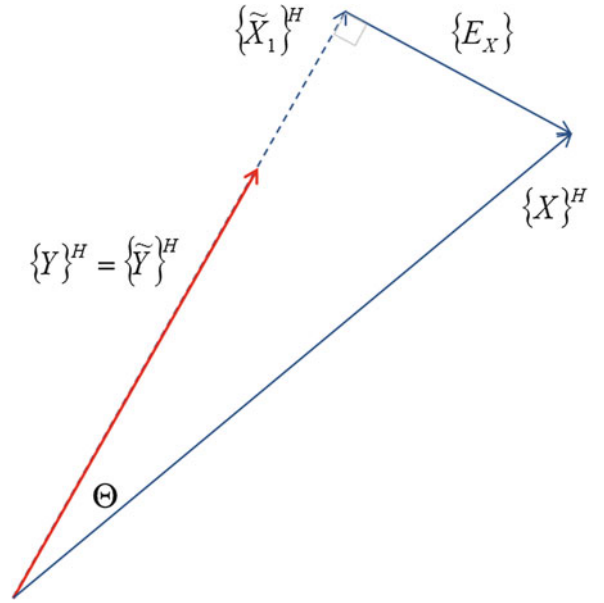
$$[H_1] = [Y] [X]^H ([X] [X]^H)^{-1} = [S_{YX}] [S_{XX}]^{-1}. \quad (2.7)$$

This derivation emphasizes that the main assumption in calculating $[H_1]$ is that both the reference and response channels are estimated from a set of basis vectors, which in this case are the columns of the matrix $\{X\}^H$. This can be represented geometrically for a single reference as in Fig. 2.1 and for multiple references as in Fig. 2.2. For H_1 , the estimate value $[\tilde{X}]^H$ is taken to be the same as the true value $\{X\}^H$, and the columns of $\{\tilde{Y}\}^H$ are a projection of the columns of $\{Y\}^H$ onto the columns of the basis vectors in $\{X\}^H$. The resulting error E_y is orthogonal to the estimate.

2.2.2 Review of H_2

The traditional description of the H_2 method assumes that there is no error on the responses and derives an expression to minimize the error on the references in a least-squares sense. Another way to consider H_2 is that the columns of $\{Y\}^H$ are the basis vectors that are used to describe both the reference and response spectral matrices. Thus, the reference channels $\{X\}^H$ are a linear combination of the responses. In practice, FRFs are calculated using H_2 when there is a single reference or in the rare case where the number of responses is equal to the number of references. However, the derivation of a generalized version of H_2 given later in this paper allows an unlimited number of responses to be used to estimate the basis vectors.

Fig. 2.3 Geometric representation of H_2 : single input. The column vector $\{X\}^H$ is projected onto the basis vector $\{Y\}^H$



First, the estimate $[\tilde{X}]^H$ must be calculated as a linear combination of the columns of $[Y]^H$, or $[\tilde{X}]^H = [Y]^H [C_X]$. The matrix $[C_X]$ can be solved for in a least-squares sense such that $[C] = ([Y] [Y]^H)^{-1} [Y] [X]^H$. Therefore

$$[\tilde{X}]^H = [Y]^H ([Y] [Y]^H)^{-1} [Y] [X]^H. \quad (2.8)$$

Since the columns of $[Y]^H$ are used as basis vectors, the matrix $[C_Y]$ is the identity matrix, and therefore

$$[\tilde{Y}]^H = [Y]^H. \quad (2.9)$$

The estimates can then be substituted into the equation for the FRF, $[\tilde{X}]^H [H_2]^H = [\tilde{Y}]^H$. Thus,

$$[Y]^H ([Y] [Y]^H)^{-1} [Y] [X]^H [H_2]^H = [Y]^H. \quad (2.10)$$

Using the same procedure as for H_1 , the final version of H_2 is

$$[H_2] = [S_{YY}] [S_{XY}]^{-1}. \quad (2.11)$$

Since an inverse is taken, the number of linearly independent columns in $[Y]^H$ must be equal to the number of linearly independent columns in $[X]^H$. Otherwise the $[S_{XY}]$ matrix is not invertible.

This can be represented graphically for a single reference as in Fig. 2.3. In this case, the vector $\{\tilde{X}\}^H$ is a projection of the columns of $\{X\}^H$ onto the vector $\{Y\}^H$.

2.3 Review of Singular Value Decomposition

Any complex rectangular matrix $[Z]$ can be decomposed such that

$$[U] [\Sigma] [V]^H = [Z] \quad (2.12)$$

where the left singular matrix $[U]$ and the right singular matrix $[V]$ are unitary; i.e., $[U]^H [U] = [I]$, and $[V]^H [V] = [I]$. The matrix $[\Sigma]$ is a diagonal matrix of real-valued singular values. The singular values represent the contribution of the columns

in $[U]$ and $[V]$ to the matrix $[Z]$. A useful application of SVD is that $[\Sigma]$ and a subset of columns of $[U]$ and $[V]$ can be used to estimate $[Z]$.

$$[\tilde{Z}] = [U_N] [\Sigma_N] [V_N]^H \quad (2.13)$$

where $[U_N]$ and $[V_N]$ are the columns of $[U]$ and $[V]$ associated with the N largest singular values and $[\Sigma_N]$ is a diagonal matrix with the N largest singular values.

2.3.1 SVD Relationship to an Eigenvalue Solution

The matrix $[Z]$ can be pre-multiplied by its Hermitian to obtain

$$[V] [\Sigma] [U]^H [U] [\Sigma] [V]^H = [Z]^H [Z]. \quad (2.14)$$

Note that $[U]^H [U] = [I]$, and therefore

$$[V] [\Sigma]^2 [V]^H = [Z]^H [Z]. \quad (2.15)$$

If the result is pre-multiplied by $[V]^H$ and then post-multiplied by $[V]$, the equation becomes

$$[V]^H [Z]^H [Z] [V] = [\Sigma]^2. \quad (2.16)$$

Note that the eigenvalues, $[\Lambda]$, and eigenvectors, $[\Phi]$, of $[Z]^H [Z]$ can be calculated such that

$$[\Phi]^H [Z]^H [Z] [\Phi] = [\Lambda] \quad (2.17)$$

where the eigenvectors can be scaled such that $[\Phi]^H [\Phi] = [I]$, and $[\Lambda]$ is a diagonal matrix.

Therefore, the eigenvectors, $[\Phi]$, and eigenvalues, $[\Lambda]$, of the matrix product $[Z]^H [Z]$ are equivalent to the matrix $[V]$ and the square of the singular value matrix $[\Sigma]$ of the original matrix $[Z]$, respectively. Thus,

$$[\Phi] = [V], \quad \text{and} \quad [\Lambda] = [\Sigma]^2. \quad (2.18)$$

One can calculate the singular values or eigenvalues of $[Z]^H [Z]$ to obtain $[V]$ and $[\Sigma]^2$ and then use the relationship

$$[U] = [Z] [V] [\Sigma]^{-1} \quad (2.19)$$

to obtain the basis vectors $[U]$.

2.4 Derivation of H_{SVD}

The use of singular value decomposition to estimate FRF matrices is equivalent to using a total least-squares approach [8]. Since the response vectors in $[Y]^H$ should be a linear combination of the reference vectors in $[X]^H$, the total number of independent vectors of the combined set of $[X]^H$ and $[Y]^H$ should be equal to the total number of independent inputs. To find the dominant vectors associated with both the response and reference, one can assemble both reference and response vectors into a full matrix,

$$[W]^H = \begin{bmatrix} [X]^H & [Y]^H \end{bmatrix} = [U_W] [\Sigma_W] [V_W]^H, \quad (2.20)$$

and then solve for its singular values. Assume the matrices are sorted by decreasing singular value such that the largest singular value is first; these matrices can be partitioned into the first N_{ref} columns, represented by the subscript R , and the remaining columns that are represented by the subscript O . The matrices can then be partitioned as follows:

$$[U_W] = \begin{bmatrix} [U_{WR}] & [U_{WO}] \end{bmatrix}, \quad [V_W] = \begin{bmatrix} [V_{WR}] & [V_{WO}] \end{bmatrix}, \quad \text{and} \quad [\Sigma_W] = \begin{bmatrix} [\Sigma_{WR}] & [0] \\ [0] & [\Sigma_{WO}] \end{bmatrix} \quad (2.21)$$

Furthermore, $[V_W]$ can be partitioned into rows corresponding to the references and responses such that

$$[V_W] = \begin{bmatrix} [V_{WR}] & [V_{WO}] \end{bmatrix} = \begin{bmatrix} [V_{WXR}] & [V_{WYO}] \\ [V_{WYR}] & [V_{WYO}] \end{bmatrix}. \quad (2.22)$$

Since there are N_{REF} references, $[W]^H$ can be represented by the first N_{REF} singular value vectors. Thus,

$$[\tilde{W}]^H = [U_{WR}] [\Sigma_{WR}] [V_{WR}]^H. \quad (2.23)$$

Expanding this equation yields

$$\begin{bmatrix} [\tilde{X}]^H & [\tilde{Y}]^H \end{bmatrix} = [U_{WR}] [\Sigma_{WR}] \begin{bmatrix} [V_{WXR}]^H & [V_{WXR}]^H \end{bmatrix}. \quad (2.24)$$

Therefore,

$$[\tilde{X}]^H = [U_{WR}] [\Sigma_{WR}] [V_{WXR}]^H, \quad \text{and} \quad [\tilde{Y}]^H = [U_{WR}] [\Sigma_{WR}] [V_{WYR}]^H. \quad (2.25)$$

The estimated vectors can be geometrically viewed as shown in Fig. 2.4. In this case, the estimates of the reference and response vectors are projections to the basis vectors calculated from the singular value decomposition.

Note that $[\tilde{X}]^H [H_{SVD}]^H = [\tilde{Y}]^H$. Substituting in the estimates of $[X]^H$ and $[Y]^H$ leads to an estimate of $[H_{SVD}]$ such that

$$[U_{WR}] [\Sigma_{WR}] [V_{WXR}]^H [H_{SVD}]^H = [U_{WR}] [\Sigma_{WR}] [V_{WXR}]^H. \quad (2.26)$$

Pre-multiplying by $[\Sigma_{WR}]^{-1} [U_{WR}]^H$ and taking the Hermitian leads to

$$[H_{SVD}] = [V_{WYR}] [V_{WXR}]^{-1}. \quad (2.27)$$

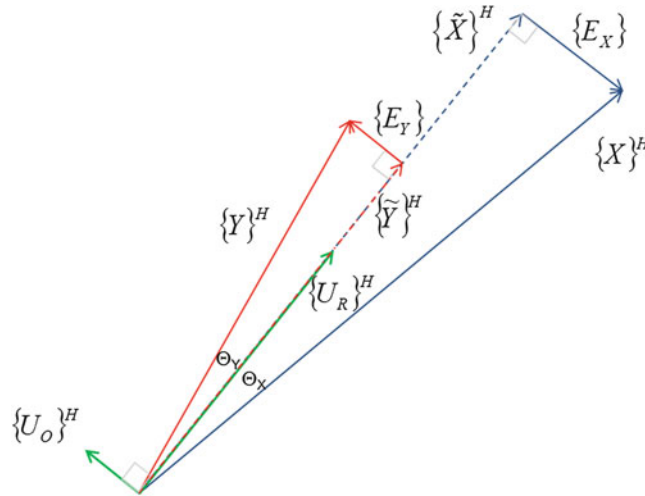


Fig. 2.4 Geometric representation of H_{SVD} . The column vectors of $[X]^H$ and $[Y]^H$ are projected onto the singular value basis vectors $[U]$

Since the left singular vectors $[U_W]$ are not used explicitly, it is more efficient to calculate the singular values (or eigenvalues) of $[W][W]^H$ to obtain the right singular vectors $[V_W]$. Thus,

$$[S_{WW}] = [W][W]^H = \begin{bmatrix} [X][X]^H & [X][Y]^H \\ [Y][X]^H & [Y][Y]^H \end{bmatrix} = \begin{bmatrix} [S_{XX}] & [S_{XY}] \\ [S_{YX}] & [S_{YY}] \end{bmatrix} = [V_W][\Sigma_W]^2[V_W]^H \quad (2.28)$$

The procedure for calculating $[H_{SVD}]$ is as follows:

1. Calculate $[S_{XX}]$, $[S_{YY}]$, and $[S_{YX}]$ and assemble them into $[S_{WW}]$ from Eq. (2.28).
2. Calculate the left singular vectors $[V_{WR}]$ from $[S_{WW}]$.
3. Partition $[V_W]$ into $[V_{WR}]$ and $[V_{WO}]$.
4. Calculate $[H_{SVD}]$ from Eq. (2.27).

2.4.1 SVD Using Partial Set of Reference and Response Channels as Basis Vectors

The previous cases show that FRFs can be calculated using different sets of vectors as basis vectors: H_1 uses the references as basis vectors, H_{SVD} uses all reference and response vectors to calculate the basis vectors, and the generalized form of H_2 uses all of the response vectors to calculate the basis vectors. The most general case is where a partial set of reference and response vectors is used to estimate the basis vectors.

The first step is to partition both the references and responses into those that are part of the singular value decomposition, identified by the subscript “P” and hereafter called the partitioned set or P-set, and those that are not part of the singular value decomposition, which are identified by the subscript “Q” and hereafter called the Q-set.

The results of the singular value decomposition on the partitioned set are used to calculate the estimated values for the responses and references that are part of the singular value decomposition. The references and responses not part of the singular value decomposition are calculated as a least-squares fit of the basis vectors, and then the contribution of the basis vector is removed by inverting the results of the singular value decomposition.

First, the reference and response channels are partitioned into P-set and Q-set such that

$$[X]^H = \begin{bmatrix} [X_P]^H & [X_Q]^H \end{bmatrix}, \quad \text{and} \quad [Y]^H = \begin{bmatrix} [Y_P]^H & [Y_Q]^H \end{bmatrix}. \quad (2.29)$$

The vectors that are part of the singular value decomposition can be assembled as

$$[W_P]^H = \begin{bmatrix} [X_P]^H & [Y_P]^H \end{bmatrix}. \quad (2.30)$$

Also, the vectors that are not part of the singular value decomposition can be assembled as

$$[W_Q]^H = \begin{bmatrix} [X_Q]^H & [Y_Q]^H \end{bmatrix}. \quad (2.31)$$

Singular value decomposition of $[W_P]^H$ leads to

$$[U_{W_P}][\Sigma_{W_P}][V_{W_P}]^H = [W_P]^H = \begin{bmatrix} [X_P]^H & [Y_P]^H \end{bmatrix}. \quad (2.32)$$

The matrices can be partitioned into the first N_{REF} columns, represented by the subscript “R,” and the remaining columns, represented by the subscript “O.”

$$[U_{W_P}] = \begin{bmatrix} [U_{W_P R}] & [U_{W_P O}] \end{bmatrix}, \quad \text{and} \quad [V_{W_P}] = \begin{bmatrix} [V_{W_P R}] & [V_{W_P O}] \end{bmatrix} \quad (2.33)$$

Furthermore, the matrix $[V_{W_P R}]$ can be partitioned into rows corresponding to the reference and response signals such that

$$[V_{W_P R}] = \begin{bmatrix} [V_{W_P R X}] \\ [V_{W_P R Y}] \end{bmatrix}. \quad (2.34)$$

The estimate of the partitioned channels used for singular values can be written as

$$\begin{aligned} [\tilde{W}_P]^H &= [U_{WPR}] [\Sigma_{WPR}] [V_{WPR}]^H \\ &= [U_{WPR}] [\Sigma_{WPR}] [V_{WPRX}]^H [V_{WPRY}]^H \\ &= [\tilde{X}_P]^H [\tilde{Y}_P]^H. \end{aligned} \quad (2.35)$$

From this, the following relationships can be extracted:

$$[U_{WPR}] = [\tilde{W}_P]^H [V_{WPR}] [\Sigma_{WPR}]^{-1}, \quad [\tilde{X}_P]^H = [U_{WPR}] [\Sigma_{WPR}] [V_{WPRX}]^H, \quad \text{and} \quad [\tilde{Y}_P]^H = [U_{WPR}] [\Sigma_{WPR}] [V_{WPRY}]^H. \quad (2.36)$$

Estimates for the references and responses that were not part of the singular value decomposition can be determined by performing a least-squares fit of the vectors and then substituting the basis vectors with measured channels in the partitioned sets.

$$\begin{aligned} [\tilde{X}_Q]^H &= [U_{WPR}] [U_{WPR}]^H [X_Q]^H \\ &= [U_{WPR}] [S_{WPR}]^{-1} [V_{WPR}]^H [W_P] [X_Q]^H \\ &= [U_{WPR}] [\Sigma_{WPR}]^{-1} [V_{WPR}]^H [S_{WPRX_Q}]. \end{aligned} \quad (2.37)$$

Likewise,

$$[\tilde{Y}_Q]^H = [U_{WPR}] [\Sigma_{WPR}]^{-1} [V_{WPR}]^H [S_{WPRY_Q}]. \quad (2.38)$$

With the estimates for all references and all responses complete, the next step is to assemble them into the estimate for the FRF matrix from the following equation:

$$[\tilde{X}]^H [H_{SVDp}]^H = [\tilde{Y}]^H. \quad (2.39)$$

Direct substitution of the partitioned matrices leads to

$$[\tilde{X}_P]^H [\tilde{X}_Q]^H [H_{SVDp}]^H = [\tilde{Y}_P]^H [\tilde{Y}_Q]^H. \quad (2.40)$$

Direct substitution of the estimates leads to

$$\begin{aligned} &[[U_{WPR}] [\Sigma_{WPR}] [V_{WPRX}]^H [U_{WPR}] [\Sigma_{WPR}]^{-1} [V_{WPR}]^H [S_{WPRX_Q}]] [H_{SVDp}]^H \\ &= [[U_{WPR}] [\Sigma_{WPR}] [V_{WPRY}]^H [U_{WPR}] [\Sigma_{WPR}]^{-1} [V_{WPR}]^H [S_{WPRY_Q}]]. \end{aligned} \quad (2.41)$$

After pre-multiplying both sides by $[U_{WPR}]^H$, the Hermitian can be taken to obtain

$$[H_{SVDp}] = \begin{bmatrix} [V_{WPRY}] [\Sigma_{WPR}] \\ [S_{Y_Q W_P}] [V_{WPR}] [\Sigma_{WPR}]^{-1} \end{bmatrix} \begin{bmatrix} [V_{WPRX}] [\Sigma_{WPR}] \\ [S_{X_Q W_P}] [V_{WPR}] [\Sigma_{WPR}]^{-1} \end{bmatrix}^{-1}. \quad (2.42)$$

The required cross-spectral matrices for this calculation are $[S_{W_P W_P}]$ and $[S_{W_P W_Q}]$.

The procedure for calculating the FRF matrix H_{SVDp} using a subset of channels to form the basis vectors is as follows:

1. Calculate the matrices $[S_{W_P W_P}]$ and $[S_{W_Q W_P}]$.
2. Calculate the singular value matrices $[V_{WPR}]$ and $[\Sigma_{WPR}]$ from $[S_{W_P W_P}]$.
3. Calculate the estimates for the reference and response channels from Eqs. (2.36), (2.37), and (2.38).
4. Perform the calculation for $[H_{SVDp}]$ from Eq. (2.42).

2.4.2 Generalized H_2

The H_2 estimator can be generalized using singular value decomposition by performing a singular value decomposition on all of the output channels and no input channels such that

$$[W_{H2}]^H = [Y]^H. \quad (2.43)$$

The singular value decomposition of the matrix $[Y]^H$ leads to

$$[U_{H2}] [\Sigma_{H2}] [V_{H2}]^H = [Y]^H \quad (2.44)$$

and the partition of the N_{REF} highest singular values as

$$[U_{H2R}] [\Sigma_{H2R}] [V_{H2R}]^H = [\tilde{Y}]^H. \quad (2.45)$$

In this case, the reference channels are not used in the formation of the basis vectors; therefore $[X]^H = [X_Q]^H$. Also, all response channels are used to calculate the basis vectors, so $[Y]^H = [Y_P]^H$. Making the proper substitutions into Eq. (2.42) leads to the equation for a generalized version of the H_2 estimator as

$$[H_2] = [V_{H2R}] [\Sigma_{H2R}] \left[[S_{XY}] [V_{H2R}] [\Sigma_{H2R}]^{-1} \right]^{-1}. \quad (2.46)$$

The procedure for calculating the generalized version of H_2 is as follows:

1. Calculate the matrices $[S_{XY}]$ and $[S_{YY}]$.
2. Calculate the singular value matrices $[V_{H2}]$ and $[\Sigma_{H2}]$ from $[S_{YY}]$.
3. Perform the calculation for $[H_2]$ from Eq. (2.46).

Note that the same two matrices $[S_{XY}]$ and $[S_{YY}]$ are used in the generalized version of H_2 as are used in the original version of H_2 .

2.4.3 Review of H_V

The H_V estimator assumes noise in both the input and the output channels [2]. The row of the FRF matrix associated with a given channel “i” is found by solving the following eigenvalue problem:

$$\begin{bmatrix} [S_{XX}] & \{S_{XY_i}\} \\ \{S_{Y_iX}\} & S_{Y_iY_{il}} \end{bmatrix} = [V] [\Lambda] [V]^H \begin{bmatrix} [V_{XR}] & \{V_{XO}\} \\ \{V_{Y_iR}\} & V_{Y_iO} \end{bmatrix} \begin{bmatrix} [\Lambda_R] & \{0\} \\ \{0\}^T & \Lambda_O \end{bmatrix} \begin{bmatrix} [V_{XR}] & \{V_{XO}\} \\ \{V_{Y_iR}\} & V_{Y_iO} \end{bmatrix}^H \quad (2.47)$$

where the eigenvectors and eigenvalues are the matrices $[V]$ and $[\Lambda]$, respectively. These matrices can be further partitioned as well. The subscript “R” represents the eigenvectors associated with the largest NREF eigenvalues, the subscript “O” corresponds to the lowest eigenvalue, and the subscripts “X” and “Y_i” correspond to the references and the response of a single channel “i,” respectively. The *i*th row of the FRF matrix is then solved for as follows:

$$\{H_{Vi}\} = -V_{YiO}^{-H} [V_{XO}]^H. \quad (2.48)$$

2.4.4 Equivalence of H_{SVD} and H_V

The difference in the solution between the H_{SVD} and H_V estimators is that one uses the eigenvectors associated with the largest eigenvalues and the other uses the eigenvectors associated with the lowest eigenvalues. Their equivalence can be shown by investigating the matrix formed by the identity $[V_W]^H [V_W] = [I]$. This matrix can be partitioned as

$$\begin{bmatrix} [V_{WXR}] & [V_{WYO}] \\ [V_{WYR}] & [V_{WYO}] \end{bmatrix}^H \begin{bmatrix} [V_{WXR}] & [V_{WYO}] \\ [V_{WYR}] & [V_{WYO}] \end{bmatrix} = \begin{bmatrix} [I_{RR}] & [0] \\ [0] & [I_{OO}] \end{bmatrix}. \quad (2.49)$$

The left-hand side can be expanded to

$$\begin{bmatrix} [V_{WXR}]^H [V_{WXR}] + [V_{WYR}]^H [V_{WYR}] & [V_{WXR}]^H [V_{WYO}] + [V_{WYR}]^H [V_{WYO}] \\ [V_{WYO}]^H [V_{WXR}] + [V_{WYO}]^H [V_{WYR}] & [V_{WYO}]^H [V_{WYO}] + [V_{WYO}]^H [V_{WYO}] \end{bmatrix} = \begin{bmatrix} [I_{RR}] & [0] \\ [0] & [I_{OO}] \end{bmatrix} \quad (2.50)$$

The lower-left side can be broken out to obtain

$$[V_{WYO}]^H [V_{WXR}] + [V_{WYO}]^H [V_{WYR}] = [0]. \quad (2.51)$$

After post-multiplying by $[V_{WXR}]^{-1}$ and pre-multiplying by $[V_{WYO}]^{-H}$, the equation becomes

$$[V_{WYR}] [V_{WXR}]^{-1} + [V_{WYO}]^{-H} [V_{WYO}]^H = [0]. \quad (2.52)$$

Moving the second term to the right-hand side yields the following relationship:

$$[H_{SVD}] = [V_{WYR}] [V_{WXR}]^{-1} = -[V_{WYO}]^{-H} [V_{WYO}]^H. \quad (2.53)$$

If the singular value decomposition is applied to a single output channel, then the equation above is equivalent to the solution for H_V .

There are two important items to note. The FRF matrix is more efficiently calculated using the smallest eigenvalues if the number of responses is less than the number of references, which is the case with the H_V estimator where the FRF is calculated one response degree of freedom at a time. However, if all response channels are calculated at once, then the FRF matrix is more efficiently calculated using the H_{SVD} method.

Second, since the two solutions are equivalent, the basis vectors used in H_V are determined from the singular value decomposition of $[X]^H [Y_i]^H$. Note that the basis vectors change for each row of $[H_V]$ since the eigenvalue solution is calculated for each response.

2.5 Multiple Coherence

Multiple coherence is defined as the correlation coefficient describing the linear relationship between an output and all known inputs [9, 10]. The references used to calculate FRFs are not necessarily the inputs, and the responses are not necessarily the outputs. However, multiple coherence is used to assess the linear relationship between a single response and all measured references as well. The definition of multiple coherence is

$$MCOH_{Y_i} = \frac{\{H_{Y_iX}\} [S_{XX}] \{H_{Y_iX}\}^H}{S_{Y_iY_i}}, \quad (2.54)$$

where $\{H_{Y_iX}\}$ is the i th row of the FRF matrix. This definition works well for the H_1 estimator because the basis vectors are the reference channels. However, if we use this definition for other FRF estimators, then multiple coherence values can be greater than one. The matrix $[S_{XX}]$ can be replaced with $[X][X]^H$, and if we note that the basis vectors for H_1 are equal to the references such that $[\tilde{X}]^H = [X]^H$, and note that $[\tilde{X}]^H [H]^H = [\tilde{Y}]^H$, then multiple coherence can be expanded to

$$MCOH_{Y_i} = \frac{\{H_{Y_iX}\} [\tilde{X}] [\tilde{X}]^H \{H_{Y_iX}\}^H}{S_{Y_iY_i}} = \frac{[\tilde{Y}_i] [\tilde{Y}_i]^H}{S_{Y_iY_i}} = \frac{S_{\tilde{Y}_i\tilde{Y}_i}}{S_{Y_iY_i}}. \quad (2.55)$$

If this definition is used, then no matter which FRF estimator is used, multiple coherence will range from 0 to 1. Figs. 2.2, 2.3, and 2.4 show that multiple coherence is the square of the cosine of the angle between the estimated and measured response [11]. This definition is valid for reference channels as well, since both reference and response estimates are used to calculate the FRF matrix. If we group the references and responses into a single matrix, $[W]^H$, we can define multiple coherence as

$$MCOH_{W_i} = \frac{\{\tilde{W}_i\} \{\tilde{W}_i\}^H}{\{W_i\} \{W_i\}^H} = \frac{S_{\tilde{W}_i \tilde{W}_i}}{S_{W_i W_i}}. \quad (2.56)$$

Since the calculation of any FRF matrix involves calculating the estimated reference and response vectors, the calculation of multiple coherence is straightforward and can be performed at the same time as the calculation of the FRF matrix.

2.6 Example

The FRF estimation methodology was investigated on a set of test data where traditional FRF estimators were known to produce non-optimal results. The test data was collected during a previous investigation on estimating fixed base modes by using boundary degrees of freedom as references when calculating FRFs [12]. In that investigation it was shown that the H_V method was superior to the H_1 method. However, in some cases neither method adequately estimated the FRF of low-responding channels at resonant frequencies.

The test article, shown in Fig. 2.5, was an aluminum beam vertically cantilevered from a base platform. The aluminum beam was 30.5 cm tall, 2.54 cm thick, and 7.562 cm wide. The beam was bolted to a plate 1.9 cm thick using two bolts and adhered to the plate using superglue in an attempt to mitigate any nonlinearities at the connection between the plate and the beam. The Y-axis in this example is defined as the vertical direction along the length of the beam, the X-axis is in line with the strong axis of the beam, and the Z-axis is in line with the weak axis of the beam. This plate was then bolted to a larger box beam in an attempt to simulate a seismic mass or a shake table. The test article was excited with an impact hammer at several locations, shown by arrows in Fig. 2.5, to generate time-history data. Six rigid body deformations, designated as node 9999 with directions “X+,” “Y+,” “Z+,” “RX+,” “RY+,” and “RZ+,” were estimated from accelerometers mounted on

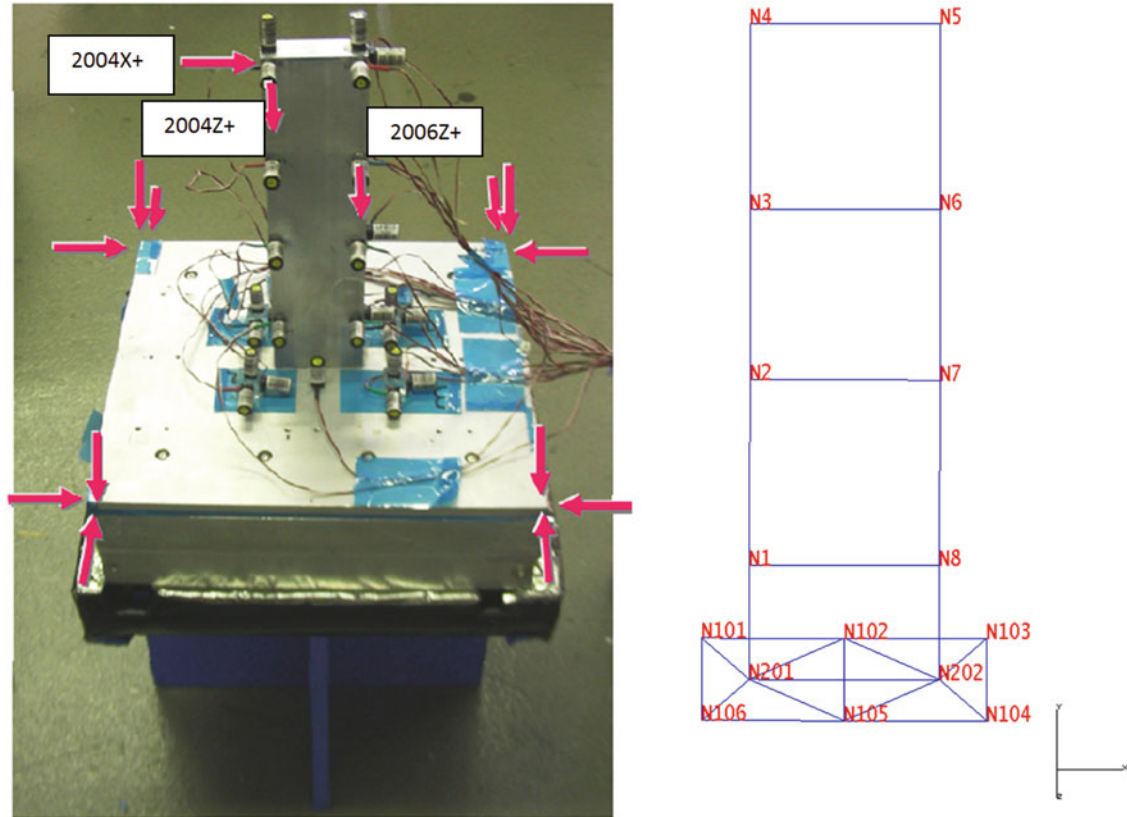


Fig. 2.5 Cantilevered beam mounted to rigid base on soft foam supports, with picture on *left* and test display model identifying test nodes on *right*. Rigid body motion of base plus three impact locations on beam were used as references to calculate FRF

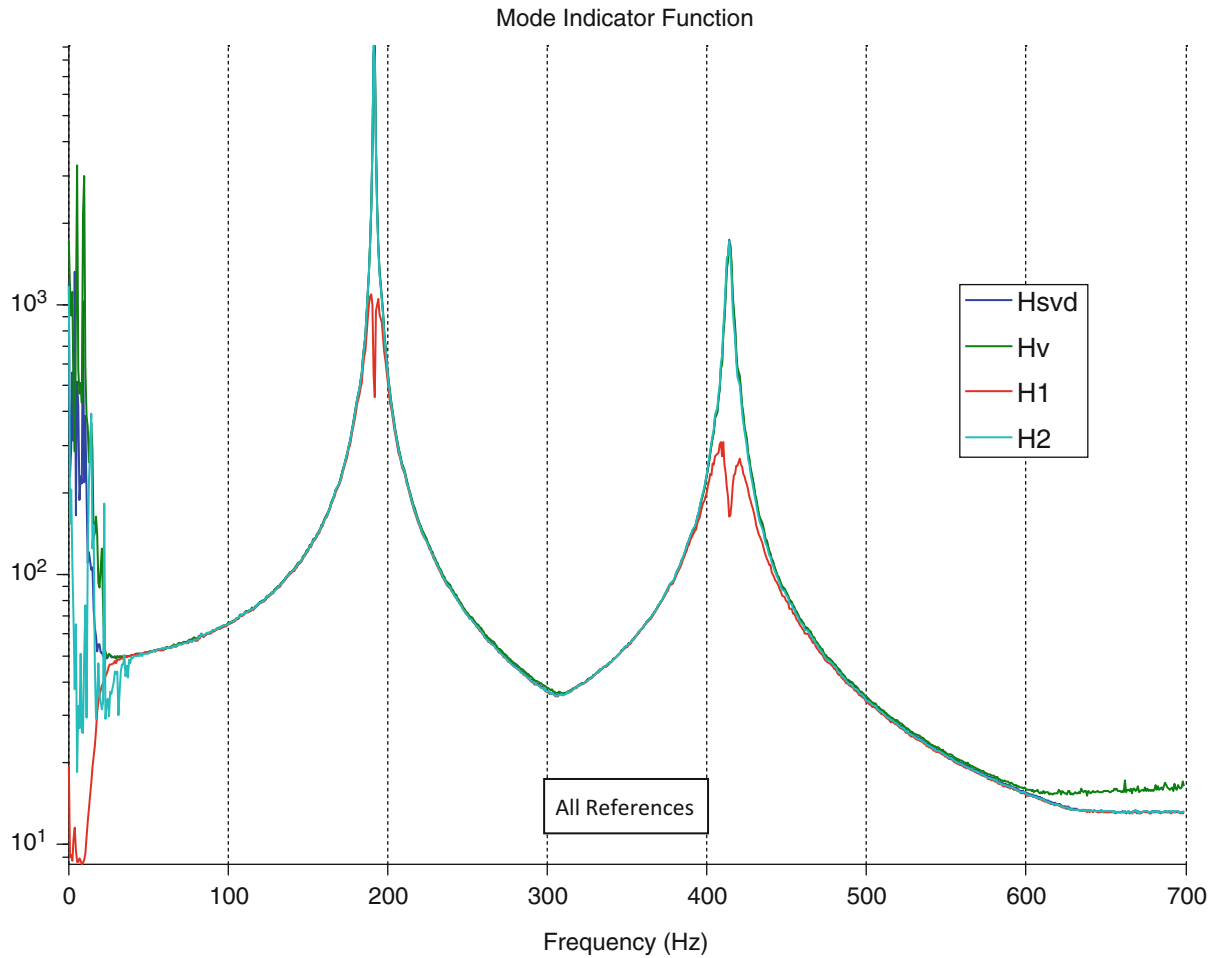


Fig. 2.6 Primary complex mode indicator function using H_{SVD} (blue), H_V (green), H_1 (red), and generalized H_2 (cyan)

the base. These six rigid body deformations plus the three forces applied to the beam were used as references when calculating FRFs, for a total of nine reference degrees of freedom. The three forces applied to the beam were near test nodes 4 and 6.

FRF matrices from the H_1 , generalized H_2 , H_V , and H_{SVD} methods were calculated and compared. The primary curve from the complex mode indicator function (CMIF) from the four estimators is presented in Fig. 2.6. Overall, the H_1 estimator provides a poor estimate of the FRF at resonance. At low frequencies, none of the estimators is able to adequately estimate the FRF. The estimators that are based on singular value decomposition overestimate the FRF. The H_1 estimator underestimates response, which should level out to a constant value of approximately 20 as the frequency approaches zero. The CMIFs were then recalculated using the FRF associated with the three force references and then using the FRF associated with the rigid body base accelerations as references. These CMIFs are shown in Fig. 2.7. The generalized H_2 method poorly estimates the response of the FRF associated with forces at low frequencies. This is likely due to the fact that there are only six independent motions of the structure at low frequencies—the six rigid body modes. To increase that number to the full set of nine independent references, the load cells would need to be part of the basis vector calculation.

The full CMIFs for the H_{SVD} and H_V methods are presented in Fig. 2.8. The CMIF associated with the H_{SVD} estimator is smooth for all singular values. However, the CMIF associated with H_V is not as well behaved near the second resonance at 410 Hz from the fourth-to-last singular value. This summary plot shows that the calculation of the basis vectors from all response coordinates is superior to calculating basis vectors one channel at a time.

Typical estimated FRFs are presented for all methods in Fig. 2.9. Overall, the generalized H_2 method fails at low frequencies, and the H_V method does a poor job of estimating low-responding channels at resonant frequencies. All SVD-based methods perform well when channel response is high. It is believed that the H_{SVD} method performs well near resonance for all channels because having all the channels available allows it to estimate a more accurate set of basis vectors.

A final case was run where one or two accelerometers were added to the references to calculate the basis vectors. The accelerometers added were located at test node 5 in the strong (X) and weak (Z) axes of the beam. Figure 2.10 presents

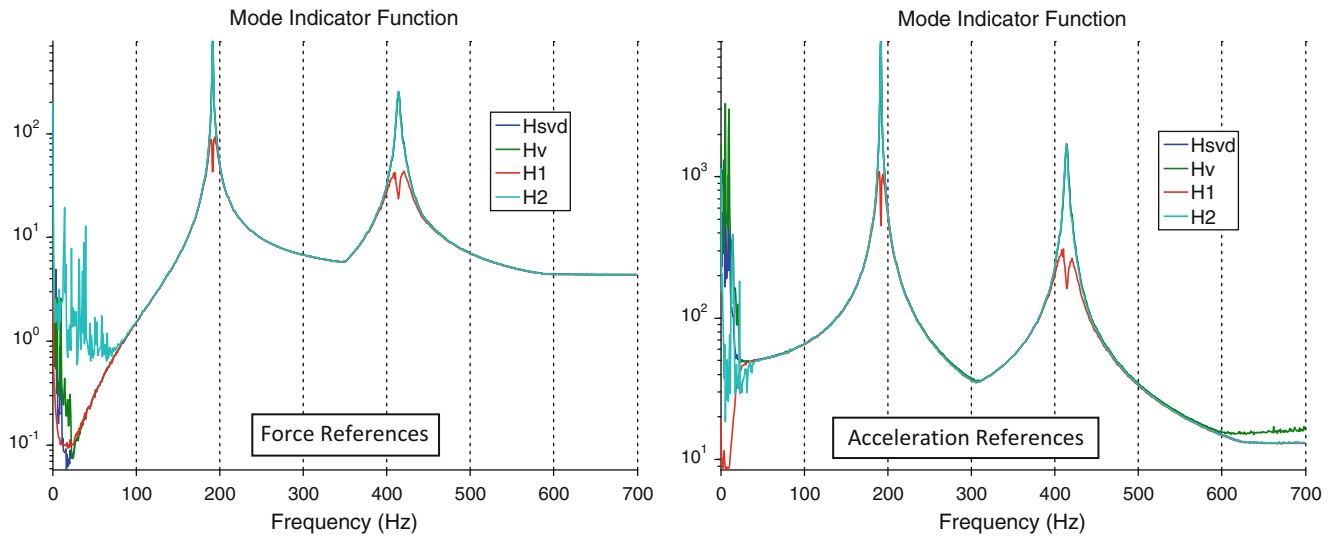


Fig. 2.7 Complex mode indicator function using H_{SVD} (black), H_V (red), H_1 (green), and generalized H_2 (blue), using FRFs associated with impact force references (left) and with rigid body accelerations (right)

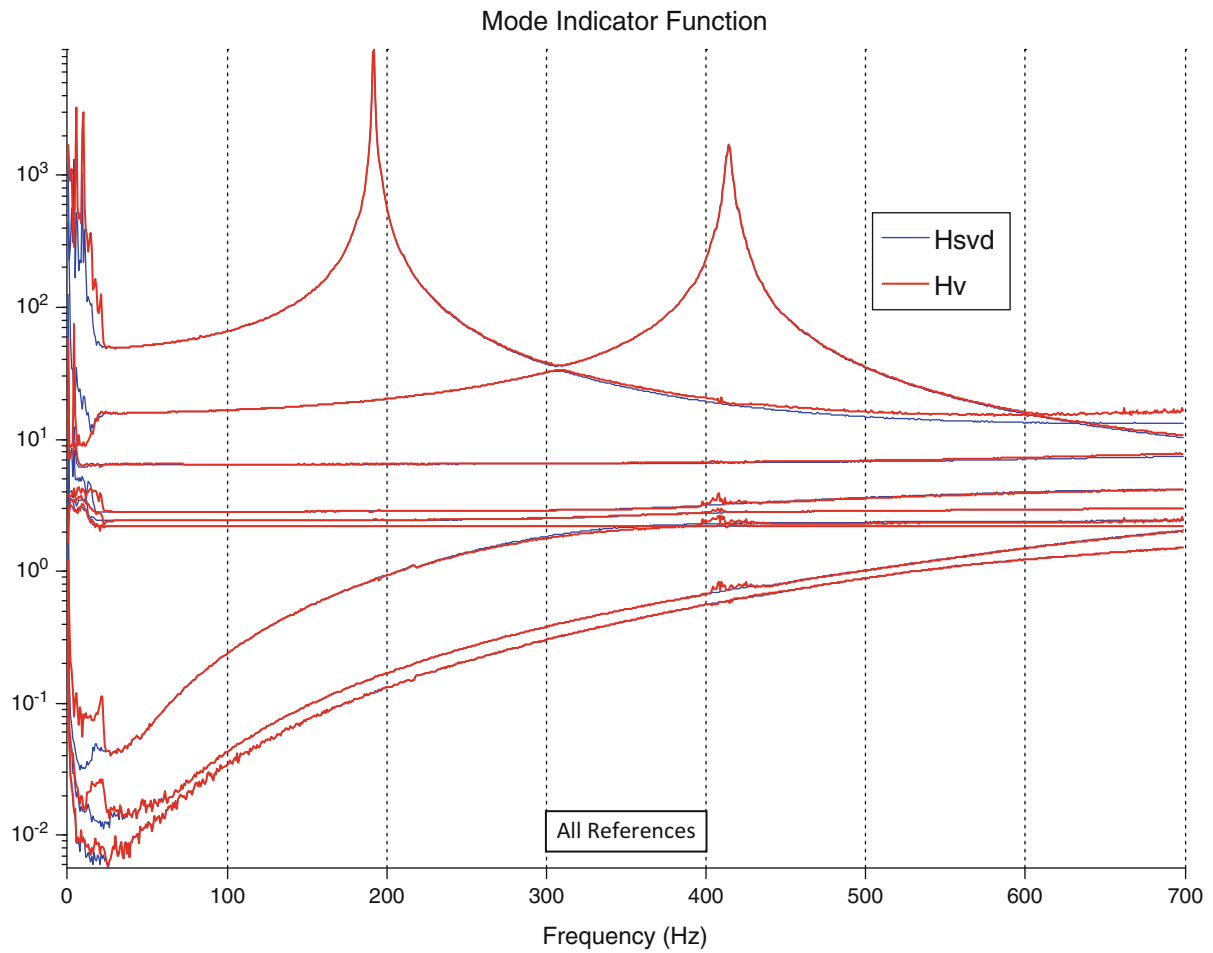


Fig. 2.8 Complex mode indicator function using H_{SVD} (blue) and H_V (red)

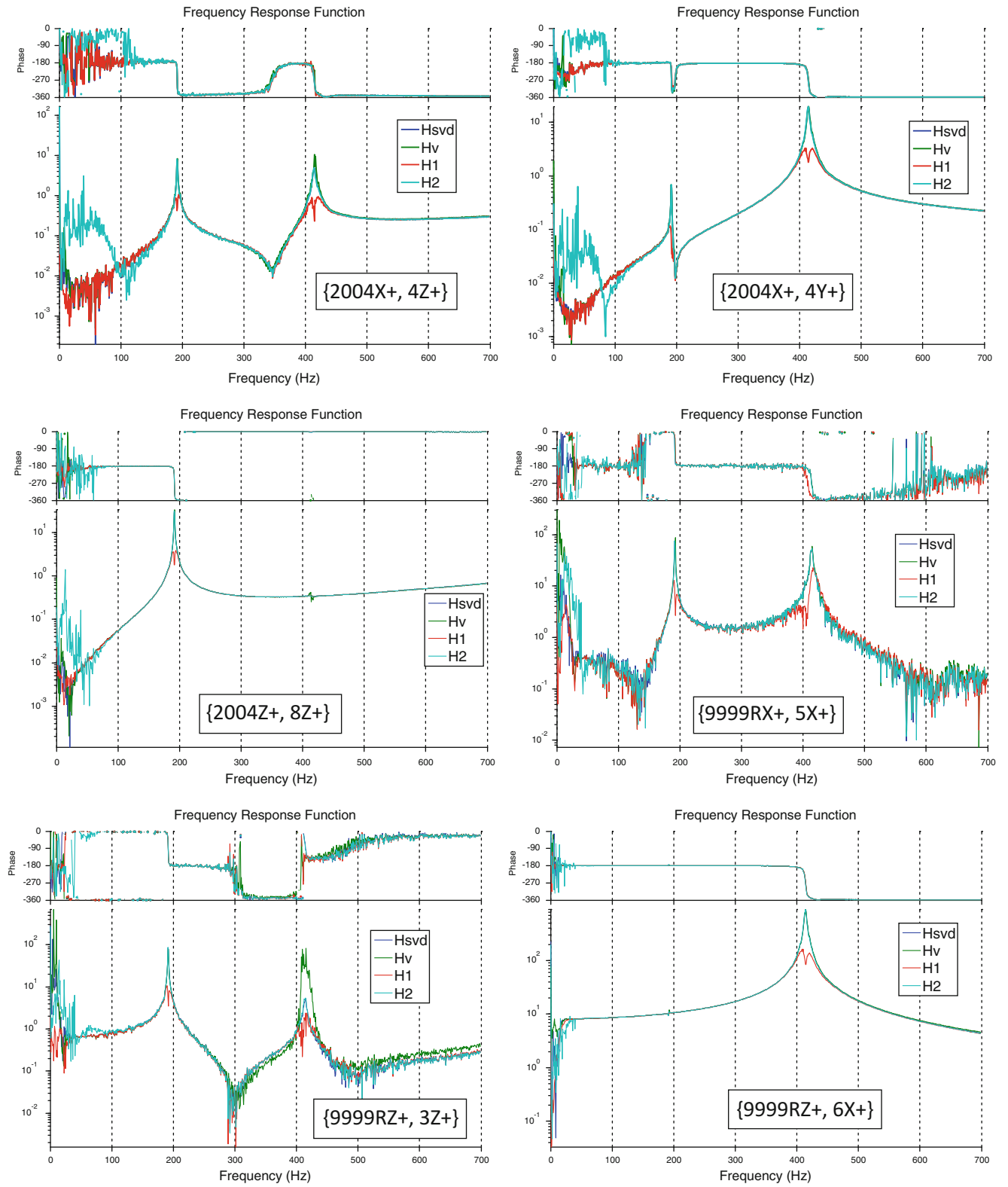


Fig. 2.9 Examples of calculated FRFs using H_{SVD} (blue), H_V (green), H_1 (red), and generalized H_2 (cyan)

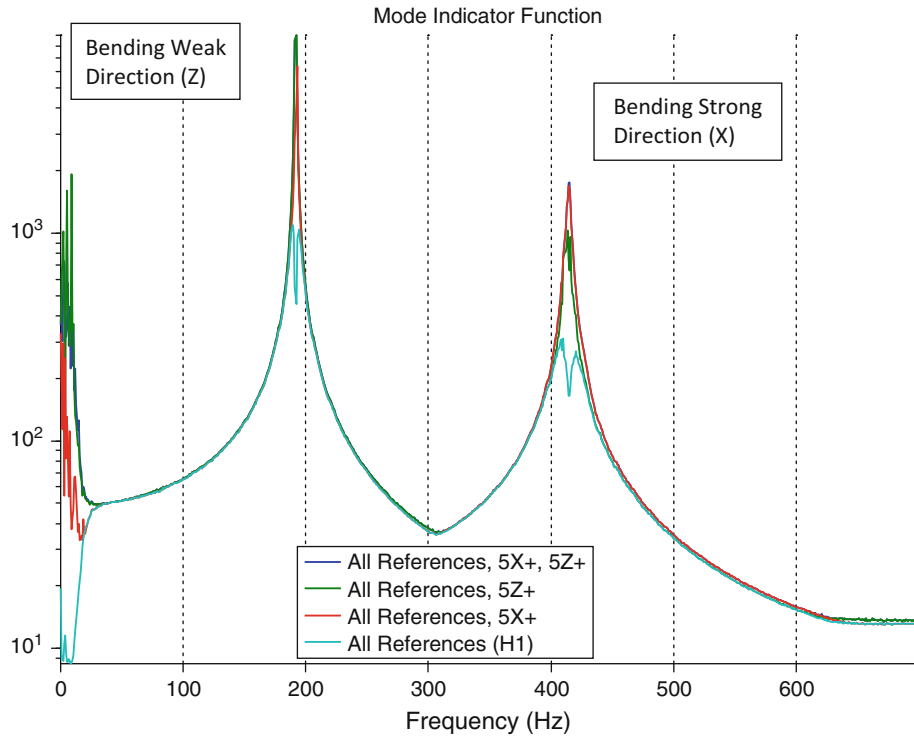


Fig. 2.10 Primary CMIF using all references and different response channels in singular value decomposition

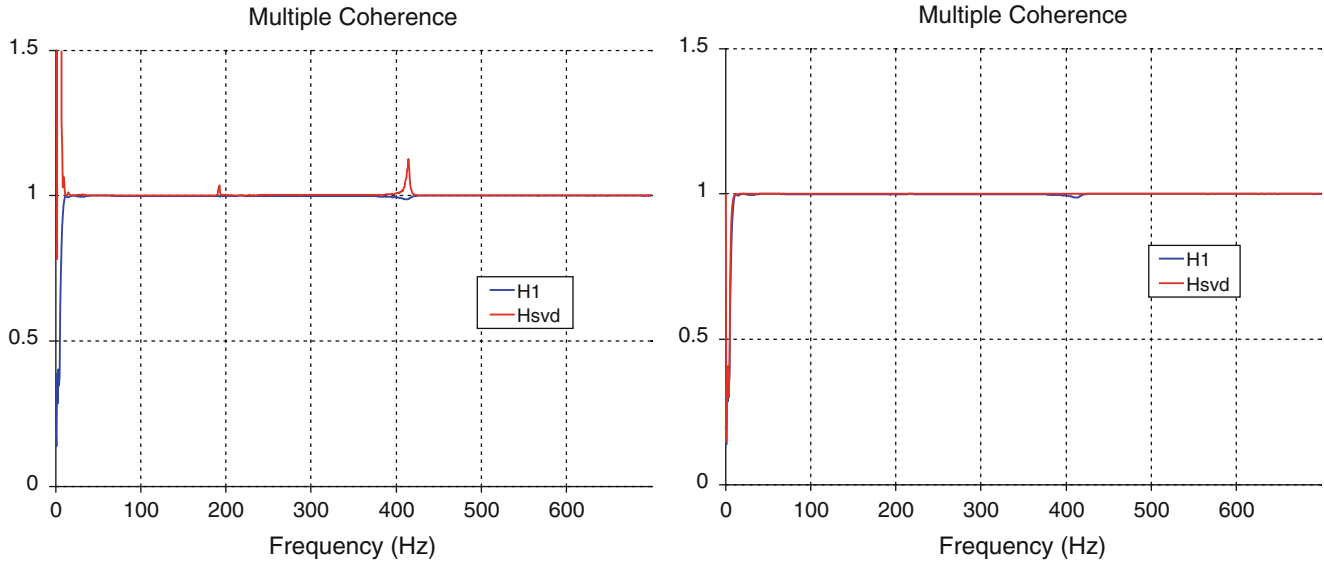


Fig. 2.11 Multiple coherence for response 6X+ using published definition (*left*) and proposed definition (*right*)

the primary CMIF for four cases: 5X+ and 5Z+, 5Z+, 5X+, and no accelerometers. Adding both accelerometers to the singular value decomposition allowed the singular value decomposition to calculate a valid set of basis vectors at resonance, and the resulting FRFs are nearly identical to adding all response channels to the singular value decomposition. There are clear deficiencies in the CMIF in the perpendicular direction when a single accelerometer is added, however. Finally, the results are equivalent to the H_1 estimator when no response channels are used to estimate the basis vectors.

Multiple coherence using the published definition (Eq. (2.54)), and the proposed definition (Eq. (2.56)) for H_1 and H_{SVD} is presented in Fig. 2.11. The published definition of multiple coherence holds for H_1 since the basis vectors in H_1 are the references. However, the published definition leads to a value greater than 1 when H_{SVD} is used because H_{SVD} uses a different set of basis vectors when calculating FRFs.

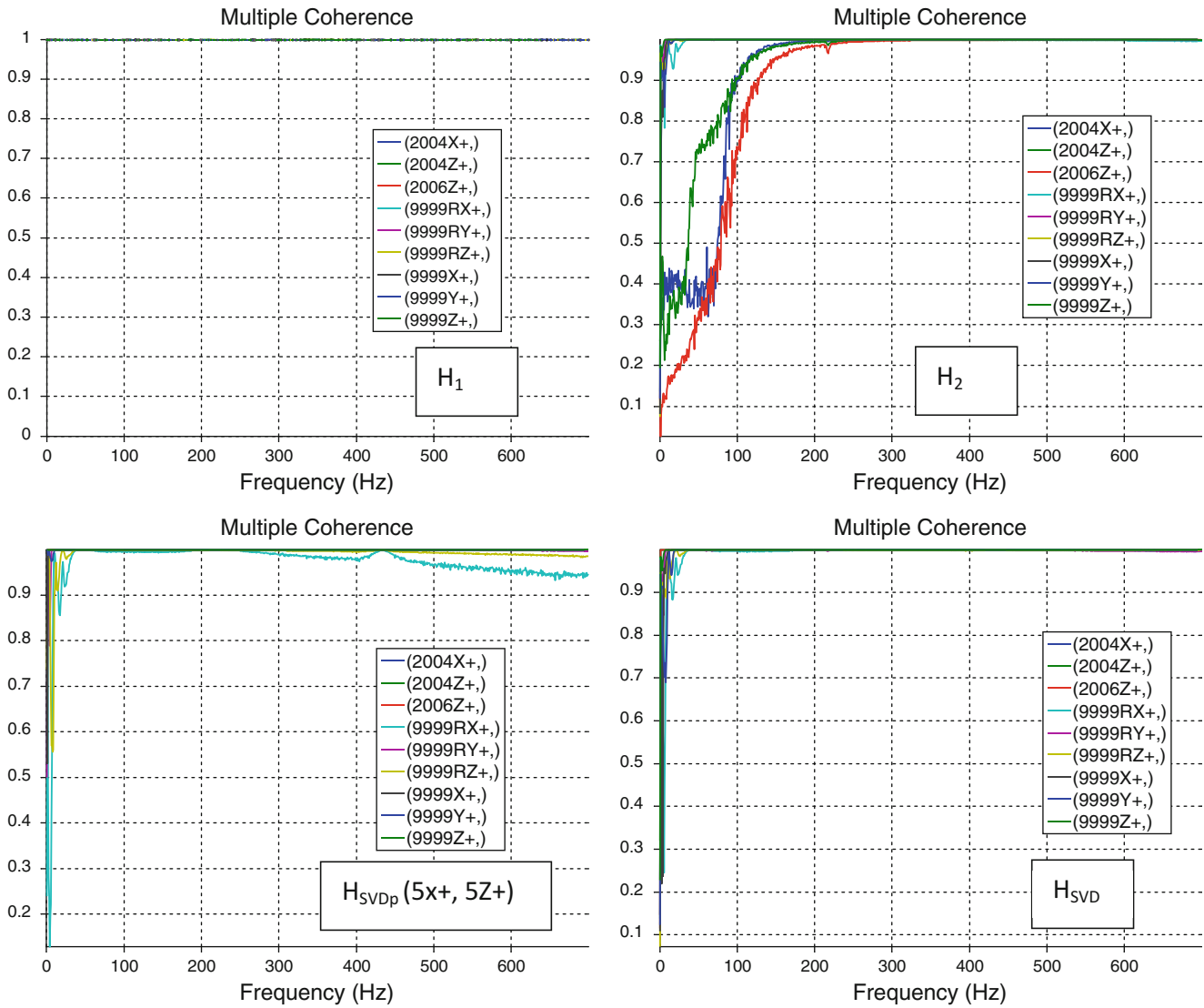


Fig. 2.12 Multiple coherence for all reference channels: H_1 (upper left), generalized H_2 (upper right), H_{SVDp} (lower left), and H_{SVD} (lower right).

Finally, multiple coherence for the reference channels is plotted in Fig. 2.12 for four cases: H_1 , generalized H_2 , H_{SVDp} using all references and $5X+$ and $5Z+$ to calculate the basis vectors, and H_{SVD} . Note that since the basis vectors of H_1 are identical to the references, the multiple coherence is exactly 1 and provides no insight. This uninteresting result is probably the reason multiple coherence is not calculated for reference channels. The multiple coherence results for the generalized H_2 function verify that the basis vectors do not adequately represent the load cell references (2004X+, 2004Y+, and 2006Z+) at low frequencies. The multiple coherence results using all references and a subset of responses are adequate, especially at the resonant frequencies, and adding all response channels to the calculation of the basis vectors improves the multiple coherence results at higher frequencies.

2.7 Summary

This paper has shown that all FRF estimation techniques based on linear algebra are derived using the same procedure:

1. Define a set of basis vectors.
2. Estimate the reference and response spectra from the set of basis vectors.
3. Use the estimated reference and response spectra to solve for the FRF matrix.

For the cases of H_1 and H_2 , the basis vectors are explicitly defined as columns of the reference or the response spectra, respectively. Singular value decomposition of a set of channels can also be used to define the basis vectors. The H_{SVD} method uses all the channels to compute a consistent set of basis vectors for all channels. Other methods such as the generalized H_2 or the H_{SVDp} method can also be defined that use singular value decomposition on a subset of channels to calculate the basis vectors. The H_V method stands apart from the other methods since it computes a different set of basis vectors for each output channel. As a consequence of developing this method, the definition of multiple coherence has been generalized to include all FRF estimators.

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