Preface

This book introduces computational proximity. Basically, **computational proximity** (CP) is an algorithmic approach to finding nonempty sets of points that are either close to each other or far apart. The basic notion of computational proximity draws its inspiration from the Preface written in 2009 by S.A. Naimpally in [1, pp. 23–28] and the Foreword in [2].

In CP, two types of near sets are considered, namely, spatially near sets and descriptively near sets. Spatially near sets contain points identified by their location and have at least one point in common. Descriptively near sets contain non-abstract points that have both locations and measurable features such as colour and gradient orientation. Connectedness, boundedness, mesh nerves, convexity, shapes and shape theory are principal topics in the study of nearness and separation of physical as well as abstract sets. CP has a hefty visual content. Applications of CP include computer vision, multimedia, brain activity, biology, social networks and cosmology.

CP leads to the study of structures in various forms in such things as sets of picture points in digital images or sets of nodes in location-based social networks. Typically in computational proximity, one starts with some form of proximity space (topological space equipped with a proximity relation) that has an inherent geometry.

A **topological space** is a nonempty set together with a collection of open sets that satisfy certain properties (for the details, see Appendix F.2). A **proximity space** is a topological space equipped with a proximity relation. Various forms of proximity relations are introduced in Sect. 1.4.

Using various algorithmic methods combined with instances of proximity spaces and computational geometry, it is possible to discover hidden objects and patterns in the selected mathematical structures. After that, one then highlights the presence of such things as connectedness, boundedness, bornologies, bornological nerves, mesh bornological nerves, convexity and shapes contained in such proximal structures, which are the focus of current research on proximity [3]. To see the importance of geometry in the study of connectedness in CP, recall what Stephen Willard succinctly observed: *The topological study of connected spaces is heavily geometric (or visual)* [4, §26, p. 191].

The region-based approach in CP represents a shift in focus from points to regions in the study of near sets. Both near (proximal) sets and classical, Zermelo–Fraenkel (ZF) sets are axiomatized. For various forms of proximity axioms, see Sect. 1.4 and Chap. 2. The ZF set theory axioms are given in Appendix E.

From a point-based perspective, sets of physical objects such as picture points are near sets, provided the sets contain points with matching descriptions. From a region-based perspective, sets of either abstract regions such as polygons in the Euclidean plane or picture regions are near sets, provided the regions have matching descriptions. This means that both purely geometric regions as well as regions occupied by sets of physical entities are susceptible to the computational proximity approach. Unlike points that have only location, regions in both types of sets of objects have features such as area, diameter, convex, concave and tessellated. Basically, region-based descriptions are helpful in detecting, characterizing, analyzing and classifying patterns in sets of objects that can be either purely geometric or physical.

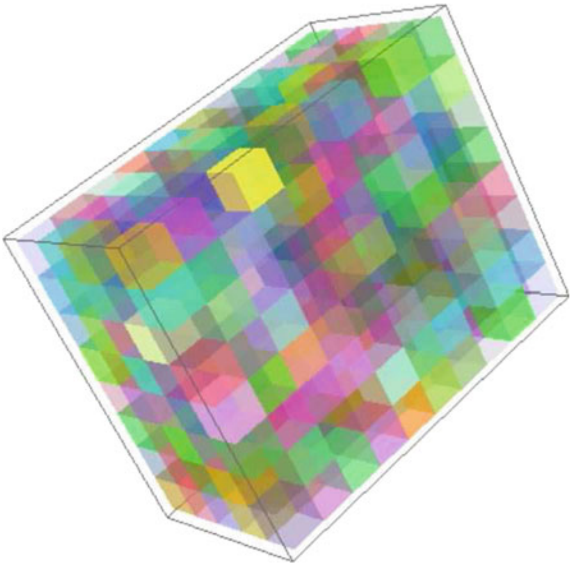

Fig. 1 $10 \times 8 \times 8 \times 3$ Array of cubes

Ultimately, computational proximity leads to various applications of topology in the study of set patterns in proximity spaces [5–7], new forms of topological spaces [6] and new forms of algebraic structures such as near groups [8–11] as well as leading to a new form of topology of digital images [12] useful in solving pattern recognition, analysis and classification problems.

CP utilizes the very rich collection of proximities such as Čech [13], Efremovič [14], Lodato [15], Wallman [16], and the more recent descriptive proximity [17–19]. Descriptive proximity grew out of a study of extensions of uniform topologies on metric spaces and descriptive forms of the traditional separation axioms. It is well known that an Efremovič proximity space [14] naturally defines a topology in the space [20]. A Leader uniform topology [21] is defined by finding all sets near each given set in a proximity space.

Computational proximity is important for the following reasons. Instantiation of proximities viewed in the context of computational geometry leads to the discovery of new forms of

(i) **Connectedness**. One need only look at a Delaunay triangulation on an annulus to find a connected space as well as collections of subsets that are strongly connected (See, e.g., Sect. 1.16). Connectedness leads to a study of adjacent as well as strongly connected mesh nerves (See, e.g. Sect. 1.17).

(ii) **Boundedness, Bornology and Bornological Nerves**. These structures are fundamental building blocks in CP (see, e.g. Sect. 1.18). A **bornology** is a boundedness $\mathscr{B}$ on a nonempty set X that it is a cover of X, i.e. $X \subseteq \mathscr{B}$. Examples of bornologies can be found collections of cubes in Euclidean 3D space such as the ones shown in Fig. 1.

(iii) **Proximal Boundedness and Proximal Boundedness Nerves**. In CP, these structures have inherent affinities that are fundamental in the detection and analysis of patterns in a topology of digital images as well as a topology of other sets of physical entities such as brain cells (see, e.g. Sect. 1.19).

(iv) **Mesh bornological nerves**. These structures tend to surround digital image as well as other forms of objects (see, e.g. Sects. 1.18, 1.19).

(v) **Local proximity spaces**. These structures provides a means of comparing, analyzing and classifying such things as mesh bornological nerves endowed with a proximity, which are not only local proximity spaces but also are a good source of tools in solving object recognition problems (see, e.g. Sect. 1.20).

(vi) **Convexity Structures**. A family of convex sets has the **convexity property**, provided the intersection of any number of sets in the family belongs to the family. In effect, a family of sets with the convexity property has a strong proximity. For more about this, see Sect. 1.7. An interest in convexity leads to the study of digital image tessellations in CP (see, e.g. Sect. 1.11).

(vii) **Homotopic Maps, Shapes and Borsuk–Ulam Theorem**. A pair of continuous maps f, g on a set X are homotopic, provided $f(X)$ continuously deforms into $g(X)$. It was K. Borsuk who first associated the geometric notion of shape and homotopies. For more about this, see Sect. 5.3.

(viii) **Manifolds**. A **manifold** is a topological space with added properties. Manifolds are very useful structures, especially if there is a need to solve dimensionality reduction problems. Digital image manifolds are a natural by-product of the excursions in the topology of digital images in this book. For more about this, see Sect. 8.1. ■

This book has been derived from my lectures in a graduate course on the topology of digital images taught over the past several years. Many of my students have provided important insights and valuable suggestions concerning topics in this book. I especially want to thank Binglin Li, Colin Gaudreau, Schail Younas, Clara Guadagni, Braden Cross, Dominic Villar, Doungrat Chitcharoen, Chido Uchime, Randima Hettiarachchi and Enoch A-iyeh for their comments and helpful observations. I especially want to thank Binglin Li for her work on Voronoï regions with a maximum number of neighbouring regions in a Dirichlet tessellation.

I am very grateful for the discussions I have had with Anna Di Concilio, Som Naimpally, Clara Guadagni, Zdzisław Pawlak, Andrzej Skowron, Mehmet Ali Öztürk, Mustafa Uçkun, Ebubekir İnan, Arturo Tozzi and Irakli Dochviri concerning a number of topics in this book.

The topics in this monograph introduce many forms of proximities with a computational flavour (especially, what has become known as the strong contact relation), many nuances of topological spaces, and point-free geometry that are a direct result of my discussions with Anna Di Concilio. Som Naimpally also introduced me to many aspects of the foundations of proximities and topology over a number of years. It was Som Naimpally who first suggested the importance of the proximity relation between digital images and original scenes represented in digital picture form. Both of these researchers have provided a great number of insights concerning these topics that underlie the foundations of computational proximity. My discussions with Clara Guadagni and Anna Di Concilio led to the introduction and deeper understanding of the various forms of strong proximity [Di Concilio strong contact] (see, e.g. [22–25]). I extend my thanks to Clara Guadagni for her many suggestions and corrections for the text.

Computational Proximity also has its roots in the closely related mathematics of rough sets and nearness approximation spaces. I have had many discussions concerning these topics with Zdzisław Pawlak and Andrzej Skowron as well with many others such as Jarosław Stepaniuk, Zbigniew Suraj, Ewa Orłowska, Lech Polkowski, Marcin Wolski, Piotr Wasilewski, L. Puzio, Jerzy W. Grzymała-Busse, Maria do C. Nicoletti, Roman Slowiński, Sankar K. Pal, Yiyu Yao, A. Czyżewski, B. Kostek, A. Gomolińska and G. Cattaneo. It was Zdzisław Pawlak who introduced me to rough sets and granular computing during FUZZ IEEE 1996 in New Orleans and it was Andrzej Skowron who introduced me to many of the nuances of rough set theory and Som Naimpally's and B.D. Warrack's 1970 book on proximity spaces [26] during one of my visits to the Warsaw University. A series of articles on nearness approximation spaces grew out of my discussions with Andrzej Skowron and Jarosław Stepaniuk (see, e.g. [27–28]), leading to [29–31].

Another offshoot of the original work on near sets[1] is the work by Tiwari [34], Wasilewski [35], Wolski [36, 37], Gomolińska [29], Fashandi [38], Henry [39], Ramanna [40] and Meghdadi [41, 42].

Eventually, the work on nearness approximation spaces led to a number of papers on near algebraic structures such as groups and semigroups in nearness approximation spaces by E. İnan, M. Uçkun and M.A. Öztürk (see, e.g. [8–10, 43]).

[1]For an overview of near set theory, see https://en.wikipedia.org/wiki/Near_sets and [32, 33].

Many insights concerning the algebra component of CP stem from my discussions with Mehmet Ali Öztürk, Mustafa Uçkun and Ebubekir İnan. This has led to a number of publications (see, e.g. [44–47]). I extend my thanks to Ebubekir İnan for his many suggestions and corrections for the text.

CP has benefited hugely from my discussions with Arturo Tozzi about the various applications of the Borsuk–Ulam Theorem (BUT). It was Arturo Tozzi who pointed out the many important applications of BUT in the study of brain activity [48], quantum entanglement [49], fractals and power laws [50] and multisensory neurons [51].

It was Irakli Dochviri who suggested the study of topological sorts of near sets in proximity spaces [52, 53]. It is not enough to identify sets that are near each other either spatially or descriptively. For near set theory and computational proximity to be useful in many applications, it is necessary to determine to what extent one set A is a near a particular set B relative to other sets that are near the same set B. This problem is directly related to the work by A. Gomolińska and M. Wolski on graded nearness [29]. A very interesting counterpart of the near sets sorting problem is the far sets sorting problem: *To what extent are various sets far (remote, distant) from a particular set*? The solution of this problem leads to the identification of separated patterns found in collections of sets that are far from each other.

At various times, a number of researchers have provided important insights concerning topics in this book. Among these researchers, I extend *many thanks* (*Grazi mille, Çok Teşekküler*) to

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This book contains a generous selection of implementations of CP algorithms in Mathematica scripts and a few implementations in Matlab scripts. For the most part, these scripts are given in Appendix A.

Chapter problems have been classified. Those problems that begin with 🚲 are the kind you can run with, and probably will not take much time to solve. Problems that begin with ☕ are the kind you can probably solve in about the time it takes to drink a cup of tea or coffee. The remaining problems will need varying lengths of time to solve.

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